International Journal of Theoretical Physics, Vol. 44, No. 11, November 2005 (© 2005)
DOI: 10.1007/s10773-005-0342-8

An Algebra of Effects in the Formalism of Quantum Mechanics on Phase Space*

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Defining an addition of the effects in the formalism of quantum mechanics on phase space, we obtain a new effect algebra that is strictly contained in the effect algebra of all effects. A new property of the phase space formalism comes to light, namely that the new effect algebra does not contain any pair of noncommuting projections. In fact, in this formalism, there are no nontrivial projections at all. We illustrate this with the spin-1/2 algebra and the momentum/position algebra. Next, we equip this algebra of effects with the sequential product and get an interpretation of why certain properties fail to hold.

KEY WORDS: effect algebra; quantum mechanics on phase space.

PACS: 02.10.Gd, 03.65.Bz.

1. INTRODUCTION

An effect in quantum mechanics on a Hilbert space $\mathfrak{H}$ is an operator A on $\mathfrak{H}$ such that $0 \leq A \leq I$.

In the standard quantum mechanics, two effects, A and B, are added to get $A \oplus B$ iff $0 \leq A \oplus B \leq I$, where $A \oplus B$ is the operator addition. But if we have effects given with a complete interpretation of how to measure A and how to measure B, then the problem of how to measure $A \oplus B$ is vague or nonexistant, especially when A and B do not commute. The best we can say is that, statistically, $Tr(\rho[A \oplus B]) = Tr(\rho A) + Tr(\rho B)$, ρ a density operator. In this paper we will give a definition of A, B, and $A \oplus B$ in the formalism of quantum mechanics on phase space (Schroeck, 1996) in which an interpretation of $\oplus$ is transparent.

We begin with two situations: First, we take $\mathfrak{H} = \mathbb{C}^2$ for a spin-1/2 system, and $A = S_z^+$, $B = S_{(z+x)/2^{1/2}}^+$ where $S_u^{\pm}$ is the projection onto the set of vector states, ψ, such that $S_u^{\pm}\psi = \psi$. Note that $\{S_u^+, S_u^-\}$ is a projection valued measure, and thus contains only effects. What does $A \oplus B$ mean in this case? Note that

*This paper was a submission to the Fifth International Quantum Structure Association Conference (QS5), which took place in Cesena, Italy, March 31–April 5, 2001.

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$0 \leq A \oplus B \leq 2I$; so, we can take $aA \oplus bB$, $a + b \leq 1$, $a, b \geq 0$. Then we still ask what measurement do we perform to measure $aA \oplus bB$?

For our second situation, take $\mathfrak{H} = L^2(\mathbb{R})$, $A = E^P(\Delta_1)$, E^P being the spectral measure for P, the momentum, Δ_1 a Borel set in $\mathbb{R}$, and $B = E^Q(\Delta_2)$, E^Q being the spectral measure for the position, Q, etc. (Note: $E(\Delta) = \int_\Delta dE_\lambda$.) Thus, these two operators do not commute either, but they are effects, since they are both projections. We have again only $A \oplus B \leq 2I$, and to get anywhere in the usual formalism we would have to take $aA \oplus bB$, $a + b \leq 1$, $a, b \geq 0$. But what experiment would we have to perform to measure $aA \oplus bB$?

2. THE FORMALISM OF QUANTUM MECHANICS ON PHASE SPACE

In the formalism of quantum mechanics on phase space (Schroeck, 1996), an observable (a self-adjoint operator on $\mathfrak{H}$) is associated with a positive operator valued measure as follows:

i) Take all the generating observables of the theory. (Say, take S_x, S_y, S_z if the system has spin, take P_x, P_y, P_z, Q_x, Q_y, Q_z if the system has momentum and position, etc.) Form them into a Lie algebra. This may require that additional observables such as I may be needed in the Lie algebra in the case of the P's and Q's, etc. Let these observables be labelled $B_1, B_2, \ldots, B_n$. Then, for $\alpha_i \in \mathbb{R}$, $\alpha_1 B_1 + \cdots + \alpha_n B_n$ is also in the Lie algebra.

ii) From the Lie algebra, form the (connected) Lie group by obtaining

$$W(\alpha_1, \ldots, \alpha_n) = W(\alpha) = \exp\{i(\alpha_1 B_1 + \cdots + \alpha_n B_n)\}, \alpha_i \in \mathbb{R}.$$

$W(\alpha)$ is a unitary operator in $\mathfrak{H}$ since B_i is self-adjoint for each i.

iii) Take an $\eta \in \mathfrak{H}$, $\|\eta\| = 1$, and form

$$T^\eta(\alpha) = |W(\alpha)\eta\rangle\langle W(\alpha)\eta \mid .$$

Note the important property

$$T^\eta(\alpha)T^\eta(\beta) \neq 0, \quad \text{for} \quad \alpha \neq \beta,$$

as

$$\langle W(\alpha)\eta \mid W(\beta)\eta\rangle \neq 0 \quad \text{for} \quad \alpha \neq \beta$$

in general. The η is fixed, represents how you are going to do the measurement, and may satisfy additional constraints to get special properties for the T^η's. For an interpretation, if ψ is a unit vector in $\mathfrak{H}$, then

$$Tr(P_\psi T^\eta(\alpha)) = \langle \psi, W(\alpha)\eta\rangle\langle W(\alpha)\eta, \psi\rangle$$

is the transition probability to the vector state given by $W(\alpha)\eta$. We are choosing an η and then translating, thereby getting an interpretation of what we are doing here.

Notice that if we had an I in our Lie algebra, then if α_{n+1} corresponded to $I = B_{n+1}$ in the general form of an element of the Lie algebra, we would have

$$Tr(P_\psi T^\eta(\alpha_1, \ldots, \alpha_n, 0)) = Tr(P_\psi T^\eta(\alpha_1, \ldots, \alpha_{n+1})).$$

Thus, we may drop α_{n+1} from further consideration. Alternatively, we can use a "Borel section," σ, of the group, and denote the relevant α_i's by, for example when α_{n+1} is irrelevant,

$$\sigma(\alpha) = (\alpha_1, \ldots, \alpha_n, 0).$$

We can choose the $\sigma(\alpha)$ so that they form coordinates for a general phase space, $\Gamma = \{x = \sigma(\alpha) \mid \alpha \in \text{group parameter space}\}$. See Schroeck (1996).

iv) Define

$$T^\eta(\Delta) = \int_\Delta T^\eta(x)\, dx,\ \Delta \text{ a Borel set in } \Gamma.$$

Then $\Delta \to T^\eta(\Delta)$ is a positive operator valued measure (Schroeck, 1996) called a localization operator for the phase space Γ.

Next take

$$A^\eta(F) = \int_\Gamma F(x) T^\eta(x)\, dx$$

for any measurable function F on Γ. Note that $A^\eta(\chi_\Delta) = T^\eta(\Delta)$. Also note that $A^\eta(1) = I$. From this, we get

$$\|A^\eta(F)\| \le \sup_{x\in\Gamma} \mid F(x) \mid$$

and

$$0 \le A^\eta(F) \le I \quad \text{for} \quad 0 \le F(x) \le 1, \text{ a.e. } x.$$

A function only of the observable B_1 in the Lie algebra is given by

$$A^\eta(f) = \int_\Gamma f(x_1) T^\eta(x)\, dx,$$

f a function of x_1 only. Similarly for B_i. In fact, for each i,

$$B_i = c_i \int_\Gamma x_i T^\eta(x)\, dx,\ 0 < c_i \le 1,$$

and for each measurable function f on B_i, there is a function $\mathcal{F}$ such that

$$f(B_i) = \int_\Gamma \mathcal{F}(x_i) T^\eta(x)\, dx = A^\eta(\mathcal{F}).$$

(See the proof in Schroeck, 1996. It depends on the "informational completeness" of the A^η—that we can distinguish between all states with the $Tr(\rho A^\eta(f))$. This is equivalent to the condition on η that $\langle W(\alpha)\eta \mid W(\beta)\eta\rangle \neq 0$ for $\alpha \neq \beta$. The proof holds in the sense of being in the limit of the $A^\eta(\mathcal{F})$.) Note that if $f(B_i) = E^{B_i}(\Delta_i)$, then $f(B_i) \neq A^\eta(\chi_{I\times\Delta_i\times I})$ although for η "sharply peaked" it will be approximately equal. We shall return to this later, in Sec. VI.

Definition 1. We take as the definition of the set of (the relevant) effects, the set $\{A^\eta(F) \mid F \text{ is measurable}, 0 \leq F(x) \leq 1 \text{ a.e. } x\}$. Also define, with $+$ denoting operator addition,

$$A^\eta(F) \oplus A^\eta(G) = A^\eta(F) + A^\eta(G) = A^\eta(F+G), \tag{1}$$

which in turn is an effect in our sense if and only if $(F+G)(x) \leq 1$, a.e. x.

We then have the following:

$$Tr(\rho A^\eta(F)) = \int_\Gamma F(x) Tr(\rho T^\eta(x))\, dx = \int_\Gamma F(x) \sum_i d_i Tr(P_{\psi_i} T^\eta(x))\, dx$$

where $\rho = \sum d_i P_{\psi_i}$ is a general density matrix. In this fashion we obtain the statistical agreement for $\oplus$ that we had before. Furthermore, we have the interpretation that when we measure $A^\eta(\chi_\Delta)$ in density state ρ we get the probability that ρ will be in the state $P_{W(x)\eta}$ for some $x \in \Delta$.

We note that we could have alternatively defined the set of effects as the set of $A^\eta(F)$'s with $\|A^\eta(F)\| \leq 1$, but that may not have the nice properties forthcoming.

We should also note that what we will prove about the algebra of effects in our formalism is, in fact, true about any image in any Hilbert space of the set of fuzzy sets! The setting of the formalism of quantum mechanics in phase space is just an example.

3. EXAMPLES

We will work out the example of spin-$\frac{1}{2}$ explicitely. See (Schroeck, 1982 and 1996, Chap. II.3.A.). In $\mathbb{C}^2$ there is only one form of a non trivial projection, namely

$$T(x) = \frac{1}{2}(I + x\cdot\sigma),\ x \in \mathbb{R}^3,\ \|x\| = 1,\ \sigma = (\sigma_1, \sigma_2, \sigma_3)$$

a representation of the Pauli spin algebra. Thus, the only freedom in choosing an η is in the direction it "points" on the unit sphere $\mathcal{S}^2$. We have $\mathcal{S}^2$ as a homogeneous space of the rotation group. With the invariant measure μ on $\mathcal{S}^2$, normalized so that $\mu(\mathcal{S}^2) = 2$, and with η chosen to point in the direction of the North Pole, then we have

$$A^\eta(1) = \int_{\mathcal{S}^2} T(x)\, d\mu(x) = I,$$

and

$$A^\eta(x_i) = \int_{\mathcal{S}^2} x_i T(x)\, d\mu(x) = \int_{\mathcal{S}^2} x_i \frac{1}{2}(I + x\cdot\sigma)\, d\mu(x) = \frac{1}{3}\sigma_i.$$

Hence, we get all the generators σ_i of the rotation group on $\mathcal{S}^2$. Furthermore, we can get all effects in the set $\{A^\eta(F) \mid F \text{ is measurable}\}$. The set is informationally complete in $\mathbb{C}^2$.

Take the example of momentum and position in $L^2(\mathbb{R})$. Then $\{P, Q, I\}$ are the elements of the Lie algebra for the Heisenberg group. Now take η to stand for a vector state that has

$$\langle \eta, P\eta\rangle = 0, \langle \eta, Q\eta\rangle = 0.$$

Then, for $\sigma(p, q) = (p, q, 0)$,

$$\langle W(\sigma(p, q))\eta, PW(\sigma(p, q))\eta\rangle = p \quad \text{and} \quad \langle W(\sigma(p, q))\eta, QW(\sigma(p, q))\eta\rangle = q;$$

that is, $T^\eta(p, q)$ is the state that corresponds to moving η by (p, q) in the phase space. $A^\eta(\chi_\Delta)$ corresponds to a measurement of a particle by asking if it would transist to any of the states $T^\eta(p, q)$ for $(p, q) \in \Delta$. Similarly for $A^\eta(F)$. We next take $B = f(P) = A^\eta(\mathcal{F})$ and $C = h(Q) = A^\eta(\mathcal{H})$ for some $\mathcal{F}$ and $\mathcal{H}$ between 0 and 1. (Note that $\mathcal{F}(p, q) = \mathcal{F}(p, 0)$ and $\mathcal{H}(p, q) = \mathcal{H}(0, q)$.) Thus

$$B \oplus C = A^\eta(\mathcal{F} + \mathcal{H})$$

as long as $(\mathcal{F} + \mathcal{H})(x) \leq 1$. It corresponds to the experiment in which you will describe the particle transisting to the state $T^\eta(p, q)$ located in the fuzzy set $\mathcal{F} + \mathcal{H}$.

Thus, for the examples, we have answered the question of "what experiments do we perform when we measure $A + B$" for A and B in our effect algebra.

4. THE EFFECT ALGEBRA FOR THE FORMALISM OF QUANTUM MECHANICS ON PHASE SPACE

To obtain an effect algebra, we have only to check if, for F, G, H, $F + G$, $G + H$, and $F + G + H$ between 0 and 1,

1) $A^\eta(F) \oplus A^\eta(G) = A^\eta(G) \oplus A^\eta(F)$,

2) $[A^\eta(F) \oplus A^\eta(G)] \oplus A^\eta(H) = A^\eta(F) \oplus [A^\eta(G) \oplus A^\eta(H)]$,
3) $A^\eta(1) = I$,
4) $A^\eta(F)$ has a supplement $A^\eta(F)'$, namely $A^\eta(1 - F)$.

These are valid since $A^\eta(F) \oplus A^\eta(G) = A^\eta(F + G)$ and A^η is a (normalized) positive operator valued measure. Thus, we obtain an effect algebra in terms of this addition.

We have the following general

Definition 2. Let A and B be effects in $\mathfrak{H}$. Then A and B are comeasurable iff we can write $A = A_1 \oplus C$, $B = B_1 \oplus C$, for A_1, B_1, and C effects and $A_1 \oplus B_1 \oplus C$ is an effect.

In particular, for $A = A^\eta(F)$, and $B = A^\eta(H)$, then $A_1 = A^\eta(F - \min(F, H))$, $C = A^\eta(\min(F, H))$, and $B_1 = A^\eta(H - \min(F, H))$.

Theorem 3. *In the phase space approach, every pair of effects, $A^\eta(F)$ and $A^\eta(G)$, is comeasurable.*

Proof: Let $H(x) = \min(F(x), G(x))$ a.e. x. Then $A^\eta(F) = A^\eta(F - H) \oplus A^\eta(H)$, $A^\eta(G) = A^\eta(G - H) \oplus A^\eta(H)$, and $A^\eta(F - H) \oplus A^\eta(G - H) \oplus A^\eta(H) = A^\eta((F - H) + (G - H) + H)$. Thus, we must show that, for fuzzy sets F and G, then $(F - H) + (G - H) + H$ is located between 0 and 1. But $F - H$, $G - H$, and H are ≥ 0, and hence so is their sum. To get the sum ≤ 1, consider the sum pointwise: first suppose $F(x) \geq G(x)$ for some x. Then $H(x) = G(x)$, so, $(F - H)(x) + (G - H)(x) + H(x) = (F - G)(x) + G(x) = F(x) \leq 1$. Similarly if $G(x) \geq F(x)$. Thus, $(F - H) + (G - H) + H \leq 1$. □

Next, we present a well-known theorem from a general Hilbert space approach which asserts that there are noncomeasurable effects. See, for example Busch *et al.* (1995).

Theorem 4. *In Hilbert space, two projections are comeasurable iff they commute.*

We have an immediate corollary of the two theorems:

Theorem 5. *In the formalism of quantum mechanics on phase space, the set $\{A^\eta(F) \mid 0 \leq F \leq 1, F$ measurable$\}$ does not contain any two projections that do not commute.*

As an example, we return to the case of spin-1/2. With the notation previously established, we have $A^\eta(F(x)) = A^\eta(\frac{1}{2}(1 + 3b \cdot x)) = T(b)$. But $F(x) = \frac{1}{2}(1 + 3b \cdot x)$ is not a fuzzy set function. In fact $-1 \leq F(x) \leq 2$. There are other forms of F that will also give $T(b)$, but they are worse in that the corresponding F has a larger range. Consequently, I and 0 are the only projections in $\{A^\eta(F) \mid 0 \leq F(x) \leq 1\}$. None-the-less, we have an effect algebra quite different from the algebra of all effects in $\mathbb{C}^2$. Thus, we have a proper subset of the set of all effects, one that contains no noncommuting projections.

Now in general, if we have informational completeness for the A^η, as we also do for the nonrelativistic spinless quantum mechanics with $\eta =$ a Gaussian for example, we have a curious situation: From the informational completeness, we can approximate every (bounded, self-adjoint) operator by an operator of the form $A^\eta(F)$, for some measurable F, in a topology that comes from the trace. (See Schroeck (1996) Section III.3.D, and Healy and Schroeck (1995).) But, we can never get all the operators directly from the effect algebra in which $0 \leq F \leq 1$.

Conclusion 6. *We conclude from the theorem that there are noncomeasurable effects, and all other similar theorems that are based on noncommuting projections, perhaps come from an idealization that may not hold in the laboratory.*

We can make this last theorem stronger by applying the Naimark dilation theorem (Schroeck, 1996) [Chap. II.11.F.] to the $\{A^\eta(F) \mid 0 \leq F \leq 1\}$ by converting from the positive operator valued measure to a projection valued measure on phase space. From this point of view, the $\{A^\eta(F) \mid 0 \leq F \leq 1\}$ will be represented by a host of projections via the spectral theorem, but these projections will all commute.

We can make this theorem even stronger:

Theorem 7. *In the formalism of quantum mechanics on phase space with the covariance condition with respect to group G, the set of effects $\{A^\eta(F) \mid 0 \leq F \leq 1$, F measurable$\}$ does not contain any projections other than the trivial ones.*

Proof: Suppose $A^\eta(F)$ was a nontrivial projection and W is a representation of G. Then we have

$$W(g)A^\eta(F)W(g)^{-1} = A^\eta(g \cdot F),$$

$$g.F(x) = F(g \cdot x),$$

for any g in the symmetry group of the system (Schroeck, 1996). Pick a g that is not accidentally a symmetry for F. Then you get another nontrivial projection that does not commute with $A^\eta(F)$. $\square$

We have separated this theorem from the previous one because it depends explicitly on the covariance condition.

5. THE SET OF EFFECTS FOR THE FORMALISM OF QUANTUM MECHANICS ON PHASE SPACE AS AN M. V. EFFECT ALGEBRA

While we are considering $\mathcal{E}$ = the set of effects for the formalism of quantum mechanics on phase space with respect to η, we may consider exactly what structure $\mathcal{E}$ has within the chain: effect algebra—interpolation algebra—Riesz decomposition algebra—lattice ordered effect algebra—distributive algebra—M. V. effect algebra—Heyting effect algebra—Boolean algebra. [See Foulis (2000) for the definitions.] We will show in a subsequent paper that $\mathcal{E}$ satisfies the axioms of an M. V. effect algebra, and in fact a Heyting effect algebra, so that it also satisfies the axioms of all the intermediate algebras as well. It is not a Boolean algebra.

6. AN ALGEBRA OF EFFECTS WITH THE SEQUENTIAL PRODUCT

There also must be a multiplication in the effect algebra for it to be an algebra in the sense of mathematics. This can be accomplished in several ways to yield a nonassociative algebra. We will take a physically motivated definition of our product and then interpret it physically from the standpoint of quantum mechanics on phase space.

A Hilbert space effect A, $0 \leq A \leq 1$, is related to a state in the following way: For any state ρ in $\mathfrak{H}$, and any positive operator valued measure (POVM) A, (an observable), defined on the elements Δ of some σ-field, then the probability that a measurement of A on this system will lead to a result in Δ is

$$p_\rho^A(\Delta) = Tr(\rho A(\chi_\Delta)).$$

Given a POVM A and a state ρ and assuming that $Tr(\rho A(\chi_\Delta)) \neq 0$, then the probability that when measuring POVM B in Δ' is

$$p^B_{A(\Delta)\rho}(\Delta') = \frac{Tr(A(\chi_\Delta)^{1/2}\rho A(\chi_\Delta)^{1/2}B(\chi_{\Delta'}))}{Tr(\rho A(\chi_\Delta))},$$

whereby $A(\chi_\Delta)^{1/2}$ we mean the unique positive square root of $A(\chi_\Delta)$. $A(\chi_\Delta)^{1/2}$ is again a Hilbert space effect (Gudder and Nagy, 2001). Thus we define a multiplication by

$$A(\chi_\Delta) \circ B(\chi_{\Delta'}) = A(\chi_\Delta)^{1/2}B(\chi_{\Delta'})A(\chi_\Delta)^{1/2}.$$

This multiplication takes two Hilbert space effects into a Hilbert space effect. More generally, the product $A \circ B \equiv A^{1/2}BA^{1/2}$ is called the sequential product of Hilbert space effects. It has the property that

$$(A \circ B) \circ C \neq A \circ (B \circ C)$$

in general. In this fashion, we can hope to get a nonassociative algebra for the effect algebra (of all effects). But we have a subeffect algebra in the phase space approach to quantum mechanics. What will happen in this case?

Take $A = A^{\eta}(F) = \int_{\Gamma} F(x)T^{\eta}(x)$. Notice that $T^{\eta}(x) = P_{W(x)\eta}$ for $||\eta|| = 1$. Then $T^{\eta}(x)T^{\eta}(y) = \mid W(x)\eta\rangle \ \langle W(x)\eta \mid W(y)\eta\rangle \ \langle W(y)\eta \mid$; i.e., the $T^{\eta}(x)$'s are projections, but not orthogonal projections because $\langle W(x)\eta \mid W(y)\eta\rangle$ is not zero in general for $x \neq y$ and never zero in the case of informational completeness. So, the transform $A^{\eta}(F) \longmapsto A^{\eta}(F)^{1/2}$ is not given by $A^{\eta}(F)^{1/2}$ " $=$ "$A^{\eta}(F^{1/2})$ or any $A^{\eta}(H)$, H any measurable function, in general. (We do have the approximation

$$A^{\eta}(F)^{1/2} \approx A^{\eta}(H)$$

in the case of informational completeness, H being some measurable function. For a vector η that is highly peaked in the sense that

$$\langle W(x)\eta \mid W(y)\eta > \approx \delta^{x}_{y},$$

then we would have $H \approx F^{1/2}$. This "highly peaked" phenomena may not occur in the limit, as it violates the uncertainty principle, for example.)

Even if $A^{\eta}(F)^{1/2} = A^{\eta}(H)$, we would next be concerned with

$$\begin{aligned} A^{\eta}(F) \circ A^{\eta}(G) &= A^{\eta}(F)^{1/2}A(G)A(F)^{1/2} \\ &\approx \int_{\Gamma\times\Gamma\times\Gamma} H(x)G(y)H(z)T^{\eta}(x)T^{\eta}(y)T^{\eta}(z)dxdydz. \end{aligned}$$

We are again faced with the same difficulty, and the same impossible resolution in terms of the highly peaked phenomena. (In the case of "highly peaked η," we have

$$A^{\eta}(F) \circ A^{\eta}(G) \approx A^{\eta}(FG) = A^{\eta}(GF) \approx A^{\eta}(G) \circ A^{\eta}(F),$$

which means that they "almost commute."

We next check the distributive properties between our $\oplus$ and the sequential $\circ$. We have

$$\begin{aligned} A^{\eta}(F) \circ [A^{\eta}(G) \oplus A^{\eta}(H)] &= A^{\eta}(F)^{1/2}[A^{\eta}(G) + A^{\eta}(H)]A^{\eta}(F)^{1/2} \\ &= A^{\eta}(F)^{1/2}A^{\eta}(G)\, A^{\eta}(F)^{1/2} + A^{\eta}(F)^{1/2}A^{\eta}(H)\, A^{\eta}(F)^{1/2} \\ &= A^{\eta}(F) \circ A^{\eta}(G) \oplus A^{\eta}(F) \circ A^{\eta}(H). \end{aligned}$$

Thus, it is right distributive. For the left distributive property, we have

$$\begin{aligned} [A^{\eta}(F) \oplus A^{\eta}(G)] \circ A^{\eta}(H) &= A^{\eta}(F+G) \circ A^{\eta}(H) \\ &= A^{\eta}(F+G)^{1/2}A^{\eta}(H)A^{\eta}(F+G)^{1/2} \end{aligned}$$

which we compare with $[A^{\eta}(F) \circ A^{\eta}(H)] \oplus [A^{\eta}(G) \circ A^{\eta}(H)]$.

These two expressions will fail to be equal on two counts:

i) $A^{\eta}(F+G)^{1/2} \neq A^{\eta}(F)^{1/2} + A^{\eta}(G)^{1/2}$ since the T^{η}'s are not orthogonal (and the freshman's dream is false);
ii) $A^{\eta}(F)^{1/2} A^{\eta}(H) A^{\eta}(G)^{1/2} \neq 0$ since the T^{η}'s are not orthogonal.

Thus, in all cases everything boils down to essentially the same reason (modulo the freshman's dream), the T^{η}'s are not orthogonal. This is the price you pay for having a nonlocal theory.

Conclusion 8. *The effect algebra in phase space is not an algebra in the sense of mathematics with respect to the sequential product, the failure being a consequence of the nonlocal nature of the formalism.*

ACKNOWLEDGMENTS

I would like to thank Dr. Stan Gudder for suggesting this topic, Dr. Rick Ball for relevant discussions on M. V. algebras, and Drs. Jim Brooke and Paul Busch for many discussions.

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