Informational completeness, positive operator valued measures and the search for an optimal landmine detector.

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Abstract

We study the statistical propsects and applicability of quantum state discrimination techniques for the analysis of data from nuclear quadrupole resonance experiments. The target application in mind is the remote detection of explosive content in anti-personnel landmines. The method is aimed at incorporation of results from multiple sensing modalities. In particular we introduce partial informational completeness and show that increasing the number of observables in a partially complete set in order to diminish the trace-distance between the reconstructed and the full density matrix of the NQR subsystem, will allow for a lower Bayes-risk.

1 Nuclear quadrupole resonance and landmines

Detecting landmines is a dangerous task. The sensitivity required of a metal detector to detect the sometimes subgram quantities of metal in plastic landmines results in an enormous number of false alarms [3]. A possible solution to to this problem involves the use of nuclear quadrupole resonance techniques. Nuclear quadrupole resonance (NQR) signals result from the relaxation of nuclear quadrupole momenta to their original thermal equilibrium position after an initial, high power RF pulse has been applied. The thermal equilibrium configuration of the nuclear spins is a function of the electromagnetic field in the vicinity of the quadrupole active nuclei. As a result, the NQR spectrum is very specific with respect to chemical compounds in the substance involved and can serve as a fingerprint to identify that substance. Because of its high potential value in remote explosive detection, there is renewed interest in NQR methods for landmine and UXO detection, as well as

for securing high risk areas such as airports by non-intrusive means. Quite incidentally, NMR and NQR systems have recently also received great attention for their applicability in the fast growing field of quantum information [18] and state of the art quantum computers are currently based on NMR. In NMR a high intensity homogenous magnetic field introduces a preferred axis of quantization and causes the energy levels to undergo a Zeeman splitting. In NQR, this splitting is caused by the interaction of the nuclear quadrupole with the electric field gradient. A necessary condition for the use of NQR, is the presence of a substance with a nuclear quadrupole moment. An ideal candidate is the naturally stable nitrogen isotope ^{14}N, (with a natural abundance of 99.64 %) with nuclear spin 1 and corresponding nuclear quadrupole moment. All mass-produced landmines contain ^{14}N, so that, in principle, it is possible to detect any non-metallic mine by NQR [1]. The NQR spectrum for ^{14}N has transitions in the frequency range between 0 and 6 MHz, actual values depending mostly on the electric field gradient tensor, which is primarily determined by the charge distribution of the electrons that bind the nitrogen to the rest of the explosive. The resulting NQR signal is therefore highly dependent on the chemical structure of the sample, and delivers a potentially very reliable classification with an accompanying very low false alarm rate. Compared with other popular mine detection techniques such as the metal detector and the ground penetrating radar, NQR-based detector performance is not very sensitive with respect to weather conditions. Add to this that it is possible to construct a hand held NQR detector, and it seems that NQR is an ideal candidate for explosive detection [10]. The main challenge for NQR techniques, is the inherently low energy content of the signal, resulting in a very low signal to noise ratio (SNR). To improve the SNR, many repetitions of the experiment are necessary. Rather than just measuring the free induction decay of a single excitation, one can set up an appropriate sequence of RF pulses, and measure the returned echo after each such pulse. In this way we obtain a larger data set from which inferences can be made. The rate at which repetition is physically informative, is bound from below in a fundamental way by the physical parameters of the relaxation process. The nuclear relaxation is a result of two different mechanisms, called the spin-spin relaxation and the spin-lattice relaxation. The relaxation time that characterizes the spin-lattice relaxation, denoted $T_{1\rho}$, determines the time necessary for the system to regain its original thermal equilibrium state, and gives a bound on how quickly a pulse sequence can be

[1] An exception is the PFM-1 landmine which contains a liquid explosive, which is outside the scope of current NQR techniques.

initiated after another. The spin-spin relaxation time, denoted T_2, is indicative of the decoherence as a result of spin-spin interactions and determines the length of the spin echo sequence. Spin-spin relaxation times are generally (much) shorter than spin-lattice relaxation times. For most explosives, the relaxation times are short enough so that NQR detection becomes feasible. Unfortunately, about 60% of the landmines contain $\alpha-$trinitrotoluene (TNT), which has relaxation times that lead to prohibitively long detection times within the operational limits of landmine detection. It is therefore projected that an NQR based landmine detector will probably serve mainly as a confirmation sensor, i.e. a detector that is employed to decrease the false alarm rate only after a metal detector or a ground penetrating radar system has detected a potential landmine. Whether used as a confirmation or as a primary detector, NQR detection efficiency for TNT will benefit from a reduction in the time necessary for reliable detection. Because one cannot shorten the relaxation parameters of TNT, much effort has gone into cleverly designing the emitted RF pulse and increasing the sensitivity of the receiver. Besides these efforts, it is worthwhile to pursue better signal analytic detection techniques.

2 Quantum operations and the evolution of the NQR signal

It is not feasible to describe the entire quantum-physical state of the landmine, nor would this be very interesting. What causes the NQR signal, is only the change in the net magnetization along the direction of the solenoid. In the case of ^{14}N, we are dealing with a spin-1 system so that the relevant quantum mechanical subspace is spanned by just three orthogonal vectors. A full determination of the state in this three-dimensional subspace could, in principle, lead to efficient strategies for detection and classification of the NQR signal. We will briefly show how quantum operations can serve as a framework to relate the measured quadrature components of the current in the coil to quantum state discrimination tools. In theoretical descriptions of NQR ([8], [14], and [17]), the state of the system is a classical statistical mixture of pure quantum states, described by a density operator ρ belonging to the class of linear, positive operators that sum to one when they act upon a complete set of eigenvectors. If we consider as system the landmine, its immediate surroundings, and the NQR detector, the detection system can be considered as closed and the dynamics of the total density operator ρ_{closed}

is governed by the unitary evolution that solves the Schrödinger equation

$$\frac{d\rho_{closed}(t)}{dt} = -\frac{i}{\hbar}[\mathcal{H}, \rho_{closed}(0)] \tag{1}$$

Here $\rho_{closed}(0)$ is the initial density operator and $\mathcal{H} = \mathcal{H}_{rf} + \mathcal{H}_Q$, with $\mathcal{H}_Q$ is the nuclear quadrupole Hamiltonian, $\mathcal{H}_{rf}$ the Hamiltonian corresponding to the RF pulse and [,] is the commutator. The strength of the quadrupolar Hamiltonian depends mainly on the coupling between the electric field gradient (EFG) and the quadrupolar moment. The efficiency of the excitation by an RF field depends on the relative orientation between the incident radiation and the EFG principal axis frame. Because the EFG principal axis frame depends on the molecular orientation, it is not possible to excite all quadrupole levels with the same efficiency in a powder crystalline sample. A calculation shows that the signal strength resulting from a crystalline powder is approximately only 43% the strength of a signal stemming from a single crystal with the same number of NQR active nuclei [14]. In absence of the RF pulse, a canonical ensemble of NQR spin-1 systems at temperature T, is described by a density operator $\rho_{thermal}$:

$$\rho_{thermal} = \frac{\exp(-\mathcal{H}_Q/k_BT)}{Tr\exp(-\mathcal{H}_Q/k_BT)]} = \frac{1}{Z}(1 - \mathcal{H}_Q/k_BT) + O(\frac{1}{T^2}) \tag{2}$$

Here k_B is the Boltzmann constant, T the temperature in Kelvin, and Z the partition function, which acts as a normalization. The second form for the density operator in thermal equilibrium Eq.(2), is generally a good approximation for demining applications, as $1/k_BT$ is small at room temperature in comparison to $\mathcal{H}_Q$. The RF pulse perturbs the thermal equilibrium state $\rho_{thermal}$ and it is the relaxation from this perturbed state to Eq.(2), according to Eq.(1), that is responsible for the NQR signal that we are interested in. The Hamiltonian $\mathcal{H}_{rf}$ is $\mathcal{H}_{pulse}$ for a period of time, followed by absence of a pulse interaction for another period of time, after which $\mathcal{H}_{pulse}$ is switched on again , and so on. It is usual to approximate this as a series $\mathcal{H}_0$, $\mathcal{H}_1$, $\mathcal{H}_2$, $\mathcal{H}_3, \ldots$ describing the Hamiltonians at the time instances $t_0, t_1, t_{2,\ldots}$ The evolution Eq.(1) can then be formally solved for ρ to yield

$$\rho(t_0 + t_1 + \ldots) = e^{-i\mathcal{H}_n t_n} \ldots e^{-i\mathcal{H}_0 t_0} \rho(0) e^{i\mathcal{H}_0 t_0} \ldots e^{i\mathcal{H}_n t_n}$$

Because our data comes from the electron current in the coil, we need a way to connect the state of the mixture of the quadrupole active spins to this current. The coil used is a Faraday detector, and the electron current in the coil is the direct result of the load of the preamplifier connected to the coil

and the change of the magnetic flux inside the coil. The expectation of the magnetization in the direction of the axis of symmetry of the solenoid (say, the z-axis), is obtained by tracing over the product of the state ρ_{sys} (the mixture of quadrupole active spin-1 states) with the magnetization operator μ_z along that spatial axis:

$$\langle M_z \rangle = Tr(\mu_z \rho_{sys}) \tag{3}$$

Such a tracing operation, is an example of a so-called *quantum operation*. A quantum operation offers the most general possible description of an evolution [18], and is defined as a mapping ε that transforms an initial stateρ_0 to a final state ρ

$$\rho = \varepsilon(\rho_0) \tag{4}$$

such that there exists a set $\mathcal{O}$, called *operation elements*,

$$\mathcal{O} = \{E_k : \sum_k E_k E_k^\dagger = I, \forall \rho : Tr(E_k \rho) \geq 0\}, \tag{5}$$

for which ε can be written as

$$\varepsilon(\rho_0) = \sum_k E_k \rho_0 E_k^\dagger \tag{6}$$

The operations satisfy $\sum E_k E_k^\dagger = I$ by definition (5) and are hence *trace-preserving*. Important examples of operations that are trace-preserving are projective measurements, unitary evolutions and partial tracing. If the quantum operation is a general description of a quantum measurement (or evolution) then to each outcome k we associate one member E_k of the collection of measurement operators $\mathcal{O} = \{E_k, k = 1, 2, ...\}$ that act on the state space. If the state is ρ immediately before the measurement, then the probability that the outcome k occurs is

$$p(k|\rho) = Tr(E_k \rho E_k^\dagger) \tag{7}$$

and the state after the interaction if k occurs is

$$\rho_{fin} = \frac{E_k \rho E_k^\dagger}{Tr(E_k \rho E_k^\dagger)} \tag{8}$$

The two most common examples of quantum operations, are unitary transformations ($\varepsilon(\rho_0) = U \rho_0 U^\dagger$, U a unitary transformation) and von Neumann projective measurements ($\varepsilon_m(\rho_0) = P_m \rho_0 P_m^\dagger$, with P_m a projector on the

subspace labelled m). Many more examples, such as in quantum computation, can be found in [18] and modern descriptions of quantum experiments as in [6], [20]. In the latter, a set $\{M_k\}$ of positive operators satisfying $\sum M_k = I$ and $E_k = M_k^{1/2}$ is used.

Quantum operations are also a natural way to describe quantum noise and the evolution of an open system. The mathematical prescription of a quantum operation arises when one considers the system to be in interaction with an environment that together form a closed system, for which Eq. (1) applies. To see how this applies here, we denote the initial state of the system under investigation by ρ_{sys}, and the state of the environment (soil and interfering RF fields) as ρ_{env}, then the compound system can be written as a tensor product of those states: $\rho_{sys} \otimes \rho_{env}$. Following the standard rules of quantum mechanics, the expected mixture ρ is the partial trace over the degrees of the environment of the time evolved state of the closed system:

$$\rho = Tr_{env}(U(\rho_{sys} \otimes \rho_{env})U^\dagger) \tag{9}$$

It can be shown [18] that Eq. (9) is only slightly more general than Eq. (6), hence ρ can be described as resulting from a quantum operation acting on the system density matrix. Depending on whether the system contains TNT or not, the examined system has a density matrix written as ρ_{sys}^{tnt}, or ρ_{sys}^{1}. We expect either of two generic types of operation to have occurred:

$$\begin{aligned} \varepsilon'(\rho_{sys}^{tnt}) &= Tr_{env}(U(\rho_{sys}^{tnt} \otimes \rho_{env})U^\dagger) = \rho^0 \\ \varepsilon'(\rho_{sys}^{1}) &= Tr_{env}(U(\rho_{sys}^{1} \otimes \rho_{env})U^\dagger) = \rho^1 \end{aligned} \tag{10}$$

Here ρ^0 is the resulting mixture that produces the magnetization in the presence of TNT, and ρ^1 is the resulting mixture after the interaction, in absence of TNT. An optimal detection of TNT, hence entails optimally distinguishing the two quantum states ρ^0 and ρ^1. As mentioned above, we do not posses detailed knowledge of the states ρ^0 and ρ^1 in practice, but we have the quadrature components $V(t)$. The quadrature components are a result of the change in the magnetization M Eq.(3). With N the number of turns in a solenoid of area A, and Q the quality factor of the coil, we have

$$V(t) = QN\frac{d(\mu_0 MA)}{dt} \tag{11}$$

The quadrature components are induced by the magnetization in Eq.(3) can always be modelled by means of another quantum operation acting on the unknown mixture ρ_{sys}:

$$V(t) = \varepsilon_{qc}(\rho_{sys}) \tag{12}$$

Quantum operations are closed under conjunction; two consecutive quantum operations can always be represented as a single quantum operation. What we need to distinguish in the laboratory then, is to which type the measured $V(t)$ belongs:

$$\begin{aligned} V_0(t) &= \varepsilon_{qc}(\varepsilon'(\rho_{sys}^{tnt})) = \varepsilon(\rho_{sys}^{tnt}) \\ V_1(t) &= \varepsilon_{qc}(\varepsilon'(\rho_{sys}^{1})) = \varepsilon(\rho_{sys}^{1}) \end{aligned} \tag{13}$$

3 Bayesian decisions for NQR data

In essence, Bayesian detection deals with the optimal decision of a hypothesis from a set of mutually exclusive hypotheses. Consider the binary decision problem

$$\begin{aligned} H_0 &: \text{the signal indicates TNT presence} \\ H_1 &: \text{the signal indicates no TNT presence} \end{aligned}$$

If a given set of data is compatible only with one of the two hypotheses, the decision problem becomes trivial. However, in practice, the data generally supports both hypotheses, albeit with a different probability, and the decision task is consequently complicated by this fact. If we are given data x_i from a set of possible outcome results $X = \{x_1, x_2, \ldots, x_i, \ldots, x_n\}$, and the factual occurrence of x_i supports both hypotheses, we need to infer what the probability was of getting the result x_i as a result of either hypothesis being true. That is, we need some means to evaluate $p(x_i|H_0)$ and $p(x_i|H_1)$. Any additional (prior) information can be included under the label D and then we compare $p(x_i|H_0, D)$ and $p(x_i|H_1, D)$. What we are after is the probability of H_0 or H_1 being true, on the condition that D holds and x_i was the outcome of the experiment. By the use of Bayes' theorem [13] we have

$$p(H_j|x_i, D) = p(H_j|D)\frac{p(x_i|H_j, D)}{p(x_i|D)}, j = 0, 1 \tag{14}$$

We eliminate the denominator by calculating the ratio of Eq. (14) for $j = 0$ and $j = 1$:

$$\frac{p(H_0|x_i, D)}{p(H_1|x_i, D)} = \frac{p(H_0|D)}{p(H_1|D)}\frac{p(x_i|H_0, D)}{p(x_i|H_1, D)} \tag{15}$$

In absence of any preference which of the two hypotheses is more likely than the other on the basis of the prior information we set $\frac{p(H_0|D)}{p(H_1|D)} = 1$. In complete absence of any prior information, we omit dependence on D. The

quantity of interest for optimally choosing between two alternative hypotheses is the likelihood ratio (also called *the odds* in the binary case):

$$\Lambda_i = \frac{p(H_0|x_i)}{p(H_1|x_i)} \tag{16}$$

We call the detector a maximum-likelihood detector (MLD hereafter) iff the obtained outcome x_i is the outcome that maximizes the odds Eq.(16) that the outcome given pertains to the system under investigation rather than to noise in the detection system. It turns out that this is a model for quantum as well as classical observation [1]. Assuming our detector is MLD, allows for an optimal detection strategy by reversing the logic of the detector[2]. As with any type of detection technique, we have no a priori knowledge whether the physical detector satisfies the statistical assumptions we imposed on the detector, and actual performance will depend on how well this condition will be met. In accordance with quantum mechanics, we assume the probability $p(H_0|x_i)$ (and $p(H_1|x_i)$) is a monotone function of the trace distance between the actually measured signal, and the ideal (averaged over many samples) signal obtained in the presence (absence) of TNT. Numerator and denominator in Eq.(16) can be substituted by the corresponding trace distance, as the outcome for which the likelihood ratio is maximal, is invariant under monotone transformations. A second rationale for taking the trace distance, is that it arises naturally when one considers the Bayes risk in the binary state discrimination problem.

3.1 Trace distance and Bayes risk of distinguishing quantum states

If we are given two states ρ^0 and ρ^1 with *a priori* probabilities p_0 and $p_1 = 1 - p_0$, then, following Eq. (4), we look for two operations elements $\mathcal{O} = \{E_0, E_1\}$ such that $E_0 + E_1 = I$ and $E_0, E_1 \geq 0$ that minimize the *Bayes risk* or probability of error [9] :

$$R_{\mathcal{O}}(p_0) = p_0 Tr(\rho^0 E_1) + p_1 Tr(\rho^1 E_0) \tag{17}$$

rewriting Eq.(17) once with $E_1 = I - E_2$ and once with $E_2 = I - E_1$, adding and dividing, yields

$$R_{\mathcal{O}}(p_0) = \frac{1}{2}[1 - Tr[(p_0\rho^0 - p_1\rho^1)(E_0 - E_1)]$$

[2] A very similar approach to observation with the same name, is proposed in several papers that deal with visual perception by humans. We refer to [11] and the references found there.

To proceed, we define the trace distance between $\rho^0$and ρ^1, as

$$D(\rho^0, \rho^1) = \frac{1}{2} Tr \sqrt{(\rho^0 - \rho^1)(\rho^0 - \rho^1)^\dagger} \tag{18}$$

The trace distance is symmetric in its arguments, positive iff $\rho^0 \neq \rho^1$, zero iff $\rho^0 = \rho^1$, and satisfies the triangle inequality. In other words, it is a bona-fide distance measure on the set of density matrices. Another important property of the trace distance, is given by

$$D(\rho^0, \rho^1) = \max_{E_i \in \mathcal{O}} Tr(E_i(\rho^0 - \rho^1))$$

With this we can show [18] that the minimum value the Bayes risk $\min_{\mathcal{O}} R_{\mathcal{O}}(p_0)$ can attain does not depend on the E_k and equals

$$\min_{\mathcal{O}} R_{\mathcal{O}}(p_0) = R_{Bayes}(p_0) \tag{19}$$

$$= \frac{1}{2} - \frac{1}{2} Tr \sqrt{(p_0\rho^0 - p_1\rho^1)(p_0\rho^0 - p_1\rho^1)^\dagger} \tag{20}$$

If sample could equally well contain TNT or not, we have as prior probabilities $p_0 = p_1 = 1/2$:

$$R_{Bayes}(p_0) = \frac{1}{2} - D(\rho^0, \rho^1) \tag{21}$$

We see the minimal Bayes risk is attained for two states that maximize the trace distance. Trace preserving quantum operations can be shown to cause a contraction in the space of density operators [18]. Because the trace distance is a true distance measure on the space of density operators, it can only decrease as a result of an arbitrary trace-preserving quantum operation ε:

$$D(\rho^0, \rho^1) \geq D(\varepsilon(\rho^0), \varepsilon(\rho^1)) \tag{22}$$

If the current in the coil is the result of Eq. (10), then being able to distinguish the currents reliably (i.e., the trace distance is greater than can be explained from fluctuations), indicates we have successfully distinguished the situations represented by H_0 and H_1.Quantum operations can only have the effect of reducing the trace distance, which in turn will increase the minimal Bayes risk associated with distinguishing the two situations. Hence MLD detection using the quadrature components induces some loss in detector performance in comparison with the same procedure applied to a reconstruction of the state ρ_{sys}; because we skip one quantum operation in

Eq.(13), this would lead to a lower Bayes-risk. We will make this idea more precise in the next section.

In actual demining applications, the necessary acquisition time will further increase as a result of RF interference, other NQR active soil constituents such as piezoelectric ceramics, and the fact that only single sided (as opposed to the sample being *within* the coil, as is the case for our data), remote acquisition is possible.

4 Informationally complete measurements

The most straightforward and useful definition for examining whether a set of observations can determine the state of the system is provided by Prugovečki's concept [19] of informational completeness.

Definition (informational completeness): Let $\mathcal{F} = \{A_s | s \in I\}$ be a collection of bounded operators on a Hilbert space $\mathcal{H}$. Let S be the set of density operators on $\mathcal{H}$. Then we say $\mathcal{F}$ is *informationally complete* iff $\forall A \in \mathcal{F}, \rho \in S$:

$$Tr(A_s\rho) = Tr(A_s\rho') \Rightarrow \rho = \rho' \tag{23}$$

Another name for an informationally complete set of observables is a *quorum* [4], [5]. To make sure that $\mathcal{F}$ does not contain superfluous observables, we introduce

Definition (irreducible informational completeness): We will call a set of observables $\mathcal{F} = \{A_s | s \in I\}$ *irreducibly informational complete*, iff $\mathcal{F}$ is informational complete and no proper subset of $\mathcal{F}$ is informational complete.

If we measure the expectation values of all observables in a quorum infinitely precise then there is only one state compatible with that data. However, in the actual state determination problem we face two different types of knowledge. In practice we often do not have a full quorum. In practice we do not have infinitely precise estimates for the expectation values. Hence, only approximate state determination is possible. We first focus on not having a full quorum. Using the trace distance D, we can make this notion precise.

Definition ($\delta-$informational completeness): Let $\mathcal{F} = \{A_s | s \in I\}$ be an irreducibly informational complete collection of bounded operators on a Hilbert space $\mathcal{H}$. Let ρ be the unique density operator that is associated with expectation values of the observables in $\mathcal{F}$. Let S be the set of density operators on $\mathcal{H}$. Let $\mathcal{Z}$ be a proper subset of $\mathcal{F}$. Then we say $\mathcal{Z}$ is

$\delta-$*informational complete* iff $\forall A \in \mathcal{F}, \rho, \in \rho' \in S, \exists \delta \in \mathbb{R}^+$:

$$\begin{aligned} \forall A_s \quad &\in \quad \mathcal{Z} : Tr(A_s\rho) = Tr(A_s\rho') && (24) \\ &\Rightarrow \quad D(\rho, \rho') = \delta && (25) \end{aligned}$$

Note that ε cannot be zero as a result of the demand that $\mathcal{Z}$ is defined as a proper subset of an irreducibly informational complete set. Let us see what happens if we pick an even smaller subset of observables.

Theorem: Let $\mathcal{F} = \{A_s | s \in I\}$ be an irreducibly informational complete collection of bounded operators on a Hilbert space $\mathcal{H}$. Let ρ be the unique density operator that is associated with expectation values of the observables in $\mathcal{F}$. Let $\mathcal{Z}$ be a proper subset of $\mathcal{F}$ that is δ_z-*informational complete.* Let $\mathcal{W}$ be a proper subset of $\mathcal{Z}$ that is δ_w-*informational complete.* Then we have

$$\mathcal{W} \subset \mathcal{Z} \Rightarrow \delta_z < \delta_w$$

Proof: Let S be the set of density operators on $\mathcal{H}$. Let ρ be the unique density operator that is associated with expectation values of the observables in $\mathcal{F}$, let ρ' be a density operator such that $\forall A_s \in \mathcal{Z} : Tr(A_s\rho) = Tr(A_s\rho') \Rightarrow D(\rho, \rho') = \delta_z$ and let $\rho^\sharp$ be a density operator such that $\forall A_s \in \mathcal{W} : Tr(A_s\rho) = Tr(A_s\rho^\sharp) \Rightarrow D(\rho, \rho^\sharp) = \delta_w$. Because D is a true distance measure, the triangle inequality holds:

$$\begin{aligned} D(\rho, \rho') + D(\rho', \rho^\sharp) &\geq D(\rho, \rho^\sharp) \\ \delta_z + D(\rho', \rho^\sharp) &\geq \delta_w \end{aligned}$$

Since δ_z, δ_w and D are not negative, we have shown $\delta_z \leq \delta_w$. D is a true distance measure so $D(\rho', \rho^\sharp)$ can only be zero if $\rho' = \rho^\sharp$, implying $\delta_z = \delta_w$. Let us denote by $\mathcal{Y}$ the set of observables that are in $\mathcal{Z}$ but not in $\mathcal{W}$. This set is not empty by the assumption that $\mathcal{W}$ is a proper subset of $\mathcal{Z}$. If $\delta_z = \delta_w$ then the observables in $\mathcal{Y}$ do not constrain the density operator any further than the observables in $\mathcal{W}$ already do. Elimination of $\mathcal{Y}$ from $\mathcal{F}$ would yield the same density operator as one obtained from the full set $\mathcal{F}$, contrary to the assumption that $\mathcal{F}$ was irreducible.■

By Eq. (17) we see that the Bayes risk associated with discriminating between states determined by the set of observables in $\mathcal{Z}$ is smaller than the same risk for $\mathcal{W}$. Thus the theorem is a reformulation of the intuitive idea that measuring more observables helps to determine the state, and hence increases ones ability of discrimination between states.

Let us now briefly discuss the second type of lack of knowledge [2]. In general a higher number of measurements will produce a better estimate for

the expectation values of the observables, and hence a more precise state determination is possible. This idea was made precise by Summhammer [21] with a remarkable result. It turns out that the mathematical structure of the random variables in quantum theory is such, that indeed, the measurement uncertainty decreases *strictly* with the number of measurements. In particular, the uncertainty does not depend on the expectation value, or the probability of any particular outcome, but only on the number of measurements we choose to make. In NQR detection of explosives, the usual way to combat the extremely low signal-to-noise-ratio, is to increase the number of measurements. Without any additional means, such as the use of entanglement in quantum metrology, the expected error goes as the square root of the number of measurements. This implies the gain in the precision of state determination decreases with each measurement. We have shown above there is another way to decrease the number of false detection as expressed through the Bayes-risk, by measuring other observables. What we propose then, is to use estimate other observables, when the rewards of re-measuring the same observable becomes smaller and smaller. Not every observable is equally simple to detect, and noise considerations may have widely varying characteristics depending on which observable is being measured. Hence it remains to be determined experimentally at which point one method can be superseded by the other.

5 Concluding remarks

We have examined the applicability of a Bayesian quantum state discrimination technique to investigate potential improvement of remote TNT detection capability by NQR measurements. The Bayesian technique allows to include data from primary detectors (such as a metal detector or ground penetrating radar) in the form of prior probabilities, so that the NQR detector becomes a confirmation sensor. Weakness of the NQR detector lies mainly in the inherent ultra weak signal emitted by the landmine. To improve the signal-to-noise-ratio, it is common to increase the number of measurements. The expected error decreases as the square root of the numbe rof measurements. Hence repetition takes time, and the benefits gained with each additional measurement decrease. Hence we have argued for the inclusion of data to reveal other elements of the density operator. For example, it would be interesting to combine different pulse sequences that allow for a more complete reconstruction of the full density matrix of the spin-1 NQR system, and see whether this leads to a better detector as a result of

the further decreased minimal Bayes risk. To facilitate the communciation about the performance of such an approach, we introduced the concept of $\delta-$informational completeness. As the state contains all information about the system, a detector based on the reconstructed density operator, yields an approximation to a truly optimal detector. It remains to be investigated whether an implementation of a more complete state reconstruction offers practical improvements in terms of the necesary data acquisition time for demining applications.

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