SEQUENTIAL PRODUCTS OF QUANTUM MEASUREMENTS

Stan Gudder
Department of Mathematics
University of Denver
Denver, Colorado 80208
sgudder@math.du.edu

Abstract

Sequential products of quantum measurements are defined and studied. Two types of measurement equivalence are considered and their relationships with compatibility and the sequential product are discussed. It is shown that a measurement $\mathcal{A}$ is sharp if and only if $\mathcal{A}$ is equivalent to the sequential product of $\mathcal{A}$ with itself. Refinements of measurements are defined and it is shown that they produce a partial order on the set of measurements. Lattice properties of this partially ordered set are briefly discussed. Finally we consider convex combinations and conditioning for quantum measurements.

1 Introduction

A sequential product of quantum effects has recently been introduced and studied [4, 5, 6, 7]. The present paper extends this concept to a sequential product for quantum measurements. Since an effect can be thought of as a two-valued measurement, the sequential product of measurements generalizes that of effects in a natural way. In this work we only consider discrete quantum measurements which are measurements with at most a countable number of outcomes. Concepts such as sharp, compatible and coexistent measurements directly generalize the corresponding notions for effects. If $\mathcal{A}$ and $\mathcal{B}$ are quantum measurements on the same Hilbert space, we denote their sequential product by $\mathcal{A} \circ \mathcal{B}$. We interpret $\mathcal{A} \circ \mathcal{B}$ as performing the measurement $\mathcal{A}$ first and the measurement $\mathcal{B}$ next.

Two types of measurement equivalence are considered and their relationships with compatibility and the sequential product are discussed. In the first type of equivalence, $\mathcal{A} \approx \mathcal{B}$ if $\mathcal{A}$ and $\mathcal{B}$ are identical except for a possible ordering of their outcomes. It is shown that if $\mathcal{A}$ is sharp and $\mathcal{B}$ is arbitrary, then $\mathcal{A} \circ \mathcal{B} \approx \mathcal{B} \circ \mathcal{A}$ if and only if $\mathcal{A}$ and $\mathcal{B}$ are compatible. It is also shown that $\mathcal{A} \circ \mathcal{A} \approx \mathcal{A}$ if and only if $\mathcal{A}$ is sharp. Refinements of measurements are defined and we demonstrate that they produce a partial order on the set of measurements. Lattice properties of this partially ordered set are discussed. For example, we prove that if $\mathcal{A}$ is sharp and $\mathcal{B}$ is arbitrary, then the least upper bound $\mathcal{A} \wedge \mathcal{B}$ exists if and only if $\mathcal{A}$ and $\mathcal{B}$ are compatible and in this case $\mathcal{A} \wedge \mathcal{B} = \mathcal{A} \circ \mathcal{B}$.

We say that two measurements $\mathcal{A}$ and $\mathcal{B}$ coexist if they have a common refinement and show that this generalizes coexistence of effects. It is shown that if $\mathcal{A}$ is sharp and $\mathcal{B}$ is arbitrary, then $\mathcal{A}$ and $\mathcal{B}$ coexist if and only if $\mathcal{A}$ and $\mathcal{B}$ are compatible. We next define convex combinations of measurements and show that they have properties analogous to those of effects.

The last section discusses conditioning one measurement relative to another. This concept is defined directly in terms of the sequential product. Properties of conditioning are given and its relationship to conditional probability is discussed. Throughout the paper, various conjectures and open problems are presented.

2 Notation and Definitions

In classical probability theory we may think of an event as a characteristic function χ_A. If χ_B is another event, then $\chi_A\chi_B = \chi_{A\cap B}$ is the event that occurs when χ_A and χ_B both occur. A more complete description is given by the two-valued measurement $\mathcal{A} = \{\chi_A, \chi_{A'}\}$ where $\chi_{A'} = 1 - \chi_A$ is the complement (negation) of χ_A. We interpret $\mathcal{A}$ as having the value (outcome) 1 if χ_A occurs and the value (outcome) 0 if χ_A does not occur. If $\mathcal{B} = \{\chi_B, \chi_{B'}\}$ is another two-valued measurement, we can form their **sequential product**

$$\begin{aligned}\mathcal{A} \circ \mathcal{B} &= \{\chi_A\chi_B, \chi_A\chi_{B'}, \chi_{A'}\chi_B, \chi_{A'}\chi_{B'}\}\\ &= \{\chi_{A\cap B}, \chi_{A\cap B'}, \chi_{A'\cap B}, \chi_{A'\cap B'}\}\end{aligned}$$

We interpret $\mathcal{A}\circ\mathcal{B}$ as the measurement resulting from first performing measurement $\mathcal{A}$ and then performing measurement $\mathcal{B}$. Then $\mathcal{A}\circ\mathcal{B}$ has four values (outcomes) that we arbitrarily set to $1, 2, 3, 4$. These values correspond to whether χ_A and χ_B, χ_A and $\chi_{B'}$, $\chi_{A'}$ and χ_B or $\chi_{A'}$ and $\chi_{B'}$ both occur. Notice that just as $\chi_A + \chi_{A'} = 1$ we have

$$\chi_A\chi_B + \chi_A\chi_{B'} + \chi_{A'}\chi_B + \chi_{A'}\chi_{B'} = 1$$

Since $\mathcal{A}\circ\mathcal{B} = \mathcal{B}\circ\mathcal{A}$ the order of performing the measurements is irrelevant in classical probability theory. As we shall see, because of quantum interference, this is no longer true in quantum mechanics.

We can extend this discussion to n-valued measurements. In this case, $\mathcal{A} = \{f_1, \ldots, f_n\}$ where $f_1, \ldots, f_n$ are random variables satisfying $f_i f_j = 0$, $i \neq j$, and $\sum f_i = 1$. It follows that $f_i = \chi_{A_i}$ are characteristic functions with $A_i \cap A_j = \emptyset$, $i \neq j$, and $\cup A_i$ is the entire sample space Ω. That is $\{A_i \colon i = 1, \ldots, n\}$ is a partition of Ω. If $\mathcal{B} = \{g_1, \ldots, g_m\}$ is an m-valued measurement, we can form the sequential product.

$$\mathcal{A}\circ\mathcal{B} = \{f_i g_j \colon i = 1, \ldots, n, j = 1, \ldots, m\}$$

In general this gives an mn-valued measurement corresponding to a finer partition of Ω. Of course, we can also consider a countable partition of Ω corresponding to a discrete measurement $\{f_1, f_2, \ldots\}$.

This discussion can be further extended to classical fuzzy probability theory [5]. In this case our fuzzy events are represented by functions in $[0,1]^X$ for some nonempty set X. An n-valued measurement is given by $\mathcal{A} = \{f_1, \ldots, f_n\}$ where $f_i \in [0,1]^X$ satisfy $\sum f_i = 1$. In general, the fuzzy events f_i need not be characteristic functions. For the particular case in which they are characteristic functions ($f_i^2 = f_i$) we say that $\mathcal{A}$ is a **sharp** measurement. As before, if $\mathcal{B} = \{g_1, \ldots, g_m\}$ is an m-valued measurement then

$$\mathcal{A}\circ\mathcal{B} = \{f_i g_j \colon i = 1, \ldots, n, j = 1, \ldots, m\}$$

Notice that $f_i g_j \in [0,1]^X$ and $\sum f_i g_j = 1$ so $\mathcal{A}\circ\mathcal{B}$ is again a measurement. Again, $\mathcal{A}\circ\mathcal{B} = \mathcal{B}\circ\mathcal{A}$ so the order in which the measurements are performed is irrelevant.

We now consider quantum measurements. The formalism is similar to that of classical fuzzy probability theory except now the fuzzy events are represented by positive operators on a complex Hilbert space H. In this case the order in which measurements are performed is relevant and this is an essential feature of quantum mechanics. We denote the set of bounded linear operators on H by $\mathcal{B}(H)$. We use the notation

$$\mathcal{E}(H) = \{A \in \mathcal{B}(H) \colon 0 \le A \le I\}$$
$$\mathcal{P}(H) = \left\{P \in \mathcal{E}(H) \colon P^2 = P\right\}$$
$$\mathcal{D}(H) = \{\rho \in \mathcal{E}(H) \colon \operatorname{tr}(\rho) = 1\}$$

The elements of $\mathcal{E}(H)$ correspond to fuzzy quantum events and are called **effects**. The elements of $\mathcal{P}(H)$ are projections corresponding to quantum events and are called **sharp effects**. The elements of $\mathcal{D}(H)$ are density operators corresponding to probability measures and are called **states**. If $\rho \in \mathcal{D}(H)$, $A \in \mathcal{E}(H)$ then $\operatorname{tr}(\rho A)$ is the probability that A occurs (is observed) in the state ρ. For $A, B \in \mathcal{E}(H)$, the **sequential product** of A and B is $A \circ B = A^{1/2}BA^{1/2}$. It is easy to show that $A \circ B \in \mathcal{E}(H)$. We interpret $A \circ B$ as the effect that occurs when A occurs first and B occurs second [4, 5, 6, 7]. It can be shown that $A \circ B = B \circ A$ if and only if $AB = BA$ [7]. In this case we say that A and B are **compatible**.

A **measurement** is a finite or infinite sequence $\{A_i\}$ where $A_i \in \mathcal{E}(H)$ satisfy $\sum A_i = I$. If $\mathcal{A} = \{A_i\}$ is a measurement, then A_i is the effect observed when $\mathcal{A}$ is performed and the result is the ith outcome. We call $A_1, A_2, \ldots$ the **elements** of $\mathcal{A}$. If the system is in the state ρ and $\mathcal{A}$ is performed, then the probability that the result is the ith outcome is given by $\operatorname{tr}(\rho A_i)$. Notice that $i \mapsto \operatorname{tr}(\rho A_i)$ is a probability distribution because

$$\sum \operatorname{tr}(\rho A_i) = \operatorname{tr}\left(\rho \sum A_i\right) = \operatorname{tr}(\rho) = 1$$

A measurement is also called a **discrete positive operator-valued measure** (POVM). If $A_i \in \mathcal{P}(H)$ then $\mathcal{A} = \{A_i\}$ is a **sharp measurement** which is also called a **discrete projection-valued** measure (PVM). We denote the set of measurements by $\mathcal{M}(H)$ and the set of sharp measurements by $\mathcal{S}(H)$. A **submeasurement** is a finite or infinite sequence $\{A_i\}$ where $A_i \in \mathcal{E}(H)$ with $\sum A_i \le I$. We denote the set of submeasurements by sub-$\mathcal{M}(H)$. We define the set of sharp submeasurements in a similar way and denote this set by sub-$\mathcal{S}(H)$. Of course, if $\mathcal{A} = \{A_i\} \in$ sub-$\mathcal{M}(H)$ then we can extend $\mathcal{A}$ to a sequence containing $I - \sum A_i$ to obtain a measurement.

For $\mathcal{A}, \mathcal{B} \in$ sub-$\mathcal{M}(H)$ with $\mathcal{A} = \{A_i\}$, $\mathcal{B} = \{B_j\}$ we define the **sequential product** of $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{A} \circ \mathcal{B} = \{A_i \circ B_j\}$. If $\mathcal{A}, \mathcal{B} \in \mathcal{M}(H)$ we interpret

$\mathcal{A}\circ\mathcal{B}$ to be the measurement obtained when $\mathcal{A}$ is performed first and $\mathcal{B}$ is performed second. Moreover, we indeed have that $\mathcal{A}\circ\mathcal{B}\in\mathcal{M}(H)$ because

$$\sum_{i,j} A_i\circ B_j=\sum_{i,j} A_i^{1/2}B_jA_i^{1/2}=\sum_i A_i^{1/2}\sum_j B_jA_i^{1/2}=\sum A_i=I$$

In a similar way, if $\mathcal{A},\mathcal{B}\in\text{sub-}\mathcal{M}(H)$ then $\mathcal{A}\circ\mathcal{B}\in\text{sub-}\mathcal{M}(H)$. The sequential product is noncommutative and nonassociative in general. Defining the **identity measurement** $\mathcal{I}=\{I\}$, we have that $\mathcal{A}\circ\mathcal{I}=\mathcal{I}\circ\mathcal{A}=\mathcal{A}$ for all $\mathcal{A}\in\text{sub-}\mathcal{M}(H)$.

3 Equivalence

For $\mathcal{A},\mathcal{B}\in\text{sub-}\mathcal{M}(H)$, we write $\mathcal{A}\approx\mathcal{B}$ if the nonzero elements of $\mathcal{A}$ are a permutation of the nonzero elements of $\mathcal{B}$. Of course, $\approx$ is an equivalence relation and when $\mathcal{A}\approx\mathcal{B}$ we say that $\mathcal{A}$ and $\mathcal{B}$ are **equivalent**. In this case the two submeasurements are identical up to an ordering of their outcomes. It is clear that if $\mathcal{A}\approx\mathcal{B}$ then $\mathcal{C}\circ\mathcal{A}\approx\mathcal{C}\circ\mathcal{B}$ and $\mathcal{A}\circ\mathcal{C}\approx\mathcal{B}\circ\mathcal{C}$ for all $\mathcal{C}\in\text{sub-}\mathcal{M}(H)$.

For $\mathcal{A}\in\mathcal{M}(H)$, $\rho\in\mathcal{D}(H)$ we define

$$\mathcal{A}(\rho)=\sum A_i\circ\rho=\sum A_i^{1/2}\rho A_i^{1/2}$$

Notice that $\mathcal{A}(\rho)\in\mathcal{D}(H)$ and we interpret $\mathcal{A}(\rho)$ as the post-measurement state when the input state is ρ and the outcome of the measurement is not observed. We write $\mathcal{A}\sim\mathcal{B}$ and say that $\mathcal{A}$ and $\mathcal{B}$ are **weakly equivalent** if $\mathcal{A}(\rho)=\mathcal{B}(\rho)$ for every $\rho\in\mathcal{D}(H)$. It is clear that $\sim$ is an equivalence relation and that $\mathcal{A}\approx\mathcal{B}$ implies $\mathcal{A}\sim\mathcal{B}$. In contrast to $\mathcal{A}\approx\mathcal{B}$, if $\mathcal{A}\sim\mathcal{B}$ we cannot consider $\mathcal{A}$ and $\mathcal{B}$ to be essentially identical. This is because $\text{tr}(\rho A_i)$ may not coincide with $\text{tr}(\rho B_j)$ for any j. In this case the probability of outcome i for measurement $\mathcal{A}$ does not equal the probability of any outcome of measurement $\mathcal{B}$. By the unitary freedom theorem [11], $\mathcal{A}\sim\mathcal{B}$ if and only if there exists a (possibly infinite) unitary matrix $[u_{ij}]$ such that $A_i^{1/2}=\sum u_{ij}B_j^{1/2}$ where we augment $\mathcal{A}$ or $\mathcal{B}$ with 0s if necessary so they have the same number of elements.

Lemma 3.1. *If $\mathcal{A},\mathcal{B}\in\mathcal{S}(H)$ then $\mathcal{A}\sim\mathcal{B}$ if and only if $\mathcal{A}\approx\mathcal{B}$.*

Proof. Of course, $\mathcal{A} \approx \mathcal{B}$ implies $\mathcal{A} \sim \mathcal{B}$. If $\mathcal{A} \sim \mathcal{B}$ there exists a unitary matrix $[u_{ij}]$ such that $A_i = \sum u_{ij} B_j$ for all i. Since u_{ij} is in the spectrum of A_i, $u_{ij} = 0$ or 1. Since $[u_{ij}]$ is unitary, there is a j such that $u_{ij} = 1$ and $u_{ik} = 0$ for $k \neq j$. It follows that $[u_{ij}]$ is a permutation matrix. Hence $\mathcal{A} \approx \mathcal{B}$. □

We say that $\mathcal{A} = \{A_i\}$ and $\mathcal{B} = \{B_j\}$ are **compatible** if $A_i B_j = B_j A_i$ for all i and j. It follows from the unitary freedom theorem that $\sim$ preserves compatibility; that is, if $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{C}$ is compatible with $\mathcal{A}$, then $\mathcal{C}$ is compatible with $\mathcal{B}$. It is clear that if $\mathcal{A}$ and $\mathcal{B}$ are compatible, then $\mathcal{A} \circ \mathcal{B} \approx \mathcal{B} \circ \mathcal{A}$. The converse does not hold. Indeed, $\mathcal{A} \circ \mathcal{A} \approx \mathcal{A} \circ \mathcal{A}$ and yet $\mathcal{A}$ need not be compatible with itself. In general, sequential products do not preserve $\sim$; that is $\mathcal{A} \sim \mathcal{B}$ does not imply that $\mathcal{C} \circ \mathcal{A} \sim \mathcal{C} \circ \mathcal{B}$ or that $\mathcal{A} \circ \mathcal{C} \sim \mathcal{B} \circ \mathcal{C}$. However, we do have the following result.

Lemma 3.2. *If $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{C}$ is compatible with $\mathcal{A}$ and $\mathcal{B}$, then $\mathcal{C} \circ \mathcal{A} \sim \mathcal{C} \circ \mathcal{B}$.*

Proof. For $\rho \in \mathcal{D}(H)$ we have

$$\begin{aligned}(\mathcal{C} \circ \mathcal{A})(\rho) &= \sum_{i,j} (C_i A_j)^{1/2} \rho (C_i A_j)^{1/2} = \sum_{i,j} C_i^{1/2} A_j^{1/2} \rho A_j^{1/2} C_i^{1/2} \\ &= \sum_i C_i^{1/2} \sum_j A_j^{1/2} \rho A_j^{1/2} C_i^{1/2} = \sum_i C_i^{1/2} \sum_j B_j^{1/2} \rho B_j^{1/2} C_i^{1/2} \\ &= \sum_{i,j} (C_i B_j)^{1/2} \rho (C_i B_j)^{1/2} = (\mathcal{C} \circ \mathcal{B})(\rho)\end{aligned}$$

Hence, $\mathcal{C} \circ \mathcal{A} \sim \mathcal{C} \circ \mathcal{B}$. □

The next lemma will be useful for proving Theorem 3.4.

Lemma 3.3. *For $A \in \mathcal{B}(H)$, $B \in \mathcal{E}(H)$, if $ABA^* = 0$ then $BA^* = 0$.*

Proof. Since $ABA^* = 0$, for every $x \in H$ we have

$$\|B^{1/2} A^* x\|^2 = \left\langle B^{1/2} A^* x, B^{1/2} A^* x \right\rangle = \langle ABA^* x, x \rangle = 0$$

Hence, $B^{1/2} A^* x = 0$. so $BA^* x = 0$ for all $x \in H$. Thus, $BA^* = 0$. □

Theorem 3.4. *If $\mathcal{P} \in$ sub-$\mathcal{S}(H)$ and $\mathcal{A} \in$ sub-$\mathcal{M}(H)$ and $\mathcal{P} \circ \mathcal{A} \approx \mathcal{A} \circ \mathcal{P}$, then $\mathcal{P}$ and $\mathcal{A}$ are compatible.*

Proof. Letting $\mathcal{P} = \{P_i\}$ and $\mathcal{A} = \{A_i\}$ we have that $\{P_i \circ A_j\} \approx \{A_j \circ P_i\}$. If $P_i \circ A_j = A_j \circ P_i$ then $P_iA_j = A_jP_i$ [7]. Suppose $P_i \circ A_j = A_r \circ P_s$, $s \neq i$, so that $P_iA_jP_i = A_r^{1/2}P_sA_r^{1/2}$. Pre and post multiplying by P_s gives $(P_sA_r^{1/2})P_s(P_sA_r^{1/2})^* = 0$. By Lemma 3.3 we have

$$P_sA_r^{1/2}P_s = P_s(P_sA_r^{1/2})^* = 0$$

Hence, $P_sA_r = A_rP_s$. Finally, suppose $P_i \circ A_j = A_r \circ P_i$ so that $P_iA_jP_i = A_r^{1/2}P_iA_r^{1/2}$. Pre and Post multiplying by P_k, $k \neq i$, gives

$$(P_kA_r^{1/2})P_i(P_kA_r^{1/2})^* = 0$$

By Lemma 3.3 we have $P_iA_r^{1/2}P_k = 0$, $k \neq i$. Summing over k we obtain

$$P_iA_r^{1/2}(I - P_i) = 0$$

Hence, $P_iA_r^{1/2} = P_iA_i^{1/2}P_i$ so that $P_iA_r^{1/2} = A_r^{1/2}P_i$. It follows that $P_iA_r = A_rP_i$. We conclude that $\mathcal{P}$ and $\mathcal{A}$ are compatible. □

We define the **supplement** of $\mathcal{A} \in \mathcal{E}(H)$ by $A' = I - A$.

Lemma 3.5. *Let* $A, B \in \mathcal{E}(H)$ *and suppose that* $\dim H < \infty$. *Then* $A \circ B + A' \circ B = B'$ *if and only if* $B = \frac{1}{2}I$.

Proof. It is clear that $B = \frac{1}{2}I$ implies $A \circ B + A' \circ B = B'$. Conversely, assume that $A \circ B + A' \circ B = B'$. Then $B = I - A \circ B - A' \circ B$. Hence,

$$\begin{aligned} B &= I - A \circ (I - A \circ B - A' \circ B) - A' \circ (I - A \circ B - A' \circ B) \\ &= A^2 \circ B + 2(A \circ A') \circ B + (A')^2 \circ B \end{aligned}$$

Since

$$A^2 + 2(A \circ A") + (A')^2 = I$$

we conclude from [1] that $BA^2 = A^2B$ so that $AB = BA$. Therefore,

$$B = I - AB - A'B = I - B$$

so that $B = I/2$. □

Although $\mathcal{A}\circ\mathcal{B}\approx\mathcal{B}\circ\mathcal{A}$ does not imply that $\mathcal{A}$ and $\mathcal{B}$ are compatible in general, the next result shows that this does hold for two-valued measurements when $\dim H\infty$. We do not know if the condition $\dim H<\infty$ can be relaxed.

Theorem 3.6. *Suppose that* $\dim H<\infty$ *and* $\mathcal{A}=\{A,A'\}$, $\mathcal{B}=\{B,B'\}$. *If* $\mathcal{A}\circ\mathcal{B}\approx\mathcal{B}\circ\mathcal{A}$, *then* $AB=BA$.

Proof. If $\mathcal{A}\circ\mathcal{B}\approx\mathcal{B}\circ\mathcal{A}$ we have that

$$\{A\circ B,A\circ B',A'\circ B,A'\circ B'\}\approx\{B\circ A,B\circ A',B'\circ A,B'\circ A'\}$$

If $A\circ B=B\circ A$, then $AB=BA$ [7] and similarly for the other corresponding terms. We can therefore assume equality for noncorresponding terms.
Case 1. If $A\circ B=B\circ A'$ and $A\circ B'=B\circ A$, then

$$A=A\circ B+A\circ B'=B\circ A'+B\circ A=B$$

so clearly $AB=BA$.
Case 2. If $A\circ B=B\circ A'$ and $A'\circ B=B\circ A$, then

$$B=B\circ A+B\circ A'=A'\circ B+A\circ B$$

and it follows from [1] that $AB=BA$.
Case 3. If $A'\circ B=B\circ A'$ and $A'\circ B'=B'\circ A$, then

$$A'=A'\circ B+A'\circ B'=B\circ A+B'\circ A$$

Applying Lemma 3.5 we conclude that $A=\frac{1}{2}I$ so clearly $AB=BA$. The other cases are similar $\square$

Theorem 3.7. *Let* $\mathcal{A}=\{A_i\},\mathcal{B}=\{B_j\}\in\mathcal{M}(H)$. *Then* $\mathcal{A}\circ\mathcal{B}\in\mathcal{S}(H)$ *if and only if* $\mathcal{A},\mathcal{B}\in\mathcal{S}(H)$ *and* $\mathcal{A}\circ\mathcal{B}\approx\mathcal{B}\circ\mathcal{A}$.

Proof. Since $A_i\circ B_j\in\mathcal{P}(H)$ we have that $A_iB_j=B_jA_i$ for all i,j so $\mathcal{A}\circ\mathcal{B}=\mathcal{B}\circ\mathcal{A}$ [7]. Now $A_iB_j\in\mathcal{P}(H)$ and $(A_iB_j)(A_iB_k)=0$ imply that $A_i=\sum_j A_iB_j\in\mathcal{P}(H)$. Similarly, $B_j\in\mathcal{P}(H)$. The converse is clear. $\square$

4 Properties of Sequential Products

This section derives various properties for sequential products of measurements. We also discuss an order on $\mathcal{M}(H)$. The order on $\mathcal{E}(H)$ is the usual operator order.

Lemma 4.1. *Any* $\mathcal{A} \in \text{sub-}\mathcal{M}(H)$ *has a maximal element.*

Proof. Let $\mathcal{C}$ be a chain in $\mathcal{A}$. Then $\mathcal{C}$ has a largest element. Indeed, otherwise there exist $A_i \in \mathcal{C}$ with $0 < A_1 < A_2 < \cdots$. If $\langle A_1 x, x\rangle \neq 0$, then

$$0 < \langle A_1 x, x\rangle \leq \langle A_2 x, x\rangle \leq \cdots$$

and this contradicts the fact that $\sum \langle A_i x, x\rangle \leq 1$. Hence, $\mathcal{C}$ has an upper bound in $\mathcal{A}$. By Zorn's lemma, $\mathcal{A}$ has a maximal element. □

Lemma 4.2. *If* $\mathcal{A} = \{A_i\} \in \text{sub-}\mathcal{M}(H)$ *and* $A_j \in \mathcal{P}(H)$ *for some* j*, then* $A_j \circ A_k = 0$ *for all* $k \neq j$*.*

Proof. Since

$$\begin{aligned} A_j = A_j \circ I \geq A_j \circ \sum_k A_k = \sum_k A_j \circ A_k = A_j \circ A_j + \sum_{k\neq j} A_j \circ A_k \\ = A_j + \sum_{k\neq j} A_j \circ A_k \end{aligned}$$

we have that $\sum_{k\neq j} A_j \circ A_k = 0$. Hence, $A_j \circ A_k = 0$ for $k \neq j$. □

The next result shows that sharp measurements are precisely the idempotents in $\mathcal{M}(H)$ under sequential products.

Theorem 4.3. *For* $\mathcal{A} \in \text{sub-}\mathcal{M}(H)$ *we have that* $\mathcal{A} \circ \mathcal{A} \approx \mathcal{A}$ *if and only if* $\mathcal{A} \in \text{sub-}\mathcal{S}(H)$*.*

Proof. It follows from Lemma 4.2 that for $\mathcal{A} \in \text{sub-}\mathcal{S}(H)$ we have $\mathcal{A} \circ \mathcal{A} \approx \mathcal{A}$. Conversely, suppose $\mathcal{A} = \{A_i\}$ and $\mathcal{A} \circ \mathcal{A} \approx \mathcal{A}$. Then $\{A_i \circ A_j\} \approx \{A_i\}$ and we may assume that $A_i \neq 0$ for all i. Let $B_1, B_2, \ldots$ be all the projections in $\mathcal{A}$. By Lemma 4.2, $B_i \circ A_j = 0$ whenever $A_j \neq B_i$. Let $\mathcal{A}_1$ be the submeasurement $\mathcal{A} \smallsetminus \{B_i \colon i = 1, 2, \ldots\}$ and assume that $\mathcal{A}_1 \neq \emptyset$. Writing $\mathcal{A}_1 = \{C_i\}$ we have that $\mathcal{A}_1 \circ \mathcal{A}_1 \approx \mathcal{A}_1$ so that $\{C_i \circ C_j\} \approx \{C_i\}$. By Lemma 4.1, $\{C_i = C_j\}$ has a maximal element, say $C_i \circ C_j$. Since $C_i \circ C_j \leq C_i$

and C_i is an element of $\{C_i \circ C_j\}$ we have that $C_i \circ C_j = C_i$. It follows from [7] that

$$C_i \circ C_j = C_j \circ C_i = C_i$$

But now $C_j \circ C_i$ is a maximal element and $C_j \circ C_i \le C_j$. Hence, $C_j = C_j \circ C_i = C_i$. We conclude that $C_i \circ C_i = C_i$ so $C_i \in \mathcal{P}(H)$. Since this is a contradiction, $\mathcal{A}_1 = \emptyset$. Hence, $\{A_i\} = \{B_i\} \in \text{sub-}\mathcal{S}(H)$. $\square$

We now give a simple example which shows that Theorem 4.3 does not hold for weak equivalence. Let $\mathcal{A} = \left\{\frac{1}{2}I, \frac{1}{2}I\right\}$. Then

$$\mathcal{A} \circ \mathcal{A} = \left\{\tfrac{1}{4}I, \tfrac{1}{4}I, \tfrac{1}{4}I, \tfrac{1}{4}I\right\} \sim \mathcal{A}$$

but $\mathcal{A}$ is not sharp. However, $\mathcal{A} \sim \mathcal{I} \in \mathcal{S}(H)$. We say that $\mathcal{A} \in \text{sub-}\mathcal{M}(H)$ is **finite** if $\mathcal{A}$ has a finite number of elements.

Theorem 4.3 is easier to prove for finite $\mathcal{A} \in \mathcal{M}(H)$. Indeed, let $\mathcal{A} = \{A_1, \dots, A_n\}$ where $A_i \ne 0$, $i = 1, \dots, n$, and suppose $\mathcal{A} \circ \mathcal{A} \approx \mathcal{A}$. Then $\{A_i \circ A_j\} \approx \{A_i\}$ and since $A_i \ne 0$, $A_i^2 \ne 0$, $i = 1, \dots, n$. Since $\mathcal{A} \circ \mathcal{A}$ and $\mathcal{A}$ have the same number of nonzero elements, we conclude that $A_i \circ A_j = 0$, $i \ne j$. It follows that

$$A_i = A_i \circ I = A_i \circ \sum_j A_j = \sum_j A_i \circ A_j = A_i^2$$

so $A_i \in \mathcal{P}(H)$, $i = 1, \dots, n$. Hence, $\mathcal{A} \in \mathcal{S}(H)$.

For $\mathcal{A} = \{A_i\} \in \text{sub-}\mathcal{M}(H)$ and $\mathcal{B} = \{B_j\} \in \text{sub-}\mathcal{M}(H)$ we call $\mathcal{A}$ a **refinement** of $\mathcal{B}$ and write $\mathcal{A} \le \mathcal{B}$ if we can adjoin 0s to $\mathcal{A}$ if necessary and organize the elements of $\mathcal{A}$ so that $\mathcal{A} \approx \{A_{ij}\}$ and $B_i = \sum_j A_{ij}$ for all i. For example, $\mathcal{A} \circ \mathcal{B} \le \mathcal{A}$. Indeed, $\mathcal{A} \circ \mathcal{B} = \{A_i \circ B_j\}$ and $A_i = \sum_j A_i \circ B_j$ for all i. We now show that the converse does not hold.

This example shows that $\mathcal{A} \le \mathcal{B}$ does not imply that $\mathcal{A} \approx \mathcal{B} \circ \mathcal{C}$ for some $\mathcal{C} \in \text{sub-}\mathcal{M}(H)$. Let $A \in \mathcal{E}(H)$ have the properties that A and A' are invertible and $A \ne \lambda I$ for $0 < \lambda < 1$. Letting $\mathcal{B} = \{A, A'\}$ and $\mathcal{A} = \left\{\frac{1}{2}A, \frac{1}{2}A, A'\right\}$ we have that $\mathcal{A} \le \mathcal{B}$. Suppose that $\mathcal{A} \approx \mathcal{B} \circ \mathcal{C}$ where $\mathcal{C} = \{C_1, C_2, \dots\}$. Assume that $A' = A' \circ C_i$ for some i. Since A' is invertible, we have $C_i = I$. But then $\mathcal{C} = \mathcal{I}$ and $\mathcal{A} \approx \mathcal{B}$ which is a contradiction. Otherwise, $A' = A \circ C_i$ for some i. Since A is invertible, we have that $C_i = A^{-1} - I$. Hence,

$$A' \circ C_i = (I - A)(A^{-1} - I) = A^{-1} + A - 2I$$

If $A' \circ C_i = \frac{1}{2}A$, then $A^{-1} + \frac{1}{2}A - 2I = 0$ or $A^2 - 4A + 2I = 0$. It follows that $A = \left(2 - \sqrt{2}\right) I$ which is a contradiction. If $A' \circ C_i = A'$, then $A^{-1} + A - 2I = I - A$ or $2A^2 - 3A + I = 0$. Hence, $(I - 2A)(I - A) = 0$ which implies that $A = \frac{1}{2}I$. Since this is again a contradiction, we conclude that $\mathcal{A} \not\approx \mathcal{B} \circ \mathcal{C}$ for any $\mathcal{C} \in \text{sub-}\mathcal{M}(H)$.

We now show that $\le$ gives a partial order on sub-$\mathcal{M}(H)$. Strictly speaking, we are considering equivalence classes because we use $\approx$ instead of equality.

Theorem 4.4. (sub-$\mathcal{M}(H), \le$) *is a poset in which* $\mathcal{A} \le \mathcal{B}$ *implies* $\mathcal{C} \circ \mathcal{A} \le \mathcal{C} \circ \mathcal{B}$.

Proof. It is clear that $\le$ is reflexive and it is not hard to show that $\le$ is transitive. To prove anti-symmetry, let $\mathcal{A} = \{A_i\}$, $\mathcal{B} = \{B_j\}$ and suppose that $\mathcal{A} \le \mathcal{B}$ and $\mathcal{B} \le \mathcal{A}$. Eliminate from $\mathcal{A}$ all the elements that have a singleton sum decomposition $A_i = B_j$ in the definition of refinement and denote the resulting submeasurement by $\mathcal{A}_1$. After eliminating these elements from $\mathcal{B}$ we have the submeasurement $\mathcal{B}_1$ and $\mathcal{A}_1 \le \mathcal{B}_1$, $\mathcal{B}_1 \le \mathcal{A}_1$. By Lemma 4.1, $\mathcal{A}_1$ has a maximal element A_i. Then some $B_j \in \mathcal{B}_1$ satisfies $B_j = A_i + A_{j_1} + A_{j_2} + \cdots$. Hence, $B_j \ge A_i$. Now there exists an $A_k \in \mathcal{A}_1$ such that $A_k = B_j + B_{k_1} + B_{k_2} + \cdots$. Hence, $A_i \ge B_j \ge A_i$. Since A_i is maximal in $\mathcal{A}_1$, we have that $A_k = B_j = A_i$. Hence, A_i has a singleton sum decomposition which is contradiction. We conclude that $\mathcal{A}_1 = \mathcal{B}_1 = \emptyset$ and $\mathcal{A} \approx \mathcal{B}$. Finally, suppose that $\mathcal{A} \le \mathcal{B}$. We then have that $B_i = \sum_j A_{ij}$ for all i so that

$$C_k \circ B_i = C_k \circ \sum_j A_{ij} = \sum_j C_k \circ A_{ij}$$

It follows that $\mathcal{C} \circ \mathcal{B} \le \mathcal{C} \circ \mathcal{A}$. □

It follows from Theorem 4.4 that $(\mathcal{M}(H), \le)$ is a poset with largest element $\mathcal{I}$.

Theorem 4.5. (a) *If* $\mathcal{A} \in \text{sub-}\mathcal{S}(H)$, $\mathcal{B} \in \text{sub-}\mathcal{M}(H)$ *and* $\mathcal{B} \le \mathcal{A}$, *then* $\mathcal{B} \approx \mathcal{A} \circ \mathcal{B} \approx \mathcal{B} \circ \mathcal{A}$. (b) *If* $\mathcal{A} \in \text{sub-}\mathcal{S}(H)$, $\mathcal{B} \in \text{sub-}\mathcal{M}(H)$ *and* $\mathcal{A} \le \mathcal{B}$, *then* $\mathcal{B} \in \text{sub-}\mathcal{S}(H)$ *and* $\mathcal{A} \approx \mathcal{A} \circ \mathcal{B} \approx \mathcal{B} \circ \mathcal{A}$. (c) *If* $\mathcal{A}, \mathcal{B} \in \text{sub-}\mathcal{M}(H)$ *and* $\mathcal{A} \circ \mathcal{B} \approx \mathcal{A}$, *then* $\mathcal{A}$ *and* $\mathcal{B}$ *are compatible,* $\mathcal{B} \in \text{sub-}\mathcal{S}(H)$ *and* $\mathcal{A} \le \mathcal{B}$.

Proof. (a) Since $\mathcal{B} \leq \mathcal{A}$, we can write $\mathcal{B} = \{B_{ij}\}$, $\mathcal{A} = \{A_i\}$ where $A_i = \sum_j B_{ij}$ for all i. Since $B_{ij} \leq A_i$ we have that

$$B_{ij} \circ A_i = A_i \circ B_{ij} = B_{ij}$$

for all i and $B_{ij} \circ A_k = 0$ for all j and $k \neq i$. It follows that $\mathcal{B} \approx \mathcal{A} \circ \mathcal{B} \approx \mathcal{B} \circ \mathcal{A}$. (b) It is clear that $\mathcal{B} \in \text{sub-}\mathcal{S}(H)$ and the result follows from (a). (c) By Lemma 4.1, $\mathcal{A}$ has a maximal element $A_k \in \mathcal{A}$. Now $A_k = A_i \circ B_j$ for some i and j. Since $A_k = A_i \circ B_j \leq A_i$ we have that $A_k = A_i$. Hence, $A_k = A_k \circ B_j$. It follows that $A_k \circ B_j = B_j \circ A_k$ [7]. Since $A_k = A_k \circ B_j = \sum_r A_k \circ B_r$ we have that $A_k \circ B_r = 0$ for $r \neq j$. Call A_k type 1 if $A_k \circ B_j = A_k$ for some j. Let $\mathcal{A}_1 = \{A_{i_1}, A_{i_2}, \dots\}$ be the non-type 1 elements of $\mathcal{A}$. If $\mathcal{A}_1 \neq \emptyset$, then by Lemma 4.1, $\mathcal{A}_1$ contains a maximal element A_{i_k}. As before, there is a B_s such that $A_{i_k} \circ B_s = A_{i_k}$ which is a contradiction. Hence, all elements of $\mathcal{A}$ are type 1. No for every B_j we have $A_{j_1} \circ B_j = A_{j_1}, A_{j_2} \circ B_j = A_{j_2}, \cdots$ and $A_i \circ B_j = 0$, $i \neq j_1, j_2, \dots$. Hence,

$$B_j = \sum_i B_j \circ A_{j_i} = \sum_i A_{j_i} \circ B_j = \sum_i A_{j_i}$$

Therefore, $\mathcal{A} \leq \mathcal{B}$ and $\mathcal{A}$, $\mathcal{B}$ are compatible. Moreover,

$$B_j^2 = B_j \circ B_j = \sum_i B_j \circ A_{j_i} = B_j$$

Hence, $\mathcal{B} \in \text{sub-}\mathcal{S}(H)$. $\square$

We now briefly discuss the lattice properties of the poset $\mathcal{M}(H)$. As we shall show, $\mathcal{A} \wedge \mathcal{B}$ does not always exist. The characterization of pairs $\mathcal{A}$, $\mathcal{B}$ for which $\mathcal{A} \wedge \mathcal{B}$ (or $\mathcal{A} \vee \mathcal{B}$) exist is an open problem. Denoting the set of two-element measurements by $\mathcal{M}_2(H)$, it is clear that the members of $\mathcal{M}_2(H)$ are anti-atoms. Hence, if $\mathcal{A}, \mathcal{B} \in \mathcal{M}_2(H)$ and $\mathcal{A} \neq \mathcal{B}$, then $\mathcal{A} \vee \mathcal{B} = I$. More generally, for $\mathcal{A} \in \mathcal{M}(H)$, $\mathcal{B} \in \mathcal{M}_2(H)$, if $\mathcal{A} \leq \mathcal{B}$ then $\mathcal{A} \vee \mathcal{B} = \mathcal{B}$ and if $\mathcal{A} \not\leq \mathcal{B}$ then $\mathcal{A} \vee \mathcal{B} = I$

We shall show in Section 5 that for $\mathcal{A} \in \mathcal{M}(H)$, $\mathcal{B} \in \mathcal{S}(H)$, $\mathcal{A} \wedge \mathcal{B}$ exists if and only if $\mathcal{A}$ and $\mathcal{B}$ are compatible and in this case $\mathcal{A} \wedge \mathcal{B} = \mathcal{A} \circ \mathcal{B}$. In general, even when $\mathcal{A}$ and $\mathcal{B}$ are compatible, $\mathcal{A} \circ \mathcal{B} \neq \mathcal{A} \wedge \mathcal{B}$. For example, if $\mathcal{A} \in \mathcal{M}(H) \smallsetminus \mathcal{S}(H)$ then $\mathcal{A} \wedge \mathcal{A} = \mathcal{A}$ and $\mathcal{A} \circ \mathcal{A} \neq \mathcal{A}$. As another example, let

$$\mathcal{A} = \left\{\tfrac{1}{4}I, \tfrac{1}{4}I, \tfrac{1}{2}I\right\}$$
$$\mathcal{B} = \left\{\tfrac{1}{2}I, \tfrac{1}{2}I\right\}$$

Then $\mathcal{A} \le \mathcal{B}$ so that $\mathcal{A} \wedge \mathcal{B} = \mathcal{A}$. However,

$$\mathcal{A} \circ \mathcal{B} = \left\{\tfrac{1}{8}I, \tfrac{1}{8}I, \tfrac{1}{4}I, \tfrac{1}{8}I, \tfrac{1}{8}I, \tfrac{1}{4}I\right\} \ne \mathcal{A}$$

For a slightly more complicated example, let

$$\mathcal{A} = \left\{\tfrac{1}{2}I, \tfrac{1}{3}I, \tfrac{1}{6}I\right\}$$
$$\mathcal{B} = \left\{\tfrac{1}{2}I, \tfrac{1}{2}I\right\}$$
$$\mathcal{C} = \left\{\tfrac{2}{3}I, \tfrac{1}{3}I\right\}$$

Then $\mathcal{A} = \mathcal{B} \wedge \mathcal{C}$. Indeed, $\mathcal{A} \le \mathcal{B}, \mathcal{C}$ and if $\mathcal{D} \le \mathcal{B}, \mathcal{C}$ with $\mathcal{D} = \{\mathcal{D}_i\}$, then $\frac{1}{2}I = \sum_r D_{i_r}$, $\frac{1}{3}I = \sum_s D_{i_s}$, so that $\frac{1}{6}I = \sum_t D_{i_t}$. Hence, $\mathcal{D} \le \mathcal{A}$ so that $\mathcal{A} = \mathcal{B} \wedge \mathcal{C}$. However,

$$\mathcal{A} \ne \mathcal{B} \circ \mathcal{C} = \left\{\tfrac{1}{3}I, \tfrac{1}{3}I, \tfrac{1}{6}I, \tfrac{1}{6}I\right\}$$

5 Coexistence

For simplicity of notation, in Section 5 and 6 we shall write $\mathcal{A} = \mathcal{B}$ for $\mathcal{A} \approx \mathcal{B}$. In this way, we are implicitly working with equivalence classes. Two effects $A, B \in \mathcal{E}(H)$ **coexist** if there exist effects $C_1, C_2, C_3 \in \mathcal{E}(H)$ such that $C_1 + C_2 + C_3 \le I$ and $A = C_1 + C_2$, $B = C_2 + C_3$. Coexistence has been thoroughly studied in the literature [2, 3, 8, 9, 10]. We say that $A \in \mathcal{E}(H)$ is **associated** with $\mathcal{A} = \mathcal{M}(H)$ if $\mathcal{A} \le \{A, A'\}$.

Lemma 5.1. *Two effects $A, B \in \mathcal{E}(H)$ coexist if and only if A, B are associated with a common measurement $\mathcal{A} \in \mathcal{M}(H)$.*

Proof. If A and B coexist, let C_1, C_2, C_3 be as in the definition. Letting $C_4 = I - C_1 - C_2 - C_3$ we have the measurement $\mathcal{A} = \{C_1, C_2, C_3, C_4\}$ and since $A = C_1 + C_2$, $B = C_2 + C_3$, A and B are both associated with $\mathcal{A}$.

Conversely, suppose A and B are both associated with $\mathcal{A} = \{A_i\} \in \mathcal{M}(H)$. We can then write

$$A = \sum_r A_{i_r} + \sum_s A_{i_s}$$
$$B = \sum_s A_{i_s} + \sum_t A_{i_t}$$

where $i_r \neq i_t$ for all s, t. Letting $C_1 = \sum_r A_{i_r}$, $C_2 = \sum_s A_{i_s}$, $C_3 = \sum_t A_{i_t}$ we have that $A = C_1 + C_2$, $B = C_2 + C_3$ and $C_1 + C_2 + C_3 \leq I$. Hence A and B coexist. □

The following lemma summarizes some of the elementary properties of coexistence [9, 10].

Lemma 5.2. (a) *If $A, B \in \mathcal{E}(H)$ and $A + B = P \in \mathcal{P}(H)$, then A, B are compatible.* (b) *IF $A, B \in \mathcal{E}(H)$ are compatible, then A, B coexist.* (c) *If $A \in \mathcal{E}(H)$, $P \in \mathcal{P}(H)$ and A, P coexist, then A, P are compatible.*

Proof. (a) Since $A \leq P$ we have that $AP = PA = A$. Hence,

$$A^2 + AB = AP = A$$

Therefore, $AB = A - A^2$ and taking adjoints gives $AB = BA$. (b) If A, B are compatible we have $AB \in \mathcal{E}(H)$ and we can write

$$A = (A - AB) + AB$$
$$B = (B - AB) + AB$$

Moreover,

$$(A - AB) + (B - AB) + AB = A(I - B) + B \leq (I - B) + B = I$$

(c) As in the definition, $A = C_1 + C_2$, $P = C_2 + C_3$ where $C_1 + C_2 + C_3 \leq I$. Now there exists a $C_4 \in \mathcal{E}(H)$ with $C_1 + C_2 + C_3 + C_4 = I$. By (a) we have $C_2C_3 = C_3C_2$ and $C_1C_4 = C_4C_1$. Hence,

$$C_1(C_1 + P) = (C_1 + P)C_1$$

so that $C_1P = PC_1$. Therefore, $AP = PA$. □

Notice that coexistence is a much weaker property than compatibility. In fact, it is easy to find $A, B, C \in \mathcal{E}(H)$ such that $A+B+C=I$ and no pair of the A, B, C commute. Letting $D = A + B$, $E = B + C$ we have that D and E coexist and $DE \neq ED$ in general. We say that $\mathcal{A}, \mathcal{B} \in \mathcal{M}(H)$ **coexist** if they have a common refinement, $\mathcal{C} \leq \mathcal{A}, \mathcal{B}$. (This definition is different than, but similar to, the definition of coexistence of observables [9, 10]. Notice that if $\mathcal{A} \circ \mathcal{B} = \mathcal{B} \circ \mathcal{A}$, then $\mathcal{A} \circ \mathcal{B} \leq \mathcal{A}, \mathcal{B}$ so $\mathcal{A}$, $\mathcal{B}$ coexist. The next lemma shows that this definition generalizes the definition of coexistence of effects.

Lemma 5.3. *Two effects $A, B \in \mathcal{E}(H)$ coexist if and only if $\{A, A'\}, \{B, B'\}$ coexist.*

Proof. This follows directly from Lemma 5.1. □

Notice that if $\{A_i\}$, $\{B_j\}$ coexist, then $\{A_i, A_i'\}$, $\left\{B_j, B_j'\right\}$ coexist so A_i, B_j coexist for all i, j. The example at the end of this section shows the converse does not hold. Since $\mathcal{A} \leq \mathcal{B}$ implies $\mathcal{C} \circ \mathcal{A} \leq \mathcal{C} \circ \mathcal{B}$, it follows that if $\mathcal{A}$, $\mathcal{B}$ coexist, then $\mathcal{C} \circ \mathcal{A}$, $\mathcal{C} \circ \mathcal{B}$ coexist. In contrast, if $\mathcal{A}$ and $\mathcal{B}$ are compatible, then $\mathcal{C} \circ \mathcal{A}$ and $\mathcal{C} \circ \mathcal{B}$ need not be compatible.

Theorem 5.4. *If $\mathcal{A} \in \mathcal{M}(H)$ and $\mathcal{B} \in \mathcal{S}(H)$ coexist, then $\mathcal{A}$ and $\mathcal{B}$ are compatible.*

Proof. Suppose $\mathcal{A} = \{A_i\}$ and $\mathcal{B} = \{B_j\}$ coexist so they have common refinement. Then $\{A_i, A_i'\}$ and $\left\{B_j, B_j'\right\}$ have a common refinement for all i, j. Hence $\{A_i, A_i'\}$ and $\left\{B_j, B_j'\right\}$ coexist so by Lemma 5.3, A_i and B_j coexist. Applying Lemma 5.2(c), A_i, B_j are compatible so $\mathcal{A}$, $\mathcal{B}$ are compatible. □

Theorem 5.5. *For $\mathcal{A} \in \mathcal{M}(H)$ and $\mathcal{B} \in \mathcal{S}(H)$, $\mathcal{A} \wedge \mathcal{B}$ exists if and only if $\mathcal{A}$ and $\mathcal{B}$ are compatible. In this case $\mathcal{A} \wedge \mathcal{B} = \mathcal{A} \circ \mathcal{B}$.*

Proof. If $\mathcal{A} \wedge \mathcal{B}$ exists, then $\mathcal{A}$ and $\mathcal{B}$ coexist so by Theorem 5.4, $\mathcal{A}$ and $\mathcal{B}$ are compatible. Conversely, suppose $\mathcal{A}$ and $\mathcal{B}$ are compatible. Then $\mathcal{B} \circ \mathcal{A} \leq \mathcal{A}, \mathcal{B}$. Suppose $\mathcal{C} \leq \mathcal{A}, \mathcal{B}$. By Theorem 4.5, $\mathcal{C} = \mathcal{C} \circ \mathcal{B} = \mathcal{B} \circ \mathcal{C}$. Since $\mathcal{C} \leq \mathcal{A}$ we have that

$$\mathcal{C} = \mathcal{B} \circ \mathcal{C} \leq \mathcal{B} \circ \mathcal{A}$$

Hence, $\mathcal{B} \circ \mathcal{A} = \mathcal{A} \wedge \mathcal{B}$. □

Example Let $P, Q \in \mathcal{P}(H)$ with $PQ \neq QP$ and let $A = \frac{1}{2}P$, $B = \frac{1}{2}Q$. Then $A, B \in \mathcal{E}(H)$ and since $A + B \leq I$ we have that A and B coexist. Since $B + B = Q \in \mathcal{P}(H)$ and A is not compatible with $B + B$, it follows from Lemma 5.2(c) that A and $B+B$ do not coexist. We conclude that an effect A can coexist with two effects B_1, B_2 where $B_1 + B_2 \leq I$ and yet A and $B_1 + B_2$ do not coexist. Letting $A = \frac{1}{2}P$, $B = \frac{1}{2}Q$ as before, define the measurements $\mathcal{A} = \{A, A'\}$, $\mathcal{B} = \left\{B, B, \frac{1}{2}I - B, \frac{1}{2}I - B\right\}$. Then the elements of $\mathcal{A}$ and $\mathcal{B}$ mutually coexist but since A does not coexist with $B + B$ we conclude that $\mathcal{A}$ and $\mathcal{B}$ do not coexist. (The corresponding problem for observables is unsolved [9, 10].)

6 Convexity

If $\mathcal{A} = \{A_i\}$ and $\mathcal{B} = \{B_j\}$ are in sub-$\mathcal{M}(H)$ we say that $\mathcal{A} \oplus \mathcal{B}$ is **defined** if $\sum A_i + \sum B_j \leq I$ and in this case we define the submeasurement

$$\mathcal{A} \oplus \mathcal{B} = \{A_1, A_2, \dots, B_1, B_2, \dots\}$$

Defining the zero submeasurement $\mathcal{O} = \{0\}$ we see that $\{\text{sub-}\mathcal{M}(H), \mathcal{O}, \oplus\}$ is a generalized effect algebra [3]. That is, whenever $\oplus$ is defined, we have that $\oplus$ is commutative, associative, $\mathcal{O} \oplus \mathcal{A} = \mathcal{A}$ for all $\mathcal{A}$, $\mathcal{A} \oplus \mathcal{B} = \mathcal{A} \oplus \mathcal{C}$ implies $\mathcal{B} = \mathcal{C}$ and $\mathcal{A} \oplus \mathcal{B} = \mathcal{O}$ implies $\mathcal{A} = \mathcal{B} = \mathcal{O}$. We also have that if $\mathcal{A} \in$ sub-$\mathcal{M}(H)$, then there exists a $\mathcal{B} \in$ sub-$\mathcal{M}(H)$ such that $\mathcal{A} \oplus \mathcal{B} \in \mathcal{M}(H)$. Moreover, if $\mathcal{A} \oplus \mathcal{B}$ is defined and $\mathcal{C} \in$ sub-$\mathcal{M}(H)$ then $\mathcal{C} \circ (\mathcal{A} \oplus \mathcal{B}) = \mathcal{C} \circ \mathcal{A} \oplus \mathcal{C} \oplus \mathcal{B}$.In a similar way, we defined $\oplus \mathcal{A}_i$, if $\mathcal{A}_i$ is a countable set in sub-$\mathcal{M}(H)$ satisfying the analogous condition. Again we have that $\mathcal{A} \circ (\oplus \mathcal{B}_i) = \oplus \mathcal{A} \circ \mathcal{B}_i$ whenever $\oplus \mathcal{B}_i$ is defined. For $0 \leq \lambda \leq 1$ and $\mathcal{A} \in$ sub-$\mathcal{M}(H)$, define $\lambda \mathcal{A} \in$ sub-$\mathcal{M}(H)$ by $\lambda \{A_i\} = \{\lambda A_i\}$. If $\lambda_i \geq 0$, $\sum \lambda_i = 1$, then $\oplus \lambda_i \mathcal{A}_i$ is always defined and if $\mathcal{A}_i \in \mathcal{M}(H)$, then $\oplus \lambda_i \mathcal{A}_i \in \mathcal{M}(H)$. Moreover,

$$\mathcal{A} \circ (\oplus \lambda_i B_i) = \oplus \lambda_i \mathcal{A} \circ \mathcal{B}_i$$
$$(\oplus \lambda_i B_i) \circ \mathcal{A} = \oplus \lambda_i B_i \circ \mathcal{A}$$

It is clear that $\oplus (\lambda_i \mathcal{A}) \sim \mathcal{A}$. We now show that convex combinations preserve $\sim$.

Lemma 6.1. *If $\mathcal{A} \sim \mathcal{B}$, $\mathcal{C} \sim \mathcal{D}$ and $0 \leq \lambda \leq 1$, then*

$$\lambda \mathcal{A} \oplus (1 - \lambda) \mathcal{C} \sim \lambda \mathcal{B} \oplus (1 - \lambda) \mathcal{D}$$

Proof. We have that $\mathcal{A}(\rho) = \mathcal{B}(\rho)$ and $\mathcal{C}(\rho) = \mathcal{D}(\rho)$ for all $\rho \in \mathcal{D}(H)$. Hence,

$$\begin{aligned}\left[\lambda\mathcal{A} \oplus (1-\lambda)\mathcal{C}\right](\rho) &= \lambda\mathcal{A}(\rho) + (1-\lambda)\mathcal{C}(\rho) = \lambda\mathcal{B}(\rho) + (1-\lambda)\mathcal{D}(\rho)\\ &= \left[\lambda\mathcal{B} \oplus (1-\lambda)\mathcal{D}\right](\rho)\end{aligned}$$

for every $\rho \in \mathcal{D}(H)$. □

The next lemma shows that convexity preserves the important concepts of $\mathcal{E}(H)$.

Lemma 6.2. *Suppose that $\lambda_i \geq 0$, $\sum \lambda_i = 1$ and $A_i, B_i \in \mathcal{E}(H)$ $i = 1, \ldots, n$.* (a) $(\sum \lambda_i A_i)' = \sum \lambda_i A_i'$. (b) *If A_i, B_j are compatible for all i, j, then $\sum \lambda_i A_i$ is compatible with $\sum \lambda_i B_i$.* (c) *If A_i and B_i coexist, $i = 1, \ldots, n$, then $\sum \lambda_i A_i$ coexists with $\sum \lambda_i B_i$.*

Proof. (a) Since

$$\sum \lambda_i A_i' + \sum \lambda_i A_i = \sum \lambda_i (A_i' + A_i) = \sum \lambda_i I = I$$

we have that $\sum \lambda_i A_i' = (\sum \lambda_i A_i)'$. (b) This is clear. (c) We prove this by induction on n. Let $n = 2$ and suppose A_i and B_i coexist, $i = 1, 2$. Then $A_1 = C_1 + D_1$, $B_1 = D_1 + E_1$, $A_2 = C_2 + D_2$, $B_2 = D_2 + E_2$ where $C_1 + D_1 + E_1 \leq I$ and $C_2 + D_2 + E_2 \leq I$. We then have that

$$\begin{aligned}\lambda_1 A_1 + \lambda_2 A_2 &= \lambda_1 C_1 + \lambda_1 D_1 + \lambda_2 C_2 + \lambda_2 D_2\\ &= (\lambda_1 C_1 + \lambda_2 C_2) + (\lambda_1 D_1 + \lambda_2 D_2)\\ \lambda_1 B_1 + \lambda_2 B_2 &= \lambda_1 D_1 + \lambda_1 E_1 + \lambda_2 D_2 + \lambda_2 E_2\\ &= (\lambda_1 D_1 + \lambda_2 D_2) + (\lambda_1 E_1 + \lambda_2 E_2)\end{aligned}$$

Since

$$\begin{aligned}\lambda_1 C_1 + \lambda_2 C_2 + &\lambda_1 D_1 + \lambda_2 D_2 + \lambda_1 E_1 + \lambda_2 E_2\\ &= \lambda_1 (C_1 + D_1 + E_1) + \lambda_2 (C_2 + D_2 + E_2) \leq I\end{aligned}$$

we conclude that $\lambda_1 A_1 + \lambda_2 A_2$ and $\lambda_1 B_1 + \lambda_2 B_2$ coexist. Next suppose the result holds for n and assume that A_i and B_i coexist, $i = 1, \ldots, n+1$. Letting $\alpha = \sum_{i=1}^n \lambda_i$, by the induction hypothesis, $\frac{1}{\alpha} \sum_{i=1}^n \lambda_i A_i$ and $\frac{1}{\alpha} \sum_{i=1}^n \lambda_i B_i$ coexist. Hence, $\sum_{i=1}^n \lambda_i A_i$ and $\sum_{i=1}^n \lambda_i B_i$ coexist. Again by the induction hypothesis,

$$\frac{1}{2} \sum_{i=1}^n \lambda_i A_i + \frac{1}{2} \lambda_{n+1} A_{n+1}$$

and

$$\frac{1}{2}\sum_{i=1}^{n}\lambda_i B_i + \frac{1}{2}\lambda_{n+1}B_{n+1}$$

coexist. Hence, $\sum_{i=1}^{n+1}\lambda_i A_i$ and $\sum_{i=1}^{n+1}\lambda_i B_i$ coexist. $\square$

If $\mathcal{A},\mathcal{B}\in$ sub-$\mathcal{M}(H)$ and $\mathcal{A}\oplus\mathcal{B}\in\mathcal{M}(H)$, we call $\mathcal{B}$ a **supplement** of $\mathcal{A}$ and write $\mathcal{B}=\mathcal{A}'$. Notice that $\mathcal{A}'$ is not unique. The proof of the next result is analogous to that of Lemma 6.2.

Theorem 6.3. *Suppose that $\lambda_i\geq 0$, $\sum\lambda_i=1$ and $\mathcal{A}_i,\mathcal{B}_i\in$ sub-$\mathcal{M}(H)$, $i=1,\ldots,n$.* (a) *$(\oplus\lambda_i\mathcal{A}_i)'=\sum\lambda_i\mathcal{A}_i'$.* (b) *If $\mathcal{A}_i$, $\mathcal{B}_j$ are compatible for all i, j, then $\oplus\lambda_i\mathcal{A}_i$ is compatible with $\oplus\lambda_i\mathcal{B}_i$.* (c) *If $\mathcal{A}_i$ and $\mathcal{B}_i$ coexist, $i=1,\ldots,n$, then $\oplus\lambda_i\mathcal{A}_i$ and $\oplus\lambda_i\mathcal{B}_i$ coexist.*

7 Conditioning

For $\mathcal{A}=\{A_i\},\mathcal{B}=\{B_j\}\in\mathcal{M}(H)$ we defined $\mathcal{A}\mid\mathcal{B}\in\mathcal{M}(H)$ by

$$\mathcal{A}\mid\mathcal{B}=\left\{\sum_j B_j\circ A_i\colon i=1,2,\ldots\right\}$$

and call $\mathcal{A}\mid\mathcal{B}$ the measurement $\mathcal{A}$ **conditioned by** the measurement $\mathcal{B}$. Notice that $\mathcal{B}\circ\mathcal{A}\leq\mathcal{A}\mid\mathcal{B}$ and if $\mathcal{A}$ and $\mathcal{B}$ are compatible, then $\mathcal{A}\mid\mathcal{B}\approx\mathcal{A}$. IT follows from a result in [1] that the converse does not hold. That is, $\mathcal{A}\mid\mathcal{B}\approx\mathcal{A}$ does not imply that $\mathcal{A}$ and $\mathcal{B}$ are compatible. If $\mathcal{A}\mid\mathcal{B}\approx\mathcal{A}$ we say that $\mathcal{A}$ is **unaffected** by $\mathcal{B}$. It follows from a result in [1] that $\mathcal{A}$ is unaffected by $\mathcal{B}$ does not imply that $\mathcal{B}$ is unaffected by $\mathcal{A}$. If $\mathcal{A}$ is compatible with itself we say that $\mathcal{A}$ is **commutative**.

Lemma 7.1. *If $\dim H<\infty$ and $\{A,A'\}$ is unaffected by $\{B,B'\}$, then $\{A,A'\}$ and $\{B,B'\}$ are compatible.*

Proof. If $\{A,A'\}\mid\{B,B'\}=\{A,A'\}$, then

$$\{B\circ A+B'\circ A,B\circ A'+B'\circ A'\}\approx\{A,A'\}$$

If $B\circ A+B'\circ A=A$, then $AB=BA$. If $B\circ A+B'\circ A=A'$, then by Lemma 3.5 we have that $A=\frac{1}{2}I$. In either case, $\{A,A'\}$ and $\{B,B'\}$ are compatible. $\square$

We do not know if the condition $\dim H < \infty$ can be relaxed.

Theorem 7.2. *Let $\mathcal{A} \in \mathcal{M}(H)$ and $\mathcal{B} \in \mathcal{S}(H)$. If $\mathcal{A}$ is unaffected by $\mathcal{B}$ or if $\mathcal{B}$ is unaffected by $\mathcal{A}$, then $\mathcal{A}$ and $\mathcal{B}$ are compatible.*

Proof. Let $\mathcal{A} = \{A_i\}$, $\mathcal{B} = \{B_j\}$. If $\mathcal{A} \mid \mathcal{B} \approx \mathcal{A}$ then for every i we have $\sum_j B_j A_i = A_k$ for some k. Hence,

$$B_r A_i B_r = B_r A_k = A_k B_r$$

for every r so that $\mathcal{A}$ and $\mathcal{B}$ are compatible. If $\mathcal{B} \mid \mathcal{A} \approx \mathcal{B}$ then for every i we have $\sum_j A_j^{1/2} B_i A_j^{1/2} = B_k$ for some k. If $k = i$, then from a result in [1] we have $A_j B_i = B_i A_j$ for every j. Otherwise, $k \neq i$ and then

$$B_i A_j^{1/2} B_i A_j^{1/2} B_i = 0$$

Hence $(B_i \circ A_j)^2 = 0$ which implies that $B_i \circ A_j = 0$. It follows that $B_i A_j = A_j B_i$ [7]. But then $B_i = B_k$ which is impossible. Hence, $k = i$ and the result follows. □

We conjecture that if $\mathcal{A}$ does not affect itself, then $\mathcal{A}$ is commutative. The next example shows that this conjecture holds in some special cases. Let $\mathcal{A} = \{A_1, A_2, A_3\}$ with $A_i \neq 0$, $i = 1, 2, 3$ and suppose that $\mathcal{A} \mid \mathcal{A} \approx \mathcal{A}$ and $\dim H < \infty$. We then have that $\{A_1 \circ A_1 + A_2 \circ A_1 + A_3 \circ A_1, A_1 \circ A_2 + A_2 \circ A_2 + A_3 \circ A_2, A_1 \circ A_3 + A_2 \circ A_3 + A_3 \circ A_3\} \approx \{A_1, A_2, A_3\}$. If $\sum A_i \circ A_1 = A_1$, then $A_1 A_2 = A_2 A_1$ and $A_1 A_3 = A_3 A_1$ [1]. But then

$$A_2 A_3 = A_2(I - A_1 - A_2) = (I - A_1 - A_2)A_2 = A_3 A_2$$

so that $\mathcal{A}$ is commutative. Next, suppose that $\sum A_i \circ A_1 = A_2$. We then obtain

$$\begin{aligned} A_1 \circ A_1 + A_3 \circ A_1 &= A_2 - A_2 \circ A_1 = A_2 \circ (I - A_2 - A_3) \\ &= A_2 - A_2 \circ A_2 - A_2 \circ A_3 \end{aligned}$$

Hence,

$$A_2 = A_1 \circ A_1 + A_3 \circ A_1 + A_2 \circ A_2 + A_2 \circ A_3 = A_1 \circ A_1 + A_2 \circ A_1 + A_3 \circ A_1$$

We conclude that

$$A_2 \circ A_1 - A_2 \circ A_2 - A_2 \circ A_3 = 0$$

Since $\sum A_2 \circ A_i = A_2$ we have that $A_2 \circ A_1 = \frac{1}{2}A_2$. Similarly, $A_3 \circ A_2 = \frac{1}{2}A_3$ and $A_1 \circ A_3 = \frac{1}{2}A_1$. In general, if $A \circ B = \frac{1}{2}A$ then we need not have that $AB = BA$. For example, let P_x be the one-dimensional projection onto the span of the unit vector x, and let $B \in \mathcal{E}(H)$ satisfy $\langle Bx, x\rangle = \frac{1}{2}$ and $BP_x \neq P_x B$. Then B and P_x are not compatible, but $P_x \circ B = \frac{1}{2}P_x$. However, suppose that A_1 is invertible. Then $A_1 \circ A_3 = \frac{1}{2}A_1$ implies that $A_3 A_1^{1/2} = \frac{1}{2}A^{1/2}$ so that $A_3 A_1 = A_1 A_3$. If in addition, A_2 is invertible, then $A_2 A_1 = A_1 A_2$ and $\mathcal{A}$ is commutative. Or suppose that $A_1 \leq \frac{1}{2}I$. Then $2A_1 \in \mathcal{E}(H)$ and $A_2 \circ (2A_1) = A_2$ implies that $(2A_1)A_2 = A_2(2A_1)$ [7]. Hence, A_1 and A_2 are compatible. If in addition, $A_2 \leq \frac{1}{2}I$, then A_2 and A_3 are compatible and $\mathcal{A}$ is commutative.

It is easy to check that conditioning preserves convex combinations. That is, $(\oplus\lambda_i\mathcal{A}_i) \mid \mathcal{B} = \oplus\lambda_i(\mathcal{A}_i \mid \mathcal{B})$. It is also easy to check that $\mathcal{A} \leq \mathcal{B}$ implies that $\mathcal{A} \mid \mathcal{C} \leq \mathcal{B} \mid \mathcal{C}$.

For $A \in \mathcal{E}(H)$, $\rho \in \mathcal{D}(H)$ we denote the probability of A in the state ρ by $P_\rho(A) = \mathrm{tr}(\rho A)$. If $A, B \in \mathcal{E}(H)$, $\rho \in \mathcal{D}(H)$, then the **conditional probability of** A **given** B in the state ρ is

$$P_\rho(A \mid B) = \frac{\mathrm{tr}(\rho B \circ A)}{\mathrm{tr}(\rho B)} = \frac{\mathrm{tr}\left[(B \circ \rho)A\right]}{\mathrm{tr}(\rho B)}$$

assuming that $\mathrm{tr}(\rho B) \neq 0$ [7]. Since

$$\rho \mid B = \frac{B \circ \rho}{\mathrm{tr}(\rho B)} \in \mathcal{D}(H)$$

we can write $P_\rho(A \mid B) = P_{\rho|B}(A)$. Analogous to our definition of conditioned measurements, it is natural to define $A \mid \mathcal{B} = \sum B_i \circ A$ for $A \in \mathcal{E}(H)$, $\mathcal{B} = \{B_i\} \in \mathcal{M}(H)$. It is clear that $A \mid B \in \mathcal{E}(H)$ and we call $A \mid B$ the effect A **conditioned** by the measurement $\mathcal{B}$. Conditions under which $A \mid B = A$ have been studied in [1]. We now have the formulas

$$\begin{aligned} P_\rho(A \mid \mathcal{B}) &= \mathrm{tr}\left(\rho \sum B_i \circ A\right) = \sum \mathrm{tr}\left(\rho B_i \circ A\right) = \sum \mathrm{tr}\left[(B_i \circ \rho)A\right] \\ &= \mathrm{tr}\left[\left(\sum B_i \circ \rho\right) A\right] = \mathrm{tr}\left[(\rho \mid \mathcal{B})A\right] = P_{\rho|\mathcal{B}}(A) \end{aligned}$$

and

$$P_\rho(A \mid \mathcal{B}) = \sum \operatorname{tr}(\rho B_i \circ A) = \sum \operatorname{tr}(\rho B_i) P_\rho(A \mid B_i) = \sum P_\rho(B_i) P_\rho(A \mid B_i)$$

In the case when $P_\rho(A) = P_\rho(A \mid \mathcal{B})$ we say the **law of total probability** holds. Although this law always holds in classical probability theory it does not hold in general for quantum mechanics. [1, 7].

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