BUCHSTEINER LOOPS

PIROSKA CSÖRGŐ, ALEŠ DRÁPAL, AND MICHAEL K. KINYON

Abstract. Buchsteiner loops are those which satisfy the identity $x\backslash(xy \cdot z) = (y \cdot zx)/x$. We show that a Buchsteiner loop modulo its nucleus is an abelian group of exponent four, and construct an example where the factor achieves this exponent.

A *loop* $(Q, \cdot)$ is a set Q together with a binary operation $\cdot$ such that for each $a, b \in Q$, the equations $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x, y \in Q$, and such that there is a neutral element $1 \in Q$ satisfying $1 \cdot x = x \cdot 1 = x$ for every $x \in Q$. Standard references in loop theory are [4, 6, 32].

The variety (*i.e.*, equational class) of all loops being too broad for a detailed structure theory, most investigations focus on particular classes of loops. In this paper we investigate a variety of loops which has not hitherto received much attention, despite the fact that it is remarkably rich in structure, namely the variety defined by the identity

$$x\backslash(xy \cdot z) = (y \cdot zx)/x. \tag{B}$$

Here $a\backslash b$ denotes the unique solution x to $a \cdot x = b$, while b/a denotes the unique solution y to $y \cdot a = b$. We call (B) the *Buchsteiner law* and a loop satisfying it a *Buchsteiner loop* since Hans-Hennig Buchsteiner seems to have been the first to notice their importance [7].

The Buchsteiner law (B) is easily seen to be equivalent to each of the following:

$$\text{(B')} \qquad xy \cdot z = xu \quad \iff \quad y \cdot zx = ux \qquad \text{for all } x, y, z, u\text{, and}$$

$$\text{(B'')} \qquad xy \cdot z = xu \cdot v \quad \iff \quad y \cdot zx = u \cdot vx \qquad \text{for all } x, y, z, u, v\,.$$

Both the identity (B) and the implications (B'), (B") will prove useful in what follows.

Buchsteiner loops can be understood in terms of coinciding left and right principal isotopes or in terms of autotopisms that identify the left and the right nucleus. Among the equalities that can be obtained by nuclear identification this is the only one which has not been subjected to a systematic study. (The other equalities are the left and right Bol laws, the Moufang laws, the extra laws, and the laws of left and right conjugacy closedness [18].)

All Buchsteiner loops are G-loops (a loop Q is said to be a G-loop if every loop isotope of Q is isomorphic to Q). Groups are G-loops, and it is well-known and

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easy to see that conjugacy closed loops are G-loops as well. There are also various subclasses of Moufang loops which turn out to be G-loops, such as the class of all simple Moufang loops, or the variety of all M_k-loops for $k \not\equiv 1 \pmod 3$ [8]. Buchsteiner loops seem to be the only other known class of G-loops with a concise equational definition. Such classes are of considerable interest since it is known that G-loops cannot be described by first-order sentences [28].

Our main results are the statement that in every Buchsteiner loop Q, the nucleus N is a normal subloop such that the factor loop Q/N is an abelian group of exponent 4, and the construction of an example in which that exponent is achieved.

Our investigations have been helped immensely by recent progress on conjugacy closed loops [28, 14, 26, 11, 27, 16, 17]. Buchsteiner did not work with the notion of conjugacy closedness, and it turns out that the examples he constructed in [7] are all conjugacy closed. This paper thus seems to present the first example of a proper (non-CC) Buchsteiner loop. A deeper understanding of this will require further study. We are inclined to believe that Buchsteiner's speculations in [7] concerning the connections to nonassociative division algebras (and, indirectly, to projective planes) may prove to have been prescient.

To obtain the present results we had to use the concept of *doubly weak inverse property* (WWIP, for short), which naturally generalizes the classical weak inverse property (WIP) of Osborn [31]. WWIP loops can be also seen as a special case of the more general notion of m-inverse loops [22, 23]. We believe that this is the first instance where this concept found a highly nontrivial natural application in an equational theory. The main result of §4 is the proof that each Buchsteiner loop is a WWIP loop.

In §5, we apply properties of WWIP loops to show that each Buchsteiner loop Q is a G-loop and that Q/N is an abelian group. In §6, we consider the special case of Buchsteiner loops with the WIP, and show that these are exactly WIP CC loops.

To get the aforementioned restriction on the exponent of Q/N we use associator calculus, which is developed in §7. We suspect that future results on Buchsteiner loops will require further and finer calculations with associators. In §8 and §9 we construct an example of a Buchsteiner loop Q on 1024 elements such that Q/N is indeed of the minimal exponent 4. Further, Q has a factor of order 64 with this same property.

We will discuss the relationship of our work to that of Buchsteiner [7] and Basarab [3] in §10, and we conclude by announcing some further results and stating several problems in §11.

We have tried to write this paper in a way that is accessible to researchers who are not specialists in loop theory. We define all notions we need and when mentioning basic properties which are easy to show we usually offer a proof.

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1. Preliminaries

In this section we introduce many of the basic tools of loop theory, and examine how they are used in Buchsteiner loops.

1.1. **Multiplication groups.** Let Q be a loop. For each $x \in Q$, the *left* and *right translation* maps $L_x, R_x : Q \to Q$ are defined by $L_x y = xy$ and $R_x y = yx$, respectively. For any $S \subseteq Q$, set

$$L_{(S)} = \{L_s \mid s \in S\} \qquad \text{and} \qquad R_{(S)} = \{R_s \mid s \in S\} .$$

The group generated by both types of translations

$$\operatorname{Mlt} Q = \langle L_{(Q)}, R_{(Q)} \rangle = \langle L_x, R_x \mid x \in Q \rangle$$

is called the *multiplication group*, while the *left* and *right multiplication groups* are defined, respectively, by

$$\mathcal{L} = \mathcal{L}(Q) = \langle L_{(Q)} \rangle = \langle L_x \mid x \in Q \rangle \quad \text{and} \quad \mathcal{R} = \mathcal{R}(Q) = \langle R_{(Q)} \rangle = \langle R_x \mid x \in Q \rangle .$$

Working with left and right translation maps allows many computations in loop theory to be carried out in groups. For instance, rewriting the Buchsteiner law (B) in terms of translations immediately yields

Lemma 1.1. *In a loop Q, the Buchsteiner law* (B) *is equivalent to each of the following:*

$$L_x^{-1} R_z L_x = R_x^{-1} R_{zx} \text{ for all } x, z \in Q \,, \tag{1.1}$$

$$R_x^{-1} L_y R_x = L_x^{-1} L_{xy} \text{ for all } x, y \in Q \,. \tag{1.2}$$

The following alternative form of Lemma 1.1 is also useful.

Proposition 1.2. *In a loop Q, the Buchsteiner law* (B) *is equivalent to each of the following:*

$$R_x R_{(Q)}^{L_x} = R_{(Q)} \text{ for all } x \in Q \,, \tag{1.3}$$

$$L_x L_{(Q)}^{R_x} = L_{(Q)} \text{ for all } x \in Q \,. \tag{1.4}$$

Proof. If (1.1) holds, then so does (1.3). Conversely, if (1.3) holds, then for each $y \in Q$, there exists $z \in Q$ such that $L_x^{-1} R_y L_x = R_x^{-1} R_z$. Applying both sides to $1 \in Q$, we get $yx = z$, and so $L_x^{-1} R_y L_x = R_x^{-1} R_{yx}$, which is (1.1). The proof of the equivalence of (1.2) and (1.4) is similar. □

As subgroups of $\operatorname{Mlt} Q$, neither $\mathcal{L}$ nor $\mathcal{R}$ has to be normal in the general case. However, both Lemma 1.1 and Proposition 1.2 have the following immediate consequence.

Corollary 1.3. *In a Buchsteiner loop Q, $\mathcal{L}$ and $\mathcal{R}$ are normal subgroups of* $\operatorname{Mlt} Q$.

1.2. **Inner mappings.** The stabilizer in $\operatorname{Mlt} Q$ of the neutral element of a loop Q is called the *inner mapping group* of Q:

$$\operatorname{Inn} Q = (\operatorname{Mlt} Q)_1 = \{\varphi \in \operatorname{Mlt} Q \mid \varphi(1) = 1\},$$

while the *left* and *right inner mapping groups* are defined, respectively, by

$$\mathcal{L}_1 = \mathcal{L} \cap \operatorname{Inn} Q \qquad \text{and} \qquad \mathcal{R}_1 = \mathcal{R} \cap \operatorname{Inn} Q \,.$$

Generators for these groups are defined as follows:

$$L(x,y) = L_{xy}^{-1} L_x L_y \,, \qquad T_x = R_x^{-1} L_x \,, \qquad R(x,y) = R_{yx}^{-1} R_x R_y$$

for $x, y \in Q$. Then it turns out [4, 6] that

$$\operatorname{Inn} Q = \langle T_x, L(x,y), R(x,y) \mid x, y \in Q \rangle \,,$$

$$\mathcal{L}_1 = \langle L(x,y) \mid x, y \in Q \rangle \,, \qquad \mathcal{R}_1 = \langle R(x,y) \mid x, y \in Q \rangle \,.$$

In Buchsteiner loops, matters are somewhat simpler. By rewriting (1.1) and (1.2) in the form

$$[R_z, L_x] = R_z^{-1} R_x^{-1} R_{zx} \quad \text{and} \quad [L_y, R_x] = L_y^{-1} L_x^{-1} L_{xy} \,, \tag{1.5}$$

we get

Theorem 1.4. *Let Q be a Buchsteiner loop. Then for all $x, y \in Q$,*

$$R(x, y) = [L_x, R_y] = L(y, x)^{-1} \,.$$

In particular, $\mathcal{L}_1 = \mathcal{R}_1$.

1.3. **Nuclei, the center, and related subloops.** The *left*, *middle*, and *right* nucleus of a loop Q are defined, respectively, by

$$N_\lambda = N_\lambda(Q) = \{a \in Q \,|\, a \cdot xy = ax \cdot y \quad \forall x, y \in Q\} \,,$$
$$N_\mu = N_\mu(Q) = \{a \in Q \,|\, x \cdot ay = xa \cdot y \quad \forall x, y \in Q\} \,,$$
$$N_\rho = N_\rho(Q) = \{a \in Q \,|\, x \cdot ya = xy \cdot a \quad \forall x, y \in Q\}, ,$$

while the intersection

$$N = N(Q) = N_\lambda \cap N_\mu \cap N_\rho$$

is called the *nucleus* of Q. We observe that N_λ is the set of fixed points of $\mathcal{R}_1$, N_ρ is the set of fixed points of $\mathcal{L}_1$, and N_μ is the set of fixed points of the group $\langle [R_x, L_y] \,|\, x, y \in Q \rangle$.

It is easy to prove that each of the nuclei is a subgroup (associative subloop) of Q. However, in general, the nuclei need not coincide.

Lemma 1.5. *Let Q be a Buchsteiner loop. For all $x, y, z \in Q$,*

$$x \cdot yz = xy \cdot z \qquad \Longleftrightarrow \qquad y \cdot zx = yz \cdot x \qquad \Longleftrightarrow \qquad z \cdot xy = zx \cdot y \,.$$

Proof. This is an immediate consequence of (B'). □

Corollary 1.6. *In a Buchsteiner loop, $N_\lambda = N_\mu = N_\rho$.*

This corollary also follows from Theorem 1.4 and the characterizations of the nuclei as fixed point sets of subgroups of $\operatorname{Inn} Q$.

A subloop S of a loop Q is *normal* if it is invariant under the action of $\operatorname{Inn} Q$, or equivalently, if it is a block of $\operatorname{Mlt} Q$ containing the neutral element. Then one can define a quotient structure Q/S over the associated block system. In general, the nuclei of a loop are not necessarily normal subloops.

Let Q be a loop, and suppose $\phi \in \operatorname{Mlt} Q$ satisfies $\phi R_x = R_x \phi$ for all $x \in Q$. Applying both sides to 1 gives $\phi(x) = ax$ where $a = \phi(1)$, and so $\phi = L_a$. However, $L_a R_x = R_x L_a$ for all $x \in Q$ exactly when $a \in N_\lambda$. Similarly, $\psi \in \operatorname{Mlt} Q$ satisfies $\psi L_x = L_x \psi$ for all $x \in Q$ if and only if $\psi = R_b$ for some $b \in N_\rho$.

Then we can express the observations of the preceding paragraph as part (i) of the following.

Lemma 1.7. *Let Q be a loop. Then*

(i) $C_{\operatorname{Mlt} Q}(\mathcal{R}) = L_{(N_\lambda)}$ *and* $C_{\operatorname{Mlt} Q}(\mathcal{L}) = R_{(N_\rho)}$.
(ii) *If* $\mathcal{R} \trianglelefteq \operatorname{Mlt} Q$, *then* $L_{(N_\lambda)} \trianglelefteq \operatorname{Mlt} Q$ *and* $N_\lambda \trianglelefteq Q$.
(iii) *If* $\mathcal{L} \trianglelefteq \operatorname{Mlt} Q$, *then* $R_{(N_\rho)} \trianglelefteq \operatorname{Mlt} Q$ *and* $N_\rho \trianglelefteq Q$.

Proof. Parts (ii) and (iii) follow because the centralizer of a normal subgroup is again a normal subgroup, and orbits of normal subgroups of permutation groups form block systems. □

Corollary 1.8. *Let Q be a Buchsteiner loop. Then N is a normal subloop of Q, and $L_{(N)}$ and $R_{(N)}$ are normal subgroups of* $\mathrm{Mlt}\, Q$.

Proof. This follows from Corollaries 1.3 and 1.6, and Lemma 1.7. □

In an arbitrary loop Q, the set

$$C(Q) = \{a \in Q \mid ax = xa,\ \forall x \in Q\} = \{a \in Q \mid T_a = \mathrm{id}_Q\}$$

is not necessarily a subloop. The *center* of Q is defined as

$$Z(Q) = N(Q) \cap C(Q)\,.$$

The center is exactly the set of fixed points of the inner mapping group $\mathrm{Inn}\, Q$, and thus is a normal, abelian subgroup of Q.

Lemma 1.9. *In a Buchsteiner loop Q, $Z(Q) = C(Q)$.*

Proof. Fix $c \in C(Q)$, and in (1.1), take $x = c$. Since $L_c = R_c$, we have $R_c^{-1} R_z R_c = R_c^{-1} R_{zc}$, and so $R_z R_c = R_{cz}$ for all $z \in Q$. This says that $c \in N_\mu = N$ (Corollary 1.6), and so $c \in Z(Q)$. □

1.4. **Isotopisms.** An *isotopism* of loops $(Q, *)$ and $(Q, \cdot)$ with the same underlying set Q is a triple (α, β, γ) of permutations of Q satisfying

$$\alpha(x) \cdot \beta(y) = \gamma(x * y) \tag{1.6}$$

for all $x, y \in Q$. In this case, $(Q, *)$ and $(Q, \cdot)$ are said to be *isotopic*. Two types of isotopisms are of primary interest in loop theory, namely, principal isotopisms and autotopisms. We begin with the former.

An isotopism (α, β, γ) is called *principal* if $\gamma = \mathrm{id}_Q$. In such a case, if $1 \in Q$ is the neutral element of $(Q, *)$, and if we set $a = \alpha(1)$ and $b = \beta(1)$, then (1.6) becomes

$$x * y = (x/b) \cdot (a\backslash y)$$

for all $x, y \in Q$. Here $\backslash$ and $/$ are the left and right division operations in $(Q, \cdot)$. The loop $(Q, *)$ is then called a *principal isotope* of $(Q, \cdot)$.

Let $(Q, \cdot)$ be a loop and fix $e \in Q$. Define on Q loop operations $*$ and $\circ$ by

$$x * y = (x/e) \cdot y \qquad \text{and} \qquad x \circ y = (x \cdot ye)/e\,. \tag{1.7}$$

Thus $(Q, *)$ is a principal isotope of Q with neutral element e, while 1 is the neutral element of $(Q, \circ)$.

Lemma 1.10. *The right translation $R_e : x \mapsto xe$ yields an isomorphism $(Q, \circ) \cong (Q, *)$.*

Proof. Indeed, $R_e(x \circ y) = x \cdot ye = ((xe)/e) \cdot ye = R_e(x) * R_e(y)$. □

Lemma 1.10 clearly has a mirror version, which associates the operations $x \cdot (e\backslash y)$ and $e\backslash(ex \cdot y)$. In the loop theory literature, the operations $e\backslash(ex \cdot y)$ and $(x \cdot ye)/e$ are sometimes called the "left derivative" and "right derivative" at e, respectively [32]. However, invoking Lemma 1.10, we will call the operation $(x \cdot ye)/e$ the *right isotope at e* and the operation $e\backslash(ex \cdot y)$ the *left isotope at e*. We obviously have

Lemma 1.11. *A loop Q satisfies the Buchsteiner law* (B) *if and only if for every $e \in Q$, the left and right isotopes at e coincide.*

When dealing with a Buchsteiner loop, we can thus drop the left/right distinction and refer simply an *isotope at e*. We will denote this isotope by $Q[e]$.

1.5. **Autotopisms.** An isotopism (α,β,γ) of a loop $(Q,\cdot)$ to itself is called an *autotopism*. The set $\operatorname{Atp} Q$ of all autotopisms of a loop Q is a group, and a permutation α of Q is an automorphism, that is, $\alpha \in \operatorname{Aut} Q$, if and only if $(\alpha,\alpha,\alpha) \in \operatorname{Atp} Q$.

Lemma 1.12. *A loop Q satisfies the Buchsteiner law* (B) *if and only if*

$$\mathbf{B}(x) = (L_x, R_x^{-1}, L_x R_x^{-1})$$

is an autotopism for all $x \in Q$.

Proof. The triple $\mathbf{B}(x)$ yields an autotopism if and only if $(xy)(z/x) = x((yz)/x)$ for all $x, y, z \in Q$. By replacing z with zx we get $xy \cdot z = x((y \cdot zx)/x)$, which is equivalent to (B). □

The nuclei can be characterized in terms of autotopisms.

Lemma 1.13. *Let Q be a loop. For $a \in Q$,*

$$a \in N_\lambda \iff (L_a, \mathrm{id}_Q, L_a) \in \operatorname{Atp} Q\,,$$
$$a \in N_\mu \iff (R_a, L_a^{-1}, \mathrm{id}_Q) \in \operatorname{Atp} Q\,,$$
$$a \in N_\rho \iff (\mathrm{id}_Q, R_a, R_a) \in \operatorname{Atp} Q\,.$$

Proof. For instance, $x \cdot ay = xa \cdot y$ for all $x, y \in Q$ if and only if $xy = (xa)\cdot(a\backslash y)$ for all $x, y \in Q$, which gives the characterization of N_μ. The other cases are similar. □

Let us denote the triples of permutations occurring in the preceding lemma as follows:

$$\alpha_\lambda(x) = (L_x, \mathrm{id}, L_x)\,, \qquad \alpha_\mu(x) = (R_x, L_x^{-1}, \mathrm{id})\,, \qquad \alpha_\rho(x) = (\mathrm{id}, R_x, R_x)\,.$$

Suppose that Q is a loop in which there exist $\varepsilon, \eta \in \{-1, 1\}$ and $\xi, \chi \in \{\lambda, \mu, \rho\}$ such that $\alpha_\xi^\varepsilon(a)\,\alpha_\chi^\eta(a)$ is an autotopism for all $a \in Q$. It is then clear that $a \in N_\xi(Q) \Leftrightarrow a \in N_\chi(Q)$.

Say that an identity can be obtained by *nuclear identification* if it can be expressed by an autotopism of the form $\alpha_\xi^\varepsilon(a)\,\alpha_\chi^\eta(a)$. From Lemma 1.12, we see that the Buchsteiner law can be obtained in this form, as

$$\mathbf{B}(x) = (L_x, R_x^{-1}, L_x R_x^{-1}) = (L_x, \mathrm{id}_Q, L_x)(\mathrm{id}_Q, R_x, R_x)^{-1} = \alpha_\lambda(x)\alpha_\rho(x)^{-1}.$$

In particular, this implies the already noted fact that $N_\lambda = N_\rho$ (Corollary 1.6). We have already mentioned in the introduction that the most frequently studied varieties of loops are those that can be described by identities which follow from a nuclear identification. This concept is studied in detail in [18].

The following proposition will be crucial to a later description of isotopes of Buchsteiner loops.

Proposition 1.14. *Let Q be a loop. For a permutation f of Q,*

(i) $(f, \mathrm{id}_Q, f) \in \operatorname{Atp} Q \Leftrightarrow f$ *centralizes* $R_{(Q)} \Leftrightarrow f = L_a$ *where* $a = f(1) \in N_\lambda$,
(ii) $(\mathrm{id}_Q, f, f) \in \operatorname{Atp} Q \Leftrightarrow f$ *centralizes* $L_{(Q)} \Leftrightarrow f = R_a$ *where* $a = f(1) \in N_\rho$.

Proof. (f, id_Q, f) is an autotopism if and only if for each $x, y \in Q$, $f(x)y = f(xy)$, that is, $R_y f(x) = f R_y(x)$. This shows the first equivalence of (i). Take $x = 1$ and set $a = f(1)$ to get $f(y) = ay$ for all y, that is, $f = L_a$. That $a \in N_\lambda$ follows from Lemma 1.13. Conversely, if $f = L_a$ for some $a \in N_\lambda$, then f centralizes $R_{(Q)}$ by Lemma 1.7. This establishes (i), and the proof of (ii) is similar. □

Let us give a classical example how the notion of an autotopism can be used.

Lemma 1.15. *Let Q be a loop, let $(\alpha, \beta, \gamma) \in \operatorname{Atp} Q$ satisfy $\alpha(1) = 1$ and set $c = \beta(1)$. Then*

(i) $\gamma = \beta = R_c\alpha$,
(ii) *α is an isomorphism from Q to the right isotope at c,*
(iii) *$\alpha \in \operatorname{Aut} Q$ if and only if $c \in N_\rho$.*

Proof. We have $\gamma(x) = \alpha(1)\beta(x) = \beta(x)$ for all $x \in Q$, and $\alpha(x) \cdot c = \beta(x)$, establishing (i). Next, $\alpha(x)(\alpha(y) \cdot c) = \alpha(xy) \cdot c$ for all $x, y \in Q$, which gives (ii). If $c \in N_\rho$, then we get $\alpha(x)\alpha(y) = \alpha(xy)$ for all $x, y \in Q$. On the other hand the latter equality implies $c \in N_\rho$ since α is a permutation of Q. This proves (iii). □

In the loop theory literature, the permutation α described in the preceding lemma is sometimes called a "right pseudoautomorphism" with companion c.

At this point, and throughout the rest of the paper, it will be useful to introduce notation for left and right inverses. In a loop Q, let

$$I(x) = x\backslash 1 \qquad \text{and} \qquad J(x) = 1/x$$

for all $x \in Q$. (Here we follow the notation of [4].) Thus $x \cdot I(x) = J(x) \cdot x = 1$, and $J = I^{-1}$. In general, $I(x)$ and $J(x)$ can differ, but when they coincide, we write $x^{-1} = I(x) = J(x)$. In that case, $(x^{-1})^{-1} = x$.

Corollary 1.16. *Let Q be a Buchsteiner loop, and fix $x, y \in Q$. Then*

(i) *$L(x, y)$ is an isomorphism from Q to the isotope at $(J(y)/x) \cdot xy$,*
(ii) *$L(x, y) \in \operatorname{Aut} Q$ if and only if $(J(y)/x) \cdot xy \in N$.*

Proof. We compose Buchsteiner autotopisms to get

$$\mathbf{B}(xy)^{-1}\mathbf{B}(x)\mathbf{B}(y) = (L(x, y),\ R_{xy}R_x^{-1}R_y^{-1},\ R_{xy}L_{xy}^{-1}L_xR_x^{-1}L_yR_y^{-1}).$$

The first component fixes 1 and so the lemma applies. □

Incidentally, the equality of the second and third components of the autotopism in the preceding proof gives another proof of Theorem 1.4.

A loop Q is said be an *A_ℓ-loop* if $\mathcal{L}_1 \leq \operatorname{Aut} Q$, that is, if $L(x, y) \in \operatorname{Aut} Q$ for all $x, y \in Q$. An *A_r-loop* is similarly defined, and we will say that a loop satisfying both properties is an *$A_{\ell,r}$-loop*. By Theorem 1.4, the A_ℓ and A_r properties are equivalent in Buchsteiner loops. Corollaries 1.8 and 1.16 imply that a Buchsteiner loop is an $A_{\ell,r}$-loop if and only if Q/N satisfies $(J(y)/x) \cdot xy = 1$ for all x, y. Later we will show that Q/N satisfies a much stronger property, and so every Buchsteiner loop will turn out to be an $A_{\ell,r}$-loop.

In the meantime, Corollary 1.16 gives us a useful family of automorphisms. Let

$$E_x = L(J(x), x) = L_{J(x)}L_x\,, \tag{E}$$

for each x in a loop Q.

Lemma 1.17. *Let Q be a Buchsteiner loop. Then for each $x \in Q$,*

(i) $E_x = R(x, J(x))^{-1} = [L_x, R_{J(x)}]^{-1}$,
(ii) $E_x \in \operatorname{Aut} Q$,
(iii) $E_{J(x)} = E_x = E_{I(x)}$,
(iv) $E_xI^n(x) = I^{n-2}(x)$ *and* $E_xJ^n(x) = J^{n+2}(x)$ *for each integer n,*
(v) $E_x = [L_x^{-1}, R_x^{-1}]$,
(vi) $E_x = L_xR_xR(x, x)R_x^{-1}L_x^{-1} = R_xL_xR(x, x)L_x^{-1}R_x^{-1}$, *and*
(vii) $E_x^{-1} = L_xR_xL(x, x)R_x^{-1}L_x^{-1} = R_xL_xL(x, x)L_x^{-1}R_x^{-1}$

Proof. Part (i) is a specialization of Theorem 1.4 to the present setting. Part (ii) follows from Corollary 1.16(ii). For (iii), we use part (ii) to compute

$$J(x)(x \cdot I(x)y) = E_x(I(x)y) = E_x I(x) \cdot E_x(y) = J(x) \cdot E_x(y)\,.$$

Canceling, we have $E_x = L(x, I(x)) = E_{I(x)}$, and the other equality of (iii) follows from replacing x with $J(x)$. For (iv), we use (iii) to compute $E_x I^n(x) = E_{I^{n-2}(x)} I^n(x) = I^{n-2}(x) \cdot I^{n-1}(x) I^n(x) = I^{n-2}(x)$, and the other equality follows from $J = I^{-1}$.

Since $xy = xI(x) \cdot xy$, applying (B') gives $yx = I(x)(xy \cdot x)$, that is, $R_x = L_{I(x)} R_x L_x$. Multiplying on the left by L_x and rearranging, we have (v). For (vi), we use Theorem 1.4 and (v): $R(x,x) = L_x^{-1} R_x^{-1} L_x R_x = R_x^{-1} L_x^{-1} E_x L_x R_x$. The mirror of this argument yields the other equality of (vi). Finally, (vii) is obtained from inverting (vi) and using Theorem 1.4. □

In §2 we shall need another easy general result about autotopisms:

Lemma 1.18. *Let Q be a loop, and let α and β be permutations of Q. Suppose that $\beta(1) = 1$. Then the triple (α, β, β) is an autotopism if and only if $\alpha = \beta \in \operatorname{Aut} Q$.*

Proof. Only the direct implication needs a proof. If (α, β, β) is an autotopism, then $\alpha(x) = \alpha(x) \cdot 1 = \alpha(x)\beta(1) = \beta(x \cdot 1) = \beta(x)$ for all $x \in Q$. □

2. Special elements and CC loops

An element a of a loop Q is said to have the *left inverse property* (LIP), or to be an *LIP element*, if there exists $b \in Q$ such that $L_a^{-1} = L_b$, that is, $L_a L_b = L_b L_a = \mathrm{id}_Q$. Applying this to $1 \in Q$ gives $ab = ba = 1$, and so $b = I(a) = J(a) = a^{-1}$. The left inverse property can be thus expressed as

(LIP) $$L_a L_{a^{-1}} = L_{a^{-1}} L_a = L(a^{-1}, a) = \mathrm{id}_Q\,.$$

In particular, a is an LIP element if and only if a^{-1} is an LIP element.

Similarly, $a \in Q$ is said to have the *right inverse property* (RIP), or to be an *RIP element*, if

(RIP) $$R_a R_{a^{-1}} = R_{a^{-1}} R_a = R(a, a^{-1}) = \mathrm{id}_Q\,.$$

If $a \in Q$ is both an LIP and RIP element, then we will refer to it simply as an *inverse property* (IP) *element*.

An element a in a loop Q is said to be *flexible* if $a \cdot xa = ax \cdot a$ for all $x \in Q$, that is, if and only if

(Flex) $$L_a R_a = R_a L_a\,.$$

Setting $x = J(a)$ and canceling shows that each flexible element a satisfies $I(a) = J(a)$.

An element a in a loop Q is said to be *left alternative* if $a \cdot ax = a^2 x$ for all $x \in Q$, and *right alternative* if $xa \cdot a = xa^2$ for all $x \in Q$. Equivalently, these are given, respectively, by

(LAlt) $L_a^2 = L_{a^2}$ (RAlt) $R_a^2 = R_{a^2}\,.$

An element a in a loop Q is said to be an *extra* element if it satisfies $a(y \cdot za) = (ay \cdot z)a$ for all $y, z \in Q$, that is, if and only if the triple of permutations

(Ex) $$\mathbf{E}(a) = (L_a,\, R_a^{-1},\, R_a^{-1} L_a)$$

is an autotopism of Q. Setting $z = 1$, one has that each extra element is flexible. Setting $z = a^{-1}$ shows that each extra element is an LIP element, and setting $y = a^{-1}$ gives that each extra element is also an RIP element.

Proposition 2.1. *Let Q be a Buchsteiner loop. For an element $a \in Q$, each of the following is equivalent: (i) a has the LIP, (ii) a has the RIP, (iii) a is flexible, (iv) a is left alternative, (v) a is right alternative, (vi) a is extra.*

Proof. Since $E_a = R_{J(a)}^{-1} R_a^{-1} = [L_a^{-1}, R_a^{-1}]$ by Lemma 1.17, we have the equivalence of (i), (ii), and (iii). The equivalence of (iii), (iv), and (v) follows from Lemma 1.5.

We have already noted that (vi) implies (iii). Conversely, if (iii) holds, that is, if a is flexible, then $L_a R_a^{-1} = R_a^{-1} L_a$, and so by examining the third components of autotopisms, we have $\mathbf{B}(a) = \mathbf{E}(a)$. Thus (iii) implies (vi). □

An element a of a loop Q is said to be a *Moufang* element if it satisfies $a(xy \cdot a) = ax \cdot ya$ for all $x, y \in Q$, that is, if the triple

$$\mathbf{M}(a) = (L_a,\, R_a,\, L_a R_a)$$

is an autotopism. It is immediate that every Moufang element is flexible.

Lemma 2.2. *In a Buchsteiner loop Q, an element a is Moufang if and only if it is extra and $a^2 \in N(Q)$.*

Proof. Since a Moufang element is flexible, it is extra by Proposition 2.1. Then $\mathbf{M}(a)\mathbf{E}(a) = (L_a^2, \mathrm{id}_Q, L_a^2) = (L_{a^2}, \mathrm{id}_Q, L_{a^2})$, since a is left alternative. By Lemma 1.13 and Corollary 1.6, $a^2 \in N_\lambda = N$. Conversely, if a is extra and $a^2 \in N$, then $(L_{a^2}, \mathrm{id}_Q, L_{a^2})\mathbf{E}(a)^{-1} = (L_a^2 L_a^{-1}, R_a^{-1}, L_a^2 L_a^{-1} R_a) = \mathbf{M}(a)$ is an autotopism, and so a is Moufang. □

Neither the set of extra elements nor the set of Moufang elements in a Buchsteiner loop is necessarily a subloop; a counterexample for both cases can be found in [13].

A loop is called a *Moufang* loop if every element is Moufang, and a loop is called an *extra* loop if every element is extra. A loop is extra if and only if it is Moufang and every square is in the nucleus [9]. Thus by the lemma, a Buchsteiner loop is extra if and only if it is Moufang. On the other hand, we have the following.

Lemma 2.3. *Every extra loop is a Buchsteiner loop.*

Proof. Since each element x of an extra loop is flexible, that is, $L_x R_x = R_x L_x$, we have $\mathbf{B}(x) = \mathbf{E}(x)$. □

Are there any non-extra Buchsteiner loops? We have observed that in such a loop there have to exist elements without "nice" properties. This rules out Moufang loops, but it does not rule out CC loops. Now, a loop is extra if and only if it is both Moufang and CC. On the other hand, a CC loop Q is extra if and only if (A) $x^2 \in N(Q)$ for all $x \in Q$, and (B) every element is IP (or LIP, or RIP, or flexible, or left or right alternative). We shall show below that a CC loop Q is a Buchsteiner loop if and only if condition (A) holds. This gives a vast class of examples of Buchsteiner loops, since every loop with $|Q : N| = 2$ is a CC loop [21]. These loops can be derived from groups in a constructive way [15] and they are never extra. A non-CC Buchsteiner loop will be constructed in §9.

CC loops were defined independently by Soikis [33] and by Goodaire and Robinson [21]. Proofs for the basic properties established in these papers can be also found elsewhere, see e.g. [14]. CC loops are those loops satisfying the LCC and RCC laws:

$$\text{(LCC)} \qquad x \cdot yz = ((xy)/x) \cdot xz$$

$$\text{(RCC)} \qquad zy \cdot x = zx \cdot (x\backslash(yx))\,.$$

These are respectively equivalent to certain triples being autotopisms:

$$\text{(LCC}'\text{)} \qquad \mathbf{L}(x) = (R_x^{-1}L_x,\, L_x,\, L_x)$$

$$\text{(RCC}'\text{)} \qquad \mathbf{R}(x) = (R_x,\, L_x^{-1}R_x,\, R_x)$$

Lemma 2.4. *A Buchsteiner loop is an LCC loop if and only if it is an RCC loop.*

Proof. Using (B”), we have that in Buchsteiner loops, (LCC) is equivalent to $zx \cdot y = z((xy)/x) \cdot x$. Replacing y with $x\backslash(yx)$, we get $zy \cdot (x\backslash(yx)) = zx \cdot y$, which is (RCC). The argument is clearly reversible. □

Proposition 2.5. *Let Q be a conjugacy closed loop. Then Q is a Buchsteiner loop if and only if $x^2 \in N(Q)$ for every $x \in Q$.*

Proof. The CC loop Q is a Buchsteiner loop if and only if

$$\mathbf{L}(x)\mathbf{B}(x)^{-1}\mathbf{R}(x) = (I\,, L_xR_xL_x^{-1}R_x\,, L_xR_xL_x^{-1}R_x)$$

is an autotopism for each $x \in Q$. By Proposition 1.14 and Corollary 1.6, this holds if and only if $L_xR_xL_x^{-1}R_x = R_{x^2}$ where $x^2 \in N(Q)$. □

3. Inverse properties

In a loop Q, the conditions

$$I^m(xy)I^{m+1}(x) = I^m(y) \qquad \text{and} \qquad J^{m+1}(x)J^m(yx) = J^m(y)\,, \tag{3.1}$$

for all $x, y \in Q$, are equivalent. Indeed, if the former one holds, then for $x' = I^{m+1}(x)$ and $y' = I^m(xy)$ we obtain $J^{m+1}(x')J^m(y'x') = J^{m+1}(x')J^m(I^m(y)) = xy = J^m(y')$.

A loop satisfying these conditions is called an *m-inverse loop*. These loops were introduced by Karkliňš and Karkliň [22] as a generalization of the weak [31] and cross [1] inverse properties, which correspond to the cases $m = -1$ and $m = 0$, respectively.

By reading $I^m(xy)I^{m+1}(x) = I^m(y)$ as $R_{I^{m+1}(x)}I^mL_x = I^m$, we get another pair of equivalent forms

$$R_{I^{m+1}(x)} = I^mL_x^{-1}J^m \quad \text{and} \quad L_{J^{m+1}(x)} = J^mR_x^{-1}I^m\,, \tag{3.2}$$

for all $x \in Q$. In particular, we have the following expressions for the left and right division operations in m-inverse loops:

$$x\backslash y = J^m(I^m(y)I^{m+1}(x)) \qquad \text{and} \qquad y/x = I^m(J^{m+1}(x)J^m(y)). \tag{3.3}$$

Let us reformulate some basic properties as they appear in [22] and in later works.

Lemma 3.1. *Let Q be an m-inverse loop. Then*

(i) *Q is also a $(-2m-1)$-inverse loop, and*
(ii) $I^{3m+1} \in \operatorname{Aut} Q$.

Proof. We have $J^{-(2m+1)+1}(x)J^{-(2m+1)}(yx) = I^{2m}(x)I^{2m+1}(yx) = I^m(I^m(yx) \cdot I^{m+1}(y)) \cdot I^{m+1}(I^m(yx)) = I^m(I^{m+1}(y)) = J^{-(2m+1)}(y)$, which yields (i). Setting $x' = I^{m+1}(x)$ and $y' = I^m(xy)$ in $I^{2m}(x')I^{2m+1}(y'x') = I^{2m+1}(y')$ yields $I^{3m+1}(x)I^{3m+1}(y) = I^{3m+1}(xy)$, and so (ii) holds. □

Lemma 3.2. *Let Q be an m-inverse loop. If (α, β, γ) is an autotopism, then so are*

$$(J^{m+1}\beta I^{m+1}, J^m\gamma I^m, J^m\alpha I^m) \quad \textit{and} \quad (I^m\gamma J^m, I^{m+1}\alpha J^{m+1}, I^m\beta J^m)\,.$$

Proof. In $\alpha(u)\beta(v) = \gamma(uv)$, take $u = I^m(xy)$, $v = I^{m+1}(x)$, to obtain $\alpha I^m(xy) \cdot \beta I^{m+1}(x) = \gamma I^m(y)$. Apply J^m and then multiply on the left by $J^{m+1}\beta I^{m+1}(x)$ to get $J^{m+1}\beta I^{m+1}(x) \cdot J^m\gamma I^m(y) = J^{m+1}\beta I^{m+1}(x) \cdot J^m(\alpha I^m(xy) \cdot \beta I^{m+1}(x)) = J^m\alpha I^m(xy)$. The other case is similar. □

A loop is called an *IP-loop* if it consists solely of IP-elements. In an IP-loop $(xy)^{-1} = y^{-1}x^{-1}$ since $(xy)^{-1}x = (xy)^{-1}(xy \cdot y^{-1}) = y^{-1}$, and so IP-loops satisfy the identities of the weak inverse property:

(WIP) $$xI(yx) = I(y) \quad \text{and} \quad J(xy)x = J(y).$$

However the cross inverse property $(xy)I(x) = y$ holds in an IP-loop only when the loop is commutative. While general m-inverse loops have some common algebraic properties [22], they nevertheless seem rather to be a topic of a combinatorial nature (e.g., see [23] and the subsequent generalizations to quasigroups [24, 25]).

We shall call a loop a *W^{k}IP*, $k \geq 1$, if it is an m-inverse loop, where $m = ((-2)^k - 1)/3$. (Note that for $k = 1$ we get WIP.)

Proposition 3.3. *Let Q be a W^{k}IP loop for some $k \geq 1$. Then*
- (i) *Q is a W^{h}IP loop for every $h \geq k$,*
- (ii) *$I^{2^k} \in \operatorname{Aut} Q$.*

Proof. If $3m = (-2)^k - 1$, then $|3m + 1| = 2^k$ and $3(-2m - 1) = -2(3m) - 3 = (-2)^{k+1} - 1$. The statement thus follows from Lemma 3.1. □

We shall refer to W^{2}IP as the *doubly weak inverse property* and write WWIP:

(WWIP) $$I(xy)I^2(x) = I(y) \qquad \text{and} \qquad J^2(x)J(yx) = J(y)\,.$$

In the next section we shall show that every Buchsteiner loop is a WWIP loop. In §6, we will examine Buchsteiner loops that satisfy WIP.

4. Doubly weak inverse property

Throughout this section, let Q be a Buchsteiner loop. Our main goal is to show that Q has WWIP.

Setting $z = I(xy)$ in (B) and rearranging gives the first equality of

(4.1) $$I(x)x = y \cdot I(xy)x \qquad \text{and} \qquad xJ(x) = xJ(yx) \cdot y\,,$$

and the second equality is verified similarly.

Lemma 4.1. *For all $x \in Q$,*
- (i) *$J(x)^2 \cdot x = I(x)$ and $x \cdot I(x)^2 = J(x)$,*
- (ii) *$I(x)/x = x\backslash J(x)$,*
- (iii) *$J(x)^2 = I(x)^2$,*
- (iv) *$I(x)x = I(x)^2x^2$ and $xJ(x) = x^2J(x)^2$,*

(v) *$I(x)x = J(x)J^2(x)$ and $xJ(x) = I^2(x)I(x)$,*
(vi) *$I(x) \cdot xJ(x) = J(x)$ and $I(x)x \cdot J(x) = I(x)$,*
(vii) *$J(x)I(x) \cdot x = J(x)$ and $x \cdot J(x)I(x) = I(x)$,*
(viii) *$J(x)/x = x\backslash I(x)$,*
(ix) *$I^2(x) = xJ(x) \cdot x$ and $J^2(x) = x \cdot I(x)x$.*

Proof. For (i): using Lemma 1.17(i) and 1.17(iv), we compute

$$J(x)^2 \cdot x = R_x R_{J(x)} J(x) = E_x^{-1} J(x) = J^{-1}(x) = I(x)\,.$$

The other equality follows similarly.

We have $L_x^{-1} R_{J(x)} L_x = R_x^{-1} R_{J(x)x} = R_x^{-1}$, by (1.1). Thus $L_x^{-1} R_{J(x)} = R_x^{-1} L_x^{-1}$. Applying both sides to $1 \in Q$, we obtain $I(x)/x = x\backslash J(x)$, which is (ii).

We obtain (iii) from (i) and (ii).

For (iv): In (4.1), set $y = x\backslash J(x)$ to get $I(x)x = (x\backslash J(x)) \cdot I(J(x))x = (x\backslash J(x)) \cdot x^2$. Since (i) gives $x\backslash J(x) = I(x)^2$, we have $I(x)x = I(x)^2 x^2$. The other equality follows from replacing x with $J(x)$.

For (v): Using (iv) and (iii), $I(x)x = I(x)^2 x^2 = J(x)^2 x^2$. Now (iii), with x replaced by $J(x)$, is $x^2 = J^2(x)^2$, and so $I(x)x = J(x)^2 J^2(x)^2$. Now the second equality of (iv), with x replaced by $J(x)$, is $J(x)J^2(x) = J(x)^2 J^2(x)^2$, and so $I(x)x = J(x)J^2(x)$. The other equality is obtained by replacing x with $I(x)$.

For (vi): Using (v) and Lemma 1.17(iii) and 1.17(iv), we compute

$$I(x) \cdot xJ(x) = I(x) \cdot I^2(x)I(x) = L_{I(x)} L_{I^2(x)} I(x) = E_{I^2(x)} I(x) = E_x I(x) = J(x)\,.$$

The other equality is given by the mirror of this argument.

For (vii): Apply (B') to (vi).

For (viii): This follows immediately from (vii).

For (ix): Compute $I^2(x) = E_x^{-1}(x) = R_x R_{J(x)}(x) = xJ(x) \cdot x$, using Lemma 1.17(iv) and 1.17(i). □

Lemma 4.2. *For all $x \in Q$, $I(x)x = xJ(x)$.*

Proof. Using Lemma 4.1(viii) and (B), we get

$$I(x)\backslash I^2(x) = x/I(x) = (x \cdot xI(x))/I(x) = I(x)\backslash(I(x)x \cdot x)\,,$$

and so $I^2(x) = I(x)x \cdot x$. But also $I^2(x) = xJ(x) \cdot x$ by Lemma 4.1(ix). Thus $I(x)x \cdot x = xJ(x) \cdot x$, and the proof is complete after canceling. □

In view of the preceding lemma and many subsequent calculations, it will be useful to set

$$(\eta) \qquad \eta(x) = xJ(x) = I(x)x$$

for all $x \in Q$.

Lemma 4.3. *For all $x \in Q$, $L(x,x) \in \mathrm{Aut}(Q)$ and $R(x,x) \in \mathrm{Aut}(Q)$.*

Proof. Set $u = (J(x)/x) \cdot x^2$. Since $J(x)/x = J(x)I(x)$ (Lemma 4.1(vii)), we have $u \cdot J(x)x = J(x)I(x) \cdot x^2$. Applying (B”), we get $xu \cdot J(x) = (x \cdot J(x)I(x)) \cdot x = I(x)x = \eta(x) = xJ(x)$, using Lemmas 4.1(vii) and 4.2. Canceling $J(x)$ and then x, we have $u = 1$. By Corollary 1.16(ii), $L(x,x) \in \mathrm{Aut}(Q)$. The other claim follows from $R(x,x) = L(x,x)^{-1}$ (Theorem 1.4). □

Lemma 4.4. *For all $x \in Q$,*

$$\begin{array}{rrclcrcl} \text{(i)} & L_{I^2(x)} & = & L_{\eta(x)} L_x & \textit{and} & R_{\eta(x)} R_x & = & R_{J^2(x)}\,, \\ \text{(ii)} & L_x L_{\eta(x)} & = & L_{J^2(x)} & \textit{and} & R_x R_{\eta(x)} & = & R_{I^2(x)}\,, \\ \text{(iii)} & L_x R_{\eta(x)} & = & R_{\eta(x)} L_x & \textit{and} & R_x L_{\eta(x)} & = & L_{\eta(x)} R_x\,. \end{array}$$

Proof. For (i): We will prove the first equality; the second will follow by the mirror of the argument. By Theorem 1.4, $L_{I^2(x)} = L_{I^2(x)} L(x^2, I(x)) R(I(x), x^2)$, and it will be useful to compute $L_{I^2(x)} L(x^2, I(x))$ and $R(I(x), x^2)$ separately.

Firstly, we use Lemma 4.1(i) to compute

$$L_{I^2(x)} L(x^2, I(x)) = L_{I^2(x)} L^{-1}_{x^2 \cdot I(x)} L_{x^2} L_{I(x)} = L_{I^2(x)} L^{-1}_{I^2(x)} L_{x^2} L_{I(x)} = L_{x^2} L_{I(x)}\,.$$

By (1.2), $L_{x^2} = L_x R_x^{-1} L_x R_x$, while by Lemma 1.17(iii), $L_{I(x)} = L_x^{-1} E_{I(x)} = L_x^{-1} E_x$. Thus

$$L_{I^2(x)} L(x^2, I(x)) = L_x R_x^{-1} L_x R_x L_x^{-1} E_x\,. \tag{4.2}$$

Next, using Lemma 4.1(i), $R(I(x), x^2) = R^{-1}_{x^2 \cdot I(x)} R_{I(x)} R_{x^2} = R^{-1}_{I^2(x)} R_{I(x)} R_{x^2}$. By (1.1), $R_{x^2} = R_x L_x^{-1} R_x L_x$. Thus $R(I(x), x^2) = R^{-1}_{I^2(x)} R_{I(x)} R_x L_x^{-1} R_x L_x = R^{-1}_{I^2(x)} E^{-1}_{I(x)} L_x^{-1} R_x L_x$. We apply Lemma 1.17(iii) to conclude

$$R(I(x), x^2) = R^{-1}_{I^2(x)} E_x^{-1} L_x^{-1} R_x L_x\,. \tag{4.3}$$

Now we put (4.2) and (4.3) together to get

$$L_{I^2(x)} = L_x R_x^{-1} L_x R_x L_x^{-1} E_x R^{-1}_{I^2(x)} E_x^{-1} L_x^{-1} R_x L_x\,.$$

Now $E_x R^{-1}_{I^2(x)} = R^{-1}_{E_x I^2(x)} E_x = R_x^{-1} E_x$, using Lemma 1.17(ii) and 1.17(iv). Thus $L_{I^2(x)} = L_x R_x^{-1} (L_x R_x L_x^{-1} R_x^{-1}) L_x^{-1} R_x L_x$. Applying Lemma 1.17(v) to the parenthesized expression, $L_{I^2(x)} = L_x R_x^{-1} E_x L_x^{-1} R_x L_x$. Now $E_x L_x^{-1} = L_{J(x)}$, so $L_{I^2(x)} = L_x R_x^{-1} L_{J(x)} R_x L_x$. Finally, by (1.2), $L_{\eta(x)} = L_x R_x^{-1} L_{J(x)} R_x$, and so $L_{I^2(x)} = L_{\eta(x)} L_x$, as claimed.

For (ii) and (iii): Using Lemma 4.1(ix), we rewrite the first equality of (i) as $\eta(x) \cdot xy = \eta(x)x \cdot y$. By Lemma 1.5, we also have $x \cdot y\eta(x) = xy \cdot \eta(x)$ and $x \cdot \eta(x)y = x\eta(x) \cdot y$. The first of these is the first equality of (iii), while the second can be seen to be the first equality of (ii) once we have observed that $x\eta(x) = x \cdot I(x)x = J^2(x)$, by Lemmas 4.2 and 4.1(ix). The other equalities of (ii) and (iii) similarly follow from the second equality of (i). □

Recall the inner mapping notation $T_x = R_x^{-1} L_x$.

Lemma 4.5. *For all $x \in Q$,*

(i) $T_{\eta(x)} = R(x,x) E_x^{-1}$, *and*
(ii) $T_{\eta(x)} \in \mathrm{Aut}(Q)$.

Proof. We compute

$$\begin{aligned} T_{\eta(x)} &= R^{-1}_{\eta(x)} L_{\eta(x)} = R^{-1}_{\eta(J^2(x))} L_{\eta(J^2(x))} = R_x^{-1} R_{J^2(x)} L_{\eta(J^2(x))} \\ &= R_x^{-1} L_{\eta(J^2(x))} R_{J^2(x)} = R_x^{-1} L_{\eta(x)} R_{J^2(x)} = R_x^{-1} L_x^{-1} L_{J^2(x)} R_{J^2(x)}\,, \end{aligned}$$

using Lemmas 4.2, 4.4(ii), 4.4(iii), 4.2 again, and 4.4(ii) again. Now $E_x L_x R_x = L_{J^2(x)} R_{J^2(x)} E_x$ using Lemma 1.17(ii) and 1.17(iv). Thus

$$T_{\eta(x)} = R_x^{-1} L_x^{-1} E_x L_x R_x E_x^{-1} = R(x,x) E_x^{-1}\,,$$

using Lemma 1.17(vi). This establishes (i). Part (ii) follows from (i), from $E_x \in \mathrm{Aut}(Q)$ (Lemma 1.17(ii)), and from $R(x,x) \in \mathrm{Aut}(Q)$ (Lemma 4.3). ☐

Lemma 4.6. *For $a \in Q$,*

$$L_a R_a^{-1} \in \mathrm{Aut}\, Q \quad \Leftrightarrow \quad T_a \in \mathrm{Aut}\, Q \quad \Leftrightarrow \quad a \in N.$$

Proof. Indeed, if $L_a R_a^{-1} \in \mathrm{Aut}\, Q$, then $a((xy)/a) = a(x/a) \cdot a(y/a)$ for all $x, y \in Q$. Setting $y = a$ yields $ax = a(x/a) \cdot a$, which is equivalent to $a \cdot xa = ax \cdot a$. Thus $L_a R_a = R_a L_a$.

Similarly, $L_a^{-1} R_a \in \mathrm{Aut}\, Q$ means $a\backslash(xy \cdot a) = (a\backslash(xa)) \cdot (a\backslash(ya))$ for all $x, y \in Q$. Again, setting $x = a$ gives $ay \cdot a = a \cdot ya$, that is, $L_a R_a = R_a L_a$. Thus $L_a R_a^{-1} \in \mathrm{Aut}\, Q \Leftrightarrow L_a^{-1} R_a \in \mathrm{Aut}\, Q$.

Finally, $L_a R_a^{-1} \in \mathrm{Aut}\, Q$ if and only if

$$\mathbf{B}(a)(L_a R_a^{-1},\, L_a R_a^{-1},\, L_a R_a^{-1})^{-1} = (L_a R_a L_a^{-1},\, L_a^{-1},\, \mathrm{id}_Q) \in \mathrm{Atp}\, Q\,.$$

This last expression is an autotopism if and only if it is equal to $(R_a,\, L_a^{-1},\, \mathrm{id}_Q)$. But this is equivalent to $a \in N_\mu = N$, by Corollary 1.6. ☐

Lemma 4.7. *For all $x \in Q$, $\eta(x) \in N(Q)$.*

Proof. By Lemma 4.5(ii), $T_{\eta(x)} \in \mathrm{Aut}(Q)$. By Lemma 4.6, $\eta(x) \in N(Q)$. ☐

We are now ready for the main result of this section.

Theorem 4.8. *Every Buchsteiner loop has the doubly weak inverse property.*

Proof. By (4.1) and Lemmas 4.7, 4.4(ii) (with $I^2(x)$ in place of x), and 4.2,

$$\eta(x) = y \cdot R_x I(xy) = y \cdot R_{\eta(I^2(x))} R_{I^2(x)} I(xy) = y \cdot (I(xy) I^2(x) \cdot \eta(x))\,.$$

By Lemma 4.7, $\eta(x) = (y \cdot I(xy) I^2(x)) \cdot \eta(x)$. Canceling, we get (WWIP). ☐

5. Calculations in isotopes

Many important results in loop theory are obtained by considering a given equality within the principal isotopes of a loop Q. In this section we shall follow this pattern. We first obtain that, in fact, a Buchsteiner loop is isomorphic to all of its isotopes. At the end we shall be able to verify that the factor of every Buchsteiner loop by its nucleus is an abelian group.

Recall that for a Buchsteiner loop Q, we denote by $Q[b]$ the isotope at $b \in Q$. The operation in $Q[b]$ is defined as $b\backslash(bx \cdot y) = (x \cdot yb)/b$, cf. §1. Also recall that a *G-loop* is a loop which is isomorphic to all of its loop isotopes. In fact, for a loop to be a G-loop, it is sufficient for it to be isomorphic to all of its left and right loop isotopes [4].

Theorem 5.1. *Let Q be a Buchsteiner loop. Then for each $x \in Q$, there exists an isomorphism from Q to $Q[x]$. In particular, every Buchsteiner loop is a G-loop.*

Proof. By Theorem 4.8, Q has WWIP, and so starting with the autotopism $\mathbf{B}(u)$, we obtain the autotopism $\hat{\mathbf{B}}(u) = (IL_u R_u^{-1} J, I^2 L_u J^2, I R_u^{-1} J)$, using Lemma 3.2. Now consider the autotopism $(\alpha_u, \beta_u, \gamma_u) = \mathbf{B}(I(\eta(u)))^{-1} \hat{\mathbf{B}}(u)$. We have

$$\alpha_u(1) = L_{I(\eta(u))}^{-1} I L_u R_u^{-1} J(1) = I(\eta(u))\backslash I(uJ(u)) = 1\,,$$

using direct computation. Also, since $\eta(u) \in N(Q)$ (Lemma 4.7), we use Lemmas 4.2 and 4.1(ix) (with $x = I(u)$) to compute

$$\begin{aligned}\beta_u(1) &= R_{I(\eta(u))} I^2 L_u J^2(1) = I^2(u) I(\eta(u)) \\ &= I(\eta(u) \cdot I(u)) = I(\eta(I(u)) \cdot I(u)) = I^4(u)\,,\end{aligned}$$

where in the third equality, we are using the identity $I(x)a^{-1} = I(ax)$ for any $a \in N(Q)$. Now applying Lemma 1.15, we have $(\alpha_u, \beta_u, \gamma_u) = (\alpha_u, R_{I^4(u)}\alpha_u, R_{I^4(u)}\alpha_u)$. Therefore $\alpha_{J^4(x)}$ is the desired isomorphism from Q to $Q[x]$. □

Proposition 5.2. *Let Q be a loop. The following are equivalent.*

(i) *Q is a Buchsteiner loop,*

(ii) *For all $x, y, u, v \in Q$,*

$$(\hat{\mathrm{B}}) \qquad (xy)\backslash((xy \cdot u)v) = (u(v \cdot yx)])/(yx)\,,$$

(iii) *For all $x, y, z, u, v \in Q$,*

$$(\hat{\mathrm{B}}') \qquad (xy \cdot u)v = xy \cdot z \qquad \Rightarrow \qquad u(v \cdot yx) = z \cdot yx\,.$$

(iv) *For all $x, y \in Q$,*

$$\tilde{\mathbf{B}}(x, y) = (L_{xy}, R_{yx}^{-1}, L_{xy} R_{yx}^{-1})$$

is an autotopism.

Proof. The equivalence of (ii) and (iii) is clear, as is the fact that $(\hat{\mathrm{B}})$ implies (B), and so (ii) implies (i). Also, (iv) is just a rewrite of (ii). To finish the proof, we shall show that (i) implies (ii).

Suppose Q is a Buchsteiner loop, and fix $x \in Q$. The isotope $Q[x]$ is also a Buchsteiner loop by Theorem 5.1. Denote its operation by $\circ$, and its left and right translations as $\tilde{L}_y$ and $\tilde{R}_y$, $y \in Q$, respectively. Then $\tilde{L}_y(z) = L_x^{-1} L_{xy}(z)$ and $\tilde{R}_y(z) = R_x^{-1} R_{yx}(z)$. Thus

$$\tilde{L}_y^{-1}((y \circ u) \circ v) = L_{xy}^{-1} L_x L_x^{-1} L_{x(y \circ u)}(v) = (xy)\backslash((xy \cdot u)v])\,,$$

and

$$\tilde{R}_y^{-1}(u \circ (v \circ y)) = R_{yx}^{-1} R_x R_x^{-1} R_{(v \circ y)x}(u) = (u(v \cdot yx))/(yx)\,.$$

Therefore $(\hat{\mathrm{B}})$ holds. □

Theorem 5.3. *Let Q be a Buchsteiner loop. Then Q/N is an abelian group.*

Proof. We compute the autotopism

$$\tilde{\mathbf{B}}(x, y)\mathbf{B}(xy)^{-1} = (I, R_{yx}^{-1} R_{xy}, L_{xy} R_{yx}^{-1} R_{xy} L_{xy}^{-1})\,.$$

By Proposition 1.14, $R_{yx}^{-1} R_{xy} = R_a$ for some $a \in N_\rho = N$. Evaluating at $1 \in Q$, we have $a = (xy)/(yx) \in N$, that is, $xy \equiv yx \bmod N$ for all $x, y \in Q$. Thus Q/N is an abelian group. □

Corollary 5.4. *A Buchsteiner loop Q is an $A_{l,r}$-loop.*

Proof. Since Q/N is an abelian group, we certainly have $(J(y)/x) \cdot xy \equiv 1 \bmod N$. But then Q is an A_l-loop by Corollary 1.16. The rest follows from Theorem 1.4. □

We conclude this section with a description of a normal subloop of a Buchsteiner loop which characterizes the centers of the left and right multiplication groups.

For a loop Q, set

$$M(Q) = \{a \in Q \,|\, L_a \in \mathcal{R}\} = \{a \in Q \,|\, T_a \in \mathcal{R}_1\}\,. \tag{M}$$

To see that these define the same set, note that for $a \in Q$, $L_a = R_a\psi$ for some $\psi \in \mathcal{R}_1$ if and only if $T_a \in \mathcal{R}_1$. In addition, let

$$\Gamma(Q) = \{a \in Q \,|\, a = \phi(1) \text{ for some } \phi \in \mathcal{L} \cap \mathcal{R}\}\,,$$

that is, $\Gamma(Q)$ is the orbit of $\mathcal{L} \cap \mathcal{R}$ containing the neutral element $1 \in Q$. Note that $M(Q) \subseteq \Gamma(Q)$.

Proposition 5.5. *Let Q be a Buchsteiner loop. Then*

(i) $M(Q) = \Gamma(Q) = \{a \in Q \,|\, R_a \in \mathcal{L}\} = \{a \in Q \,|\, T_a \in \mathcal{L}_1\}$,
(ii) $M(Q)$ *is a normal subloop of* Q,
(iii) $M \le Z(N)$, $Z(\mathcal{L}) = R_{(M)}$, *and* $Z(\mathcal{R}) = L_{(M)}$

Proof. Part (i) follows from $\mathcal{L}_1 = \mathcal{R}_1$ (Theorem 1.4). Since $\mathcal{L} \cap \mathcal{R}$ is a normal subgroup of $\operatorname{Mlt} Q$ (Corollary 1.3), its orbits form a block system of $\operatorname{Mlt}(Q)$, and so (ii) holds.

For $a \in M$, $L_a^{-1}R_a \in \mathcal{L}_1 \le \operatorname{Aut} Q$, since Q is an A_l-loop (Corollary 5.4). Thus $a \in N$ by Lemma 4.6, and so $M \subseteq N$. Now for $a \in M$, $c \in N$, $ca = R_ac = L_a\varphi(c) = ac$ for some $\varphi \in \mathcal{L}_1$, and so $M \le Z(N)$. The remaining assertions of (iii) follow from Lemma 1.7. □

6. Weak inverse property

In this section, we shall describe those Buchsteiner loops with the weak inverse property, and make further remarks about Buchsteiner CC loops. A parallel development can be found in [18].

Theorem 6.1. *Let Q be a WIP Buchsteiner loop. Then Q is a CC loop.*

Proof. Starting with the Buchsteiner autotopism $\mathbf{B}(x)$, we obtain from Lemma 3.2 (with $m = -1$) that $(R_x^{-1}, IL_xR_x^{-1}J, IL_xJ)$ is an autotopism for each x. Now $IL_xJ = R_{I^2(x)}^{-1}$ by WWIP, and $IR_x^{-1}J = L_x$ by WIP. Taking inverses, we have that $(R_x, L_x^{-1}R_{I^2(x)}, R_{I^2(x)})$ is an autotopism for each $x \in Q$. Now since Q/N is an abelian group, $I^2(x) = xn$ for some $n \in N$, and so $R_{I^2(x)} = R_xR_n$. Then $(R_x, L_x^{-1}R_x, R_x)$ is an autotopism for each $x \in Q$. But by (RCC'), this implies Q is an RCC loop. By Lemma 2.4, Q is a CC loop. □

By Proposition 2.5, a WIP Buchsteiner loop has every square in its nucleus. However, using, for instance, a finite model builder like Mace4 [29], it is easy to find examples of CC loops of order 16 with nuclear squares, but which do not have the WIP. Thus the variety of Buchsteiner CC loops, which we denote here by **BuchCC**, is wider than the variety of WIP Buchsteiner loops, here denoted by **BuchWIP**.

In the other direction, let Q be a Buchsteiner loop with two-sided inverses, that is, in which $J(x) = I(x)$ for all $x \in Q$. We denote the variety of such loops by **Buch2SI**. Then the identity (4.1) becomes $y \cdot I(xy)x = 1$, which is WIP. By Theorem 6.1, Q is a CC loop. In CC loops, the condition of having two-sided inverses is equivalent to power-associativity [28]. A detailed structure theory for power-associative CC loops, including those with the WIP is given in [27].

Narrower still is the variety of Buchsteiner loops with central squares, denoted by **BuchCS**. If each x^2 is central, then $x^2x = xx^2$, and this identity is equivalent in CC loops to power associativity [28]. A power associative CC loop has nuclear squares if and only if it has the WIP, but there exist power associative CC loops with WIP which squares which are not central [27].

Summarizing, we have the following proper inclusions among varieties of Buchsteiner CC loops:

$$\mathbf{BuchCS} \quad \subset \quad \mathbf{Buch2SI} \quad \subset \quad \mathbf{BuchWIP} \quad \subset \quad \mathbf{BuchCC}$$

7. Associator calculus

Let Q be a loop. For $x, y, z \in Q$ define the *associator* $[x, y, z]$ by

$$(x \cdot yz)\,[x, y, z] = xy \cdot z \tag{7.1}$$

We define the *associator subloop* $A(Q)$ to be the smallest normal subloop of Q such that $Q/A(Q)$ is a group. Equivalently, $A(Q)$ is the smallest normal subloop of Q containing all associators.

The following observations, as well as Lemma 7.3 below, are based upon [26, Lemma 4.2].

Lemma 7.1. *Let Q be a loop.*

(i) $[ax, y, z] = [x, y, z]$ *for all* $a \in N_\lambda$, $x, y, z \in Q$,
(ii) $[x, y, z] = [x, y, za]$ *for all* $a \in N_\rho$, $x, y, z \in Q$,
(iii) $[xa, y, z] = [x, ay, z]$ *for all* $a \in N_\mu \cap N_\lambda$, $x, y, z \in Q$,
(iv) $[x, ya, z] = [x, y, az]$ *for all* $a \in N_\mu \cap N_\rho$, $x, y, z \in Q$.

If $N(Q) \trianglelefteq Q$, then $[x, y, z]$ depends only on xN, yN, and zN.

Proof. Parts (i) and (ii) are immediate from the definitions. For (iii), we have $(xa\cdot yz)[xa, y, z] = (xa\cdot y)z = (x\cdot ay)z = (x\cdot(ay)z)[x, ay, z]$. Since $xa\cdot yz = x\cdot(ay)z$, we may cancel to get $[xa, y, z] = [x, ay, z]$, as claimed. The proof of (iv) is similar.

Now if $N \trianglelefteq Q$, then $xN = Nx$ for all $x \in Q$. Thus for every $a \in N$ there exist $a_1, a_2 \in N$ such that $[xa, y, z] = [a_1x, y, z]$ and $[x, y, az] = [x, y, za_2]$. Applying (i)-(iv), we have the remaining assertion. □

In particular, if $N \trianglelefteq Q$, we may regard $[-,-,-]$ as a mapping $(Q/N)^3 \to A(Q)$. If Q/N is, in addition, a group, then this allows us to write, for instance, $[x^{-1}, y, z]$ instead of $[I(x), y, z]$ or $[J(x), y, z]$. Similarly, we can ignore how elements in arguments are associated; for instance, we may write $[xyz, u, v]$ instead of $[x \cdot yz, u, v]$.

In a loop Q with normal nucleus $N = N(Q)$, for each $x \in Q$, we denote the restriction of T_x to N by

$$\tau_x = T_x|_N\,.$$

The first two parts of the following lemma have been observed many times.

Lemma 7.2. *Let Q be a loop with $N(Q) \trianglelefteq Q$. Then*

(i) *for each* $x \in Q$, $\tau_x \in \mathrm{Aut}(N(Q))$,
(ii) $\tau : Q \to \mathrm{Aut}(N(Q)); x \mapsto \tau_x$ *is a homomorphism,*
(iii) $A(Q) \le \ker(\tau)$,
(iv) τ *descends to a homomorphism* $\bar{\tau} : Q/A(Q) \to \mathrm{Aut}(N(Q))$.

Proof. If $a, b \in N$, then for all $x \in Q$, $T_x(a)T_x(b)x = T_x(a)(xb) = (T_x(a)x)b = x \cdot ab = T_x(ab)x$. This establishes (i).

Next, we have $T_xT_y(a) \cdot xy = T_x(T_y(a))x \cdot y = xT_y(a)y = xy \cdot a = T_{xy}(a) \cdot xy$ for $a \in N$, $x, y \in Q$. Thus $T_xT_y(a) = T_{xy}(a)$, and so also $T_{x\backslash y}(a) = T_x^{-1}T_y(a)$ and $T_{x/y}(a) = T_xT_y^{-1}(a)$. This proves (ii).

Now set $s = [x, y, z]$. Then for $a \in N(Q)$, we compute $(x \cdot yz) \cdot as = (x \cdot yza)s = xy \cdot za = (xy \cdot z)a = (x \cdot yz) \cdot sa$. Canceling, we have $as = sa$, that is, $s \in \ker(\tau)$. Since $A(Q)$ is the smallest normal subloop containing every associator, we have (iii).

Finally, (iv) follows immediately from (ii) and (iii). □

It will be convenient to introduce exponent notation for the action of the group $Q/A(Q)$ on $N(Q)$ as follows:

$$a^x = a^{xA(Q)} = \tau_x^{-1}(a) = T_x^{-1}(a) = x\backslash(ax)\,,$$

for $x \in Q$, $a \in N$. The notation a^x allows acting elements of Q to appear as group term. For instance, we may write a^{xyz} instead of giving the exponent an explicit association. Similarly, we will write $a^{x^{-1}}$ instead of specifying which of $I(x)$ or $J(x)$ is meant, since both have the same action upon a modulo $A(Q)$. Also, note that

$$ax = x(x\backslash(ax)) = xa^x \quad \text{and} \quad xa = ((xa)/x)x = a^{x^{-1}}x \tag{7.2}$$

for all $x \in Q$, $a \in N \trianglelefteq Q$.

Now we adjoin to these considerations the condition that $Q/N(Q)$ is a group, that is, $A(Q) \le N(Q)$. In this case, $A(Q)$ coincides with the smallest subloop containing all associators, by [27, Lemma 2.5]. This gives some insight into our situation, but we shall not need this result in what follows.

We denote the restriction to $A(Q)$ of the homomorphism τ by

$$\tilde{\tau}_x = \tau_x|_{A(Q)}\,.$$

for each $x \in Q$.

Lemma 7.3. *Let Q be a loop with $A(Q) \le N(Q) \trianglelefteq Q$. Then*

(i) $A(Q) \le Z(N(Q))$,
(ii) *for each* $x \in Q$, $\tilde{\tau}_x \in \mathrm{Aut}(A(Q))$,
(iii) $\tilde{\tau} : Q \to \mathrm{Aut}(A(Q)); x \mapsto \tilde{\tau}_x$ *is a homomorphism,*
(iv) $N(Q) \le \ker(\tilde{\tau})$,
(v) $\tilde{\tau}$ *descends to a homomorphism* $\bar{\tilde{\tau}} : Q/N(Q) \to \mathrm{Aut}(A(Q))$.

Proof. Part (i) follows from Lemma 7.2(iii). Parts (ii) and (iii) follow from parts (i) and (ii) of that same lemma. Part (iv) follows from (i), and (v) follows from (iii) and (iv). □

As with the action of $Q/A(Q)$ upon $N(Q)$, it will be helpful to use exponent notation for the action of $Q/N(Q)$ upon $A(Q)$:

$$s^x = s^{xN(Q)} = \tilde{\tau}_x^{-1}(s) = x\backslash(sx)$$

for $x \in Q$, $s \in A(Q)$.

Let Q be a loop such that $A(Q) \le N(Q) \trianglelefteq Q$, and let $xy \cdot z = s(x \cdot yz)$. Then $xy \cdot z = sx \cdot yz = xs^x \cdot yz = x \cdot s^x yz = x \cdot ys^{xy}z = (x \cdot yz)s^{xyz}$, and so $s^{xyz} = [x, y, z]$. We can thus state:

Lemma 7.4. *Let Q be a loop with $N \trianglelefteq Q$. For all $x, y, z \in Q$,*

$$xy \cdot z = (x \cdot yz)[x,y,z] = [x,y,z]^{(xyz)^{-1}}(x \cdot yz),$$
$$x \cdot yz = (xy \cdot z)[x,y,z]^{-1} = ([x,y,z]^{-1})^{(xyz)^{-1}}(xy \cdot z).$$

Now we are ready to turn to Buchsteiner loops.

Proposition 7.5. *Let Q be a loop with $A(Q) \le N \trianglelefteq Q$. Then Q is a Buchsteiner loop if and only if*

$$[x,y,z]^x = [y,z,x]^{-1} \quad \textit{for all} \quad x,y,z \in Q.$$

Proof. We have $x\backslash(xy \cdot z) = x\backslash(x \cdot yz)[x,y,z] = yz[x,y,z]$ and $(y \cdot zx)/x = ([y,z,x]^{-1})^{x^{-1}z^{-1}y^{-1}}(yz) = yz([y,z,x]^{-1})^{x^{-1}}$, by (7.2). □

We know, by Theorem 5.3, that Q/N is an abelian group if Q satisfies the Buchsteiner law. Thus the order in which elements of Q act on $A(Q)$ is of no consequence.

Let Q be a loop with $A(Q) \le N(Q) \trianglelefteq Q$. Then $L(x,y)(z) = z[x,y,z]^{-1}$ for all $x,y,z \in Q$. If, in addition, Q is an A_ℓ-loop, then also $L(x,y)^{-1}(uv) = L(x,y)^{-1}(u)L(x,y)^{-1}(v)$, for all $x,y,u,v \in Q$, and that is the same as $uv[x,y,uv] = u[x,y,u]v[x,y,v] = uv[x,y,u]^v[x,y,v]$. Thus

$$\text{(7.3)} \qquad [x,y,uv] = [x,y,u]^v[x,y,v] \ \text{ if } A(Q) \le N(Q) \trianglelefteq Q,\ Q \text{ an } A_\ell\text{-loop.}$$

Every Buchsteiner loop is an A_ℓ-loop by Corollary 5.4. In the rest of this section Q will denote a Buchsteiner loop.

Lemma 7.6. $[z^{-1},x,y] = [x,y,z]$ *for all $x,y,z \in Q$.*

Proof. From Proposition 7.5, we have $[x,y,z^{-1}]^{-1} = [z^{-1},x,y]^{z^{-1}}$. Write this as $[x,y,z^{-1}]^z = [z^{-1},x,y]^{-1}$. Furthermore, $1 = [x,y,z^{-1}z] = [x,y,z^{-1}]^z[x,y,z] = [z^{-1},x,y]^{-1}[x,y,z]$, using (7.3). □

The mapping $(x,y,z) \mapsto (z^{-1},x,y)$ defines an action of the cyclic group $\mathbb{Z}_6$ on $(Q/N)^3$. By the preceding lemma, associators are invariant under this action. For ease of use, we record this as follows.

Corollary 7.7. *For all $x,y,z \in Q$,*

$$[x,y,z] = [z^{-1},x,y] = [y^{-1},z^{-1},x] = [x^{-1},y^{-1},z^{-1}] = [z,x^{-1},y^{-1}] = [y,z,x^{-1}]\,.$$

Lemma 7.8. *For all $x,y,z \in Q$,*

$$[x,y,z] = [x,y,z]^{x^2} = [x,y,z]^{y^2} = [x,y,z]^{z^2}\,.$$

Proof. Using Proposition 7.5 and Corollary 7.7, we get $[x,y,z]^{x^2} = ([y,z,x]^x)^{-1} = ([x,y^{-1},z^{-1}]^x)^{-1} = [y^{-1},z^{-1},x] = [x,y,z]$. Further, $[x,y,z]^{y^2} = [y,z,x^{-1}]^{y^2} = [y,z,x^{-1}] = [x,y,z]$ and $[x,y,z]^{z^2} = [z,x^{-1},y^{-1}]^{z^2} = [z,x^{-1},y^{-1}] = [x,y,z]$. □

Lemma 7.9. *For all $x,y,z \in Q$,*

$$[x,y,z]^x = [y,z,x]^{-1}\,, \quad [x,y,z]^z = [z,x,y]^{-1}\,, \quad [x,y,z]^y = [x,y^{-1},z]^{-1}\,,$$
$$[x,y^{-1},z]^x = [z,x,y]^{-1}\,, \quad [x,y^{-1},z]^y = [x,y,z]^{-1}\,, \quad [x,y^{-1},z]^z = [y,z,x]^{-1}\,.$$

Proof. All of these equalities can be easily proved by means of Proposition 7.5 and Corollary 7.7. For example, $[x,y,z]^z = [z,x^{-1},y^{-1}]^z = [x^{-1},y^{-1},z]^{-1} = [z,x,y]^{-1}$. □

From Lemmas 7.8 and 7.9 we see that x, y and z induce an action by a group of exponent two on the set $\{[x,y,z]^{\pm 1}, [y,z,x]^{\pm 1}, [z,x,y]^{\pm 1}, [x,y^{-1},z]^{\pm 1}\}$.

The next lemma is easy. We need to verify that the behavior described in (7.3) is true in every position.

Lemma 7.10. *For $x, y, u, v \in Q$,*

$$[uv,x,y] = [u,x,y]^v[v,x,y]\,, \qquad [x,uv,y] = [x,u,y]^v[x,v,y]\,,$$

$$[x,y,uv] = [x,y,u]^v[x,y,v]\,.$$

Proof. From Lemma 7.9, we get $[x,uv,y]^y = [y,x,uv]^{-1} = ([y,x,u]^v[y,x,v])^{-1} = [x,u,y]^{yv}[x,v,y]^y = ([x,u,y]^v[x,v,y])^y$. The remaining case can be proved similarly. □

Lemma 7.11. *For all $x, y \in Q$, $[x,y,y] = [y,y,x]$.*

Proof. By Lemma 7.9, $[x,y,y]^y = [y,x,y]^{-1} = [y,y,x]^y$. □

Proposition 7.12. *Let Q be a Buchsteiner loop. For all $x, y, z \in Q$,*

$$[y,z,x][x,y,z] = [z,x,y][x,y^{-1},z]$$

Proof. We have

$$[x,x,z]^y[x,y,z] = [x,xy,z] = [x,yx,z] = [x,y,z]^x[x,x,z] = [x,x,z][y,z,x]^{-1}\,,$$

by Lemmas 7.9 and 7.10. From that we obtain

$$[x,x,z]^y = [x,x,z][y,z,x]^{-1}[x,y,z]^{-1}\,,$$

while

$$[x,x,z]^y = [z,x,x]^y = [x,x,z][z,x,y]^{-1}[x,y^{-1},z]^{-1}$$

is a consequence of

$$[z,x,x]^y[z,x,y] = [z,x,xy] = [z,x,yx] = [z,x,y]^x[z,x,x] = [x,x,z][x,y^{-1},z]^{-1}\,.$$

□

Express the equality of Proposition 7.12 as

$$[x,y^{-1},z] = [y,z,x][x,y,z][z,x,y]^{-1}\,. \tag{7.4}$$

From Corollary 7.7 we know that $[x,y^{-1},z] = [y,z^{-1},x] = [y,x^{-1},z]$. The right hand side of (7.4) hence retains its value when all arguments are subjected to a cyclic shift. Thus, for example,

$$[y,z,x][x,y,z][z,x,y]^{-1} = [z,x,y][y,z,x][x,y,z]^{-1}\,,$$

and therefore

$$[x,y,z][z,x,y]^{-1} = [x,y,z][x,y,z]^z = [x,y,z^2] = [x,y,z]^{-1}[z,x,y]\,.$$

The leftmost and the rightmost terms of this equation are mutually inverse, and so the element expressed by this equation has to have exponent 2.

Proposition 7.13. *Let Q be a Buchsteiner loop. For all $x, y, z \in Q$,*

$$1 = [x^2,y,z]^2 = [x,y^2,z]^2 = [x,z,y^2]^2 = [x^4,y,z] = [x,y^4,z] = [x,y,z^4]\,.$$

Proof. By the preceding paragraph, $[x,y,z^2]^2=1$. It is clear from Lemma 7.9 that the order of an associator element does not change when the arguments are cyclically shifted. Finally, $[x^4,y,z]=[x^2,y,z]^{x^2}[x^2,y,z]=[x^2,y,z]^2=1$, by Lemmas 7.11 and 7.8. □

Theorem 7.14. *Let Q be a Buchsteiner loop with nucleus $N(Q)$. Then $Q/N(Q)$ is an abelian group of exponent 4.*

Proof. We have $x\in N(Q)$ if and only if $[x,y,z]=[y,x,z]=[y,z,x]=1$ for all $y,z\in Q$. The statement thus follows from Theorem 5.3 and Proposition 7.13. □

8. A compatible system of associators

The properties of associators with which we start this section are included to make the construction of the ensuing example more transparent.

Lemma 8.1. *Let Q be a Buchsteiner loop. For all $x,y,z\in Q$,*

(i) $[x,y,z]^2=[y,z,x]^2=[z,x,y]^2$, *and* $[x^2,y,z]=[y,z,x^2]=[z,x^2,y]$,
(ii) *if* $a=[x^2,y,z]$, *then* $a^x=a^y=a^z=a$,
(iii) $[x^2,y^2,z]=[x^2,y,z^2]=[x,y^2,z^2]=1$, *and*
(iv) $[x,y,y]^{z^2}=[x,y,y]$ *and* $[y,x,y]^{z^2}=[y,x,y]$.

Proof. From Proposition 7.13 we obtain

$$1=[x^2,y,z]^2=([x,y,z]^x[x,y,z])^2=[y,z,x]^{-2}[x,y,z]^2\,.$$

Set now $a=[x^2,y,z]$. Then $a^x=([x,y,z]^x[x,y,z])^x$ equals a by Lemma 7.8, and $a^y=([y,z,x]^{-1})^y[x,y,z]^y=[z,x,y]([z,x,y][y,z,x]^{-1}[x,y,z]^{-1})$, by Proposition 7.12 and Corollary 7.7. Since we have already proved the first part of (i), we can replace $[z,x,y]^2$ with $[x,y,z]^2$, and so $a^y=[x,y,z]^x[x,y,z]=a$. Similarly we get

$$\begin{aligned}a^z&=([y,z,x]^{-1})^z[x,y,z]^z=[z,x,y][y,z,x][x,y,z]^{-1}[z,x,y]^{-1}\\&=[x,y,z][y,z,x]^{-1}=[x^2,y,z]\,,\end{aligned}$$

where we have also used the fact that a^z is of exponent two, by Proposition 7.13. Furthermore, $[x^2,y,z]=[x^2,y,z]^{x^2}$ by Lemma 7.8, and so

$$[x^2,y,z]=[y,z,x^2]^{-1}=[y,z,x^2]=[y,z,x^2]^y=[z,x^2,y]^{-1}=[z,x^2,y]\,.$$

This finishes the proof of (i) and (ii).

Now, $[x^2,y^2,z]=[x^2,y,z]^y[x^2,y,z]=[x^2,y,z]^2=1$, which yields (iii). To get (iv), we compute:

$$[x,z^2,y][x,y,y]=[x,z^2,y]^y[x,y,y]=[x,yz^2,y]=[x,z^2,y][x,y,y]^{z^2}\,,$$

and

$$[y,x,y][z^2,x,y]=[z^2,x,y]^y[y,x,y]=[yz^2,x,y]=[y,x,y]^{z^2}[z^2,x,y]\,.$$

□

Lemma 8.2. *Let Q be a Buchsteiner loop. Then for all $x,y\in Q$,*

(i) $[x,x,y]=[y,x,x]$, $[x,x,y]^x=[x,y,x]^{-1}$, $[x,y,x]^x=[x,x,y]^{-1}$, $[x,x,y]^y=[x,x,y]^{-1}$ *and* $[x,y,x]^y=[x,y,x]^{-1}$,
(ii) $[x^2,y,x]=[x^2,x,y]=[x,x^2,y]=[x,y,x^2]=[y,x,x^2]=[y,x^2,x]$,
(iii) $[x,y,x]=[x,x,y][x^2,x,y]$
(iv) $[x,y^2,x]=[x,x,y^2]=[y^2,x,x]=1=[x^2,x,y]^2$, *and*

(v) $[x,x,x]^y = [x,x,x][x,x,y]^{-1}[x,y,x]^{-1}$.

Proof. We have $[y,x,x] = [x,x,y]$ by Lemma 7.11, and hence we obtain $[y^2,x,x] = [y,x,x]^y[y,x,x] = [x,x,y]^{-1}[y,x,x] = 1$. Now, $[x,x^2,y] = [x,x,y]^x[x,x,y] = [x^2,x,y]$. That proves (ii) since we can use Lemma 8.1(i). Next, (i) follows from Lemma 7.9 as $[x,y^{-1},x] = [x,y^3,x] = [x,y^2,x]^x[x,y,x] = [x,y,x]$. We have $[x^2,x,y]^2 = 1$ by Proposition 7.13, and that makes (iii) and (iv) clear. To prove (v) consider the equalities

$$[x,x,x][x,y,x]^{-1} = [x,x,x][x,x,y]^x = [x,x,xy] = [x,x,x]^y[x,x,y]\,.$$

□

From here on, B will be a multiplicative abelian group and A an additive abelian group, where B acts on A multiplicatively. (Thus $b(a+a') = ba+ba'$ and $(b'b)a = b'(ba)$ for all $a,a' \in A$ and $b,b' \in B$.)

We shall construct a mapping $f : B^3 \to A$ such that

$$xf(x,y,z) = -f(y,z,x) \text{ for all } x,y,z \in B, \text{ and} \tag{8.1}$$

$$f(uv,y,z) = vf(u,y,z) + f(v,y,z) \text{ for all } u,v,y,z \in B. \tag{8.2}$$

In this section we shall give a concrete example of such a mapping f. In §9 we shall then extend it to a loop Q in such a way that $B \cong Q/N$, $A \cong A(Q) = N(Q)$ and that $[x,y,z]$ coincides with $f(x,y,z)$ (when appropriate identifications are done). This will give us a Buchsteiner loop, by Proposition 7.5. If we manage to find $x,y,z \in Q$ with $f(x,y,z) \neq f(y,x,z)$, then our loop will not be conjugacy closed (we explain this in detail in §9).

Set $B = \langle e_1, e_2; e_1^4 = e_2^4 = 1\rangle$ and define A as a vector space over the two-element field with basis c_{ijk}, where $k \geq i$ and $i,j,k \in \{1,2\}$. Thus $|A| = 64$.

We shall identify c_{ijk} with c_{kji}, which will allow us to deal with vectors c_{ijk} for all $i,j,k \in \{1,2,3\}$.

To define the action of B on A consider $i,j \in \{1,2\}$, $i \neq j$, and set

$$\begin{gathered} e_ic_{iij} = c_{iji}, \quad e_ic_{iji} = c_{iij}, \quad e_jc_{iij} = c_{iij}, \quad e_jc_{iji} = e_{iji}, \\ e_ic_{iii} = c_{iii}, \quad \text{and} \quad e_jc_{iii} = c_{iii} + c_{iij} + c_{iji}. \end{gathered} \tag{8.3}$$

The actions of e_1 and e_2 clearly commute. To see that we have really obtained an action of B on A, it hence suffices to verify that e_h^4 fixes each vector for both $h \in \{1,2\}$. In fact, we already have $e_h^2c_{ijk} = c_{ijk}$ for all $i,j,k \in \{1,2\}$. This is almost clear from the definitions, with perhaps one case requiring our attention: $e_j^2c_{iii} = c_{iii} + c_{iij} + c_{iji} + c_{iij} + c_{iji} = c_{iii}$.

We have verified that the action is defined correctly. Furthermore, we can state:

Lemma 8.3. *If $b \in B$ and $u \in A$, then $b^2u = u$.*

Denote by B_2 the vector space with basis e_1 and e_2 over the two-element field. Denote by π the projection $B \to B_2; e_i \mapsto e_i$. By Lemma 8.3, the action of B on A induces an action of B_2 on A, .

Lemma 8.4. *The elements of A centralized by B form a subspace generated by $c_{121} + c_{112}$ and $c_{212} + c_{221}$.*

Proof. Consider $i,j \in \{1,2\}$, $i \neq j$. Clearly $e_i(c_{iji}+c_{iij}) = e_j(c_{iji}+c_{iij}) = c_{iji}+c_{iij}$. Suppose, on the other hand, that $u \in A$ is centralized by the action. We argue by contradiction, and assume that in the standard basis the vector u has different coefficients at c_{iij} and c_{iji}. Since $e_i c_{iji} = c_{iij}$, there must exist a base vector $c \notin \{c_{iji}, c_{iij}\}$ such that $e_i c$ has different coefficients at c_{iji} and c_{iij}. However, no such vector exists. □

Define $C : B_2^3 \to A$ in such a way that $C(a,b,c) = C(c,b,a)$ for all $a,b,c \in B_2$, $C(a,b,c) = 0$ if one of $a,b,c \in B_2$ is equal to 0, $C(e_i,e_j,e_k) = c_{ijk}$ for all $i,j,k \in \{1,2,3\}$, and the following holds when $i,j \in \{1,2\}$ and $i \neq j$:

$$\begin{gathered} C(e_i,e_i,e_i+e_j) = c_{iii} + c_{iji} \quad \text{and} \quad C(e_i,e_i+e_j,e_i) = c_{iii} + c_{iij}, \\ C(e_i,e_i+e_j,e_j) = c_{iij} + c_{jji} \quad \text{and} \quad C(e_i,e_j,e_i+e_j) = c_{ijj} + c_{iji}, \\ C(e_i,e_i+e_j,e_i+e_j) = c_{iii} + c_{iij} + c_{jji} + c_{iji}, \\ C(e_i+e_j,e_j,e_i+e_j) = c_{jjj} + c_{iji}, \text{ and} \\ C(e_i+e_j,e_i+e_j,e_i+e_j) = c_{iii} + c_{jjj} + c_{iij} + c_{jji}. \end{gathered} \tag{8.4}$$

We shall usually assume that $x \in B$ is expressed in the form $\prod e_i^{\alpha_i} e_i^{2\alpha'_i}$, $i \in \{1,2\}$, where $\alpha_i, \alpha'_i \in \{0,1\}$. Let us have also $y = \prod e_i^{\beta_i} e_i^{2\beta'_i}$ and $z = \prod e_i^{\gamma_i} e_i^{2\gamma'_i}$.

For $h \in \{1,2\}$ define a mapping $s_h : B^3 \to \{0,1\}$ so that

$$s_h(x,y,z) = \alpha'_h(\beta_2\gamma_1 + \beta_1\gamma_2) + \beta'_h(\gamma_2\alpha_1 + \gamma_1\alpha_2) + \gamma'_h(\alpha_2\beta_1 + \alpha_1\beta_2), \tag{8.5}$$

and define $f : B^3 \to A$ by

$$\begin{aligned} f(x,y,z) = C(\pi(x),\pi(y),\pi(z)) \\ + s_1(x,y,z)(c_{112} + c_{121}) + s_2(x,y,z)(c_{122} + c_{212}). \end{aligned} \tag{8.6}$$

From the definition of s_h we see immediately that

$$s_h(x_1,x_2,x_3) = s_h(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)}) \text{ for all permutations } \sigma \in S_3, \tag{8.7}$$

for every $x_1,x_2,x_3 \in Q$ and $i \in \{1,2\}$.

Lemma 8.5. *Consider $x,y,z,u \in B$ and $h \in \{1,2\}$. Then*

(i) $s_h(xu^2,y,z) = s_h(x,y,z) + s_h(u^2,y,z)$,
(ii) $s_h(u^2,xy,z) = s_h(u^2,x,z) + s_h(u^2,y,z)$, *and*
(iii) $s_h(x^2,y^2,z) = 0$.

Proof. Let us have $u^2 = \prod e_i^{2\delta_i}$. Then $s_h(xu_2,y,z) = (\alpha' h + \delta_h)(\beta_2\gamma_1 + \beta_1\gamma_2) + \beta'_h(\gamma_2\alpha_1+\gamma_1\alpha_2)+\gamma'_h(\alpha_2\beta_1+\alpha_1\beta_2)$ equals $s_h(u^2,y,z)+s_h(x,y,z)$ since $s_h(u^2,y,z) = \delta_h(\beta_2\gamma_1 + \beta_1\gamma_2)$. The rest is clear. □

As an immediate consequence we obtain:

Corollary 8.6. *For all $x,y,z,u \in B$,*

(i) $f(xu^2,y,z) = f(x,y,z) + f(u^2,y,z)$, $f(x,yu^2,z) = f(x,y,z) + f(x,u^2,z)$ *and* $f(x,y,zu^2) = f(x,y,z) + f(x,y,u^2)$,
(ii) *the vector* $f(u^2,y,z) = f(u^2,z,y) = f(y,u^2,z) = f(z,u^2,y) = f(y,z,u^2) = f(z,y,u^2)$ *is centralized by the action of B on A;*
(iii) $f(u^2,xy,z) = f(u^2,x,z) + f(u^2,y,z)$, *and*
(iv) $f(x^2,y^2,z) = 0$.

Proposition 8.7. $f(x^2,y,z) = f(x,y,z) + f(y,z,x)$ *for all $x,y,z \in B$.*

TABLE 1. The case $a = \pi(x) = e_1$

$b = \beta_1 e_1 + \beta_2 e_2$	$c = \gamma_1 e_1 + \gamma_2 e_2$	$C(a,b,c)$	$C(b,c,a)$
e_1	e_2	c_{112}	c_{121}
e_1	$e_1 + e_2$	$c_{111} + c_{121}$	$c_{111} + c_{112}$
e_2	e_1	c_{121}	c_{112}
e_2	$e_1 + e_2$	$c_{122} + c_{121}$	$c_{122} + c_{112}$
$e_1 + e_2$	e_1	$c_{111} + c_{112}$	$c_{111} + c_{121}$
$e_1 + e_2$	e_2	$c_{112} + c_{122}$	$c_{122} + c_{121}$

TABLE 2. The case $a = \pi(x) = e_2$

$b = \beta_1 e_1 + \beta_2 e_2$	$c = \gamma_1 e_1 + \gamma_2 e_2$	$C(a,b,c)$	$C(b,c,a)$
e_1	e_2	c_{212}	c_{122}
e_1	$e_1 + e_2$	$c_{112} + c_{212}$	$c_{112} + c_{122}$
e_2	e_1	c_{122}	c_{212}
e_2	$e_1 + e_2$	$c_{222} + c_{212}$	$c_{222} + c_{122}$
$e_1 + e_2$	e_1	$c_{112} + c_{122}$	$c_{112} + c_{212}$
$e_1 + e_2,$	e_2	$c_{222} + c_{122}$	$c_{222} + c_{212}$

Proof. The left hand side of the equality evaluates to $\alpha_1(\beta_2\gamma_1+\beta_1\gamma_2)(c_{112}+c_{121})+\alpha_2(\beta_1\gamma_2+\beta_2\gamma_1)(c_{221}+c_{212})$, which is equal to

$$(\beta_1\gamma_2+\beta_2\gamma_1)(\alpha_1 c_{112}+\alpha_1 c_{121}+\alpha_2 c_{221}+\alpha_2 c_{212}). \tag{8.8}$$

The right hand side yields the sum

$$C(\alpha_1 e_1+\alpha_2 e_2, \beta_1 e_1+\beta_2 e_2, \gamma_1 e_1+\gamma_2 e_2)+ C(\beta_1 e_1+\beta_2 e_2, \gamma_1 e_1+\gamma_2 e_2, \alpha_1 e_1+\alpha_2 e_2), \tag{8.9}$$

as $s_h(x,y,z)+s_h(y,z,x)=0$ for both $h \in \{1,2\}$.

To compare (8.8) and (8.9) we thus need to look only at (α_1,α_2), (β_1,β_2) and (γ_1,γ_2). The equality is clear if one of these pairs is equal to $(0,0)$. We can thus assume it is not. Furthermore, note that then $\beta_1\gamma_2+\beta_2\gamma_1$ is equal to zero if and only if $(\beta_1,\beta_2)=(\gamma_1,\gamma_2)$. In such a case (8.9) is a sum of two same values, and as such it is equal to zero as well. Hence also $(\beta_1,\beta_2)\neq(\gamma_1,\gamma_2)$ can be assumed.

There are three cases to be considered, and these are $\pi(x)=e_1$, $\pi(x)=e_2$ and $\pi(x)=e_1+e_2$. Under the assumed conditions the left hand side (8.8) is in these cases equal to $c_{112}+c_{121}$, $c_{221}+c_{212}$ and $c_{112}+c_{121}+c_{221}+c_{212}$, respectively.

To get the right hand side consider Tables 1, 2 and 3. Each of them tabulates, for the given $\pi(x)$, the values of both factors that form the sum (8.9), running through the six possible combinations of (β_1,β_2) and (γ_1,γ_2). It is easy to verify that the sum of the rightmost two columns is always equal to the value given by the left hand side. □

Lemma 8.8. *If $a,b,c \in B_2$, then $aC(a,b,c)=C(b,c,a)$.*

Proof. We first verify $aC(a,b,b)=C(a,b,b)$. We have $e_iC(e_i,e_j,e_j)=c_{ijj}=C(e_i,e_j,e_j)$, both for $i=j$ and $i\neq j$. In the latter case $e_iC(e_i,e_i+e_j,e_i+e_j)=e_i(c_{iii}+c_{iij}+c_{jji}+c_{iji})=c_{iii}+c_{iji}+c_{jji}+c_{iij}=C(e_i,e_i+e_j,e_i+e_j)$, $e_j(e_iC(e_i+e_j,e_i,e_i))=e_j(c_{iii}+c_{iij})=c_{iii}+c_{iji}=C(e_i+e_j,e_i,e_i)$ and $e_j(e_iC(e_i+$

TABLE 3. The case $a = \pi(x) = e_1 + e_2$

b	c	$C(a,b,c)$	$C(b,c,a)$
e_1	e_2	$c_{112} + c_{212}$	$c_{122} + c_{121}$
e_1	$e_1 + e_2$	$c_{111} + c_{212}$	$c_{111} + c_{112} + c_{122} + c_{121}$
e_2	e_1	$c_{122} + c_{212}$	$c_{112} + c_{121}$
e_2	$e_1 + e_2$	$c_{222} + c_{121}$	$c_{222} + c_{122} + c_{112} + c_{212}$
$e_1 + e_2$	e_1	$c_{111} + c_{112} + c_{122} + c_{121}$	$c_{111} + c_{212}$
$e_1 + e_2$	e_2	$c_{222} + c_{122} + c_{112} + c_{212}$	$c_{222} + c_{121}$

$e_j, e_i + e_j, e_i + e_j)) = e_j(c_{iii} + c_{jjj} + c_{ijj} + c_{jij} + c_{iji} + c_{ijj}) = c_{iii} + c_{iij} + c_{iji} + c_{jjj} + c_{ijj} + c_{iji} = C(e_i + e_j, e_i + e_j, e_i + e_j)$.

We can thus assume $b \neq c$, and also $0 \notin \{b, c\}$. When the penultimate column of Table 1 is multiplied by $a = e_1$, we clearly always get the rightmost column. Hence the equality holds for $a = e_1$, and the same approach can be used for the case $a = e_2$. Let us have $a = e_1 + e_2$. We shall use Table 3. It is enough to observe that $e_1(e_2 x) = y$ when (x, y) takes the values $(C(a,b,c), C(b,c,a))$ in the first four rows, by Lemma 8.3. The second and the fourth row follow from the formula $e_i(e_j(c_{iii} + c_{jij})) = e_i(c_{iii} + c_{iij} + c_{iji} + c_{ijj}) = c_{iii} + c_{iji} + c_{iij} + c_{ijj}$. Finally, $e_1(e_2(c_{112} + c_{212})) = e_1(c_{112} + c_{122}) = c_{121} + c_{122}$ and $e_1(e_2(c_{122} + c_{121})) = e_1(c_{212} + c_{121}) = c_{212} + c_{112}$. □

Corollary 8.9. *$xf(x,y,z) = f(y,z,x)$ for all $x,y,z \in B$.*

Proof. Let us express $f(x,y,z)$ and $f(y,z,x)$ following the definition (8.6). By using (8.7) and Lemma 8.4 we see that the statement holds if and only if

$$\pi(x)C(\pi(x), \pi(y), \pi(z)) = C(\pi(y), \pi(z), \pi(x))$$

for all $x, y, z \in B$. However, that is exactly what is claimed in Lemma 8.8. □

Note that we have just verified (8.1) since A has exponent 2. To get (8.2), several further verifications that go back to the definition of f are needed if we wish to obtain a direct proof. Such a proof is possible. But it is not needed, if we can show, without using (8.2), that there exists a loop Q such that Q/N corresponds to B, $A(Q)$ to A and f to $[-,-,-]$, including the actions. Indeed, in such a case (8.1) guarantees that Q will be a Buchsteiner loop, by Proposition 7.5, and so (8.2) will then follow from Lemma 7.10. We shall see that the suggested path is viable.

9. Extending the associators to a loop

Our goal now is to build a Buchsteiner loop Q with $Q/N \cong B$ and $A(Q) \cong A$ in such a way that $[-,-,-]$ corresponds to the mapping $f : B^3 \to A$ defined by (8.6).

From (7.2), we see that $xa \cdot yb = (xy)(a^y b)$ for all $x, y \in Q$ and $a, b \in N$ whenever Q is a loop with $N \trianglelefteq Q$. In our case, we know the action of Q/N on N since we intend to construct a loop with $N(Q) = A(Q)$. The issue, then, is to define the product xy for representatives of classes modulo N.

If Q is a Buchsteiner loop, then

$$\{x^2 a \mid x \in Q \text{ and } a \in N(Q)\} \trianglelefteq Q \qquad (9.1)$$

is a group. Indeed, it is a normal subloop by Theorem 7.14, and it is a group by Lemma 8.1(iii).

From considering (9.1), we arrive at the idea that the definitions of products xy can be restricted to representatives modulo this group. In other words, we might be able to consider only x and y that correspond to square-free elements of B. The next lemma gives the necessary technical basis for such an approach.

At this point, we require the notion of loop *commutator*, defined by

$$xy = yx \cdot [x, y]$$

for all x, y in a loop Q. In a Buchsteiner loop, each commutator lies in the nucleus, by Theorem 5.3.

Lemma 9.1. *Let Q be a Buchsteiner loop. For all $x, y, u, v \in Q$,*

$$xu^2 \cdot yv^2 = (xy \cdot u^2v^2)[u^2, y]^{v^2}[x, y, v^2]([x, y, u^2]^{-1}[y, x, u^2])^{v^2} .$$

Proof. We start with $xu^2 \cdot yv^2 = (xu^2 \cdot y)v^2[xu^2, y, v^2]^{-1}$, and note that $[xu^2, y, v^2] = [x, y, v^2][u^2, y, v^2]^x = [x, y, v^2]$, by Lemmas 7.10 and 8.1. Furthermore, $xu^2 \cdot y = x \cdot u^2y[x, u^2, y] = x \cdot yu^2[u^2, y][x, u^2, y] = xy \cdot u^2[x, y, u^2]^{-1}[u^2, y][x, u^2, y]$. □

In our case we get a simpler equality

$$xu^2 \cdot yv^2 = (xy \cdot u^2v^2)[u^2, y][x, y, v^2] , \tag{9.2}$$

since we assume $A(Q) = N(Q)$, and $A(Q)$ is centralized by squares, by Lemma 8.3. The other simplification follows from Corollary 8.6(ii).

The question now is how to compute $[u^2, x]$. Note that the unique $a \in Q$ satisfying $x^2 \cdot x = (x \cdot x^2)a$ is equal to both $[x, x, x]$ and $[x^2, x]$. This solves the case $x = a$. To understand the remaining cases, we start from a general statement.

Lemma 9.2. *Let Q be a loop with $A(Q) \leq N(Q)$ such that Q/N is an abelian group. For all $x, y, z \in Q$,*

$$[xy, z] = [x, z]^y[y, z][x, z, y]^{-1}[x, y, z][z, x, y] .$$

Proof. Now, $xy \cdot z = (x \cdot yz)[x, y, z] = x \cdot zy[y, z][x, y, z] = xz \cdot y[x, z, y]^{-1}[y, z][x, y, z]$, and $xz \cdot y = zx[x, z] \cdot y = zx \cdot y[x, z]^y = z \cdot xy[z, x, y][x, z]^y$. These two equalities, together with Lemma 7.3(i), imply

$$xy \cdot z = z \cdot xy[x, z]^y[y, z][x, z, y]^{-1}[x, y, z][z, x, y] .$$

This gives the desired result since $xy \cdot z$ can be also expressed as $z \cdot xy[xy, z]$. □

Corollary 9.3. *Let Q be a Buchsteiner loop. For each $x, y, z \in Q$, if $[x, z, y] = [y, z, x]$, then $[xy, z] = [x, z]^y[y, z][x, y^{-1}, z]$.*

Proof. Use Proposition 7.12 and Corollary 7.7. □

From Lemma 7.10 we see that the next statement can be easily proved by induction on the length of words in Q/N. The second part of the statement follows from Lemma 7.11.

Lemma 9.4. *Let Q be a Buchsteiner loop such that Q/N is generated by the set $\{xN;\ x \in X\}$, where $X \subseteq Q$. If $[x, y, z] = [z, y, x]$ for all $x, y, z \in X$, then this property holds for all $x, y, z \in Q$. In particular it is true whenever Q/N can be generated by two elements.*

Lemma 9.5. *Let Q be a Buchsteiner loop with elements x and y. Then*

(i) $[x^2, y] = [x, y]^x[x, y][x, y, x]$,

(ii) $[x^2, xy] = [x, y]^x[x, y][x, x, x][x, y, x]^{-1}$,
(iii) $[x^2y^2, y] = [x, y]^{xy^2}[x, y]^{y^2}[y, y, y][x, y, x]$,
(iv) $[x^2y^2, x] = [y, x]^y[y, x][x, x, x][y, x, y,]$, *and*
(v) $[x^2y^2, xy] = [x, y]^{xy^2}[y, x]^y[x, x, x][y, y, y][x, y, x]^{-1}[y, x, y]^{-1}$.

Proof. To obtain (i), use Lemma 9.4 and Corollary 9.3, and note that $[x, x^{-1}, y] = [x, y^{-1}, x] = [x, y^2, x][x, y, x] = [x, y, x]$, by Corollary 7.7 and Lemma 8.2. For (ii), first note that $[x^2, xy]$ is the inverse of $[xy, x^2] = [x, x^2]^y[y, x^2][x, y^{-1}, x^2]$. Now, $[x^2, x]^y = [x, x, x]^y = [x, x, x][x, x, y]^{-1}[x, y, x]^{-1}$, where $[x, y, x]^{-1}$ gets canceled when $[x^2, y]$ is expressed by (i). Therefore (ii) follows from Lemma 8.2, since $[x, y, x^2]^{-1} = [x, x, y][x, y, x]^{-1}$.

To prove (iii), we first employ Corollary 9.3 to get $[x^2, y]^{y^2}[y, y, y][x^2, y^2, y]$. The latter associator is trivial, and the rest follows from (i) and from Lemma 8.2. To get (iv), we start from $[x^2, x]^{y^2}[y^2, x]$, which evaluates to $[y, x]^y[y, x][y, x, y][x, x, x]$, by (ii).

Let us show (v). From Corollary 9.3 we obtain $[x^2, xy]^{y^2}[y^2, xy]$, by Lemma 8.2. One part of the expression thus follows from (ii). Furthermore, $[y^2, xy] = [y, xy]^y[y, xy][y, xy, y]$, and $[y, xy]$ is an inverse of $[xy, y] = [x, y]^y[y, x, y]$. Thus $[y^2, xy] = [y, x]^{y^2}[y, x]^y a$, where a is a product of $([y, x, y]^{-1})^y[y, x, y]^{-1}$ and of $[y, xy, y] = [y, y, y][y, x, y]^y$. Hence $a = [y, y, y][y, x, y]^{-1}$. □

The formulas of Lemma 9.5 simplify substantially if $[x, y] = 1$ is assumed. We shall do so in our construction. Under simplifying assumptions, one can express Lemma 9.5 by a single formula:

Corollary 9.6. *Let Q be a Buchsteiner loop. Suppose that $e_1, e_2 \in Q$ satisfy $e_1e_2 = e_2e_1$, and that all elements $c_{ijk} = [e_i, e_j, e_k]$ are of exponent 2, for all $i, j, k \in \{1, 2\}$. Then*

$$[e_1^{2\alpha_1}e_2^{2\alpha_2}, e_1^{\beta_1}e_2^{\beta_2}] = \prod_{i,j} c_{iji}^{\alpha_i\beta_j} ,$$

for all $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \{0, 1\}$.

We are now ready for the construction. We shall define a loop structure on $Q = B \times A$ as follows. For all $i\alpha_h, \alpha'_h, \beta_h, \beta'_h \in \{0, 1\}$, where $h \in \{1, 2\}$, and for all $a, b \in A$, we set

$$(9.3)\quad \left(\prod e_h^{\alpha_h}\prod e_h^{2\alpha'_h},\ a\right)\cdot\left(\prod e_h^{\beta_h}\prod e_h^{2\beta'_h},\ b\right) =$$
$$\left(\prod_h e_h^{\alpha_h+\beta_h}\prod_h e_h^{\alpha_h\beta_h+\alpha'_h+\beta'_h},\quad D(\alpha_1e_1+\alpha_2e_2, \beta_1e_1+\beta_2e_2) + \right.$$
$$\left.\sum_h(\alpha_1\beta_2+\alpha_2\beta_1)\beta'_h z_h + \sum_{i,j}\alpha'_i\beta_j c_{iji} + e_1^{\beta_1}e_2^{\beta_2}a + b\right),$$

where $z_1 = c_{112} + c_{121}$, $z_2 = c_{221} + c_{212}$ and $D : B_2 \times B_2 \to A$ is defined in the following way:

$$\begin{gathered} D(u,v) = 0 \text{ if } u, v \neq e_1 + e_2, \\ D(e_1, e_1+e_2) = c_{112} + c_{121} \text{ and } D(e_2, e_1+e_2) = c_{122} + c_{212}, \\ D(e_1+e_2, e_1) = c_{112},\ D(e_1+e_2, e_2) = c_{122} \text{ and} \\ D(e_1+e_2, e_1+e_2) = c_{121} + c_{212}. \end{gathered} \tag{9.4}$$

The arithmetic in the exponents computed modulo 2. Set $x = \prod e_h^{\alpha_h + 2\alpha'_h}$ and $y = \prod e_h^{\beta_h + 2\beta'_h}$. The formula (9.3) can be also expressed as

$$(x,a)\cdot(y,b) = (xy,\ g(x,y) + ya + b), \text{ where } g : B^2 \to A. \tag{9.5}$$

We shall now describe the structure of g. Put $x_0 = (e_1^{\alpha_1} e_2^{\alpha_2}, 0) = (e_1^{\alpha_1}, 0)\cdot(e_2^{\alpha_2}, 0)$ and $x_1 = (e_1^{2\alpha'_1} e_2^{2\alpha'_2}, 0) = (e_1^{2\alpha'_1}, 0)\cdot(e_2^{2\alpha'_2}, 0)$, and similarly define y_0 and y_1. We have $(x,0) = x_0 x_1$ and $(y,0) = y_0 y_1$, by (9.4). Since (9.2) remains valid when u^2 and v^2 are replaced by elements that are squares modulo N, we obtain

$$x_0x_1 \cdot y_0y_1 = x_0y_0 \cdot x_1y_1[x_1,y_0][x_0,y_0,y_1]. \tag{9.6}$$

Now, $[x_0, y_0, y_1]$ should equal $f(\prod e_i^{\alpha_i}, \prod e_i^{\beta_i}, \prod e_i^{2\beta'_i}) = \sum_h \beta'_h(\alpha_1, \beta_2 + \alpha_2\beta_1)z_h$, since we wish to construct Q in such a way that f corresponds to $[-,-,-]$. Furthermore $[x_1, y_0]$ can be enumerated by means of Corollary 9.6 as $\sum \alpha'_i\beta_j c_{iji}$. From $[e_1, e_2] = 1$, one can derive $[e_1^2, e_2^2] = 1$, and so $x_1y_1 = \prod e_i^{2(\alpha'_i + \beta'_i)}$.

The correction term $D(\pi(x), \pi(y))$ is caused by x_0y_0. Put, temporarily, $e'_i = (e_i, 0)$, $i \in \{1,2\}$. Then, for example, $e'_1e'_2 \cdot e'_1 = e'_1 \cdot e'_2e'_1[e'_1, e'_2, e'_1]$, $e'_1 \cdot e'_2e'_1 = e'_1 \cdot e'_1e'_2 = e'^2_1 e'_2[e'_1, e'_1, e'_2]$ and $e'^2_1 e'_2 = e'_2 e'^2_1 [e'^2_1, e_2]$, where the latter commutator is equal to $[e'_1, e'_2, e'_1]$, by Lemma 9.5. Therefore $e'_1e'_2 \cdot e'_1 = e_2 \cdot e'^2_1[e'_1, e'_1, e'_2]$, and that is why $D(e_1 + e_2, e_1) = c_{112}$. Other values of D can be computed similarly.

Formula (9.5) is a standard way to obtain loops from groups. If $x, y, z \in B$ and $a, b, c \in A$, then

$$[(x,a),(y,b),(z,c)] = (1, g(xy,z) + zg(x,y) - g(x,yz) - g(y,z)]. \tag{9.7}$$

Our task hence is to show

$$g(xy,z) + zg(x,y) + g(x,yz) + g(y,z) = f(x,y,z) \quad \text{for all} \quad x,y,z \in B. \tag{9.8}$$

In any of x, y or z is equal to 1, then (9.8) holds, and hence we shall assume $1 \notin \{x,y,z\}$.

Now, $g(x,y)$ consists of the *correction part* $D(\sum \alpha_h e_h, \sum \beta_h e_h)$, the *associator part* $\sum(\alpha_1\beta_2 + \beta_2\alpha_1)\beta'_h z_h$ and the *commutator part* $\sum \alpha'_i\beta_j c_{iji}$. By adding the associator parts of (9.8) we obtain

$$\begin{aligned} \sum_h \Big(&\big((\alpha_1+\beta_1)\gamma_2 + (\alpha_2+\beta_2)\gamma_1\big)\gamma'_h + \big(\alpha_1\beta_2 + \alpha_2\beta_1\big)\beta'_h + \\ &\big(\alpha_1(\beta_2+\gamma_2) + \alpha_2(\beta_1+\gamma_1)\big)(\beta'_h + \gamma'_h + \beta_h\gamma_h) + \big(\gamma_1\beta_2 + \beta_1\gamma_2\big)\gamma'_h\Big) z_h, \end{aligned} \tag{9.9}$$

and the commutator parts yield

$$\sum_{i,j} \big((\alpha'_i + \beta'_i + \alpha_i\beta_i)\gamma_j + \alpha'_i(\beta_j+\gamma_j) + \beta'_i\gamma_j\big)c_{iji} \ + \ e_1^{\gamma_1}e_2^{\gamma_2}\sum_{i,j}\alpha'_i\beta_i c_{iji}. \tag{9.10}$$

Consider now only the terms in the associator and commutator parts that involve γ'_h, $h \in \{1,2\}$. There is none in the commutator part, and so we get $\sum \gamma'_h(\alpha_1\beta_2 + \alpha_2\beta_1)z_h$. Terms with β'_h occur in the commutator part, but they cancel out. Hence for $\beta' h$ we obtain $\sum \beta'_h(\alpha_1\gamma_2 + \alpha_2\beta_1)z_h$. Let us finally consider the terms with α'_h. They occur in the commutator part only and yield.

$$\sum_{i,j} \alpha'_i\beta_j c_{iji} + e_1^{\gamma_1}e_2^{\gamma_2}\sum_{i,j}\alpha'_i\beta_j c_{iji}. \tag{9.11}$$

We shall treat (9.11) as a sum of two symmetric factors that correspond to the choice of $i \in \{1,2\}$. For $i = 1$ we get

$$\alpha'_1\beta_1 c_{111} + \alpha'_1\beta_2 c_{121} + e_2^{\gamma_2}\alpha'_1\beta_1 c_{111} + e_1^{\gamma_1}\alpha'_1\beta_2 c_{121}. \tag{9.12}$$

It can be easily verified that (9.12) evaluates to $\alpha'_1(\beta_2\gamma_1 + \beta_1\gamma_2)z_1$ for all possible choices of $\gamma_1, \gamma_2 \in \{0,1\}$. For example, for $\gamma_1 = \gamma_2 = 1$ we obtain $(\alpha'_1\beta_1 + \alpha'_1\beta_1)c_{111} + (\alpha'_1\beta_1 + \alpha'_1\beta_2)c_{112} + (\alpha'_1\beta_2 + \alpha'_1\beta_1)c_{121}$. By adding (9.12) to its counterpart where $i = 2$, we thus obtain $\sum_h \alpha'_h(\beta_2\gamma_1 + \beta_1\gamma_2)z_h$, and similarly for β'_h and γ'_h.

Recall that $f(x,y,z)$ is equal, by the definition (8.6), to the sum of

$$C(\alpha_1 e_1 + \alpha_2 e_2, \beta_1 e_1 + \beta_2 e_2, \gamma_1 e_1 + \gamma_2 e_2) \tag{9.13}$$

and $\sum_h \big(\alpha'_h(\beta_2\gamma_1 + \beta_1\gamma_2) + \beta'_h(\alpha_1\gamma_2 + \alpha_2\beta_1) + \gamma'_h(\alpha_1\beta_2 + \alpha_2\beta_1)\big)z_h$.

To finish the proof, we hence need to show that (9.13) is always equal to

$$\begin{aligned} D\Big(\sum(\alpha_h+\beta_h)e_h, \sum\gamma_h e_h\Big) + e_1^{\gamma_1}e_2^{\gamma_2}D\Big(\sum\alpha_h e_h, \sum\beta_h e_h\Big) + \\ D\Big(\sum\alpha_h e_h, \sum(\beta_h+\gamma_h)e_h\Big) + D\Big(\sum\beta_h e_h, \sum\gamma_h e_h\Big) + \\ \sum_{i\neq j}\alpha_i(\beta_j+\gamma_j)\sum\beta_h\gamma_h z_h + \sum_{i,j}\alpha_i\beta_i\gamma_j c_{iji}. \end{aligned} \tag{9.14}$$

Note that (9.14) consists of six summands, two on each line. We number them 1, 2, 3, 4, 5, and 6, and we shall use this numbering in Table 4. This table verifies the equality of (9.13) and (9.14) for every choice of $a, b, c \in B_2$, where $0 \notin \{a,b,c\}$, $a = \sum\alpha_h e_h$, $b = \sum\beta_h e_h$ and $c = \sum\gamma_h e_h$. To save space in the table we write e_3 in place of $e_1 + e_2$. Each row of the table corresponds to one choice of (a,b,c), and a digit d, $1 \le d \le 6$, placed in a column labeled c_{ijk} means that c_{ijk} belongs to the support of the vector that is obtained by enumeration of the dth summand of (9.14) for the given choice of a, b and c. The occurrence of digit 0 means that c_{ijk} belongs to the support of (9.13). The equality of (9.13) and (9.14) follows from the fact that in each column and each row one gets an even number of digits.

We can thus state

Theorem 9.7. *Let B be the abelian group $\langle e_1, e_2 \mid e_1^4 = e_2^4 = 1\rangle$ and let A be the vector space over $\{0,1\}$ with basis c_{111}, c_{222}, c_{121}, c_{212}, c_{112} and c_{122}. Furthermore, let $f : B^3 \to A$ be the mapping determined by (8.6). Denote by Q be the loop on $B \times A$ defined by (9.3). Then Q satisfies the Buchsteiner law, $N(Q) \cong A$, $Q/N(Q) \cong B$ and $[(x,a),(y,b),(z,c)] = f(x,y,z)$ for all $x,y,z \in B$ and $a,b,c \in A$.*

Our goal now is to show that the loop Q can be factorized to a loop of order 64 such that not all squares are in the nucleus. To find the appropriate normal subloop we shall start with a general description of a normal subloop $H \trianglelefteq Q$ that is contained in $N = N(Q)$.

TABLE 4. Verification of the associator formula

a	b	c	c_{111}	c_{222}	c_{121}	c_{212}	c_{112}	c_{122}
e_1	e_1	e_1	06					
e_1	e_1	e_2			36		03	
e_1	e_1	e_3	06		0456		45	
e_1	e_2	e_1			03		13	
e_1	e_2	e_2						01
e_1	e_2	e_3			01	14		04
e_1	e_3	e_1	06		25		0245	
e_1	e_3	e_2			26		02	04
e_1	e_3	e_3	06		0246	14	02	01
e_2	e_1	e_1					01	
e_2	e_1	e_2				03		13
e_2	e_1	e_3			14	01	04	
e_2	e_2	e_1				36	0	3
e_2	e_2	e_2		06				
e_2	e_2	e_3		06		0456		45
e_2	e_3	e_1				26	04	02
e_2	e_3	e_2		06		25		0245
e_2	e_3	e_3		06	14	0246	01	02
e_3	e_1	e_1	06		02			
e_3	e_1	e_2			36	30	20	
e_3	e_1	e_3	06		2456	01	45	13
e_3	e_2	e_1			03	36		02
e_3	e_2	e_2		06		02		
e_3	e_2	e_3		06	01	2456	13	45
e_3	e_3	e_1	06		01	26	0241	03
e_3	e_3	e_2		06	26	02	03	0242
e_3	e_3	e_3	06	06	46	46	02	02

Set $z_1 = c_{121} + c_{112}$, $z_2 = c_{212} + c_{221}$, $s_1 = c_{121}$, $s_2 = c_{222}$, $c_1 = c_{121}$ and $c_2 = c_{212}$. We see that $\{z_h, s_h, c_h;\ 1 \le h \le 2\}$ is a basis of A. For an element $a = \sum(\vartheta_h z_h + \sigma_h s_h + \gamma_h c_h)$ set $\varphi_h(a) = \sigma_h + \gamma_h$, $h \in \{1, 2\}$. We have thus defined two linear forms $\varphi_h : A \to \{0, 1\}$.

Lemma 9.8. *A subspace $H \le A$ is a normal subloop of Q if and only if $\varphi_h(H) = H$ for both $h \in \{1, 2\}$.*

Proof. From (8.3) we see that the action of e_1 fixes z_1, z_2, s_2 and c_2. Furthermore, e_1 sends s_1 to $s_1 + z_1$, and c_1 to $c_1 + z_1$. Hence $e_1 \cdot a = a + \varphi(e_1)z_1$ for all $a \in A$, and the condition $e_1 \cdot H = H$ is equivalent to $\varphi_1(H) = H$. The case $h = 2$ is similar. $\square$

Let $H \le A$ be spanned by c_{222}, c_{122}, c_{212} and $c_{111} + c_{121}$, i. e. by z_2, c_2, s_2, c_2 and $s_1 + c_1$. Then $H \trianglelefteq Q$, by Lemma 9.8. By (8.5) and (8.6), $[e_1^2, e_2, e_1] = z_1$, $[e_1, e_2, e_1] = c$, and $[e_1^2 e_2, e_1, e_2] = z_1 + c_2$. None of these elements belongs to H, and hence none of $e_1^2 H$, $e_2 H$ and $e_1^2 e_2 H$ lies in the nucleus of Q/H. On the other hand, $[e_2^2, a, b]$ is a multiple of $z_2 \in H$, for all $a, b \in A$, and so $e_2^2 H$ is in the nucleus of Q/H. We see that Q/H is a Buchsteiner loop of order 64 such that the factor over the nucleus is isomorphic to $C_4 \times C_2$. We can hence conclude this section with the following.

Theorem 9.9. *There exists a Buchsteiner loop Q of order 64 which contains an element x such that $x^2 \notin N(Q)$.*

10. Previous work on Buchsteiner loops

As discussed in the introduction, H. H. Buchsteiner [7] was the first to consider loops satisfying the identity (B) or equivalently, the implication (B'). He showed that all isotopes of a loop satisfy (B) if and only if the loop itself satisfies ($\hat{\mathrm{B}}$). He called those loops satisfying the latter identity "i-loops", and the bulk of [7] is about i-loops. Buchsteiner did not address nor even state the question of whether or not the isotopy invariance of (B) holds automatically. He showed that for each i-loop Q, Q/N is an abelian group. He concluded his paper by posing the problem of whether or not every i-loop is a G-loop.

A. S. Basarab [3] answered Buchsteiner's question affirmatively. He showed that every i-loop satisfies the property we have called here WWIP, and then used this to show that i-loops are G-loops.

Our approach uses some of the same ingredients as in these papers, but in a different way to get stronger results. We started with (B) itself which, on the surface, at least, is a weaker identity than ($\hat{\mathrm{B}}$). We showed that a loop Q satisfying (B) has WWIP (Theorem 4.8), then used WWIP to show that Q is a G-loop (Theorem 5.1), then used that fact to prove ($\hat{\mathrm{B}}$) (Proposition 5.2), and then finally showed that Q/N is an abelian group (Theorem 5.3).

Our construction of a Buchsteiner loop Q in which Q/N achieves exponent 4 is partially motivated by our failure to understand a claim in [7] that Q/N always has exponent 2. Some of the associator calculus of [7] is correct, but there turns out to be a gap which led Buchsteiner to the identity $[x, y, z] = [y, x, z]$. It is now known that if this identity holds in a loop Q for which Q/N_λ is an abelian group, then the loop is LCC [15]. So had Buchsteiner's argument been correct, every Buchsteiner loop would be CC. Taking the above associator identity as a hypothesis, Buchsteiner's subsequent argument that Q/N has exponent 2 is valid; *cf.* Proposition 2.5.

11. Conclusions and prospects

We hope that this paper offers sufficient evidence that the variety of Buchsteiner loops deserves a place at the loop theory table alongside other better known varieties such as Moufang and CC loops. In this section we describe additional results which will be appearing elsewhere.

As mentioned in the introduction, our understanding of Buchsteiner loops has been greatly helped by recent work in conjugacy closed loops. The two varieties are tied together more strongly by the following.

Theorem 11.1. [12] *Let Q be a Buchsteiner loop. Then $Q/Z(Q)$ is a conjugacy closed loop.*

Theorem 11.2. [13] *Let Q be a Buchsteiner loop which is nilpotent of class 2. Then Q is a conjugacy closed loop.*

The examples we constructed in this paper served two purposes: firstly, to show that there exist Buchsteiner loops which are not conjugacy closed, and secondly, to exhibit examples of Buchsteiner loops Q such that $Q/N(Q)$ has exponent 4 but not 2.

It turns out that there are other constructions of non-CC Buchsteiner loops such that the factor by the nucleus has exponent 2. The paper [13] gives a construction based on rings which produces of order 128. Precise bounds on the orders for examples of various types are now understood as well.

Theorem 11.3. [19] *Let Q be a Buchsteiner loop. If $|Q| < 32$, then Q is a CC loop. If $|Q| < 64$, then $Q/N(Q)$ has exponent 2.*

The bounds in this result are sharp. As Theorem 9.9 shows, there exists a Buchsteiner loop of order 64 such that the factor by the nucleus has an element of order 4. There exist non-CC Buchsteiner loops of order 32, and in fact, they are now all classified:

Theorem 11.4. [20] *Up to isomorphism, there exist 44 Buchsteiner loops of order 32 which are not conjugacy closed.*

By a result going back to Bruck [5], any loop of nilpotency class 2 necessary has an abelian inner mapping group. It has been recently established that the converse is false; an example of a nilpotent loop of class 3 and order 128 with an abelian inner mapping group was given in [10]. The example does not belong to any of the standard loop varieties. Such examples cannot, for instance, be LCC loops, since LCC loops with abelian inner mapping groups are always of nilpotency class at most 2 [11].

Nevertheless, we now know that examples of nilpotent loops of class 3 with abelian inner mapping groups exist even within highly structured varieties:

Theorem 11.5. [19] *There exists a nilpotent Buchsteiner loop Q of nilpotency class 3 and order 128 such that $\operatorname{Inn} Q$ is an abelian group.*

Finally, while we know several general ways to construct Buchsteiner loops Q for which Q/N has exponent 2, we do not know of any such general constructions for the case when Q/N achieves exponent 4. It is certainly possible that the specific construction of §8 and §9 might be comprehended in more general terms. Such a general construction could start from the fact that the subloop generated by all squares and the nucleus is normal and is a group; see (9.1). In this connection, we also pose the problem of characterizing those groups that can appear in such a context.

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Department of Mathematics, Eötvös University, Pázmány Péter sétány 1/C, H-1117 Budapest, Hungary
E-mail address: `ska@cs.elte.hu`

Department of Mathematics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Rep.
E-mail address: `drapal@karlin.mff.cuni.cz`

Department of Mathematics, University of Denver, 2360 S. Gaylord St., Denver, Colorado 80208, U.S.A.
E-mail address: `mkinyon@math.du.edu`