EQUIVALENCE OF CONSEQUENCE RELATIONS: AN ORDER-THEORETIC AND CATEGORICAL PERSPECTIVE

NIKOLAOS GALATOS AND CONSTANTINE TSINAKIS

Abstract. Equivalences and translations between consequence relations abound in logic. The notion of equivalence can be defined syntactically, in terms of translations of formulas, and order-theoretically, in terms of the associated lattices of theories. W. Blok and D. Pigozzi proved in [2] that the two definitions coincide in the case of an algebraizable sentential deductive system. A refined treatment of this equivalence was provided by W. Blok and B. Jónsson in [3] and [4]. Other authors have extended this result to the cases of k-deductive systems and of consequence relations on associative, commutative, classical sequents. Our main result subsumes all existing results in the literature and reveals their common character. The proofs are of order-theoretic and categorical nature.

1. introduction

The aim of the present paper is to propose an order-theoretic and categorical framework for various constructions and concepts connected with the study of logical consequence relations. Our approach places under a common umbrella a number of existing results regarding the equivalence of consequence relations and provides a road map for future research in this area.

A consequence relation is defined relative to an algebraic signature $\mathcal{L}$. The set Fm of $\mathcal{L}$-formulas is the universe of the term algebra $\mathbf{Fm}$ of signature $\mathcal{L}$ over a countably infinite set of variables. Throughout this paper, we identify the algebra $\mathbf{Eq}$ of $\mathcal{L}$-equations with the algebra $\mathbf{Fm} \times \mathbf{Fm}$, and denote by $\mathbf{\Sigma}$ the monoid of substitutions of $\mathbf{Fm}$.

W. Blok and D. Pigozzi proved in [2] that a substitution invariant, finitary consequence relation $\vdash$ on $\mathbf{Fm}$ is algebraizable if and only if there exists an algebraic consequence relation $\models$ on Eq such that the lattices $\mathbf{Th}_{\vdash}$ and $\mathbf{Th}_{\models}$ of the theories corresponding to $\vdash$ and $\models$ are isomorphic under a map that commutes with inverse substitutions. A refined treatment of this equivalence was provided by W. Blok and B. Jónsson in [3] and [4]. They observed that the definition of algebraizability of $\vdash$, given in [2], can be

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rephrased as follows: there exist (i) an algebraic consequence relation $\models$ on Eq and (ii) finitary maps $\tau : Fm \to \mathcal{P}(Eq)$, and $\rho : Eq \to \mathcal{P}(Fm)$ (referred to as *translators*), which commute with substitutions, such that for all $\Psi \cup \{\phi\} \in \mathcal{P}(Fm)$ and $\varepsilon \in Eq$,

(1) $\Psi \vdash \phi$ iff $\tau[\Psi] \models \tau(\phi)$, and
(2) $\varepsilon \mathrel{=\!|\!\models} \tau\rho(\varepsilon)$.

In addition, they extended the previously mentioned result in [2] in the setting of M-sets.

Our approach, which owes considerable intellectual debt to the cited work of Blok and Jónsson, is more general and places the aforementioned considerations on solid algebraic and categorical ground. Starting with the concrete situation above, we note that there exists a natural action of $\mathbf{\Sigma}$ on $\mathbf{Fm}$ that extends to an action of the corresponding power sets. The power set $\mathcal{P}(\Sigma)$ is a ringlike object – in which set-union plays the role of addition and complex product serves as multiplication. On the other hand, $\mathcal{P}(Fm)$ is a structure corresponding to an abelian group, with set-union playing again the role of addition. The latter action possesses the critical property of being residuated, which, in this particular instance, means that it preserves arbitrary unions in each coordinate. Analogous comments hold for the action of $\mathbf{\Sigma}$ on $\mathbf{Eq}$.

This concrete situation leads naturally to the general concept of a (left) module. The *scalars* of such a structure are the elements of a complete residuated lattice $\mathbf{A}$. The *vectors* form a complete lattice $\mathbf{P}$. The scalar multiplication $\star : \mathbf{A} \times \mathbf{P} \to \mathbf{P}$ is a bi-residuated map (i.e., a residuated map in each coordinate) that satisfies the usual properties of a monoid action. For a given complete residuated lattice $\mathbf{A}$, all $\mathbf{A}$-modules constitute the objects of a category, ${}_{\mathbf{A}}\mathcal{M}$, whose morphisms are residuated maps that preserve scalar multiplication.

The category ${}_{\mathbf{A}}\mathcal{M}$ provides an ideal environment to abstract the aforementioned concepts and identify their categorical properties. For example, the structural consequence relations on an object $\mathbf{P}$ correspond bijectively to the epimorphic images of $\mathbf{P}$. Thus, such relations may be identified with objects of this category. Not surprisingly then, we stipulate that two structural consequence relations are equivalent if the $\mathbf{A}$-modules corresponding to them are isomorphic. On the other hand, we can define equivalence of structural consequence relations by abstracting the second condition for algebraizability stated above. This condition can be described in more detail as follows. Let $\vdash_\gamma$ and $\vdash_\delta$ be two structural consequence relations on $\mathbf{P}$ and $\mathbf{Q}$, respectively, and let γ and δ be the structural closure operators on $\mathbf{P}$ and $\mathbf{Q}$ that correspond to $\vdash_\gamma$ and $\vdash_\delta$. Then, for every isomorphism f between the modules of theories $\mathbf{P}_\gamma = \mathbf{Th}_{\vdash_\gamma}$ and $\mathbf{Q}_\delta = \mathbf{Th}_{\vdash_\delta}$, there exist translators (i.e., module morphisms) $\tau : \mathbf{P} \to \mathbf{Q}$ and $\rho : \mathbf{Q} \to \mathbf{P}$ such that $\delta\tau = f\gamma$ and $\gamma\rho = f^{-1}\delta$.

The main result of this work identifies categorically the modules for which the two definitions coincide: they are precisely the *projective* objects of this category. This result subsumes the cases considered in [3] and [4], as well as those involving the equivalence of structural consequence relations on sequents.

More specifically, we prove that the $\mathcal{P}(\mathbf{\Sigma})$-modules $\mathcal{P}(\mathbf{Fm})$ of formulas and $\mathcal{P}(\mathbf{Eq})$ of equations are projective. Each of these modules is cyclic, i.e., it is generated by a single element. An interesting additional result is Theorem 8.8, which presents several characterizations of projective cyclic $\mathbf{A}$-modules.

Let Seq be a set of *sequents* (intuitionistic, classical or non-associative, multi-sequents or hypersequents; refer to [9], [6] or [1]). Unless all elements in Seq have bounded length, the $\mathcal{P}(\mathbf{\Sigma})$-module $\mathcal{P}(\mathbf{Seq})$ is not cyclic, but we prove that it is projective. This result is proved by noting that $\mathcal{P}(\mathbf{Seq})$ is a coproduct of cyclic projective modules.

J. Rebagliato and V. Verdú [13] have defined the notion of equivalence of two consequence relations on (associative) sequents. The results in [3] and [4] do not cover the case of sequents, but it follows from our results that the isomorphism of the modules of theories is equivalent to the definition of Rebagliato and Verdú.

Lastly, Corollary 9.11 guarantees that under additional natural assumptions the desired translators τ and ρ are finitary; i.e., they send compact elements to compact elements. In the case of powersets, this means that they map finite sets to finite sets.

2. Algebraizability

As usual, by a *propositional* (or *algebraic*) *language* we mean a pair $\mathcal{L} = \langle L, \alpha \rangle$ consisting of a set L and a map α from L to the natural numbers. The elements of L are called *(primitive) connectives* (or *operation symbols*) and the image of a connective under α is called the *arity* of the connective.

An $\mathcal{L}$*-algebra* is a pair $\mathbf{A} = \langle A, Op[L] \rangle$, where A is a set, Op is a map that assigns an operation $Op(f) = f^{\mathbf{A}}$ on A of arity $\alpha(f)$ to every operation symbol f of L; often the map Op is considered understood for a given algebra $\mathbf{A}$. If L is finite, we usually list the elements of $Op[L]$ in the expression $\langle A, Op[L] \rangle$.

We denote by $Fm_{\mathcal{L}}$ the set of *(propositional) formulas* (or *terms*) over the language $\mathcal{L}$ and a countably infinite set Var of *propositional variables.* Also, $\mathbf{Fm}_{\mathcal{L}}$ denotes the associated $\mathcal{L}$-algebra. We denote by $\mathbf{\Sigma}_{\mathcal{L}}$ the endomorphism monoid of $\mathbf{Fm}_{\mathcal{L}}$ and refer to its elements as *substitutions.*

An *(asymmetric) consequence relation over the set* $Fm_{\mathcal{L}}$ is a subset $\vdash$ of $\mathcal{P}(Fm_{\mathcal{L}}) \times Fm_{\mathcal{L}}$ satisfying the following conditions, for all subsets $\Phi \cup \Psi \cup \{\phi, \psi, \chi\}$ of $Fm_{\mathcal{L}}$:

(1) if $\phi \in \Phi$, then $\Phi \vdash \phi$; and

(2) if $\Phi \vdash \psi$, for all $\psi \in \Psi$, and $\Psi \vdash \chi$, then $\Phi \vdash \chi$.

Usually, we write $\phi \vdash \psi$ for $\{\phi\} \vdash \psi$. A consequence relation $\vdash$ over $Fm_{\mathcal{L}}$ is called *finitary*, if for all subsets $\Phi \cup \{\phi\}$ of $Fm_{\mathcal{L}}$, whenever $\Phi \vdash \phi$, there exists a finite subset Φ_0 of Φ such that $\Phi_0 \vdash \phi$. It is called *substitution invariant* or *structural*, if for every substitution $\sigma \in \Sigma_{\mathcal{L}}$, and for all subsets $\Phi \cup \{\phi\}$ of $Fm_{\mathcal{L}}$, $\Phi \vdash \phi$ implies $\sigma[\Phi] \vdash \sigma(\phi)$.

The deducibility (or provability) relation of a Hilbert system with finitely many rule schemes (we consider axiom schemes as special cases of rule schemes) is a finitary and substitution invariant consequence relation. For example, the deducibility (or provability) relation $\vdash_{CPL}$ of Classical Propositional Logic (CPL) is a finitary and substitution invariant consequence relation over $Fm_{\mathcal{L}}$, where $\mathcal{L}$ is the language of CPL.

Associated with a consequence relation $\vdash$ on $\mathbf{Fm}_{\mathcal{L}}$ is a closure operator $\gamma_{\vdash}$ on $\mathbf{Fm}_{\mathcal{L}}$, defined by $\gamma_{\vdash}(\Phi) = \{\psi \in Fm_{\mathcal{L}} \mid \Phi \vdash \psi\}$. Conversely, a closure operator $\mathbf{Fm}_{\mathcal{L}}$ gives rise to a consequence relation. We discuss this connection in a more general setting in Section 6.

By an *equation* over $\mathcal{L}$ we mean a pair of elements $s, t \in Fm_{\mathcal{L}}$ and we usually denote it by the expression $s \approx t$. We denote by $\mathbf{Eq}_{\mathcal{L}}$ the $\mathcal{L}$-algebra $(\mathbf{Fm}_{\mathcal{L}})^2$ of equations over $\mathcal{L}$. A substitution invariant, finitary consequence relation over $\mathbf{Eq}_{\mathcal{L}}$ is defined by analogy to the previous case. If $\mathbf{A}$ is an $\mathcal{L}$-algebra, $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ is a homomorphism and $(s \approx t) \in Eq_{\mathcal{L}}$, then we denote by $h(s \approx t)$ the pair $(h(s), h(t))$ and we refer to it as an *equality*; we say that the equality is true if $h(s) = h(t)$.

If $\mathcal{K}$ is a class of $\mathcal{L}$-algebras, and $E \cup \{\varepsilon\}$ is a subset of $Eq_{\mathcal{L}}$, $E \models_{\mathcal{K}} \varepsilon$ means that for all $\mathbf{A} \in \mathcal{K}$ and all homomorphisms $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$, if $h[E]$ is a set of true equalities, then $h(\varepsilon)$ is a true equality. It is clear that $\models_{\mathcal{K}}$ is a substitution invariant consequence relation over $Eq_{\mathcal{L}}$. It is well known, see e.g. [3], that $\models_{\mathcal{K}}$ is finitary iff $\mathcal{K}$ is closed under ultraproducts.

Let $\mathcal{BA}$ denote the class of Boolean algebras. The *strong completeness theorem* for CPL states that for every subset $\Psi \cup \{\phi\}$ of $Fm_{\mathcal{L}}$

$$\Psi \vdash_{CPL} \phi \text{ iff } \{\psi \approx 1 \mid \psi \in \Psi\} \models_{\mathcal{BA}} \phi \approx 1.$$

Here 'strong' refers to the existence of assumptions Ψ.

It is also true that for every set of equations $E \cup \{s \approx t\}$ over $Fm_{\mathcal{L}}$,

$$E \models_{\mathcal{BA}} s \approx t \text{ iff } \{u \to v, v \to u \mid (u \approx v) \in E\} \vdash_{CPL} \{s \to t, t \to s\},$$

where by $X \vdash Y$, we mean $X \vdash y$, for all $y \in Y$. This is often called the *inverse strong completeness theorem* for CPL.

Furthermore, for every $\phi \in Fm_{\mathcal{L}}$ and $(s \approx t) \in Eq_{\mathcal{L}}$,

$$s \approx t \mathrel{=\!\mid\!\models}_{\mathcal{BA}} \{s \to t \approx 1, t \to s \approx 1\}.$$
$$\phi \dashv\vdash_{CPL} \{\phi \to 1, 1 \to \phi\},$$

where $X \dashv\vdash Y$ is an abbreviation for $X \vdash Y$ and $Y \vdash X$.

If for every $\phi \in Fm_{\mathcal{L}}$ and $\varepsilon = (s \approx t) \in Eq_{\mathcal{L}}$ we define $\tau(\phi) = \{\phi \approx 1\}$ and $\rho(\varepsilon) = \rho(s \approx t) = \{s \to t, t \to s\}$, then the above take the forms below.

- $\Psi \vdash_{CPL} \phi$ iff $\tau[\Psi] \models_{\mathcal{BA}} \tau(\phi)$
- $E \models_{\mathcal{BA}} \varepsilon$ iff $\rho[E] \vdash_{CPL} \rho(\varepsilon)$
- $\varepsilon \dashv\models_{\mathcal{BA}} \tau\rho(\varepsilon)$
- $\phi \dashv\vdash_{CPL} \rho\tau(\phi)$

The last two statements mean that the application of τ and ρ one after the other might not yield the original formula or equation, but it will yield a set of such mutually provable to the original.

Our discussion in the remainder of this section draws heavily from [3] and [4]. According to Blok and Pigozzi [2], a *deductive system* is a pair $\langle \mathcal{L}, \vdash \rangle$, where $\mathcal{L}$ is a propositional language and $\vdash$ is a substitution invariant, finitary consequence relation over $Fm_{\mathcal{L}}$.

A deductive system $\langle \mathcal{L}, \vdash \rangle$ is called *algebraizable* ([2]), if there exists a class of $\mathcal{L}$-algebras $\mathcal{K}$, a finite set of equations $u_i \approx v_i$, $i \in I$, on a single variable and a finite set of binary definable connectives Δ_j, $j \in J$, such that for every subset $\Psi \cup \{\phi\}$ of $Fm_{\mathcal{L}}$ and for every equation $s \approx t$ over $Fm_{\mathcal{L}}$,

(1) $\Psi \vdash \phi$ iff $\{u_i(\psi) \approx v_i(\psi) \mid \psi \in \Psi\} \models_{\mathcal{K}} u_i(\phi) \approx v_i(\phi)$, for all $i \in I$, and
(2) $s \approx t \dashv\models_{\mathcal{K}} \{u_i(s\,\Delta_j\,t) \approx v_i(s\,\Delta_j\,t) \mid i \in I, j \in J\}$.

The class $\mathcal{K}$ is called an *equivalent algebraic semantics* for $\langle \mathcal{L}, \vdash \rangle$.

It can be shown that the combination of (1) and (2) above is equivalent to the condition that for every set of equations $E \cup \{s \approx t\}$ over $Fm_{\mathcal{L}}$ and for every $\phi \in Fm_{\mathcal{L}}$,

(3) $E \models_{\mathcal{K}} s \approx t$ iff $\{u\,\Delta_j\,v \mid u \approx v \in E, j \in J\} \vdash s\,\Delta_j\,t$, for all $j \in J$.
(4) $\phi \dashv\vdash \{u_i(\phi)\,\Delta_j\,v_i(\phi) \mid i \in I, j \in J\}$.

If we define the maps $\tau : Fm_{\mathcal{L}} \to \mathcal{P}(Eq_{\mathcal{L}})$ and $\rho : Eq_{\mathcal{L}} \to \mathcal{P}(Fm_{\mathcal{L}})$ by $\tau(\phi) = \{u_i(\phi) \approx v_i(\phi) | i \in I\}$ and $\rho(s \approx t) = \{s\,\Delta_j\,t | j \in J\}$, then conditions (1) and (2) take the more elegant form

(1) $\Psi \vdash \phi$ iff $\tau[\Psi] \models_{\mathcal{K}} \tau(\phi)$, and
(2) $\varepsilon \dashv\models_{\mathcal{K}} \tau\rho(\varepsilon)$.

Next, we identify conditions under which arbitrary maps $\tau : Fm_{\mathcal{L}} \to \mathcal{P}(Eq_{\mathcal{L}})$ and $\rho : Eq_{\mathcal{L}} \to \mathcal{P}(Fm_{\mathcal{L}})$ are of the form above. First of all, for all $\phi \in Fm_{\mathcal{L}}$ and all $\varepsilon \in Eq_{\mathcal{L}}$, both $\tau(\phi)$ and $\rho(\varepsilon)$ are finite sets; we will call maps that have this property *finitary.* Also, if $\phi \in Fm_{\mathcal{L}}$, $\varepsilon \in Eq_{\mathcal{L}}$ and $\sigma \in \Sigma_{\mathcal{L}}$ is a substitution, then $\sigma[\tau(\phi)] = \tau[\sigma(\phi)]$ and $\sigma[\rho(\varepsilon)] = \rho[\sigma(\varepsilon)]$; we will call such maps *substitution invariant.* The following result is implicit in [4].

Lemma 2.1. *For maps $\tau : Fm_{\mathcal{L}} \to \mathcal{P}(Eq_{\mathcal{L}})$ and $\rho : Eq_{\mathcal{L}} \to \mathcal{P}(Fm_{\mathcal{L}})$, the following conditions are equivalent.*

(1) *τ, ρ are finitary and substitution invariant maps.*
(2) *There exists a finite set of equations $u_i \approx v_i$, $i \in I$, on a single variable, and a finite set of binary definable connectives Δ_j, $j \in J$, satisfying the relations $\tau(\phi) = \{u_i(\phi) \approx v_i(\phi) \mid i \in I\}$ and $\rho(s \approx t) = \{s\, \Delta_j\, t \mid j \in J\}$.*

Proof. We will show that (1) implies (2). Let x, y be distinct variables in Var and assume that $\tau(x) = \{u_i \approx v_i \mid i \in I\}$ and $\rho(x \approx y) = \{t_j \mid j \in J\}$. Since τ and ρ are finitary, it follows that I and J are finite.

If $\phi \in Fm_{\mathcal{L}}$, let $\kappa_\phi \in \Sigma_{\mathcal{L}}$ be the substitution that sends all variables to ϕ. Since τ is substitution invariant, we have $\kappa_x[\tau(x)] = \tau(\kappa_x(x)) = \tau(x)$, for every variable x. In other words, if we replace all variables in $\tau(x)$ by x, we get back $\tau(x)$; i.e., all the equations $u_i \approx v_i$ contain single variable. Moreover, for all $\phi \in Fm_{\mathcal{L}}$, we have $\tau(\phi) = \tau(\kappa_\phi(x)) = \kappa_\phi[\tau(x)] = \{\kappa_\phi(u_i(x) \approx v_i(x)) \mid i \in I\} = \{u_i(\phi) \approx v_i(\phi) \mid i \in I\}$.

Let Var_1 and Var_2 be two sets that partition the set Var of all variables in a way that $x \in Var_1$ and $y \in Var_2$. For all $(s \approx t) \in Eq_{\mathcal{L}}$, let $\kappa_{s\approx t} \in \Sigma_{\mathcal{L}}$ be the substitution that sends all variables in Var_1 to s and all variables in Var_2 to t. Since τ is substitution invariant, we have $\kappa_{x\approx y}[\rho(x \approx y)] = \rho(\kappa_{x\approx y}(x \approx y)) = \rho(x \approx y)$. In other words, the terms t_j are binary and depend only on the variables x and y; we set $t_j = x\Delta_j y$. We have, for all $(s \approx t) \in Eq_{\mathcal{L}}$, $\rho(s \approx t) = \rho(\kappa_{s\approx t}(x \approx y)) = \kappa_{s\approx t}[\rho(x \approx y)] = \{\kappa_{s\approx t}(x\Delta_j y) \mid j \in J\} = \{s\Delta_j t \mid i \in I\}$. □

Corollary 2.2. *A deductive system $\langle \mathcal{L}, \vdash\rangle$ is algebraizable iff there exist finitary and substitution invariant maps $\tau : Fm_{\mathcal{L}} \to \mathcal{P}(Eq_{\mathcal{L}})$ and $\rho : Eq_{\mathcal{L}} \to \mathcal{P}(Fm_{\mathcal{L}})$, and a class of $\mathcal{L}$-algebras $\mathcal{K}$ such that, for every subset $\Phi \cup \{\phi\}$ of $Fm_{\mathcal{L}}$ and $\varepsilon \in Eq_{\mathcal{L}}$,*

(1) *$\Psi \vdash \phi$ iff $\tau[\Psi] \models_{\mathcal{K}} \tau(\phi)$, and*
(2) *$\varepsilon \mathrel{=\mid\models}_{\mathcal{K}} \tau\rho(\varepsilon)$.*

Obviously, the maps τ and ρ extend to maps $\tau' : \mathcal{P}(Fm_{\mathcal{L}}) \to \mathcal{P}(Eq_{\mathcal{L}})$ and $\rho' : \mathcal{P}(Eq_{\mathcal{L}}) \to \mathcal{P}(Fm_{\mathcal{L}})$, defined by $\tau'(\Phi) = \tau[\Phi]$ and $\rho'(E) = \rho[E]$, for $\Phi \in \mathcal{P}(Fm_{\mathcal{L}})$ and $E \in \mathcal{P}(Eq_{\mathcal{L}})$. Moreover, $\tau'(\Phi)$ and $\rho'(E)$ are finite, if $\Phi \in \mathcal{P}(Fm_{\mathcal{L}})$ and $E \in \mathcal{P}(Eq_{\mathcal{L}})$ are finite; we will call such maps *finitary*. Also, if $\Phi \in \mathcal{P}(Fm_{\mathcal{L}})$, $E \in \mathcal{P}(Eq_{\mathcal{L}})$ and $\sigma \in \Sigma_{\mathcal{L}}$, then $\sigma[\tau'(\Phi)] = \tau'(\sigma[\Phi])$ and $\sigma[\rho'(E)] = \rho'(\sigma[E])$; we will call such maps *substitution invariant*. Clearly, τ' and ρ' stem from maps τ and ρ iff they preserve unions.

Corollary 2.3. *A deductive system $\langle \mathcal{L}, \vdash\rangle$ is algebraizable iff there exist finitary and substitution invariant maps $\tau : \mathcal{P}(Fm_{\mathcal{L}}) \to \mathcal{P}(Eq_{\mathcal{L}})$ and $\rho : \mathcal{P}(Eq_{\mathcal{L}}) \to \mathcal{P}(Fm_{\mathcal{L}})$ that preserve unions, and a class of $\mathcal{L}$-algebras $\mathcal{K}$ such that, for every subset $\Phi \cup \{\phi\}$ of $Fm_{\mathcal{L}}$ and $\varepsilon \in Eq_{\mathcal{L}}$,*

(1) *$\Psi \vdash \phi$ iff $\tau(\Psi) \models_{\mathcal{K}} \tau(\phi)$, and*
(2) *$\varepsilon \mathrel{=\mid\models}_{\mathcal{K}} \tau\rho(\varepsilon)$.*

Example 2.4. Let $\vdash_{BCK}$ be the least substitution invariant consequence relation on $\mathbf{Fm}_{\{\to\}}$ satisfying the following properties for all $x, y, z \in \mathbf{Fm}_{\{\to\}}$.

(B) $\vdash_{BCK} (x \to y) \to ((y \to z) \to (x \to z))$
(C) $\vdash_{BCK} (x \to (y \to z)) \to (y \to (x \to z))$
(I) $\vdash_{BCK} x \to x$
(K) $\vdash_{BCK} x \to (y \to y)$
(MP) $\{x, x \to y\} \vdash_{BCK} y$

Actually, (I) is redundant, but we include it for later reference. It is shown in [2] that $\vdash_{BCK}$ is algebraizable and the $\{\to\}$-subreducts of commutative integral residuated lattices form an algebraic semantics for it. (Refer to [5] or [10] for a short introduction to residuated lattices, and to [7] for a comprehensive treatment of these structures.) The corresponding maps τ and ρ are given by $\tau(\phi) = \{\phi \approx (\phi \to \phi)\}$ and $\rho(u \approx v) = \{u \to v, v \to u\}$. An extension of this correspondence is obtained by the algebraizability of substructural logics via residuated lattices; see [8]. □

A *theory* of a consequence relation $\vdash$ over $Fm_{\mathcal{L}}$ is a subset T of $Fm_{\mathcal{L}}$ closed under $\vdash$; i.e., for all $\phi \in Fm_{\mathcal{L}}$, $T \vdash \phi$ implies $\phi \in T$. The set of theories of $\vdash$ forms a lattice that we denote by $\mathbf{Th}_{\vdash}$. Likewise we define the lattice of theories $\mathbf{Th}_{\models}$ of a consequence relation $\models$ over $Eq_{\mathcal{L}}$. The notions of finitarity and substitution invariance have analogues for closure operators and lattices of theories. We discuss the connections between consequence relations, closure operators and lattices of theories in a more general setting in Section 6.

The following characterization of algebraizability of a deductive system is proved in [2].

Theorem 2.5. [2] *A deductive system $\langle \mathcal{L}, \vdash \rangle$ is algebraizable with equivalent algebraic semantics a quasivariety $\mathcal{K}$ iff there exists an isomorphism between $\mathbf{Th}_{\vdash}$ and $\mathbf{Th}_{\models_{\mathcal{K}}}$ that commutes with inverse substitutions.*

We will extend this result in a more general setting and provide a categorical reason for its validity.

3. Consequence relations on sets of sequents

In this section, we consider one more example of a consequence relation.

If m, n are non-negative integers (not both equal to zero), by a *(classical, associative) sequent over $\mathcal{L}$ of type* (m, n), we understand a pair (Γ, Δ) of a sequence $\Gamma = (\phi_1, \phi_2, \ldots, \phi_m)$ of $\mathcal{L}$-formulas of length m and a sequence $\Delta = (\psi_1, \psi_2, \ldots, \psi_n)$ of $\mathcal{L}$-formulas of length n. We usually write $\phi_1, \phi_2, \ldots, \phi_m \Rightarrow \psi_1, \psi_2, \ldots, \psi_n$ for (Γ, Δ). These sequents are used in the formulation of substructural logics over **FL**; see, for example, [7]. Variants of this notion of sequent have been considered in the literature; refer to [9], [1],[6], and Section 8.

We usually consider sets of sequents *closed under type*, i.e., sets of sequents such that, for all m, n, if they contain an (m, n)-sequent, then they contain all (m, n)-sequents. If Seq is a set of sequents closed under type, then $Tp(Seq)$ denotes the set of all types of the sequents in Seq.

The set of formulas can be identified with the set of all $(0, 1)$-sequents, and the set of equations can be identified with the set of all $(1, 1)$-sequents.

If $s = \phi_1, \phi_2, \ldots, \phi_m \Rightarrow \psi_1, \psi_2, \ldots, \psi_n$ is a sequent and $\sigma \in \Sigma_{\mathcal{L}}$ is a substitution, $(\sigma(\phi_1), \sigma(\phi_2), \ldots, \sigma(\phi_m) \Rightarrow \sigma(\psi_1), \sigma(\psi_2), \ldots, \sigma(\psi_n))$ is denoted by $\sigma(s)$. If Seq is a set of sequents closed under type, then a (finitary, substitution invariant) consequence relation over Seq is defined as in the case of $Fm_{\mathcal{L}}$ and $Eq_{\mathcal{L}}$.

The notion of algebraizability of a set Seq of sequents closed under type has been defined by Rebagliato and Verdú [13]. If Seq_1 and Seq_2 are sets of sequents over $\mathcal{L}$ closed under type, and $\vdash_1$ and $\vdash_2$ are two consequence relations over Seq_1 and Seq_2, respectively, a *translation* between Seq_1 and Seq_2 is a set $\boldsymbol{\tau} = \{\boldsymbol{\tau}_{(m,n)} \mid (m, n) \in Tp(Seq_1)\}$, where $\boldsymbol{\tau}_{(m,n)}$ is a finite subset of Seq_2 in (at most) $m + n$ variables. If $s \in Seq_1$ is an (m, n)-sequent, $\boldsymbol{\tau}(s) = \boldsymbol{\tau}_{(m,n)}(s)$ denotes the result of replacing the $m+n$ formulas of s for the variables in $\boldsymbol{\tau}_{(m,n)}$.

Two consequence relations $\vdash_1$ and $\vdash_2$ over Seq_1 and Seq_2, respectively, are called *equivalent in the sense of Rebagliato and Verdú*, if there are translations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ between Seq_1 and Seq_2 such that for all subsets $S_1 \cup \{s_1\}$ of Seq_1 and all subsets $S_2 \cup \{s_2\}$ of Seq_2,

(1) $S_1 \vdash_1 s_1$ iff $\boldsymbol{\tau}[S_1] \vdash_2 \boldsymbol{\tau}(s_1)$, and
(2) $s_2 \dashv\vdash_2 \boldsymbol{\tau}\boldsymbol{\rho}(s_2)$.

It follows that

(3) $S_2 \vdash_2 s_2$ iff $\boldsymbol{\rho}[S_2] \vdash_1 \boldsymbol{\rho}(s_2)$, and
(4) $s_1 \dashv\vdash_1 \boldsymbol{\rho}\boldsymbol{\tau}(s_1)$.

Lemma 3.1. *Consider maps $\tau : Seq_1 \to \mathcal{P}(Seq_2)$ and $\rho : Seq_2 \to \mathcal{P}(Seq_1)$. The following are equivalent.*

(1) *The maps τ, ρ are finitary and substitution invariant.*
(2) *There exist translations $\boldsymbol{\tau}$ and $\boldsymbol{\rho}$ between Seq_1 and Seq_2 such that $\tau(s_1) = \boldsymbol{\tau}(s_1)$ and $\rho(s_2) = \boldsymbol{\rho}(s_2)$ for all $s_1 \in Seq_1$ and $s_2 \in Seq_2$.*

Proof. The proof is based on the ideas in the proof of Lemma 2.1. The lemma is also a consequence of more general results that we prove later; see Theorem 8.12. □

It will follow from our analysis that the analogue of Theorem 2.5 holds in the case of sequents, as well. A. Pynko [11] proves the result for finitary consequence relations and J. Raftery [12] for the general case of associative sequents.

Example 3.2. An *intuitionistic, associative, commutative sequent* on a set A is a pair (Γ, ϕ), where $\Gamma \cup \{\phi\}$ is a multiset on A; traditionally, the sequent (Γ, ϕ) is denoted by $\Gamma \Rightarrow \phi$. We denote by $Seq_{Iac}(A)$ the set of all intuitionistic, associative, commutative sequents on A. The deducibility relation $\vdash_{\mathbf{FL_{ei}}^{\{\to\}}}$ of the $\{\to\}$-fragment $\mathbf{FL_{ei}}^{\{\to\}}$ of the system $\mathbf{FL_{ei}}$ – see [9] for details – is the least structural consequence relation on $Seq_{Iac}(Fm_{\{\to\}})$ that satisfies the following conditions for all $\Gamma, \Pi, \Sigma \cup \{\alpha, \beta, \delta\} \subseteq Fm_{\{\to\}}$.

$$\frac{}{\alpha \Rightarrow \alpha}(\mathrm{id}) \qquad \frac{\Gamma \Rightarrow \alpha \qquad \Sigma, \alpha, \Pi \Rightarrow \delta}{\Sigma, \Gamma, \Pi \Rightarrow \delta}(\mathrm{cut}) \qquad \frac{\Gamma, \Sigma \Rightarrow \delta}{\Gamma, \alpha, \Sigma \Rightarrow \delta}(\mathrm{i})$$

$$\frac{\Gamma \Rightarrow \alpha \qquad \Pi, \beta, \Sigma \Rightarrow \delta}{\Pi, \Gamma, \alpha \to \beta, \Sigma \Rightarrow \delta}(\to\Rightarrow) \qquad \frac{\alpha, \Gamma \Rightarrow \beta}{\Gamma \Rightarrow \alpha \to \beta}(\Rightarrow\to)$$

Here we adopt the convention that the fraction notation $\frac{S}{s}$ means $S \vdash_{\mathbf{FL_{ei}}^{\{\to\}}} s$, where $S \cup \{s\}$ is a subset of $Seq_{Iac}(Fm_{\{\to\}})$.

It is shown in [9] that $\vdash_{\mathbf{FL_{ei}}^{\{\to\}}}$ is equivalent in the sense of Rebagliato and Verdú to $\vdash_{BCK}$; see Example 2.4. Moreover, the consequence relation $\vdash_{\mathbf{FL_{e}}^{\{\to\}}}$, obtained by removing Rule (i) from $\mathbf{FL_{ei}}^{\{\to\}}$, is equivalent in the sense of Rebagliato and Verdú to the consequence relation $\vdash_{BCI}$, which is obtained from $\vdash_{BCK}$ by removing Axiom (K). Nevertheless, the relations $\vdash_{\mathbf{FL_{e}}^{\{\to\}}}$ and $\vdash_{BCI}$ are not algebraizable; see [2]. □

4. Consequence relations on powersets

So far we have defined consequence relations on the sets $\mathcal{P}(Fm_{\mathcal{L}})$, $\mathcal{P}(Eq_{\mathcal{L}})$ and, more generally, on $\mathcal{P}(Seq)$, where Seq is a set of sequents closed under type. Before we give the general definition in the case of complete lattices, we give a preview in the case of powersets. Our presentation is based on ideas developed in [3] and [4].

4.1. **Asymmetric consequence relations.**

The definition of a (finitary) consequence relation for the three examples is a special case of the following well known definition.

Let S be a set. An *asymmetric consequence relation over* S is a subset $\vdash$ of $\mathcal{P}(S) \times S$ such that, for all subsets $X \cup Y \cup \{x, y, z\}$ of S,

(1) if $x \in X$, then $X \vdash x$, and
(2) if $X \vdash y$, for all $y \in Y$, and $Y \vdash z$, then $X \vdash z$.

An asymmetric consequence relation over S is called *finitary*, if for all subsets $X \cup \{x\}$ of S, if $X \vdash x$, then there is a finite subset X_0 of X such that $X_0 \vdash x$.

The generalization of the notion of substitution invariance to arbitrary powersets requires a new notion of substitution. Note that the monoid of substitutions $\mathbf{\Sigma}_{\mathcal{L}}$ acts on both $Fm_{\mathcal{L}}$ and $Eq_{\mathcal{L}}$ – more generally on a set Seq

of sequents over $\mathcal{L}$ closed under type – in the sense that for all $\sigma_1, \sigma_2 \in \Sigma_{\mathcal{L}}$, and s in either $Fm_{\mathcal{L}}$, $Eq_{\mathcal{L}}$ or Seq,

(1) $(\sigma_1\sigma_2)(s) = \sigma_1(\sigma_2(s))$
(2) $Id_{\mathbf{\Sigma}_{\mathcal{L}}}(s) = s$.

We say that a monoid $\mathbf{\Sigma} = \langle \Sigma, \cdot, e\rangle$ *acts* on a set S, if there exists a map $\star : \Sigma \times S \to S$ such that for all $\sigma_1, \sigma_2 \in \Sigma$, and $s \in S$,

(1) $(\sigma_1 \cdot \sigma_2) \star s = \sigma_1 \star (\sigma_2 \star s)$
(2) $e \star s = s$.

A consequence relation $\vdash$ on $\mathcal{P}(S)$ is called $\mathbf{\Sigma}$-*invariant*, if for all $X \cup \{y\} \subseteq S$ and $\sigma \in \Sigma$, $X \vdash y$ implies $\{\sigma \star x \mid x \in X\} \vdash \sigma \star y$.

Actually, if $\mathbf{\Sigma}$ acts on S, then $\mathcal{P}(\mathbf{\Sigma})$ acts on $\mathcal{P}(S)$, as well, i.e., there exists a map $\star' : \mathcal{P}(\Sigma) \times \mathcal{P}(S) \to \mathcal{P}(S)$ such that for all $A_1, A_2 \in \mathcal{P}(\Sigma)$ and $X \in \mathcal{P}(S)$,

(1) $(A_1 \cdot' A_2) \star' X = A_1 \star' (A_2 \star' X)$
(2) $\{e\} \star' X = X$,

where $A_1 \star' X = \{a \star x \mid a \in A_1, x \in X\}$ and $A_1 \cdot' A_2 = \{a_1 \cdot a_2 \mid a_1 \in A_1, a_2 \in A_2\}$.

Moreover, for all $A \in \mathcal{P}(\Sigma)$ and $X, Y \in \mathcal{P}(S)$ we have

$$A \star' X \subseteq Y \text{ iff } A \subseteq Y/X \text{ iff } X \subseteq A\backslash Y,$$

where $Y/X = \{a \in \Sigma \mid \{a\} \star' X \subseteq Y\}$ and $A\backslash Y = \{x \in S \mid A \star' \{x\} \subseteq Y\}$; equivalently $\star'$ preserves arbitrary unions. If all of the above conditions are satisfied, we say that $\mathcal{P}(S)$ is a $\mathcal{P}(\mathbf{\Sigma})$-*module*. For example $\mathcal{P}(\mathbf{Fm}_{\mathcal{L}})$, $\mathcal{P}(\mathbf{Eq}_{\mathcal{L}})$ and $\mathcal{P}(\mathbf{Seq}_{\mathcal{L}})$, where Seq is a set of sequents closed under type, are all $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$-modules.

A map $\tau : \mathcal{P}(S_1) \to \mathcal{P}(S_2)$ is called $\mathcal{P}(\mathbf{\Sigma})$-*invariant* or *structural*, if for all $A \in \mathcal{P}(\Sigma)$ and $X \in \mathcal{P}(S)$, we have $A \star \tau(X) = \tau(A \star X)$.

Assume that S_1 and S_2 are sets, and that $\vdash_1$ and $\vdash_2$ are consequence relations on $\mathcal{P}(S_1)$ and $\mathcal{P}(S_2)$, respectively. Further, assume that there exist maps $\tau : \mathcal{P}(S_1) \to \mathcal{P}(S_2)$ and $\rho : \mathcal{P}(S_2) \to \mathcal{P}(S_1)$ that preserve unions such that for every subset $X \cup \{x\}$ of S_1 and $y \in S_2$,

(1) $X \vdash_1 x$ iff $\tau(X) \vdash_2 \tau(x)$,
(2) $y \dashv\vdash_2 \tau\rho(y)$.

Then we say that $\vdash_1$ and $\vdash_2$ are *similar via* τ *and* ρ. We will show in Lemma 7.6 that, in this case, $\vdash_2$ and $\vdash_1$ are similar via ρ and τ, as well.

Assume further that $\mathbf{\Sigma}$ is a monoid and that $\mathcal{P}(S_1)$ and $\mathcal{P}(S_2)$ are $\mathcal{P}(\mathbf{\Sigma})$-modules. If $\vdash_1$ and $\vdash_2$ are similar via τ and ρ, and both τ and ρ are $\mathcal{P}(\mathbf{\Sigma})$-invariant, then we say that $\vdash_1$ and $\vdash_2$ are *equivalent via* τ *and* ρ.

It is easy to see that a consequence relation $\vdash$ on $\mathcal{P}(Seq)$, where Seq is a set of $\mathcal{L}$-sequents closed under type, is algebraizable in the sense of Rebagliato and Verdú (or in the sense of Blok and Pigozzi in the case when $Seq = Fm_{\mathcal{L}}$) iff there exists a class $\mathcal{K}$ of $\mathcal{L}$-algebras such that $\vdash$ and $\models_{\mathcal{K}}$ are equivalent via finitary, $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$-invariant maps $\tau : \mathcal{P}(Seq) \to \mathcal{P}(Eq_{\mathcal{L}})$ and $\rho : \mathcal{P}(Eq_{\mathcal{L}}) \to \mathcal{P}(Seq)$.

4.2. **Symmetric consequence relations.** We define a different, but equivalent notion of consequence relation that is amenable to generalization to arbitrary lattices.

A *symmetric consequence relation over* S is a binary relation $\vdash$ on $\mathcal{P}(S)$ that satisfies, for all $X, Y, Z \in \mathcal{P}(S)$,

(1) if $Y \subseteq X$, then $X \vdash Y$
(2) if $X \vdash Y$ and $Y \vdash Z$, then $X \vdash Z$.
(3) $X \vdash \bigcup_{X \vdash Y} Y$.

Note that $\vdash$ satisfies the first two conditions iff it is a pre-order on $\mathcal{P}(S)$ that contains the relation $\supseteq$.

A symmetric consequence relation over S is called *finitary*, if for all $X, Y \in \mathcal{P}(S)$, if $X \vdash Y$ and Y is finite, then there is a finite subset X_0 of X such that $X_0 \vdash Y$.

Assume that $\mathcal{P}(S)$ is a $\mathcal{P}(\mathbf{\Sigma})$-module. A symmetric consequence relation $\vdash$ on $\mathcal{P}(S)$ is called $\mathcal{P}(\mathbf{\Sigma})$*-invariant*, if for all $X, Y \in \mathcal{P}(S)$ and $A \in \mathcal{P}(\mathbf{\Sigma})$, $X \vdash Y$ implies $A \star X \vdash A \star Y$.

Given an asymmetric consequence relation $\vdash$, we define its symmetric counterpart $\vdash^s$, by $X \vdash^s Y$, for $X, Y \in \mathcal{P}(S)$, to mean $X \vdash y$, for all $y \in Y$. Conversely, given an asymmetric consequence relation $\vdash$, we define its asymmetric counterpart $\vdash^a$, by $X \vdash^a x$ iff $X \vdash \{x\}$, for $X \in \mathcal{P}(S)$ and $x \in S$.

Lemma 4.1. *Symmetric consequence relations on $\mathcal{P}(S)$, where S is a set, are in bijective correspondence with asymmetric consequence relations on $\mathcal{P}(S)$ via the maps $\vdash \mapsto \vdash^a$ and $\vdash \mapsto \vdash^s$. Moreover, finitarity and $\mathcal{P}(\mathbf{\Sigma})$-invariance, in the case where $\mathcal{P}(S)$ is a $\mathcal{P}(\mathbf{\Sigma})$-module, are preserved under these maps.*

In what follows we will use the term *consequence relation* on $\mathcal{P}(S)$ for both types of relations relying on the context for clarifying its use.

5. Consequence relations on complete lattices

In this section we introduce the notion of a consequence relation on an arbitrary complete lattice, and show that consequence relations on a given lattice are in bijective correspondence with closure operators on it. This correspondence restricts to one between finitary consequence relations and finitary closure operators. Next we discuss the appropriate notion of substitution invariance for both consequence relations and for closure operators in the setting where the lattice is endowed with the additional structure of a module.

5.1. **Consequence relations on complete lattices.** Symmetric consequence relations are binary relations on the powerset of a set. We generalize their definition to complete lattices. We note that the definitions and results of this section extend easily to arbitrary posets.

Let $\mathbf{P}$ be a complete lattice. A *(symmetric) consequence relation on* $\mathbf{P}$ is a binary relation $\vdash$ on $\mathbf{P}$ that satisfies the following conditions, for all $x, y, z \in \mathbf{P}$.

(1) if $y \leq x$, then $x \vdash y$
(2) if $x \vdash y$ and $y \vdash z$, then $x \vdash z$
(3) $x \vdash \bigvee_{x \vdash y} y$, for all $x \in P$

Note that $\vdash$ satisfies the first two conditions iff it is a pre-order on $\mathbf{P}$ that contains the relation $\geq$.

Recall that a subset X of P is called *(upward) directed* in $\mathbf{P}$, if for all $x, y \in X$, there exists a $z \in X$ such that $x, y \leq z$. An element x of a complete lattice $\mathbf{P}$ is called *compact*, if, for all directed $Y \subseteq P$, $x \leq \bigvee Y$ implies $x \leq y$, for some $y \in Y$. Equivalently, x is compact if for all $Z \subseteq P$ if $x \leq \bigvee Z$, then there is a finite subset Z_0 of Z such that $x \leq \bigvee Z_0$. For every subset Q of P, we denote by $K_{\mathbf{P}}(Q)$ the set of compact elements of $\mathbf{P}$ that are contained in Q. We write $K_{\mathbf{P}}$ for $K_{\mathbf{P}}(P)$.

A consequence relation on $\mathbf{P}$ is called *finitary*, if for all $x, y \in P$, if $x \vdash y$ and y is compact, then there exists a compact element $x_0 \in P$ such that $x_0 \leq x$ and $x_0 \vdash y$.

The compact elements of the powerset $\mathcal{P}(S)$ are exactly the finite subsets of S. Therefore in the case where $P = \mathcal{P}(S)$, both notions of a consequence relation and a finitary consequence relation restrict to the ones defined for powersets.

5.2. **Modules over complete lattices and invariance under the action.** To define substitution invariance of a consequence relation on a complete lattice, we need to assume that the latter is endowed with a module structure. Therefore we define modules in the case of arbitrary complete lattices.

Let $\mathbf{A}$, $\mathbf{B}$ and $\mathbf{C}$ be complete lattices. A map $\star : A \times B \to C$ (viewed as a binary map) is called *residuated* provided there exist maps $\backslash_\star : A \times C \to B$ and $/_\star : C \times B \to A$, called the *residuals* of $\star$, such that for all $x \in A$, $y \in B$

and $z \in C$,

$$x \star y \leq z \Leftrightarrow x \leq z/_\star y \Leftrightarrow y \leq x\backslash_\star z.$$

A *residuated lattice* is an algebra $\mathbf{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$ such that $\langle A, \wedge, \vee\rangle$ is a lattice, $\langle A, \cdot, 1\rangle$ is a monoid, and the operation $\cdot$ is residuated with residuals $\backslash$ and $/$.

Let $\mathbf{A}$ be a complete residuated lattice, $\mathbf{P}$ a complete lattice and $\star : A \times P \to P$ a map. We say that $\langle \mathbf{P}, \star\rangle$ is a *(left)* $\mathbf{A}$*-module*, or a *(left) module over* $\mathbf{A}$, if for all $x \in P$ and $a, b \in A$,

(M1) $1_\mathbf{A} \star x = x$,
(M2) $a \star (b \star x) = ab \star x$, and
(M3) $\star$ is residuated.

In what follows we will often suppress $\star$ in $\langle \mathbf{P}, \star\rangle$, and simply write $\mathbf{P}$ instead. Clearly, $\mathbf{A}$ is itself an $\mathbf{A}$-module. We assume that $\star$ has priority over the *division operations* $\backslash$ and $/$; so $a \star x/y$ is short for $(a \star x)/y$. In the expressions $y\backslash x$ and x/y, x is called the *numerator* and y the *denominator*

Note that if $P = \mathcal{P}(S)$, then we obtain the notion of a module for powersets.

Lemma 5.1. *If $\langle \mathbf{P}, \star\rangle$ is an $\mathbf{A}$-module, then the following hold.*

(1) *The operation $\star$ preserves arbitrary joins in both coordinates. In particular, it is order-preserving in both coordinates.*
(2) *The operations $\backslash$ and $/$ preserve arbitrary meets in the denominator; moreover, they convert arbitrary joins in the denominator into meets. In particular, they are both order-preserving in the denominator and order reversing in the denominator.*
(3) $(x/y) \star y \leq x$
(4) $a \star (a\backslash x) \leq x$
(5) $x \leq a\backslash(a \star x)$
(6) $(a\backslash x)/y = a\backslash(x/y)$
(7) $[(x/y) \star y]/y = x/y$
(8) $1_\mathbf{A} \leq x/x$
(9) $(x/x) \star x = x$

The proof of the lemma is a straightforward application of the definitions and is therefore omitted.

A consequence relation $\vdash$ on the $\mathbf{A}$ module $\mathbf{P}$ is called *structural*, if $x \vdash y$ implies $a \star x \vdash a \star y$, for all $x, y \in P$ and $a \in A$.

Note that in the case where $P = \mathcal{P}(S)$ the notions of structurality and of substitution invariance of a consequence relation coincide.

5.3. **Residuated maps on complete lattices.** Let S_1, S_2 be arbitrary sets, and let $\vdash_1$, $\vdash_2$ be consequence relations on $\mathcal{P}(S_1)$ and $\mathcal{P}(S_2)$, respectively. We have seen that the maps $\tau : \mathcal{P}(S_1) \to \mathcal{P}(S_2)$ and $\rho : \mathcal{P}(S_2) \to \mathcal{P}(S_1)$ involved in the definition of similarity of $\vdash_1$ and $\vdash_2$

were assumed to preserve unions. We have noted that this is a necessary and sufficient condition for these maps to extend maps from the sets S_1 and S_2 to the powersets $\mathcal{P}(S_1)$ and $\mathcal{P}(S_2)$ respectively. The generalization of this notion in the setting of complete lattices is that of a map that preserves arbitrary joins. We will find it convenient, however, to work with the equivalent concept of a residuated map.

Let $\mathbf{P}$ and $\mathbf{Q}$ be complete lattices. A map $\tau : P \to Q$ is called *residuated*, if there exists a map $\tau_* : Q \to P$, called the *residual* of τ, such that for all $x \in P$ and $y \in Q$,

$$\tau(x) \leq y \;\Leftrightarrow\; x \leq \tau_*(y).$$

Note that a binary map is residuated, in the sense of the previous subsection, if and only if all its unary translates (sections) are residuated in the preceding sense. We will often write $\tau : \mathbf{P} \to \mathbf{Q}$ for $\tau : P \to Q$, to indicate the dependency of the residuation property on the order structure of $\mathbf{P}$ and $\mathbf{Q}$. It is clear that the residual of a residuated map is uniquely defined by

$$\tau_*(y) = max\{x \in P \mid \tau(x) \leq y\}.$$

We will always denote it by τ_*. The following lemma states well known facts from residuation theory.

Lemma 5.2. *Assume that $\tau : \mathbf{P} \to \mathbf{Q}$ and $\rho : \mathbf{Q} \to \mathbf{R}$ are residuated maps.*

(1) *τ preserves all arbitrary joins in $\mathbf{P}$ and τ_* preserves all arbitrary meets in $\mathbf{Q}$.*
(2) *$\tau\tau_* \leq I_Q$ and $\tau_*\tau \leq I_P$.*
(3) *The composition $\rho\tau$ is residuated, as well, with residual $(\rho\tau)_* = \tau_*\rho_*$.*

We note again that for complete lattices $\mathbf{P}$ and $\mathbf{Q}$, $\tau : \mathbf{P} \to \mathbf{Q}$ is residuated iff it preserves arbitrary joins.

Example 5.3. Let A and B be sets and let $R \subseteq A \times B$ be a binary relation from A to B. The map $\tau_R : \mathcal{P}(A) \to \mathcal{P}(B)$, defined by $\tau_R(X) = R[X] = \{y \in B \mid R(x, y)$, for some $x \in X\}$, is residuated and its residual is given by $(\tau_R)_*(Y) = R^{-1}[Y] = \{x \in A \mid R(x, y)$, for some $y \in Y\}$. □

Note that if $\tau : A \to \mathcal{P}(B)$ is defined by $\tau(x) = \{y \mid (x, y) \in R\}$, then $\tau_R : \mathcal{P}(A) \to \mathcal{P}(B)$ is what we called τ' in Section 2.

Let $\mathbf{P}$, $\mathbf{Q}$ be $\mathbf{A}$-modules. A map $\tau : P \to Q$ is called *structural* if $a \star \tau(x) = \tau(a \star x)$, for all $x \in P$ and $a \in A$. Obviously, structural maps on the $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$-modules of the previous sections are exactly the substitution invariant maps.

A *module morphism* $\tau : \mathbf{P} \to \mathbf{Q}$ from $\mathbf{P}$ to $\mathbf{Q}$ is a structural residuated map. We will often use the term *translator* for such a morphism. For a fixed complete residuated lattice $\mathbf{A}$, we will denote by ${}_{\mathbf{A}}\mathcal{M}$ the category of all $\mathbf{A}$-modules and module morphisms (translators).

Recall that the compact elements of a powerset $\mathcal{P}(S)$ are the finite subsets of S. Let $\mathbf{P}$ and $\mathbf{Q}$ be complete lattices. A residuated map $\tau : \mathbf{P} \to \mathbf{Q}$ is called *finitary*, if the image of every compact element is compact.

6. Consequence relations, closure operators and theories

We now have all the required background to formulate the definition of equivalence of two consequence relations and to prove our main theorem. Nevertheless, the formulation of the results and their proofs in terms of consequence relations is more cumbersome than their formulation in terms of the equivalent concept of a closure operator. Therefore, we devote this section to establishing this connection first. Moreover, we define the lattice of theories of a consequence relation.

6.1. **Closure operators.** Recall that a *closure operator* γ on a complete lattice $\mathbf{P}$ is an *expanding* ($x \leq \gamma(x)$), *monotone* ($x \leq y \Rightarrow \gamma(x) \leq \gamma(y)$) and *idempotent* ($\gamma(\gamma(x)) = \gamma(x)$) map on P; an *interior operator* γ on $\mathbf{P}$ is an *contracting* ($\gamma(x) \leq x$), *monotone* and *idempotent* map on P. If $\gamma : P \to P$ is a map, we denote by P_γ the image $\gamma[P]$ of P under γ and by $\mathbf{P}_\gamma$ the sublattice of $\mathbf{P}$ with carrier P_γ.

It is easy to see that $\gamma : P \to P$ is a closure operator on $\mathbf{P}$ iff the map $\gamma' : P \to P_\gamma$, defined by $\gamma'(x) = \gamma(x)$, for all $x \in P$, is residuated and the inclusion map $In_{P_\gamma} : P_\gamma \to P$ is its residual. We will often identify γ and γ', with the understanding that only γ' is residuated and only γ is a closure operator.

Lemma 6.1. *Assume that $\tau : \mathbf{P} \to \mathbf{Q}$ is a residuated map between the complete lattices $\mathbf{P}$ and $\mathbf{Q}$.*

(1) *$\tau_*\tau$ is a closure operator on $\mathbf{P}$ and $\tau\tau_*$ is an interior operator on $\mathbf{Q}$.*
(2) *$\mathbf{P}_{\tau_*\tau}$ is isomorphic to $\mathbf{Q}_{\tau\tau_*}$.*

If f and g are both maps from $\mathbf{P}$ to $\mathbf{Q}$, we write $f \leq g$, if $f(x) \leq g(x)$ for all $x \in P$. It is obvious that if h is a map from $\mathbf{Q}$ to a complete lattice $\mathbf{R}$ and k is a monotone map from a poset $\mathbf{T}$ to $\mathbf{P}$, then $f \leq g$ implies $fh \leq gh$ and $kf \leq kg$. Note that if γ is a closure operator on $\mathbf{P}$ and δ is an interior operator on $\mathbf{P}$, then $\delta \leq I_P \leq \gamma$, where I_P is the identity map on P.

A closure operator γ on a complete lattice $\mathbf{P}$ is called *finitary*, if it preserves directed joins; i.e., for all directed X, $\gamma(\bigvee^{\mathbf{P}} X) = \bigvee^{\mathbf{P}} \gamma[X]$. By a *finitary lattice* we understand a complete lattice in which every element is a join of compact elements; in particular, $x = \bigvee K_{\mathbf{P}}(\downarrow x)$, for all $x \in P$. It should be noted that our choice of the terms "finitary closure operator" and "finitary lattice" is dictated by other uses of "algebraic" in this area. The most commonly used terms for these concepts are *algebraic closure operator* and *algebraic lattice*, respectively.

A closure operator on an **A**-module **P** is called *structural*, if $a \star \gamma(x) \leq \gamma(a \star x)$, for all $x \in P$ and $a \in A$. Note that Lemma 6.4 below reconciles the two notions of structurality for a closure operator $\gamma : P \to P$ that is viewed as a residuated map $\gamma : P \to P_\gamma$.

Given a consequence relation $\vdash$ on a complete lattice **P**, we define the map $\gamma_\vdash : P \to P$, by $\gamma_\vdash(x) = \bigvee_{x \vdash y} y$. Also given a closure operator $\gamma : P \to P$, we define the binary relation $\vdash_\gamma$ on P, by $x \vdash_\gamma y$ iff $y \leq \gamma(x)$.

Lemma 6.2. *Consequence relations on a complete lattice* **P** *are in bijective correspondence with closure operators on* **P** *via the maps* $\vdash \mapsto \gamma_\vdash$ *and* $\gamma \mapsto \vdash_\gamma$. *If* **P** *is finitary, then* $\vdash$ *is finitary iff* $\gamma_\vdash$ *is finitary. If* **P** *is an* **A***-module, then* $\vdash$ *is structural iff* $\gamma_\vdash$ *is structural.*

We give three more characterizations of a structural closure operator on a module.

Lemma 6.3. *Let* **P** *be an* **A***-module and let* γ *be a closure operator on* **P**. *The following are equivalent*

(1) γ *is structural.*
(2) $\gamma(a \star \gamma(x)) = \gamma(a \star x)$, *for all* $a \in A$ *and* $x \in P$.
(3) $\gamma(x)/_\star y = \gamma(x)/_\star \gamma(y)$, *for all* $x, y \in P$.
(4) $\gamma(a \backslash_\star x) \leq a \backslash_\star \gamma(x)$, *for all* $a \in A$ *and* $x \in P$.
(5) $a \backslash_\star \gamma(x) \in P_\gamma$, *for all* $a \in A$ *and* $x \in P$.

Proof. It is clear that (1) is equivalent to (2). To show that (1) implies (3), let $x, y \in P$. The inequality $\gamma(x)/_\star \gamma(y) \leq \gamma(x)/_\star y$ follows from the fact that $y \leq \gamma(y)$. For the reverse inequality, by the structurality of γ, we have

$$[\gamma(x)/_\star y] \star \gamma(y) \leq \gamma([\gamma(x)/_\star y] \star y) \leq \gamma(\gamma(x)) = \gamma(x);$$

we used Lemma 5.1(3) and the monotonicity of γ. So $\gamma(x)/_\star y \leq \gamma(x)/_\star \gamma(y)$.

For the converse implication, let $a \in A$ and $x \in P$. Since $a \star x \leq \gamma(a \star x)$, we have $a \leq \gamma(a \star x)/_\star x = \gamma(a \star x)/_\star \gamma(x)$. Thus, $a \star \gamma(x) \leq \gamma(a \star x)$.

For the equivalence of (1) and (4), let $a \in A$ and $x \in P$. We have $a \star \gamma(a \backslash x) \leq \gamma(a \star (a \backslash x)) \leq \gamma(x)$, by Lemma 5.1(4). Conversely, $a \star \gamma(x) \leq a \star \gamma(a \backslash a \star x) \leq a \star [a \backslash \gamma(a \star x)] \leq \gamma(a \star x)$, by Lemma 5.1(5,4).

To show that (1) implies (5), let $a \in A$ and $x \in P$. It suffices to show that $\gamma(a \backslash_\star \gamma(x)) \leq a \backslash_\star \gamma(x)$; i.e., $a \star \gamma(a \backslash_\star \gamma(x)) \leq \gamma(x)$. Indeed,

$$a \star \gamma(a \backslash_\star \gamma(x)) \leq \gamma(a \star (a \backslash_\star \gamma(x))) \leq \gamma(\gamma(x)) \leq \gamma(x).$$

For the converse implication, let $a \in A$ and $x \in P$. Since $a \star x \leq \gamma(a \star x)$, we have $x \leq a \backslash_\star \gamma(a \star x)$. By the hypothesis, it follows that $\gamma(x) \leq a \backslash_\star \gamma(a \star x)$, hence $a \star \gamma(x) \leq \gamma(a \star x)$. ☐

Condition (5) of Lemma 6.3, in the special case of $\mathcal{P}(\mathbf{\Sigma}_\mathcal{L})$-modules, states that the lattice of theories is closed under inverse substitutions. Indeed, if $P = \mathcal{S}$, where S is the algebra of formulas, equations or sequents, then $\mathbf{P}_\gamma$ is the lattice of theories of $\vdash_\gamma$. Note that condition (5), for a a set of substitutions, is equivalent to its restriction, where a ranges only over singletons, by Theorem 5.1(2). So, condition (5) is equivalent to the statement that $\{\sigma\}\backslash T = \sigma^{-1}(T)$ is a theory, for every substitution σ and theory T.

Lemma 6.4. *Let* $\mathbf{P}$ *be an* $\mathbf{A}$*-module and let* γ *be a structural closure operator on* $\mathbf{P}$*. Then* $\langle \mathbf{P}_\gamma, \star_\gamma \rangle$ *is an* $\mathbf{A}$*-module, where* $\star_\gamma : A \times P_\gamma \to P_\gamma$ *is the map defined by* $a \star_\gamma x = \gamma(a \star x)$*. As usual, we write* $\mathbf{P}_\gamma$ *for* $\langle \mathbf{P}_\gamma, \star_\gamma \rangle$*. Moreover,* $\gamma : \mathbf{P} \to \mathbf{P}_\gamma$ *is a module morphism.*

Proof. It is clear that the first two conditions in the definition of a module are satisfied. To show that $\star_\gamma$ is residuated, note that for all $a \in A$ and $x, y \in P_\gamma$, we have

$$a \star_\gamma x \leq y \;\Leftrightarrow\; \gamma(a \star x) \leq y \;\Leftrightarrow\; a \star x \leq y \;\Leftrightarrow\; x \leq a \backslash_\star y.$$

By Lemma 6.3(5), $a\backslash_\star y \in P_\gamma$, so $\star_\gamma$ is left residuated with left division $\backslash_\gamma$ the restriction of $\backslash$ to P_γ.

Furthermore, we have

$$a \star_\gamma x \leq y \;\Leftrightarrow\; \gamma(a \star x) \leq y \;\Leftrightarrow\; a \star x \leq y \;\Leftrightarrow\; a \leq y/_\star x.$$

Thus, $\star_\gamma$ is right residuated and $/_\gamma$ is the restriction of $/_\star$ to P_γ.

The fact that $\gamma : \mathbf{P} \to \mathbf{P}_\gamma$ is a module morphism follows from the definition of $\star_\gamma$. □

6.2. **Theories.** As we have seen closure operators and consequence relations are interdefinable. Also, the properties of being structural and finitary are preserved under this correspondence. Here we discuss yet another way of looking at the same properties.

Let $\vdash$ be a consequence relation on a complete lattice $\mathbf{P}$. An element t of P is called a *theory* of $\vdash$ if $t \vdash x$ implies $x \leq t$. Note that if t is a theory, then $x \leq t$ and $x \vdash y$ imply $y \leq t$. We denote the set of theories of $\vdash$ by $Th_\vdash$.

Lemma 6.5. *If* $\vdash$ *is a consequence relation on the complete lattice* $\mathbf{P}$*, then* $Th_\vdash = P_{\gamma_\vdash}$.

Proof. Let $t \in Th_\vdash$ and set $\gamma = \gamma_\vdash$. We will show that $t \in P_\gamma$, i.e., that $\gamma(t) = t$. We have $\gamma(t) \leq \gamma(t)$, so $t \vdash \gamma(t)$. Since t is a theory, $\gamma(t) \leq t$. The other inequality holds because γ is extensive.

Conversely, assume that $\gamma(t) = t$, and let $x \in P$ such that $t \vdash x$. Then $x \leq \gamma(t) = t$. □

We define the *lattice of theories* $\mathbf{Th}_\vdash$ of $\vdash$ to be the complete lattice $\mathbf{P}_{\gamma_\vdash}$. A natural question is whether we can characterize the lattices of theories abstractly and recover the corresponding consequence relation or closure operator.

A subset Q of P is said to be *completely meet-closed*, if whenever$X \subseteq Q$, then $\bigwedge_\mathbf{P} X \in Q$.

Lemma 6.6. *Let* $\mathbf{P}$ *be a complete lattice,* γ *be a closure operator on* $\mathbf{P}$ *and* Q *a completely meet-closed subset of* $\mathbf{P}$*. Then,* P_γ *is a completely meet-closed subset of* $\mathbf{P}$*,* γ_Q *is a closure operator on* $\mathbf{P}$*,* $\gamma_{\mathbf{P}_\gamma} = \gamma$ *and* $P_{\gamma_Q} = Q$*.*

7. Similarity and equivalence of two consequence relations

In this section we define the notions of representation, similarity and equivalence between two closure operators or two consequence relations. Our development generalizes the corresponding notions in [4].

7.1. **Representation.** Let γ and δ be closure operators on the complete lattices $\mathbf{P}$ and $\mathbf{Q}$, respectively. A *representation* of γ in δ is a residuated order embedding $f : \mathbf{P}_\gamma \to \mathbf{Q}_\delta$; i.e., a residuated map satisfying $x \le y$ iff $f(x) \le f(y)$, for all $x, y \in P_\gamma$. Clearly, if R and S are completely meet-closed subsets of the complete lattices $\mathbf{P}$ and $\mathbf{Q}$, respectively, we define a representation of R in S to be a residuated order embedding $f : R \to S$. A representation $f : \mathbf{P}_\gamma \to \mathbf{Q}_\delta$ of γ in δ is said to be *induced* by the residuated map $\tau : \mathbf{P} \to \mathbf{Q}$, if $f\gamma = \delta\tau$.

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{\tau} & \mathbf{Q} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\delta} \\ \mathbf{P}_\gamma & \xrightarrow{f} & \mathbf{Q}_\delta \end{array}$$

In view of the correspondence between consequence relations and closure operators, we will denote an arbitrary consequence relation on a poset $\mathbf{P}$ by $\vdash_\gamma$ with the understanding that γ is the associated closure operator.

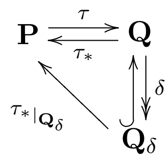
We say that a consequence relation $\vdash_\gamma$ is *represented* in the consequence relation $\vdash_\delta$ if the associated closure operator γ is represented in δ; the representation of $\vdash_\gamma$ in $\vdash_\delta$ is *induced* by a residuated map $\tau : \mathbf{P} \to \mathbf{Q}$, if the representation of the corresponding closure operators is induced by τ. Corollary 7.5 shows that $\vdash_\gamma$ is represented in $\vdash_\delta$ via τ if and only if for all $x, y \in P$,

$$x \vdash_\gamma y \text{ iff } \tau(x) \vdash_\delta \tau(y).$$

Recall that, by Lemma 6.2, a closure operator γ on an finitary lattice $\mathbf{P}$ is finitary if and only if $\vdash_\gamma$ is finitary. In other words, for all $x, y \in P$, whenever $y \le \gamma(x)$ and y is compact, there exists a compact element $x_0 \le x$ such that $y \le \gamma(x_0)$.

Lemma 7.1. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be complete lattices and let* $\tau : \mathbf{P} \to \mathbf{Q}$ *be a residuated map. If* δ *is a closure operator on* $\mathbf{Q}$*, then the map* $\delta^\tau = \tau_*\delta\tau : P \to P$ *is a closure operator on* $\mathbf{P}$*. If* $\mathbf{P}$*,* τ *and* δ *are finitary, then* δ^τ *is finitary, as well. If* $\mathbf{P}$ *and* $\mathbf{Q}$ *are* $\mathbf{A}$*-modules,* $\tau : \mathbf{P} \to \mathbf{Q}$ *is a module morphism and* δ *is a structural closure operator on* $\mathbf{Q}$*, then so is* δ^τ*.*

Proof. Note that $\delta : \mathbf{Q} \to \mathbf{Q}_\delta$ is residuated with residual the inclusion map In_{Q_δ}, so $\delta\tau : \mathbf{P} \to \mathbf{Q}_\delta$ is residuated, as well, with residual $\tau_* In_{Q_\delta} = \tau_{*|_{\mathbf{Q}_\delta}}$, by Lemma 5.2(3).



Therefore, $\delta^\tau = \tau_*\delta\tau = \tau_{*|_{\mathbf{Q}_\delta}}\delta\tau : P \to P$ is a closure operator on $\mathbf{P}$.

Assume that $\mathbf{P}$, τ and δ are finitary. If $y \leq \delta^\tau(x)$, for some compact element y, then $y \leq \tau_*\delta\tau(x)$, so $\tau(y) \leq \delta\tau(x)$. Since τ is finitary and y is compact, $\tau(y)$ is compact. Furthermore, since δ is finitary, there is a compact element $x' \leq \tau(x)$ such that $\tau(y) \leq \delta(x')$. Since $\mathbf{P}$ is finitary, $x = \bigvee K_{\mathbf{P}}(\downarrow x)$, so $\tau(x) = \bigvee \tau[K_{\mathbf{P}}(\downarrow x)]$, by Lemma 5.2(1). Since $x' \leq \tau(x)$, there exists a compact element $x_0 \leq x$ such that $x' \leq \tau(x_0)$. Consequently, $\tau(y) \leq \delta\tau(x_0)$, hence $y \leq \tau_*\delta\tau(x_0) = \delta^\tau(x_0)$, for some compact element $x_0 \leq x$. Thus, δ^τ is finitary.

For all $a \in A$ and $x \in P$, by using Lemma 5.2(2), we have

$$\tau(a \star \delta^\tau(x)) = a \star \tau\delta^\tau(x) = a \star \tau\tau_*\delta\tau(x) \leq a \star \delta\tau(x) \leq \delta(a \star \tau(x)) = \delta\tau(a \star x),$$

so $a \star \delta^\tau(x) \leq \tau_*\delta\tau(a \star x) = \delta^\tau(a \star x)$. □

We will call δ^τ the *τ-transform* of δ. Similarly, we can define the τ-transform of a consequence relation $\vdash$ on $\mathbf{Q}$ to be the relation $\vdash^\tau$ on $\mathbf{P}$ defined by $x \vdash^\tau y$ iff $\tau(x) \vdash \tau(y)$, for all $x, y \in P$. Also, we define the τ-transform of a completely meet-closed subset R of $\mathbf{Q}$ to be the subposet $\tau^{-1}[R]$ of $\mathbf{P}$. The following lemma shows that the τ-transform of a consequence relation (completely meet-closed subset) is a consequence relation (completely meet-closed subset, respectively) and the associated closure operator is the τ-transform of the original relation (meet-closed subset, respectively).

Lemma 7.2. *Let $\mathbf{P}$ and $\mathbf{Q}$ be complete lattices, $\tau : \mathbf{P} \to \mathbf{Q}$ a residuated map and δ a closure operator on $\mathbf{Q}$. The following statements are equivalent*

(1) $\gamma = \delta^\tau$
(2) *for all $x, y \in P$, $x \vdash_\gamma y$ iff $\tau(x) \vdash_\delta \tau(y)$.*
(3) $P_\gamma = \tau^{-1}[P_\delta]$

Proof. Assume (1) holds; then for all $x, y \in P$, we have $x \vdash_{\delta^\tau} y$ iff $y \leq \tau_*\delta\tau(x)$ iff $\tau(y) \leq \delta\tau(x)$ iff $\tau(x) \vdash_\delta \tau(x)$. Conversely, for all $x, y \in P$, we have $y \leq \gamma(x)$ iff $x \vdash_\gamma y$ iff $\tau(x) \vdash_\delta \tau(y)$ iff $\tau(y) \leq \delta\tau(x)$ iff $y \leq \tau_*\delta\tau(x)$. Consequently, $\gamma = \delta^\tau$.

For the equivalence of (1) and (3), note that $P_\gamma = \tau^{-1}[P_\delta]$ means that, for all $x \in P$, $x = \gamma(x)$ iff $\delta\tau(x) = \tau(x)$. Moreover, for all $x \in P$, we have $\delta\tau(x) = \tau(x)$ iff $\delta\tau(x) \leq \tau(x)$ iff $\tau_*\delta\tau(x) \leq x$ iff $\delta^\tau(x) = x$, since δ and δ^τ are closure operators. Consequently, (3) holds iff γ and δ^τ have the same fixed elements; i.e., $\gamma = \delta^\tau$. □

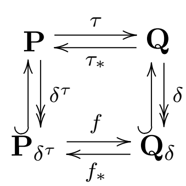
Lemma 7.3. *Let $\mathbf{P}$ be a finitary lattice, γ a finitary closure operator on $\mathbf{P}$, and X a subset of P_γ. If y is a compact element of $\mathbf{P}_\gamma$, then there exists a compact element x of $\mathbf{P}$ such that $y = \gamma(x)$.*

Proof. Let y be a compact element of $\mathbf{P}_\gamma$. Then $y = \gamma(z)$ for some $z \in P$ and $\gamma(z) = \bigvee \gamma[K_{\mathbf{P}}(\downarrow z)]$. Since y is compact and $y \leq \bigvee \gamma[K_{\mathbf{P}}(\downarrow z)]$, we have $y \leq \gamma(x)$, for some $x \in K_{\mathbf{P}}(\downarrow z)$. Thus, $y \leq \gamma(x) \leq \bigvee \gamma[K_{\mathbf{P}}(\downarrow z)] = y$, and $y = \gamma(x)$. □

Lemma 7.4. *Let $\mathbf{P}$ and $\mathbf{Q}$ be complete lattices, $\tau : \mathbf{P} \to \mathbf{Q}$ a residuated map, and δ a closure operator on $\mathbf{Q}$.*

1. *The map $f = \delta\tau_{|_{\mathbf{P}_{\delta^\tau}}} : \mathbf{P}_{\delta^\tau} \to \mathbf{Q}_\delta$ is residuated with residual $f_* = \tau_{*|_{\mathbf{Q}_\delta}} = \delta^\tau \tau_{*|_{\mathbf{Q}_\delta}} : \mathbf{Q}_\delta \to \mathbf{P}_{\delta^\tau}$.*
2. *f is a representation of δ^τ in δ induced by τ.*
3. *δ^τ is the only closure operator on $\mathbf{P}$ that is represented in δ under a representation induced by τ.*
4. *If $\mathbf{P}$, τ and δ are finitary, then f is finitary, as well.*
5. *If $\mathbf{P}$ and $\mathbf{Q}$ are $\mathbf{A}$-modules, $\tau : \mathbf{P} \to \mathbf{Q}$ is a module morphism and δ is a structural closure operator on $\mathbf{Q}$, then f is structural.*

Proof. (1) We first show that $\tau_{*|_{\mathbf{Q}_\delta}} = \delta^\tau \tau_{*|_{\mathbf{Q}_\delta}}$. Indeed, $I_P \leq \delta^\tau$, since δ^τ is a closure operator on $\mathbf{P}$, so $\tau_{*|_{\mathbf{Q}_\delta}} \leq \delta^\tau \tau_{*|_{\mathbf{Q}_\delta}}$. Conversely, $\tau\tau_* \leq I_Q$, by Lemma 5.2(2), so $\tau\tau_* In_{\mathbf{Q}_\delta} \leq I_{\mathbf{Q}} In_{\mathbf{Q}_\delta}$, that is $\tau\tau_{*|_{\mathbf{Q}_\delta}} \leq In_{\mathbf{Q}_\delta}$. By the monotonicity of $\tau_*\delta$, we have $\tau_*\delta\tau\tau_{*|_{\mathbf{Q}_\delta}} \leq \tau_*\delta In_{\mathbf{Q}_\delta}$; i.e., $\delta^\tau \tau_{*|_{\mathbf{Q}_\delta}} \leq \tau_{*|_{\mathbf{Q}_\delta}}$.



Recall that $\delta\tau : \mathbf{P} \to \mathbf{Q}_\delta$ is residuated with residual $\tau_{*|_{\mathbf{Q}_\delta}}$. For all $x \in \mathbf{P}_{\delta^\tau}$ and $y \in \mathbf{Q}_\delta$, we have

$$f(x) \leq y \iff \delta\tau(x) \leq y \iff x \leq \tau_{*|_{\mathbf{Q}_\delta}}(y) = \delta^\tau \tau_{*|_{\mathbf{Q}_\delta}}(y).$$

Since the range of $\delta^\tau \tau_{*|_{\mathbf{Q}_\delta}}$ is in $\mathbf{P}_{\delta^\tau}$, it follows that f is residuated and its residual is $f_* = \delta^\tau \tau_{*|_{\mathbf{Q}_\delta}}$.

(2) Since f is residuated with residual f_*, both f and f_* preserve order. To show that f is a representation it suffices to show that it reflects the order. Note that

$$f_* f = \tau_{*|_{\mathbf{Q}_\delta}} \delta\tau_{|_{\mathbf{P}_{\delta^\tau}}} = \tau_* I_{Q_\delta} \delta\tau I_{P_{\delta^\tau}} = \tau_*\delta\tau I_{P_{\delta^\tau}} = \delta^\tau I_{P_{\delta^\tau}} = I_{P_{\delta^\tau}}.$$

Now, for all $x, y \in P_{\delta^\tau}$, if $f(x) \leq f(y)$, then $f_* f(x) \leq f_* f(y)$, so $x \leq y$.

Moreover $f\delta^\tau = \delta\tau_{|_{\mathbf{P}_{\delta^\tau}}} \delta^\tau = \delta\tau In_{\mathbf{P}_{\delta^\tau}} \delta^\tau = \delta\tau\delta^\tau = \delta\tau$. The last equality holds because $\delta\tau \leq \delta\tau\delta^\tau$ (since δ^τ is a closure operator) and $\delta\tau\delta^\tau =$

$\delta\tau\tau_*\delta\tau \leq \delta\delta\tau = \delta\tau$ (since $\tau\tau_*$ is an interior operator). Consequently, f is induced by τ.

(3) Let γ be a closure operator on $\mathbf{P}$ that is represented in δ by a representation f induced by τ. We will show that $\gamma = \delta^\tau$.

$$\begin{array}{ccc} \mathbf{P} & \xrightarrow{\tau} & \mathbf{Q} \\ {\scriptstyle\gamma}\downarrow & & \downarrow{\scriptstyle\delta} \\ \mathbf{P}_\gamma & \xrightarrow{f} & \mathbf{Q}_\delta \end{array}$$

Since γ is a closure operator on $\mathbf{P}$, we have $\gamma_* = In_{P_\gamma}$, so $\gamma_*\gamma = \gamma$; for the same reason, we have $\delta_*\delta = \delta$. Consequently, by Lemma 5.2(3),

$$\delta^\tau = \tau_*\delta\tau = \tau_*\delta_*\delta\tau = (\delta\tau)_*\delta\tau = (f\gamma)_*f\gamma = \gamma_*\gamma = \gamma$$

The equation $(f\gamma)_*f\gamma = \gamma_*\gamma$ follows directly from the fact that f is order reflecting, since $x \leq (f\gamma)_*f\gamma(y)$ iff $f\gamma(x) \leq f\gamma(y)$ iff $\gamma(x) \leq \gamma(y)$ iff $x \leq \gamma_*\gamma(y)$, for all $x, y \in P$.

(4) Let x be a compact element of $\mathbf{P}_{\delta^\tau}$; we will show that $f(x)$ is compact in $\mathbf{Q}_\delta$. By Lemma 7.1, δ^τ is finitary. Therefore, by Lemma 7.3(3), there exists a compact element y of $\mathbf{P}$ such that $x = \delta^\tau(y)$. By the finitarity of τ and δ, we have that $f(x) = f(\delta^\tau(y)) = \delta(\tau(y)$ is compact.

(5) For all $a \in A$ and $x \in P_{\delta^\tau}$, we have $f(a \star_{\mathbf{P}_{\delta^\tau}} x) = f\delta^\tau(a \star_{\mathbf{P}} x) = \delta\tau(a \star_{\mathbf{P}} x) = \delta(a \star_{\mathbf{Q}} \tau(x)) = \delta(a \star_{\mathbf{Q}} \delta\tau(x)) = a \star_{\mathbf{Q}_\delta} \delta\tau(x) = a \star_{\mathbf{Q}_\delta} f(x)$. □

Corollary 7.5. *Let $\mathbf{P}$ and $\mathbf{Q}$ be complete lattices, and let $\vdash_\gamma$ and $\vdash_\delta$ be consequence relations on $\mathbf{P}$ and $\mathbf{Q}$, respectively. Then, $\vdash_\gamma$ is represented in $\vdash_\delta$ via a residuated map $\tau : \mathbf{P} \to \mathbf{Q}$ if and only if for all $x, y \in P$, we have $x \vdash_\gamma y$ iff $\tau(x) \vdash_\delta \tau(y)$.*

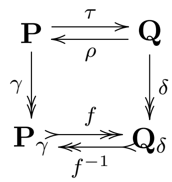
Proof. The corollary is a direct consequence of Lemma 7.2 and Lemma 7.4(3). □

It is easy to see that $\vdash_\gamma$ is represented in $\vdash_\delta$ by $f : \mathbf{Th}_{\vdash_\gamma} \to \mathbf{Th}_{\vdash_\delta}$ means that f is residuated and for all $x, y \in P$,

$$x \vdash_\gamma y \text{ iff } f\gamma(x) \vdash_\delta f\gamma(y).$$

Indeed, if $\vdash_\gamma$ is represented in $\vdash_\delta$ by f, then $x \vdash_\gamma y$ iff $y \leq \gamma(x)$ iff $f(y) \leq f(\gamma(x))$ (since f preserves and reflects order) iff $f(y) \leq \delta f(\gamma(x))$ iff $f\gamma(x) \vdash_\delta f\gamma(y)$. Conversely, to show that f reflects order, let $f\gamma(y) \leq f\gamma(x)$. Then $f\gamma(y) \leq \delta f\gamma(x)$, that is, $f\gamma(x) \vdash_\delta f\gamma(y)$; so $x \vdash_\gamma y$ that is $\gamma(y) \leq \gamma(x)$.

7.2. **Similarity.** Let γ and δ be closure operators on the complete lattices $\mathbf{P}$ and $\mathbf{Q}$, respectively. A *similarity* between γ and δ is an isomorphism $f : \mathbf{P}_\gamma \to \mathbf{Q}_\delta$. If there exists a similarity between γ and δ, then γ and δ are called *similar*. A similarity f between γ and δ is said to be *induced* by the residuated maps $\tau : \mathbf{P} \to \mathbf{Q}$ and $\rho : \mathbf{Q} \to \mathbf{P}$, if $f\gamma = \delta\tau$ and $f^{-1}\delta = \gamma\rho$. In this case we will say that γ and δ are similar *via* τ and ρ.



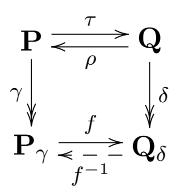
It is clear that f is a similarity between γ and δ iff f is a representation of γ in δ, f is a bijection and f^{-1} is a representation of δ in γ.

A consequence relation $\vdash_\gamma$ is called *similar* to the consequence relation $\vdash_\delta$ (via a residuated map τ) if γ is similar to δ (via τ).

Lemma 7.6. *Let γ and δ be closure operators on the complete lattices $\mathbf{P}$ and $\mathbf{Q}$, respectively. The following statements are equivalent.*

1. *γ and δ are similar via (a similarity induced by) τ and ρ.*
2. *$\gamma = \delta^\tau$ and $\delta\tau\rho = \delta$.*
3. *$\delta = \gamma^\rho$ and $\gamma\rho\tau = \gamma$.*

Proof. We will show the equivalence of the first two statements; the equivalence of the first to the third will follow by symmetry. The forward direction follows from Lemma 7.4(3) and the definition of similarity ($\delta\tau\rho = f\gamma\rho = ff^{-1}\delta = \delta$). For the converse, assume that $\gamma = \delta^\tau$ and $\delta\tau\rho = \delta$. Let f be the representation of $\gamma = \delta^\tau$ in δ given in Lemma 7.4(1). We have $f\gamma = \delta\tau$, by Lemma 7.4(2).



To show that f is onto, let $y \in Q_\delta$ and set $x = \gamma\rho(y) \in P_\gamma$. We have $f(x) = f\gamma\rho(y) = \delta\tau\rho(y) = \delta(y) = y$. Consequently, f is an order-isomorphism and γ and δ are similar. To show that the similarity f is induced by τ and ρ, we need only prove that $f^{-1}\delta = \gamma\rho$, or equivalently that $\delta = f\gamma\rho$. This is true, because $\delta = \delta\tau\rho$ and $f\gamma = \delta\tau$. □

Corollary 7.7. *Let $\mathbf{P}$ and $\mathbf{Q}$ be complete lattices and let $\vdash_\gamma$ and $\vdash_\delta$ be consequence relations on $\mathbf{P}$ and $\mathbf{Q}$, respectively. Then, $\vdash_\gamma$ is similar to $\vdash_\delta$ via the residuated maps $\tau : \mathbf{P} \to \mathbf{Q}$ and $\rho : \mathbf{Q} \to \mathbf{P}$ if and only if the following conditions hold:*

1. *for all $x, y \in P$, $x \vdash_\gamma y$ iff $\tau(x) \vdash_\delta \tau(y)$; and*
2. *for all $z \in Q$, $z \dashv\vdash_\delta \tau\rho(z)$.*

Proof. It is easy to see that $\delta\tau\rho = \delta$ iff for all $z \in Q$, $z \dashv\vdash_\delta \tau\rho(z)$. Now, the corollary follows from of Lemma 7.6(2) and Corollary 7.5. □

7.3. **Equivalence.** Let $\mathbf{P}$ and $\mathbf{Q}$ be $\mathbf{A}$-modules and let γ and δ be structural closure operators on $\mathbf{P}$ and $\mathbf{Q}$, respectively. An *equivalence* between γ and δ is a module isomorphism $f : \mathbf{P}_\gamma \to \mathbf{Q}_\delta$. Note that an equivalence is just a structural similarity. Moreover, f^{-1} is also structural. If such an isomorphism exists then γ and δ are called *equivalent*. If the equivalence is induced by module morphisms $\tau : \mathbf{P} \to \mathbf{Q}$ and $\rho : \mathbf{Q} \to \mathbf{P}$, then γ and δ are called *equivalent via* τ *and* ρ.

Theorem 7.8. *Let* $\mathbf{P}$ *and* $\mathbf{Q}$ *be* $\mathbf{A}$*-modules and let* γ *and* δ *be structural closure operators on* $\mathbf{P}$ *and* $\mathbf{Q}$*, respectively. If* γ *and* δ *are similar via the translators (i.e., module morphisms)* τ *and* ρ*, then they are equivalent via* τ *and* ρ*.*

Proof. It suffices to show that the similarity f of γ in δ is structural. Indeed, for all $a \in A$ and $x \in P_\gamma$, we have

$$\begin{aligned} f(a \star_\gamma x) &= f\gamma(a \star x) = \delta\tau(a \star x) = \delta(a \star \tau(x)) \\ &= \delta(a \star \delta\tau(x)) = a \star_\delta \delta\tau(x) = a \star_\delta f\gamma(x) \\ &= a \star_\delta f(x), \end{aligned}$$

since $\gamma(x) = x$. □

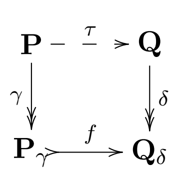
8. Equivalences induced by translators

A natural question to ask is whether every equivalence of consequence relations is induced by translators. Using small examples, it is easy to see that this is not always true. Nevertheless, we will show that this is the case for all standard situations including the powersets of formulas, equations and sequents.

In this section we present conditions on modules under which every equivalence is induced by translators. More specifically, we will prove that if $\mathbf{P}$ is an $\mathbf{A}$-module that satisfies these conditions, $\mathbf{Q}$ any $\mathbf{A}$-module and γ and δ are structural closure operators on $\mathbf{P}$ and $\mathbf{Q}$ respectively, then every representation $f : \mathbf{P}_\gamma \to \mathbf{Q}_\delta$ of γ in δ is induced by a translator.

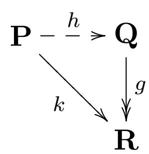
8.1. **Projective objects.** Recall that by ${}_{\mathbf{A}}\mathcal{M}$ we denote the category of $\mathbf{A}$-modules and translators (module morphisms). Every structural closure operator γ on the $\mathbf{A}$-module $\mathbf{P}$ is a translator from $\mathbf{P}$ to $\mathbf{P}_\gamma$. Assume that $\mathbf{P}$ and $\mathbf{Q}$ are $\mathbf{A}$-modules, γ and δ are structural closure operators on $\mathbf{P}$ and $\mathbf{Q}$ respectively, and $f : \mathbf{P}_\gamma \to \mathbf{Q}_\delta$ is a representation of γ in δ. We want to find a translator $\tau : \mathbf{P} \to \mathbf{Q}$ that induces f; i.e., $\delta\tau = f\gamma$. In other words we want a morphism τ in the category ${}_{\mathbf{A}}\mathcal{M}$ that completes the square.

(S)



It turns out that the objects $\mathbf{P}$ of the category ${}_{\mathbf{A}}\mathcal{M}$ for which such square can be completed are precisely the projective objects of ${}_{\mathbf{A}}\mathcal{M}$. An object $\mathbf{P}$ of ${}_{\mathbf{A}}\mathcal{M}$ is called *projective (relative to onto maps)*, if whenever there are modules $\mathbf{Q}$ and $\mathbf{R}$ and module morphisms $g : \mathbf{Q} \to \mathbf{R}$ and $k : \mathbf{P} \to \mathbf{R}$, with g onto, then there exists a morphism $h : \mathbf{P} \to \mathbf{Q}$, such that $k = gh$.

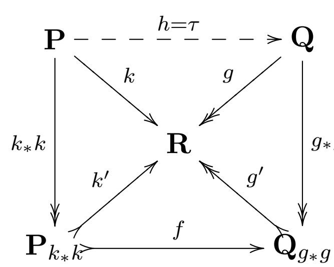
(T)



Lemma 8.1. *The objects* $\mathbf{P}$ *of the category* ${}_{\mathbf{A}}\mathcal{M}$ *for which all squares of type (S) can be completed are the projective objects of* ${}_{\mathbf{A}}\mathcal{M}$.

Proof. Obviously, if $\mathbf{P}$ is projective, then the square (S) can be completed, since we can chose $\mathbf{R} = \mathbf{Q}_\delta$, $k = f\gamma$ and $g = \delta$ in the triangle (T).

Conversely, assume that $\mathbf{P}$ is such that every square (S) can be completed and consider the triangle (T), where h is to be determined.



We know by Lemma 6.1(1,2) that k_*k is a closure operator on $\mathbf{P}$ and that $\mathbf{P}_{k_*k}$ is isomorphic to $k[\mathbf{P}]$ via the map $k' = k_{|\mathbf{P}_{k_*k}}$. Therefore, the map k factors as $k = k'(k_*k)$. Likewise, we have $g = g'(g_*g)$, where $g' = g_{|\mathbf{Q}_{g_*g}}$. Moreover, k' is an embedding and g' is an isomorphism, so the map $f = (g')^{-1}k'$ is an embedding. Since the outer square can be completed, we have $fk_*k = g_*gh$, so $g'fk_*k = g'g_*gh$, hence $k'k_*k = gh$; thus $k = gh$ and the upper triangle commutes. □

8.2. **Cyclic Modules.** We will show that the $\mathcal{P}(\mathbf{\Sigma}_\mathcal{L})$-modules discussed in Sections 3 and 4 are projective. Consequently, in view of the preceding theorem, all equivalences on these modules are induced by translators. More generally, we will identify a set of intrinsic conditions that describe cyclic projective modules. The $\mathcal{P}(\mathbf{\Sigma}_\mathcal{L})$-modules of formulas and of equations are cyclic and projective. The $\mathcal{P}(\mathbf{\Sigma}_\mathcal{L})$-module of sequents is not cyclic, but we prove that it is a coproduct of cyclic projective modules, and hence it is projective.

Let $\mathbf{A}$ be a complete residuated lattice. An $\mathbf{A}$-module $\mathbf{P}$ is called *cyclic*, if there exists an element $v \in P$, called a *generator* of $\mathbf{P}$, such that $P = A \star v = \{a \star v \mid a \in A\}$.

Lemma 8.2. *An $\mathbf{A}$-module $\mathbf{P}$ is cyclic with generator v iff $(x/v) \star v = x$, for all $x \in P$.*

Proof. If v is a generator, then for all $x \in P$, there exists an $a \in A$ such that $x = a \star v$; so $a \leq (x/v)$. We have $x = a \star v \leq (x/v) \star v \leq x$, by Lemma 5.1. So, $(x/v) \star v = x$. The converse, is obvious. □

Lemma 8.3. *If $\mathbf{A}$ is a complete residuated lattice and $\gamma : A \to A$ is a structural closure operator, then the $\mathbf{A}$-module $\langle \mathbf{A}_\gamma, \cdot_\gamma \rangle$ is cyclic with generator $\gamma(e)$.*

Proof. Obviously, $\gamma(e) \in A_\gamma$. Also, for all $\gamma(a) \in A_\gamma$, $a \cdot_\gamma \gamma(e) = \gamma(a \cdot e) = \gamma(a)$. □

Lemma 8.4. *Let $\langle \mathbf{P}, \star \rangle$ be an $\mathbf{A}$-module, $v \in P$ and $A \star v = \{a \star v \mid a \in A\}$.*

1. *Then $\mathbf{A} \star v = \langle A \star v, \star \rangle$ is an $\mathbf{A}$-module. The residual of the operation $\star$ in $\mathbf{A} \star v$ is given by $a \backslash_{\mathbf{A} \star v} q = [(a \backslash_{\mathbf{P}} q)/_{\mathbf{P}} v] \star v$.*
2. *The map $\gamma_v : A \to A$, defined by $\gamma_v(a) = a \star v/v$ is a structural closure operator.*
3. *$\mathbf{A} \star v$ is isomorphic to $\mathbf{A}_{\gamma_v}$.*

Consequently, an $\mathbf{A}$-module is cyclic if and only if it is isomorphic to a module $\mathbf{A}_\gamma$, for a structural closure operator $\gamma : \mathbf{A} \to \mathbf{A}$.

Proof. (1) First note that if $a \in A$ and $q \in A \star v$, then $q = b \star v$, for some $b \in A$, so $a \star (b \star v) = ab \star v \in A \star v$. Moreover, if $r = c \star v \in A \star v$, where $c \in A$, then $a \star r \leq q$ iff $a \star (c \star v) \leq q$ iff $c \leq (a \backslash q)/v$ iff $c \star v \leq [(a \backslash_{\mathbf{P}} q)/_{\mathbf{P}} v] \star v$. The last equivalence follows from Lemma 5.1(7).

(2) We have $a \leq \gamma_v(a)$; if $a \leq b$, then $\gamma_v(a) \leq \gamma_v(b)$ and $\gamma_v(\gamma_v(a)) = \gamma_v(a)$, by Lemma 5.1(7). Also, $a\gamma_v(b) \star v = a[(b \star v)/v] \star v \leq ab \star v$, so $a\gamma_v(b) \leq \gamma_v(ab)$. Thus, γ_v is structural.

(3) Let $f(a) = a \star v$ and $g(x) = x/v$. Note that $f : A_{\gamma_v} \to A \star v$ and $g : A \star v \to A_{\gamma_v}$, since $f(a) = a \star v \in A \star v$ and $g(a \star v) = (a \star v)/v \in A_{\gamma_v}$. For all $x \in A \star v$, we have $f(g(x)) = (x/v) \star v = x$, because of cyclicity. Also, for all $a \in A_{\gamma_v}$, $g(f(a)) = \gamma_v(a) = a$. So, $f^{-1} = g$. Moreover, both f and g are order-preserving, so they are order reflecting as well. □

Corollary 8.5. *If $\mathbf{A}$ is a complete residuated lattice and $u \in A$, then $\mathbf{A}u = \langle Au, \cdot \rangle$ is an $\mathbf{A}$-module isomorphic to $\mathbf{A}_{\gamma_u}$.*

Lemma 8.6. *Let $\mathbf{A}$ be a complete residuated lattice, $\gamma : A \to A$ a structural closure operator and $u \in A$. The following are equivalent.*

1. *$\gamma(a)u = au$, for all $a \in A$ and $\gamma(u) = \gamma(e)$*
2. *$\gamma = \gamma_u$ and $u = u^2$*

Proof. The fact that (2) implies (1) is easy to check. Conversely, from $\gamma(a)u = au$, we obtain $\gamma(a) \leq au/u = \gamma_u(a)$, for all $a \in A$. Also, from $\gamma(u) = \gamma(e)$, we obtain for all $b \in A$, $\gamma(bu) = b \star_\gamma \gamma(u) = b \star_\gamma \gamma(e) = \gamma(be) = \gamma(b)$. We have the following implications.

$$\gamma_u(a)u \leq au \;\Rightarrow\; \gamma(\gamma_u(a)u) \leq \gamma(au) \;\Rightarrow\; \gamma(\gamma_u(a)) \leq \gamma(a) \;\Rightarrow\; \gamma_u(a) \leq \gamma(a)$$

Moreover, since $\gamma = \gamma_u$, we have $\gamma_u(u) = \gamma_u(e)$, so $uu/u = u/u$, hence $(u^2/u)u = (u/u)u$. From this we obtain $u^2 = u$, because $u = eu \leq (u/u)u \leq u$ and $u^2 = uu \leq (u^2/u)u \leq u^2$, by Lemma 5.1. ☐

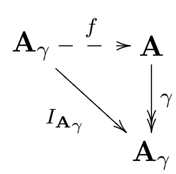
Corollary 8.7. *Let $\langle \mathbf{P}, \star \rangle$ be a cyclic $\mathbf{A}$-module with generator v. The following conditions are equivalent for an element $u \in A$.*

1. *$u \star v = v$ and $[(a \star v)/v]u = au$, for all $a \in A$.*
2. *$\gamma_v(a)u = au$, for all $a \in A$, and $\gamma_v(u) = \gamma_v(e)$.*
3. *$\gamma_v = \gamma_u$ and $u^2 = u$.*
4. *$\mathbf{P}$ is isomorphic to $\mathbf{A}u$ and $u^2 = u$.*

Proof. The equivalence (1) ⇔ (2) follows from the fact that $\gamma_v(u) = \gamma_v(e)$ iff $u \star v/v = v/v$ iff $u \star v = v$, by using Lemma 5.1. The implication (2) ⇒ (3) follows from the preceding lemma. The implication (3) ⇒ (4) follows from the facts $\mathbf{A} \star v \cong \mathbf{A}_{\gamma_v}$ (Lemma 8.4), $\mathbf{A}u \cong \mathbf{A}_{\gamma_u}$ (Corollary 8.5), and $\gamma_u = \gamma_v$. Finally, (4) ⇒ (1) follows from the fact that if $u^2 = u$, then $\mathbf{A}u$ satisfies (1) with $v = u$. ☐

Theorem 8.8. *A cyclic $\mathbf{A}$-module $\langle \mathbf{P}, \star \rangle$, with generator v, is projective if and only if there exists an element $u \in A$ that satisfies the equivalent conditions of Corollary 8.7.*

Proof. Every cyclic module is of the form $\mathbf{A}_\gamma$ for some structural closure operator $\gamma : A \to A$, by Lemma 8.4. Suppose $\mathbf{A}_\gamma$ is projective. We will verify condition (4) of Corollary 8.7. Since $\mathbf{A}_\gamma$ is projective, there exists a module morphism f that completes the diagram below.



Let $u = f(\gamma(e))$. For all $a \in A$, we have $\gamma(a) = \gamma(ae) = \gamma(a\gamma(e)) = a \cdot_\gamma \gamma(e)$, so $f(\gamma(a)) = a \cdot_{\mathbf{A}} f(\gamma(e)) = au$. Consequently, $f[A_\gamma] = Au$. Moreover, f is injective, by the diagram, so $\mathbf{A}_\gamma \cong \mathbf{A}u$. We will show that $u^2 = u$. Indeed, $u^2 = f(\gamma(e))f(\gamma(e)) = f(f(\gamma(e)) \cdot_\gamma \gamma(e)) = f(\gamma(f(\gamma(e)))) = f(\gamma(e)) = u$, because $\gamma f = Id$. We have established condition (4) of Corollary 8.7. It is straightforward to show that a cyclic module satisfying condition (4) is projective. ☐

The next result follows directly from Theorem 8.8 and Corollary 8.7 (1).

Corollary 8.9. *$\mathcal{P}(\mathbf{Fm}_\mathcal{L})$ and $\mathcal{P}(\mathbf{Eq}_\mathcal{L})$ are projective cyclic $\mathcal{P}(\mathbf{\Sigma}_\mathcal{L})$-modules.*

Proof. As noted above, we use Theorem 8.8 and Corollary 8.7 (1). In the case of the module $\mathcal{P}(\mathbf{Fm}_{\mathcal{L}})$, we let $v = \{x\}$, where x is a variable, and $u = \{\kappa_x\}$. Recall that κ_x is the substitution that maps all variables to x.

For the module $\mathcal{P}(\mathbf{Eq}_{\mathcal{L}})$, we can take $v = \{x \approx y\}$ and $u = \{\kappa_{x\approx y}\}$, where x, y are distinct variables. Here we assume that we have partitioned the set of all variables in two disjoint sets V_x, V_y with $x \in V_x$ and $y \in V_y$, and that $\kappa_{x\approx y}$ is the substitution that sends all of V_x to x and all of V_y to y. □

8.3. **Coproducts.** The preceding results do not cover the case of the $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$-module of sequents. Even though this module is not cyclic, we prove that it is a coproduct of cyclic projective modules, and hence it is projective. We start by defining coproducts in the category of **A**-modules.

Let $(\mathbf{P}_i \mid i \in I)$ be a family of **A**-modules. The *coproduct* of this family is an **A**-module **P**, denoted by $\coprod_{i\in I} \mathbf{P}_i$, together with a family of injective morphisms $(\sigma_i : \mathbf{P}_i \to \mathbf{P} \mid i \in I)$ such that for every **A**-module **Q** and every family of morphisms $(\tau_i : \mathbf{P}_i \to \mathbf{Q} \mid i \in I)$, there exists a unique morphism $\tau : \mathbf{P} \to \mathbf{Q}$ such that $\tau\sigma_i = \tau_i$.

We remark that if the coproduct of a family $(\mathbf{P}_i \mid i \in I)$ of **A**-modules exists, then the associated module morphisms σ_i are injective, and $\bigcup_{i\in I} \sigma_i(P_i)$ generates **P** as an **A**-module.

It is clear that whenever the coproduct of a family of **A**-modules exists, it is unique up to isomorphism. The next result guarantees that it always exists.

Lemma 8.10. *Let $(\mathbf{P}_i \mid i \in I)$ be a family of* **A***-modules. The* **A***-module $\coprod_{i\in I} \mathbf{P}_i$ in the definition of coproduct is the direct product $\prod_{i\in I} \mathbf{P}_i$ (with scalar multiplication defined component-wise). The associated injective module morphisms $\sigma_i : \mathbf{P}_i \to \prod_{i\in I} \mathbf{P}_i$ are defined, for each $i \in I$, by $\sigma_i(p) = (x_j)_{j\in I}$, where $x_i = p$ and $x_j = \bot$, if $j \neq i$.*

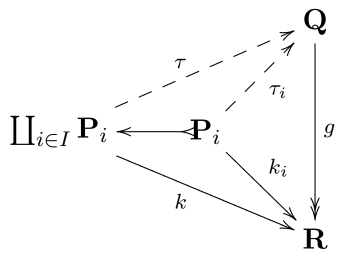
Proof. Note that the maps $\sigma_i : \mathbf{P}_i \to \prod_{i\in I} \mathbf{P}_i$ are module morphisms. If $\tau_i : \mathbf{P}_i \to \mathbf{Q}$ are module morphisms, then the map $\tau : \prod_{i\in I} P_i \to Q$, defined by $\tau((x_i)_{i\in I}) = \bigvee \tau_i(x_i)$, is residuated and its residual is $\tau_*(y) = ((\tau_i)_*(y))_{i\in I}$. It also preserves scalar multiplication, and hence it is a module morphism. □

The following standard categorical result shows why we are interested in coproducts.

Lemma 8.11. *The coproduct of a family of projective* **A***-modules is a projective* **A***-module.*

Proof. Assume that $(\mathbf{P}_i \mid i \in I)$ is a family of projective **A**-modules, let $\mathbf{Q}, \mathbf{R}$ be **A**-modules, and let $g : \mathbf{Q} \to \mathbf{R}$, $k : \coprod_{i\in I} \mathbf{P}_i \to \mathbf{R}$ be module morphisms such that g is onto. Let $\sigma_i : \mathbf{P}_i \to \coprod_{i\in I} \mathbf{P}_i$ be the injective module morphisms associated with the coproduct. Set $k_i = k\sigma_i$. Since each $\mathbf{P}_i$ is projective, there exists a module morphisms $\tau_i : \mathbf{P}_i \to \mathbf{Q}$ such that

$k_i = g\tau_i$. It follows that there exists a module morphism $\tau : \coprod_{i\in I} \mathbf{P}_i \to \mathbf{Q}$ such that $\tau_i = \tau I_i$.



Consequently, $k\sigma_i = k_i = g\tau_i = g\tau\sigma_i$ for all $i \in I$. Since for each $i \in I$ both of these morphisms are from $\mathbf{P}_i$ to $\mathbf{R}$, by the definition of the coproduct, there exists a unique morphism from $\coprod_{i\in I} \mathbf{P}_i$ to $\mathbf{R}$ such that these morphisms factor through $\mathbf{P}_i$. Since both k and $g\tau$ serve this purpose, they are equal. □

One can define different kinds of sequents. We saw intuitionistic, associative commutative sequents in Example 2.4 and we discussed classical, associative sequents. For non-associative sequents see [9], hypersequents see [1], and multi-sequents see [6]. The powersets of all these are coproducts of cyclic projective modules.

Inspired by Pynko [11], given an algebraic language $\mathcal{L}$ and a set $\mathcal{P}$ of predicate symbols, we consider atomic formulas in the language $\mathcal{L} \cup \mathcal{P}$ and we call them *$\mathcal{LP}$-sequents*. As an example, we mention that to represent associative (classical) sequents, for every pair (m, n) we define a $(m+n)$-ary predicate symbol $P_{(m,n)}$. Then, $P_{(m,n)}(\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n)$ is interpreted as the sequent $\alpha_1, \dots, \alpha_m \Rightarrow \beta_1, \dots, \beta_n$, where $\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n$ are $\mathcal{L}$-terms.

For every predicate symbol P in $\mathcal{P}$ of arity n, and a substitution σ on the terms algebra over $\mathcal{L}$, we define $\sigma(P(x_1, \dots, x_n)) = P(\sigma(x_1), \dots, \sigma(x_n))$. If $Seq_{\mathcal{LP}}$ denotes the set of the above general sequents, then clearly, $\mathcal{P}(\mathbf{Seq}_{\mathcal{LP}})$ is a $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$-module.

Theorem 8.12. *The $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$-module $\mathcal{P}(\mathbf{Seq}_{\mathcal{LP}})$ is a coproduct of cyclic projective modules. Consequently it is projective.*

Proof. We fix an enumeration of the variables and also of the elements of $\mathcal{P} = \mathcal{P}(\mathbf{Seq}_{\mathcal{LP}})$. We further enumerate the singletons v_P, containing the atomic formulas $P(x_1, \dots, x_n)$, following the above enumeration and choosing new and distinct variables. As in the proof of Corollary 8.9, for every such atomic formula $P(x_1, \dots, x_n)$, we chose a partition $V_{x_1}, \dots, V_{x_n}$ of the set of variables, with $x_i \in V_{x_i}$, and let $\kappa_{(x_1,\dots,x_n)}$ be the substitution that sends all of V_{x_i} to x_i, for all i. The elements u_P are the singletons containing the substitutions $\kappa_{(x_1,\dots,x_n)}$.

It is clear that each v_P generates a cyclic $\mathcal{P}(\mathbf{\Sigma})$-module $\mathbf{P}_P$ that is also projective. Moreover, the powerset of all the sequents $\mathcal{P}(\mathbf{Seq}_{\mathcal{LP}})$ becomes the coproduct of these modules. This is simply because $\mathbf{Seq}_{\mathcal{LP}} = \bigcup_{P\in\mathcal{P}} \mathbf{P}_P$, hence $\mathcal{P}(\mathbf{Seq}_{\mathcal{LP}}) = \mathcal{P}(\bigcup_{P\in\mathcal{P}} \mathbf{P}_P)$, which is isomorphic to $\prod_{P\in\mathcal{P}} \mathcal{P}(\mathbf{P}_P)$. In light of Lemma 8.10, the latter – together with the associated injections – is the coproduct of the family $(\mathbf{P}_P : P \in \mathcal{P})$. □

9. Finitary Translators

In the last section of the paper, we identify conditions under which an equivalence of consequence relations is induced by finitary translators. We start with the definitions of three pertinent notions.

A *finitary residuated lattice* is a finitary lattice in which the identity is a compact element, and the product of any two compact elements is compact.

A *finitary module* is an $\mathbf{A}$-module $\mathbf{P}$ such that (i) $\mathbf{A}$ is a finitary residuated lattice; (ii) $\mathbf{P}$ is a finitary lattice; and (iii) if a, v are compact elements of $\mathbf{A}$ and $\mathbf{P}$, respectively, then $a \star v$ is a compact element of $\mathbf{P}$.

For a fixed finitary residuated lattice $\mathbf{A}$, we will denote by ${}_{\mathbf{A}}\mathcal{FM}$ the category of finitary $\mathbf{A}$-modules and finitary module morphisms (finitary translators). Recall that such a morphism maps compact elements to compact elements.

Lemma 9.1. *Every finitary closure operator on a complete lattice preserves compactness.*

Proof. Assume that x is a compact element of a complete lattice $\mathbf{P}$, and let γ be a finitary closure operator on $\mathbf{P}$. Suppose that $\gamma(x) \le \bigvee^{\mathbf{P}_\gamma} \gamma[Y]$, for some directed subset $\gamma[Y]$ of $\mathbf{P}_\gamma$. Since γ is finitary, $\bigvee^{\mathbf{P}} \gamma[Y] = \bigvee^{\mathbf{P}_\gamma} \gamma[Y]$. Since $x \le \gamma(x) \le \bigvee^{\mathbf{P}_\gamma} \gamma[Y] = \bigvee^{\mathbf{P}} \gamma[Y]$ and x is compact, there is a $y \in Y$, such that $x \le \gamma(y)$; hence, $\gamma(x) \le \gamma(y)$. Consequently, $\gamma(x)$ is compact in $\mathbf{P}_\gamma$. □

Lemma 9.2. *Suppose* $\mathbf{A}$ *is a finitary residuated lattice, and let* $\mathbf{P}$ *be a cyclic* $\mathbf{A}$*-module with generator* v*. Let* $\gamma = \gamma_v$ *be the structural operator on* $\mathbf{A}$ *associated with* $\mathbf{P}$ *(Lemma 8.2). Then* $\mathbf{P}$ *is finitary iff* γ *is finitary.*

Proof. In light of Lemma 8.4, $\mathbf{P}$ is isomorphic to the module $\mathbf{A}_\gamma$, with the isomorphism mapping v to the generator $\gamma(e)$ of $\mathbf{A}_\gamma$.

Suppose first that $\mathbf{P}$ is finitary. It follows that $\mathbf{A}_\gamma$ is finitary. Let X be a directed family of elements of $\mathbf{A}_\gamma$. We must prove that $\bigvee^{\mathbf{P}} X = \bigvee^{\mathbf{P}_\gamma} X$, or, equivalently, that $\bigvee^{\mathbf{P}_\gamma} X \le \bigvee^{\mathbf{P}} X$. Let $\gamma(a)$ be a compact element of $\mathbf{A}_\gamma$ such that $\gamma(a) \le \bigvee^{\mathbf{P}_\gamma} X$. Then $\gamma(a) \le x$, for some $x \in X$, and so $\gamma(a) \le \bigvee^{\mathbf{P}} X$. Since $\bigvee^{\mathbf{P}_\gamma} X$ is the join of compact elements below it, we have $\bigvee^{\mathbf{P}_\gamma} X \le \bigvee^{\mathbf{P}} X$.

Conversely, suppose that γ is finitary. We must show that $\mathbf{P}$ is finitary, or, what amounts to the same, that $\mathbf{A}_\gamma$ is a finitary $\mathbf{A}$-module. Let $\gamma(a)$ be an arbitrary element of $\mathbf{A}_\gamma$. Let X be the set of compact elements of $\mathbf{A}$ below a. We have $a = \bigvee^{\mathbf{P}} X$, and hence, since X is directed, $\gamma(a) = \bigvee^{\mathbf{P}} \gamma[X]$. By Lemma 9.1, each $\gamma(x)$ $(x \in X)$ is compact in $\mathbf{A}_\gamma$. Thus, $\mathbf{A}_\gamma$ is finitary. □

Lemma 9.3. *Suppose that* $\mathbf{A}$ *is a finitary residuated lattice. Then any generator of a finitary cyclic* $\mathbf{A}$*-module is compact.*

Proof. Let $\mathbf{P}$ be a finitary cyclic $\mathbf{A}$-module, and let v be a generator of $\mathbf{P}$, and let γ is the associated structural closure operator. In light of Lemma 8.2, $\mathbf{P}$ is isomorphic to $\mathbf{A}_\gamma$, and, under this isomorphism, the generator $\gamma(e)$ of $\mathbf{A}_\gamma$ is mapped to v.

By Lemma 9.2, γ is finitary. It follows from Lemma 9.1 that $\gamma(e)$ is compact in $\mathbf{A}_\gamma$, since the unit e is assumed to be compact in $\mathbf{A}$. Hence, v is compact in $\mathbf{P}$. □

Lemma 9.4. *Let* $\mathbf{A}$ *be a finitary residuated lattice and let* $\mathbf{P}$ *be a finitary cyclic* $\mathbf{A}$*-module. Then,* $\mathbf{P}$ *is projective iff there exists an idempotent element* u *of* $\mathbf{A}$ *such that* $\mathbf{P} \cong \mathbf{A}u$ *(and* u *is compact in* $\mathbf{A}u$*).*

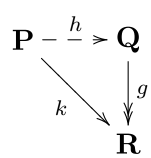
Proof. In view of Theorem 8.8, it suffices to show that u is compact in $\mathbf{A}u$, if $\mathbf{P}$ is finitary. If γ is as in the proof of the preceding result, the generator $\gamma(e)$ of $\mathbf{A}_\gamma$ is compact, by the same result. The isomorphism between $\mathbf{A}_\gamma$ and $\mathbf{A}u$ (see proof of Theorem 8.8) sends $\gamma(e)$ to u. Thus, u is compact in $\mathbf{A}u$. □

In what follows, we will use the term *regular module* for a projective, finitary cyclic $\mathbf{A}$-module for which the element u is compact in $\mathbf{A}$ – in the notation of Lemma 9.4.

Lemma 9.5. *The* $\mathcal{P}(\mathbf{\Sigma}_\mathcal{L})$*-modules* $\mathcal{P}(\mathbf{Fm}_\mathcal{L})$ *and* $\mathcal{P}(\mathbf{Eq}_\mathcal{L})$ *are regular.*

Proof. It was noted in the proof of Corollary 8.9 that $u = \{\kappa_x\}$ for $\mathcal{P}(\mathbf{Fm}_\mathcal{L})$ and $u = \{\kappa_{x\approx y}\}$ for $\mathcal{P}(\mathbf{Eq}_\mathcal{L})$, both of which are finite, hence compact. □

Lemma 9.6. *Suppose the* $\mathbf{A}$*-modules* $\mathbf{P}, \mathbf{Q}, \mathbf{R}$*, the module morphism* $k : \mathbf{P} \to \mathbf{R}$*, and the surjective module morphism* $g : \mathbf{Q} \to \mathbf{R}$ *are all finitary. Suppose, further, that* $\mathbf{P}$ *is regular. Then there exists a finitary module morphism* $h : \mathbf{P} \to \mathbf{Q}$ *such that* $gh = k$*.*



Proof. In view of the preceding result, we may assume that $\mathbf{P} = \mathbf{A}u$, where u is an idempotent element of $\mathbf{A}$ that is compact in $\mathbf{A}u$. Consider the element $y = k(u)$ of $\mathbf{R}$. It is clear that y is a compact element of $\mathbf{R}$. We claim that there exists compact w in $\mathbf{Q}$ such that $y = g(w)$. Indeed there exists x in $\mathbf{Q}$ such that $y = g(x)$. Now, $x = \bigvee^{\mathbf{Q}} X$, for some set X of compact elements of $\mathbf{Q}$, and so $g(x) = \bigvee^{\mathbf{R}} g[X]$. By the compactness of y in $\mathbf{R}$, there exists a finite subset Y of X such that $\delta(x) = \bigvee^{\mathbf{R}} g[Y]$. But then, if w denotes the compact element $\bigvee^{\mathbf{Q}} Y$ in $\mathbf{Q}$, we get $y = g(w)$, as was to be shown. Let

$z = u \star_{\mathbf{Q}} w$. Then z is a compact element of $\mathbf{Q}$. We claim that the map $\tau_z : P \to Q$, defined by $au \mapsto a \star_{\mathbf{Q}} z$, is a finitary module morphism from $\mathbf{P}$ to $\mathbf{Q}$ such that $g\tau_z = k$.

We first note that τ_z is a well-defined map. Indeed, suppose that $au = bu$, for $a, b \in A$. Then $a \star_{\mathbf{Q}} z = a \star_{\mathbf{Q}} (u \star_{\mathbf{Q}} w) = (au \star_{\mathbf{Q}} w) = (bu \star_{\mathbf{Q}} w) = \ldots = b \star_{\mathbf{Q}} z$. We next show that τ_z is residuated. We have for all $a \in A$ and $q \in Q$, $\tau_z(au) \leq q \Rightarrow a \star_{\mathbf{Q}} z \leq q \Rightarrow a \leq q/_{\mathbf{Q}} z \Rightarrow au \leq (q/_{\mathbf{Q}} z)u \Rightarrow (au) \star_{\mathbf{Q}} z \leq ((q/_{\mathbf{Q}} z)u) \star_{\mathbf{Q}} z \Rightarrow a \star_{\mathbf{Q}} z \leq (q/_{\mathbf{Q}} z) \star_{\mathbf{Q}} z \Rightarrow a \star_{\mathbf{Q}} z \leq q$. We have shown that $\tau_z(au) \leq q$ iff $au \leq (q/_{\mathbf{Q}} z)u$. Thus, τ_z is residuated and its residual is the map $(\tau_z)_* : \mathbf{Q} \to \mathbf{P}$, defined by $(\tau_z)_*(q) = (q/_{\mathbf{Q}} z)u$. To prove that τ_z is a module morphism, consider $a, b \in A$. We have, $a\tau_z(bu) = a \star_{\mathbf{Q}} (b \star_{\mathbf{Q}} z) = (ab) \star_{\mathbf{Q}} z = \tau_z(a(bu))$.

It remains to verify that τ_z is finitary. To this end, we first verify that for every compact element au of $\mathbf{A}u$ there exists a compact element c of $\mathbf{A}$ such that $au = cu$. Indeed, since $\mathbf{A}$ is finitary, $a = \bigvee^{\mathbf{A}} C$, where C is the set of compact elements of $\mathbf{A}$ below a. Thus, $au = \bigvee^{\mathbf{A}}\{cu : c \in C\} = \bigvee^{\mathbf{A}u}\{cu : c \in C\}$. Note that $\{cu : c \in C\}$ is a directed set of compact elements of $\mathbf{A}u$. Since au is compact in $\mathbf{A}u$, there exists $c \in C$ such that $au = cu$, as was to be shown. Lastly, if cu is a compact element of $\mathbf{A}u$, with compact $c \in \mathbf{A}$, then $\tau_z(cu) = c \star_{\mathbf{Q}} z$, which is a compact element of $\mathbf{Q}$, since $\mathbf{Q}$ is finitary, c is a compact element of $\mathbf{A}$ and z is a compact element of $\mathbf{Q}$. □

Corollary 9.7. *Every isomorphism between consequence relations on the sets $\mathcal{P}(\mathbf{Fm}_{\mathcal{L}})$ and $\mathcal{P}(\mathbf{Eq}_{\mathcal{L}})$ is induced by a finitary translator.*

We will not need the following result, but we state it since it is relevant to our discussion. Its proof follows ideas similar to the ones in the proof of the Lemma 9.6. Let $\mathbf{Q}$ be an $\mathbf{A}$-module and $a \in A$. An element y of Q is called *a-invariant*, if $a \star y = y$.

Theorem 9.8. *Assume that the $\mathbf{A}$-module $\mathbf{P}$ is cyclic projective with respect to the elements v and u, and that $\mathbf{Q}$ is also an $\mathbf{A}$-module. Then, there is a bijection between module morphisms τ from $\mathbf{P}$ to $\mathbf{Q}$ and u-invariant elements y of Q, given by $\tau \mapsto \tau(v)$ and $y \mapsto \tau_y$, where $\tau_y(x) = (x/v) \star y$.*

Lemma 9.9. *Let $\mathbf{P}_i$ be finitary lattices, for all $i \in I$. An element of $\prod_{i\in I} \mathbf{P}_i$ is compact iff it has finitely many non-zero coordinates and those are occupied by compact elements of the corresponding factors.*

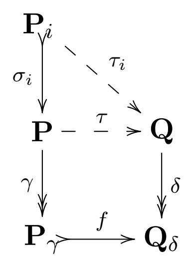
Proof. Let x_i be a compact element of $\mathbf{P}_i$, for some $i \in I$, and let $\bar{x}_i$ be the element of $\mathbf{P}$ with i-th coordinate equal to x_i and all other coordinates equal to $\perp$. Clearly, $\bar{x}_i$ is compact in $\mathbf{P}$, as any directed join exceeding it contains elements with all but the i-th coordinate equal to $\perp$. The directed join in $\mathbf{P}_i$ of the elements in the i-th coordinate exceed x_i, so one of them exceeds x_i. The corresponding element of $\mathbf{P}$ exceeds $\bar{x}_i$. Since the finite join of compact elements is also compact, we have one direction of the lemma.

Conversely, assume that $x = (x_i)_{i\in I}$ is a compact element of $\mathbf{P}$. Clearly, $x = \bigvee_{i\in I} \bar{x}_i$, so there is a finite subset I_0 of I such that $x = \bigvee_{i\in I_0} \bar{x}_i$. □

We are now ready to prove the main result of this section.

Theorem 9.10. *Let* $\mathbf{P}$ *be the coproduct of a family of regular* $\mathbf{A}$*-modules,* $\mathbf{Q}$ *an* $\mathbf{A}$*-module,* γ *a structural closure operator on* $\mathbf{P}$*,* δ *a finitary structural closure operator on* $\mathbf{Q}$ *and* f *a finitary representation of* γ *in* δ*. Then,* f *is induced by a finitary module morphism* $\tau : \mathbf{P} \to \mathbf{Q}$*.*

Proof. For each $i \in I$, let $\sigma_i : \mathbf{P}_i \to \mathbf{P}$ be the injective module morphism associated with the coproduct.



Note that the map $f\gamma\sigma_i$ is a finitary module morphism, and hence Lemma 9.6 implies that there exists a finitary module morphism $\tau_i : \mathbf{P}_i \to \mathbf{Q}$ such that $f\gamma\sigma_i = \delta\tau_i$. Now by the universal property of the coproduct, there exists $\tau : \mathbf{P} \to \mathbf{Q}$ such that $\tau\sigma_i = \tau_i$, for all $i \in I$.

To show that τ is finitary, let $x = (x_i)_{i \in I}$ be a compact element of $\mathbf{P}$. By Lemma 9.9, there is a finite subset I_0 of I such that $x_j = \bot$ for all $j \notin I_0$, and x_i is compact in $\mathbf{P}_i$, for all $i \in I_0$. Since τ_i is compact, $\tau_i(x_i)$ is compact in $\mathbf{Q}$, for $i \in I_0$. Also, $\tau_j(x_j) = \tau_j(\bot) = \bot$, for $j \notin I_0$. Therefore, by Lemma 8.10 we have, $\tau((x_i)_{i \in I}) = \bigvee_{i \in I} \tau_i(x_i) = \bigvee_{i \in I_0} \tau_i(x_i)$, which is compact, being a finite join of compact elements. □

Corollary 9.11. *Let* $\mathbf{P}$*,* $\mathbf{Q}$ *be each a coproduct of regular* $\mathbf{A}$*-modules, and let* γ*,* δ *be finitary structural closure operators on* $\mathbf{P}$ *and* $\mathbf{Q}$*, respectively. Then every equivalence of* γ *and* δ *is induced by finitary module morphism.*

In view of Theorem 8.12, Corollary 9.5 and Corollary 9.11, we have the following result.

Corollary 9.12. *Every isomorphism between consequence relations on the* $\mathcal{P}(\mathbf{\Sigma}_{\mathcal{L}})$*-modules* $\mathcal{P}(\mathbf{Seq}_{\mathcal{L}\mathcal{P}_1})$ *and* $\mathcal{P}(\mathbf{Seq}_{\mathcal{L}\mathcal{P}_2})$ *is induced by a finitary translator, where* $\mathcal{L}$ *is any algebraic language and* $\mathcal{P}_1$ *and* $\mathcal{P}_2$ *are any predicate-only languages.*

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Department of Mathematics, University of Denver, 2360 S. Gaylord St., Denver, CO 80208, USA

E-mail address: ngalatos@du.edu

URL: http://www.math.du.edu/~ngalatos

Department of Mathematics, Vanderbilt University, 1326 Stevenson Center, Nashville, TN 37240, USA

E-mail address: Constantine.Tsinakis@vanderbilt.edu

URL: http://www.math.vanderbilt.edu/people/tsinakis