REDUCED DECOMPOSITIONS WITH ONE REPETITION AND PERMUTATION PATTERN AVOIDANCE

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Abstract. In 2007, Tenner established a connection between pattern avoidance in permutations and the Bruhat order on permutations by showing that the downset of a permutation in the Bruhat order is a Boolean algebra if and only if the permutation is 3412 and 321 avoiding. Tenner mentioned, but did not prove, that if the permutation is 321 avoiding and contains exactly one 3412 pattern, or if the permutation is 3412 avoiding and contains exactly one 321 pattern, then there exists a reduced decomposition with precisely one repetition. This property actually characterizes permutations with precisely one repetition. The goal of this paper is to prove this equivalence as a first step in our program to understand Bruhat downsets by means of pattern avoidance.

1. Notation, Basic Definitions and Previous Results

We will denote the set of all permutations of $\{1,\dots,n\}$ by S_n and we will write an element $\pi \in S_n$ in one-line notation as $\pi = \pi_1\pi_2\dots\pi_n$, where the image of i under π is π_i. So, in the permutation 3214, $\pi(1) = 3$, $\pi(2) = 2$, etc. We compose permutations from left to right.

Definition 1.1. We say a permutation $\pi = \pi_1\pi_2\dots\pi_n \in S_n$ *contains a permutation* $\sigma = \sigma_1\dots\sigma_m \in S_m$ *as a pattern* if there exist $i_1 < i_2 < \cdots < i_m$ such that $\pi_{i_j} < \pi_{i_k}$ if and only if $\sigma_j < \sigma_k$. If π does not contain σ then we say π *avoids* σ.

Example 1.2. The permutation 7651324 contains the pattern 321, but avoids 231.

Permutation patterns are well-studied and we refer the interested reader to [5] or [2] for more information. We will now briefly review some of the results on reduced decompositions that are important here. Recall that S_n is generated by the transpositions $(i, i+1)$ for $1 \le i < n$.

Definition 1.3. A *reduced decomposition* of $\pi \in S_n$ is a word $j_1\dots j_l$ where each j_k is a transposition of the form $(i, i+1)$ such that $\pi = j_1\dots j_l$ and l is as small as possible.

Example 1.4. One reduced decomposition of 43152 is (34)(23)(12)(34)(23)(45)

We will, from now on, condense the notation of reduced decompositions and just write the first element i of the transposition $(i, i+1)$ in reduced decompositions. In the previous example, the reduced decomposition will be condensed to [321324]. We put brackets around the reduced decompositions to distinguish them from the actual permutations.

Note, reduced decompositions of permutations are not unique and it is a well known result [4] that any reduced decomposition of π may be obtained from any other by the use of the two braid moves:

$$[ij] = [ji] \qquad \text{if } |i-j| > 1$$
$$[i(i+1)i] = [(i+1)i(i+1)] \qquad \text{for all } i$$

We will denote by $R(\pi)$ the set of all reduced decompositions of π and use bold lower-case letters to refer to an element of $R(\pi)$. Ex: $\mathbf{j} \in R(\pi)$, $\mathbf{j} = [j_1\dots j_l]$.

The number of letters in any reduced decomposition is called the *length* of the permutation and corresponds to the number of *inversions*, or occurrences of 21 patterns, in the permutation. We use $l(\pi)$ to denote the length of the permutation π.

Definition 1.5. A *subword* of a reduced decomposition $[j_1 \dots j_l]$ is a word $[j_{i_1} \dots j_{i_k}]$ such that $1 \leq i_1 \leq \cdots \leq i_k \leq l$. A *factor* of a reduced decomposition is a subword such that $i_q = i_{q-1} + 1$ for all $1 < q \leq k$ (the letters of the subword occur consecutively in the reduced decomposition).

We now define the Bruhat order on the symmetric group.

Definition 1.6. Given π, $\sigma \in S_n$, and a reduced decomposition for $\sigma = [j_1 \dots j_q]$, we define $\pi \leq \sigma$ if and only if there exists a reduced decomposition $[j_{i_1} \dots j_{i_k}] = \pi$ with $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq q$. This order is called the *(Strong) Bruhat Order.*

Example 1.7. One can see $42135 \leq 45123$ by looking at any reduced decomposition for 45123 and finding in it a subword which is a reduced decomposition for 42135. One reduced decomposition for 45123 is [**321**4**32**]. Looking at the elements in bold we have the subword [3212] which is a reduced decomposition for 42135.

The Bruhat order has many properties and many equivalent definitions. In the specific case of permutations, the Bruhat order may also be viewed as $\sigma \leq \pi$ if σ can be obtained from π by applying a sequence of operations each of which interchanges two elements of an inversion.

Example 1.8. To show $42135 \leq 45123$, begin with 45123, interchanging 5 and 2 gives 42153, and then interchanging 5 and 3 gives 42135 .

Since our ultimate goal in this paper is to analyze the permutations that have a reduced decomposition with exactly one repeated element and to characterize them based on the patterns they avoid and contain, we now note some earlier results involving reduced decompositions and permutation patterns due to Tenner. For proofs of these results, see [7].

Definition 1.9. A permutation is *vexillary* if it avoids 2143.

Definition 1.10 (Tenner)**.** Let $[j_1 \dots j_l]$ be a reduced decomposition of $\pi = \pi_1\pi_2 \dots \pi_n$ and let $M \in \mathbb{N}$. The *shift* of $[j_1 \dots j_l]$ by M is the reduced decomposition $[(j_1 + M)(j_2 + M) \dots (j_l + M)]$ of the permutation $1 \dots M(\pi_1 + M)(\pi_2 + M) \dots (\pi_n + M)$.

Theorem 1.11 (Tenner)**.** The permutation π is vexillary if and only if for every permutation σ containing a π-pattern there exists a reduced decomposition of σ containing some shift of a reduced decomposition of π as a factor.

Definition 1.12 (Tenner)**.** Let $\pi \in S_n$ and $\mathbf{i} \in R(\pi)$. Write $\mathbf{i} = \mathbf{abc}$ where $\mathbf{a} \in R(\sigma)$ and $\mathbf{c} \in R(\tau)$. Suppose $\mathbf{b}$ contains only letters in $S = \{1 + M, 2 + M, \dots, k - 1 + M\}$ for some $M \in \mathbb{N}$. If no element of $R(\sigma)$ has an element of S as its rightmost character and no element of $R(\tau)$ has an element of S as its leftmost character, then $\mathbf{b}$ is *isolated* in $\mathbf{i}$.

Example 1.13. Consider the permutation 351462 with reduced decomposition [4213245]. 2132 is isolated, but 13 is not.

Lemma 1.14 (Tenner)**.** If a reduced decomposition of π contains an isolated shift of a reduced decomposition of σ, then π contains σ.

Corollary 1.15 (Tenner)**.** The converse is true if σ is vexillary.

2. Additional Connections Between Permutation Patterns and Reduced Decompositions

We now prove some preliminary results that will aid us in our analysis of reduced decompositions of permutations with exactly one repeated letter. Note that if π is a permutation and $\tau = (\tau_1, \tau_2)$ is a transposition then $\pi\tau$ will be the permutation π with the two elements in positions τ_1 and τ_2 switched.

Example 2.1. If we have the reduced decomposition [2132] in S_4 then the effect this decomposition will have on the identity permutation is as follows: $1234 \to 1324 \to 3124 \to 3142 \to 3412$. [2132] is a reduced decomposition of 3412.

Lemma 2.2. Let $\pi \in S_n$ have reduced decomposition $[j_1 \dots j_l]$. Let x and y be the two values in positions j_i and $j_i + 1$, respectively, when j_i is applied. Then $x < y$. In particular, if $\pi_a > \pi_b$ for some $a < b$, then there is precisely one j_i that interchanged the values π_a and π_b.

Proof. If $x > y$ when j_i is applied, the permutation $[j_1 \dots j_i]$ would have one less inversion than $[j_1 \dots j_{i-1}]$ and thus it would not be reduced, a contradiction. The rest is clear, since the two values $\pi_a > \pi_b$ have to switch positions at one point. □

Lemma 2.3. Let $\mathbf{j} = [j_1 \dots j_l]$ be a reduced decomposition of $\pi \in S_n$. If the permutation with reduced decomposition $[j_1 \dots j_m]$ $(m < l)$ has p occurrences of 321, then π has at least p occurrences of 321.

Proof. It is enough to demonstrate that if $[j_1 \dots j_m]$ is a reduced decomposition with p 321 patterns, then $[j_1 \dots j_{m+1}]$ has at least p 321 patterns. j_{m+1} switches positions j_{m+1} and $j_{m+1} + 1$, so assume x is in position j_{m+1} and y is in position $j_{m+1} + 1$ before the application of the transposition (j_m, j_{m+1}). $x < y$ by Lemma 2.2. Any 321 pattern in $[j_1 \dots j_m]$ will occur in $[j_1 \dots j_{m+1}]$ because interchanging two values in consecutive positions will not affect any existing 321 patterns. □

Lemma 2.4. Let $\mathbf{j} = [j_1 \dots j_q]$ be a reduced decomposition of $\pi \in S_n$ and let $m < q$. If $[j_1 \dots j_m]$ has a 3412 pattern, then π has the same 3412 pattern or positions of the 3412 pattern are interchanged to create at least two 321 patterns.

Proof. Consider the permutation $[j_1 \dots j_m]$ with a 3412 pattern. If none of the elements in the 3412 pattern are interchanged by $[j_{m+1} \dots j_q]$, then π will have a 3412 pattern. If elements of the 3412 pattern are interchanged, then either the "34" or the "12" must be interchanged first. If j_k, $k > m$, interchanges the "34", then $[j_1 \dots j_m \dots j_k]$ has a 4312 pattern, which has two 321 patterns. If j_k, $k > m$, interchanges the "12", then $[j_1 \dots j_m \dots j_k]$ has a 3421 pattern which also has two 321 patterns. By Lemma 2.3, this means π has at least two 321 patterns. □

Lemma 2.5. If there exists a reduced decomposition of π which contains factors $i(i+1)i$ and $j(j+1)j$, for $i \neq j$, then π contains at least two 321 patterns.

Proof. Without loss of generality, assume $i(i+1)i$ occurs before $j(j+1)j$ in the reduced decomposition, so the reduced decomposition looks like $[\dots i(i+1)i \dots j(j+1)j \dots]$. Consider the permutation σ formed by the elements to the left of $i(i+1)i$ and assume $\sigma_i = a$, $\sigma_{i+1} = b$ and $\sigma_{i+2} = c$. Lemma 2.2 implies that $a < b < c$. After applying $i(i+1)i$, cba will be a 321 pattern in σ. Now consider the permutation σ' formed by all elements in the reduced decomposition occurring before $j(j+1)j$. By Lemma 2.3, σ' has at least one 321 pattern. Assume $\sigma'_j = d$, $\sigma'_{j+1} = e$ and $\sigma'_{j+2} = f$. As before, we have $d < e < f$. Apply $j(j+1)j$ to σ' to get σ'' and note that Lemma 2.2 implies cba still forms a 321 pattern in σ'' (if it did not then two of a, b and c would have to interchange positions at least twice). If $\{a,b,c\} \cap \{d,e,f\} = \emptyset$ then fed forms another 321 pattern in σ'' completely disjoint from the 321 pattern cba. If $\{a,b,c\} \cap \{d,e,f\} \neq \emptyset$, then $|\{a,b,c\} \cap \{d,e,f\}| = 1$ since $d < e < f$ and so switching two nonshared numbers will not affect the 321 pattern cba and we will have two 321 patterns in σ''.. By Lemma 2.3, we have at least two 321 patterns in π. □

3. Trajectories

Definition 3.1. Given a reduced decomposition $[j_1 \dots j_l]$ of $\pi \in S_n$ and an element $x \in \{1 \dots n\}$, define the *trajectory of* x as the sequence of positions at which x is found by applying the permutations $()$, $[j_1]$, $[j_1 j_2]$, … $[j_1 \dots j_l]$ to x.

Example 3.2. Consider the permutation 43251 with reduced decomposition [3213234]. We will map out the trajectory of 3.

position of 3	permutation
3	()
4	[3]
4	[32]
4	[321]
3	[3213]
2	[32132]
2	[321323]
2	[3213234]

Therefore the trajectory of 3 is the sequence $< 3, 4, 4, 4, 3, 2, 2, 2 >$.

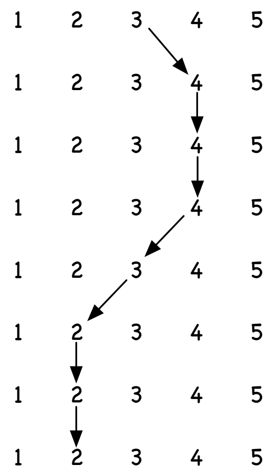

FIGURE 1

Here is a picture of all of the trajectories of 43251:

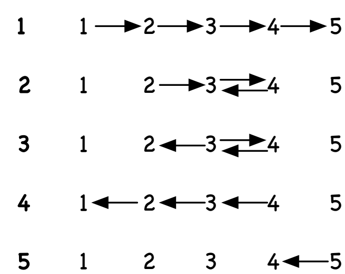

FIGURE 2. Trajectories of 43251

Our trajectories in Figure 2 are just a "flattening" of the standard braid diagram in Figure 1.

Note, if we have trajectories for all points $x \in \{1 \dots n\}$ for a permutation π, we can project them onto the x-axis and count the multiplicities of arrows coming in to position i from the right and leaving position i moving to the right. We denote this multiplicity m_i. Also note that arrows come in pairs, i.e., for each arrow that moves a point to the right, there must exist an arrow in the same position moving to the left.

The multiplicities in the example permutation are: $m_1 = 2$, $m_2 = 4$, $m_3 = 6$, and $m_4 = 2$.

An immediate observation is:

Lemma 3.3. *i is repeated k times in a reduced decomposition if and only if $m_i = 2k$.*

4. One Repetition of the Form $i(i+1)i$

We will now consider the case where we have a permutation $\pi \in S_n$ with a reduced decomposition with a factor of the form $i(i+1)i$ for some $i \in \{1, \ldots, n-1\}$ and no other repetitions. We will show that such permutations are precisely those that contain exactly one copy of 321 and avoid 3412.

Consider a permutation which contains exactly one 321 pattern and avoids 3412. The vexillary characterization of Tenner (Theorem 1.11), implies that there exists a reduced decomposition of π containing $i(i+1)i$ as a factor. Call this reduced decomposition $\mathbf{j}$. In any theorem or lemma that uses a permutation that contains exactly one 321 and avoids 3412, we will always assume $\mathbf{j}$ is the reduced decomposition of the appropriate form guaranteed by Theorem 1.11.

Lemma 4.1. *If $\pi \in S_n$ contains exactly one 321 pattern and avoids 3412, then in the reduced decomposition $\mathbf{j}$ there is no $j \in \{1, \ldots, n\} \setminus \{i, i+1\}$ appearing twice on the left or twice on the right of the factor $i(i+1)i$.*

Proof. Assume the condition of the lemma does not hold. Without loss of generality, assume the repeated element j appears twice on the left of $i(i+1)i$ (the argument will be analogous for the other case), so $\mathbf{j}$ has the form $[\ldots j \ldots j \ldots i(i+1)i \ldots]$. Let us pick the repeated element j such that the length of the factor $j \ldots j$ is minimal. It is not hard to see that there are only four possible factors that can occur between the two occurrences of j. We have either $j(j+1)(j-1)j$, $j(j-1)(j+1)j$, $j(j-1)j$ or $j(j+1)j$. If we have either $j(j+1)(j-1)j$ or $j(j-1)(j+1)j$, then consider the permutation formed by the first elements of $\mathbf{j}$ up through the second j. Lemma 2.4 gives that π contains either 3412 or multiple 321 patterns, both of which are contradictions. If we have a factor of the form $j(j+1)j$ or $j(j-1)j$, then Lemma 2.5 implies π has multiple 321 patterns, a contradiction. □

Lemma 4.2. *If $\pi \in S_n$ contains exactly one 321 pattern and avoids 3412, then in the reduced decomposition $\mathbf{j}$, neither i nor $i+1$ appears outside of the factor $i(i+1)i$.*

Proof. Again, we assume the repeated element occurs to the left of the factor $i(i+1)i$ in $\mathbf{j}$. $\mathbf{j}$ will have the form $[\ldots x \ldots i(i+1)i \ldots]$ where $x = i$ or $x = i+1$. There may be many occurrences of i or $i+1$ to the left of $i(i+1)i$, so let us pick the one such that the length of the factor $x \ldots i(i+1)i$ is minimal and apply braid moves to minimize the factor $x \ldots i(i+1)i$ as much as possible. Note that there has to be at least one element in between x and i, since neither $ii(i+1)i$ nor $(i+1)i(i+1)i$ is reduced. If there is one element in between x and i, the possibilities are $i(i-1)i(i+1)i$ or $(i+1)(i+2)i(i+1)i$. Applying a braid move to the first case gives $(i-1)i(i-1)(i+1)i$ which means π has a reduced decomposition with $i(i-1)(i+1)i$ as a factor. It was noted in Lemma 4.1 that such a factor leads to a contradiction. The second case has the factor $(i+1)(i+2)i(i+1) = (i+1)i(i+2)(i+1)$, which also gives a contradiction by Lemma 4.1.

Thus the number of elements between x and i is greater than one. First, consider the case where $x = i$. We can assume there are no extra occurrences of $i+1$ in between x and i because if there were, we could apply the braid move $i(i+1)i = (i+1)i(i+1)$ and obtain a factor $(i+1) \ldots (i+1)i(i+1)$ with a smaller number of elements in between the $(i+1)$'s. So, the factor of the reduced decomposition we are interested in is of the form $it_1 \ldots t_m i(i+1)i$ where $i \neq t_j \neq i+1$ for all t_j. $t_1 = i-1$ and since there are no repetitions among the t_j, we must have $t_m = i+2$. Continuing in this fashion we have $t_2 = i-2$, $t_{m-1} = i+3$, etc. At some point we must stop. If m is even, we have $it_1 \ldots t_m i(i+1)i = i(i-1)(i-2) \ldots (i-k)(i-k-1)(i+k) \ldots (i+2)i(i+1)i$. If m is odd, we have $it_1 \ldots t_m i(i+1)i = i(i-1)(i-2) \ldots (i-k)(i+k) \ldots (i+2)i(i+1)i$. In either case, apply braid moves to obtain the factor $i(i-1)i(i+1)i$, which is impossible by the first paragraph. The case where $x = i+1$ is similar (apply the braid move $i(i+1)i = (i+1)i(i+1)$ and look at the factor $(i+1)t_1 \dots t_m(i+1)i(i+1)$. □

Lemma 4.3. If $\pi \in S_n$ contains exactly one copy of 321 and avoids 3412, then no factor of the form $ji(i+1)ij$ can occur in $\mathbf{j}$.

Proof. Note, j cannot be moved through $i(i+1)i$ by braid moves, so $j = i-1$ or $j = i+2$; therefore, we only need to check these two possibilities for j. Note one can apply braid moves to the factor, $(i+2)i(i+1)i(i+2)$ to get the factor $i(i+1)(i+2)(i+1)i$ which is the same ordering of elements as $(i-1)i(i+1)i(i-1)$. It therefore suffices to check that no factor of the form $(i-1)i(i+1)i(i-1)$ can occur in $\mathbf{j}$. Consider the permutation π' formed by all the elements before the factor $(i-1)i(i+1)i(i-1)$ in $\mathbf{j}$ and consider applying the factor $(i-1)i(i+1)i(i-1)$ to π'. If $\pi'_{i-1} = a$, $\pi'_i = b$, $\pi'_{i+1} = c$, and $\pi'_{i+2} = d$, then applying $(i-1)i(i+1)i(i-1)$ to π' gives π'' where $\pi''_{i-1} = d$, $\pi''_i = b$, $\pi''_{i+1} = c$ and $\pi''_{i+2} = a$, as described in the following table. Recall that Lemma 2.2 forces order relations between elements every time a transposition in a reduced decomposition is applied to a permutation π.

i-1	i	i+1	i+2	action	$<$
a	b	c	d	initial positions	—
b	a	c	d	apply $i-1$	a $<$ b
b	c	a	d	apply i	a $<$ c
b	c	d	a	apply $i+1$	a $<$ d
b	d	c	a	apply i	c $<$ d
d	b	c	a	apply $i-1$	b $<$ d

Thus the ordering of a, b, c, and d is determined except for the relation between b and c. If $b < c$, then $a < b < c < d$, and we have a 4231 pattern which gives π at least two 321 patterns. If $c < b$ then $a < c < b < d$, and we have a 4321 pattern which gives π at least six 321 patterns. Both are contradictions with π having exactly one 321 pattern. □

Lemma 4.4. Assume π contains exactly one 321 pattern and avoids 3412. Then, there can be no element j which occurs both to the left and right of the factor $i(i+1)i$ in $\mathbf{j}$.

Proof. Note, by Lemma 4.2, $j \neq i$ and $j \neq i+1$. We may also assume at this point, by Lemmas 4.1 and 4.2, that there are no repeated elements between the two occurrences of j other than those in the factor $i(i+1)i$. Since braid moves cannot move $i(i+1)i$ to the left of j, we must have an occurrence of $i-1$ or $i+2$ to the left of $i(i+1)i$. Similarly, we must have an occurrence of $i-1$ or $i+2$ to the right of $i(i+1)i$. If $i-1$ occurs both to the left and right, then $j = i-1$. Similarly if $i+2$ occurs on both sides. The only other possibility is that $i-1$ occurs on one side and $i+2$ occurs on the other. Assume $i-1$ occurs on the left and $i+2$ occurs on the right. The factor then looks like $j \dots (i-1) \dots i(i+1)i \dots (i+2) \dots j$. Note that neither $i-1$ nor $i+2$ can be moved outside of the j's by braid moves, therefore our reduced decomposition looks like $j \dots (i-2) \dots (i-1) \dots i(i+1)i \dots (i+2) \dots (i+3) \dots j$. We can continue this reasoning and force on the left $j = i-k$ for some k and on the right $j = i+k'$ for some k', a contradiction.

We will now consider all factors of the form $(i-1)t_1 \dots t_m i(i+1)iu_1 \dots u_p(i-1)$. For convenience let us assume that all elements that can be moved to the left of $i(i+1)i$ by braid moves have been so moved. We claim there can be no u_j's at all in this factor. To see this, consider u_1. Since u_1 cannot be moved to the left of $i(i+1)i$, $u_1 = i-1$ or $u_1 = i+2$. If the former, then we're done, so assume $u_1 = i+2$. This means $u_k = i+k+1$ for all $2 \leq k \leq p$ and all of these u_k can be moved out to the right of $(i-1)$ by braid moves. Our factor is now of the form $(i-1)t_1 \dots t_m i(i+1)i(i-1)$. t_1 must be $i-2$, $t_2 = i-3$, ..., $t_m = i-m-1$. Any factor of this form can be reduced to a factor of the form $(i-1)(i-2)i(i+1)i(i-1)$.

One may make a similar argument for $j = i+2$ and so we must only verify that $(i-1)(i-2)i(i+1)i(i-1)$ and $(i+2)(i+3)i(i+1)i(i+2)$ cannot occur as a factor in a reduced decomposition of π. This is easily verified by an argument similar to that of Lemma 4.3 and we shall leave it to the reader. □

Lemmas 4.1, 4.2 and 4.4 combine to yield:

Proposition 4.5. If a permutation $\pi \in S_n$ contains exactly one 321 pattern and avoids 3412, then there exists a reduced decomposition of π containing $i(i+1)i$ as a factor with no other repetitions.

We now consider what patterns are contained and avoided by a permutation $\pi \in S_n$ with reduced decomposition $\mathbf{j} = [j_1 \dots j_l]$ where $\mathbf{j}$ has a factor of the form $i(i+1)i$ and no other repetitions. In such a $\mathbf{j}$, Theorem 1.11 implies the existence of at least one 321 pattern. Moreover, since there are no other repetitions, there cannot be a 3412 pattern, as we cannot transform $\mathbf{j}$ into a reduced decomposition with factor $j(j+1)(j-1)j$. We now show there cannot be more than one 321 pattern in π.

Lemma 4.6. If cba forms a 321 pattern in $\pi = \pi_1 \dots \pi_n$, then b must have reversed direction in its trajectory and so there exists an i such that $m_i > 2$.

Proof. cba is a 321 pattern so $a < b < c$. At some point a and b must switch positions, so this means that b must travel to the left in its trajectory. Similarly, at some point b and c must switch positions so this means that b must travel to the right in its trajectory. This means that b reversed directions at some position k and $m_i \geq 4$ follows by Lemma 3.3. □

Lemma 4.7. Assume π has a reduced decomposition with only one repetition. There cannot be cba and fed with $b \neq e$ as 321 patterns in π.

Proof. If there were such 321 patterns, then b and e must turn at some point in their trajectories by Lemma 4.6, so either there exists $m_i > 2$ and $m_j > 2$ with $i \neq j$ (this corresponds to b and e turning at different positions) or there exists $m_i > 4$ (which corresponds to b and e turning at the same position). In either case we have a contradiction, for Lemma 3.3 gives either two elements repeated or an element repeated at least three times. A word should be said about why $m_i \neq 4$ in the second case. If $m_i = 4$, then b and e switch positions at position i twice, a contradiction by Lemma 2.2. □

Lemma 4.8. Assume π has a reduced decomposition with only one repetition. Then 321 appears at most once in π.

Proof. Assume there are at least two 321 patterns, cba and fed, in π. If $b \neq e$, then we're done by Lemma 4.7. Therefore our two 321 patterns are cba and fbd and either $c \neq f$ or $a \neq d$. Assume $c \neq f$ and $c > f$ (one can make an analogous argument if $a \neq d$). If b reverses directions twice in its trajectory then we are done, so assume b reverses directions only once. We have $a < b < f < c$. a and b must switch positions so b must go left at least once in its trajectory. b must also switch positions with c and f so b must go right at least twice. Since $m_i > 2$ for exactly one i, we can only have that b goes left once, turns and goes right for at least two positions. b going to the left at the beginning means that a and b switched at that position. a itself must also switch positions with c, so a must go to the right at least once more, a contradiction with only one multiplicity greater than 2. □

Thus:

Proposition 4.9. If there exists a reduced decomposition of $\pi \in S_n$ containing $i(i+1)i$ as a factor and no other repetitions, then π contains exactly one 321 pattern and avoids 3412.

Propositions 4.5 and 4.9 yield the following result:

Theorem 4.10. $\pi \in S_n$ contains exactly one 321 pattern and avoids 3412 if and only if there exists a reduced decomposition of π containing $i(i+1)i$ as a factor and no other repetitions.

5. One Repetition of the Form $i(i+1)(i-1)i$

The goal of this section is to show that those permutations which have reduced decompositions with factors of the form $i(i+1)(i-1)i$ and no other repetitions are precisely those that avoid 321 and contain exactly one copy of 3412.

Similarly to the previous section, we will associate with permutation $\pi \in S_n$ which contains 3412 and avoids 321, the reduced decomposition $\mathbf{j}$, guaranteed by the vexillary characterization (Theorem 1.11), which has a factor of the form $i(i+1)(i-1)i$. Again, we always assume $\mathbf{j}$ to be this reduced decomposition in the lemmas and theorems that follow.

Lemma 5.1. If $\pi \in S_n$ avoids 321 and contains exactly one 3412, then no element in the set $\{i, i+1, i-1\}$ occurs outside of the factor $i(i+1)(i-1)i$.

Proof. Assume not. Let us choose the element in the set $\{i, i+1, i-1\}$ which occurs closest to the factor $i(i+1)(i-1)i$ and occurs to the left of the factor (analogous argument for occurring to the right). Call this element a and let us consider each possibility for a.

(1) $a = i$. This implies we have a factor of the form $it_1 \dots t_m i(i+1)(i-1)i$ in $\mathbf{j}$ with each $t_j \neq i+1, i-1$, but t_1 can only be $i+1$ or $i-1$, a contradiction.

(2) $a = i+1$. This implies we have a factor of the form $(i+1)t_1 \dots t_m i(i+1)(i-1)i$. If $m = 0$, then we have $(i+1)i(i+1)(i-1)i$, but this gives a factor of the form $(i+1)i(i+1)$, which means π has a 321 pattern by Lemma 1.14. We cannot have $m > 1$ since that would mean $t_1 = i+2$, $t_m = i-2$, $t_2 = i+3$, ... and we would be able to move everything outside of the factor by braid moves with the exception of t_1. It therefore remains to consider the possibility of a factor of the form $(i+1)(i+2)i(i+1)(i-1)i$. Let π' be formed from the reduced decomposition consisting of all elements of $\mathbf{j}$ before the factor $(i+1)(i+2)i(i+1)(i-1)i$. Assume $\pi'_{i-1} = a$, $\pi'_i = b$, $\pi'_{i+1} = c$, $\pi'_{i+2} = d$ and $\pi'_{i+3} = e$. The application of $(i+1)(i+2)i(i+1)(i-1)i$ on π' will give π'' where $\pi''_{i-1} = d$, $\pi''_i = e$, $\pi''_{i+1} = a$, $\pi''_{i+2} = b$ and $\pi''_{i+3} = c$. The reduced decomposition forces the relations $c < d$, $c < e$, $b < d$, $b < e$, $a < d$, and $a < e$, so the relations between a and b, a and c, b and c, and d and e have not been determined. Consider what the relation could be between d and e. If $e < d$, then there must be a 321 pattern in $deabc$ which means π will have at least one 321 pattern. If $d < e$, then unless $a < b < c$, there will also be a 321 pattern in $deabc$. Now, $d < e$ and $a < b < c$ means that $deabc$ forms a 45123 pattern. 45123 contains two 3412 patterns and so either they remain in π, which is a contradiction, or Lemma 2.4 gives multiple 321 patterns which is also a contradiction.

(3) $a = i-1$. This implies a factor of the form $(i-1)t_1 \dots t_m i(i+1)(i-1)i$ in the reduced decomposition. If $m = 0$ we have $(i-1)i(i+1)(i-1)i = (i-1)i(i-1)(i+1)i$. This has $(i-1)i(i-1)$ as a factor so π would have a 321 pattern, a contradiction. If $m = 1$, then $(i-1)(i-2)i(i+1)(i-1)i$ is a possibility. One can verify using a method similar to the case $a = i+1$ that this is impossible, and similarly that $m > 1$ is also impossible.

□

Lemma 5.2. If π avoids 321 and contains exactly one 3412, then no element $j \in \{1, \dots, n\}$ can appear twice on one side of the factor $i(i+1)(i-1)i$ in $\mathbf{j}$.

Proof. Assume there exists a repetition that appears twice to the left of the factor $i(i+1)(i-1)i$. (there is an analogous argument for the right). This means there exists an additional factor of the form $j(j+1)j$ or $j(j+1)(j-1)j$ where $\{j, j+1, j-1\} \cap \{i, i+1, i-1\} = \emptyset$, by Lemma 5.1. There cannot be a factor of the form $j(j+1)j$ since the occurrence of such a pattern means the occurrence of a 321 pattern. So the only possibilities are a factor of the form $j(j+1)(j-1)j$. Applying a factor $j(j+1)(j-1)j$ in a reduced decomposition of a permutation that avoids 321 means that the elements in positions $j-1$, j, $j+1$ and $j+2$ after applying the factor must form a 3412 pattern (and hence formed a 1234 pattern before the application of the factor). If they do not, there will be a 321 pattern. This means that after applying the $j(j+1)(j-1)j$ factor there will be a 3412 pattern and after applying the $i(i+1)(i-1)i$ factor there will be another 3412 pattern. Hence there will be two 3412 patterns after the application of the $i(i+1)(i-1)i$. Lemma 2.4 implies that if the two 3412 patterns do not occur in π, there will be multiple 321 patterns in π. Either option gives a contradiction. □

Lemma 5.3. If π avoids 321 and contains exactly one 3412, then no element $j \in \{1, \dots, n\}$ can appear both on the left and right of the factor $i(i+1)(i-1)i$.

Proof. We use the same reasoning as in Lemma 4.4. $\mathbf{j}$ has the form $[\dots j \dots i(i+1)(i-1)i \dots j \dots]$. To keep $i(i+1)(i-1)i$ in between the j's, we must have that $j = i-2$ or $j = i+2$. One may then reduce the form of the factor $j \dots i(i+1)(i-1)i \dots j$ to four possibilities: $(i-2)i(i+1)(i-1)i(i-2)$, $(i-2)(i-3)i(i+1)(i-1)i(i-2)$, $(i+2)i(i+1)(i-1)i(i+2)$ and $(i+2)(i+3)i(i+1)(i-1)i(i-2)$. We eliminate $(i-2)i(i+1)(i-1)i(i-2) = i(i+1)(i-2)(i-1)(i-2)i$ and $(i+2)i(i+1)(i-1)i(i+2) = i(i+2)(i+1)(i+2)(i-1)i$ because they contain a factor of the form $j(j-1)j$. The other two may be eliminated by computing what happens to the elements switched by those elements, and observing that they either cause the occurrence of 321 patterns or two 3412 patterns.

□

Lemmas 5.1, 5.2, and 5.3 prove the following proposition.

Proposition 5.4. If $\pi \in S_n$ avoids 321 and contains exactly one 3412, then there exists a reduced decomposition of π with $i(i+1)(i-1)i$ as a factor with no other repetitions.

Consider what permutations are avoided and contained by a permutation $\pi \in S_n$ with reduced decomposition $\mathbf{j} = [j_1 \dots j_l]$ where $\mathbf{j}$ has a factor of the form $i(i+1)(i-1)i$ and no other repetitions. In such a $\mathbf{j}$, Theorem 1.11 implies the existence of at least one 3412 pattern, as we cannot transform $\mathbf{j}$ into a reduced decomposition with factor $j(j+1)j$. We now show there cannot be more than one 3412 pattern in π.

Let us consider what happens to the elements in positions $i-1$, i, $i+1$, $i+2$ at the time the factor $i(i+1)(i-1)i$ is applied on a permutation avoiding 321. Assume the elements in these positions are a, b, c and d.

The trajectories at these four positions are:

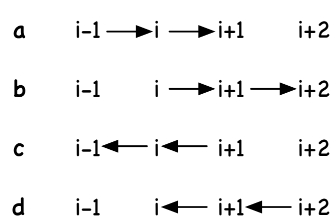

Figure 3

Note, because there can be no 321 patterns, the cdba pattern formed by the $i(i+1)(i-1)i$ factor must be a 3412 pattern in π.

Lemma 5.5. If π has a reduced decomposition with $i(i+1)(i-1)i$ as a factor and no other repetitions, then all trajectories are straight lines (no elements turn).

Proof. Assume an element turns. Let x turn at position $j+1$. x moved into position $j+1$ through the action of j in the reduced decomposition. In order for x to turn again, we must have another occurrence of j, therefore $i = j$; however, no element turns during the factor $i(i+1)(i-1)i$. □

Lemma 5.6. Assume only one repetition of the form $i(i+1)(i-1)i$ in a reduced decomposition of π. Then, if $cdab$ forms a 3412 pattern in π, $c = b + 1$.

Proof. Consider the trajectories and recall the trajectories of such a permutation must be straight lines by Lemma 5.5. Assume b must move to the right at least once before it switches positions with c. Now c and a must also switch positions, therefore either c must move to the right again which gives a position of multiplicity 4, or a must move to the left, which also gives a position of multiplicity 4. Either way, the current pattern is $cabd$. a cannot move any farther to the left because doing so would create two positions of multiplicity 4. Therefore a and d cannot switch. Contradiction. There is a similar argument if c has to move left at least once before switching with b. Therefore, b and c must have been next to each other to begin with and the result follows. □

It is worth noting at this point that since the factor $i(i+1)(i-1)i$ creates a 3412 pattern, then $b = i$ and $c = i + 1$ and also that a must remain in position $i + 1$ and d in position i.

Lemma 5.7. Assume $cdab$ and $ghef$ form 3412 patterns in π, then $c = g$ and $b = f$.

Proof. Assume not. Note that this means $c \neq g$ and $b \neq f$. Then $\{b, c\} \cap \{f, g\} = \emptyset$. We have by Lemma 5.6, that $c = b + 1$ and $g = f + 1$. b and c must switch positions at position b and f and g must switch positions at position f. a and d must swap positions and this cannot occur without at least one of them going through position b, so $m_b = 4$. Similarly h and e must swap positions and this cannot occur without one of them going through position f, so $m_f = 4$. $b \neq f$, so we have two positions of multiplicity 4. □

Lemma 5.8. There cannot exist two 3412 patterns in π of the form $cdab$ and $cfeb$.

Proof. Assume the factor $i(i+1)(i-1)i$ forms the $cdab$ pattern in π. We know that $m_b = 4$ by looking at the trajectory formed by the factor $i(i+1)(i-1)i$. Suppose there exists two extra elements e and f such that $cfeb$ forms a 3412 pattern (i.e. $e < b < c < f$). e and f must cross. One is on the left of position i and the other is on the right of position i. They cannot cross without one of them going through position i. This means $m_i = 6$. Contradiction. □

Lemmas 5.7 and 5.8 imply the following theorem:

Proposition 5.9. If $\pi \in S_n$ has a reduced decomposition with $i(i+1)(i-1)i$ as a factor and no other repetitions, then π contains exactly one 3412 pattern and avoids 321.

Propositions 5.4 and 5.9 yield the following theorem:

Theorem 5.10. $\pi \in S_n$ avoids 321 and contains exactly one 3412 pattern if and only if there exists a reduced decomposition of π containing $i(i+1)(i-1)i$ as a factor and no other repetitions.

6. Bruhat Intervals

To conclude the paper, recall that our motivation for studying reduced decompositions with one repetition was to better understand downsets in the Bruhat order in terms of pattern avoidance. In section 5 of [6], Tenner describes algorithms for constructing downsets of permutations with exactly one occurrence of 321 and 3412 avoiding and exactly one occurrence of 3412 and 321 avoiding. Here is the downset of a permutation with exactly one occurrence of 3412 that avoids 321.

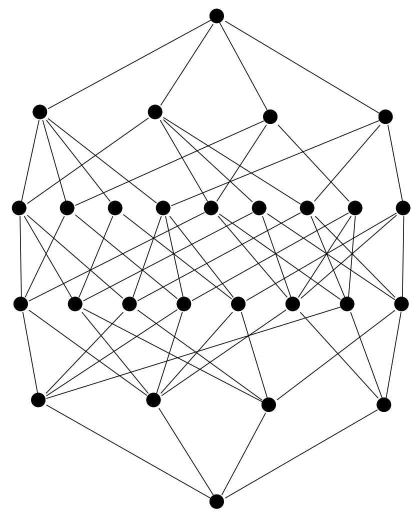

Figure 4. Downset of 41523

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