FINITE
QUANTUM MEASURE SPACES

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1 Introduction

Measure and integration theory is a well established field of mathematics that is over a hundred years old. The theory possesses many deep and elegant theorems and has important applications in functional analysis, probability theory and theoretical physics. Measure theory can be applied whenever you are measuring something whether it be length, volume, probabilities, mass, energy, etc. Although finite measure theory, in which the measure space has only a finite number of elements, is much simpler than the general theory, it also has important applications to probability theory, combinatorics and computer science. In this article we shall discuss a generalization called finite quantum measure spaces. Just as quantum mechanics possesses a certain "quantum weirdness," these spaces lack some of the simplicity and intuitive nature of their classical counterparts. Although there is a general theory of quantum measure spaces, we shall only consider finite spaces to keep technicalities to a minimum. Nevertheless, these finite spaces still convey the flavor of the subject and exhibit some of the unusual properties of quantum objects. Much of this unusual behavior is due to a phenomenon called quantum interference which is a recurrent theme in the present article.

2 Classical and Quantum Worlds

We first discuss finite measure theory in the classical world. Let $X = \{x_1, \ldots, x_n\}$ be a finite nonempty set and denote the power set of X, consisting of all subsets of X, by $\mathcal{P}(X)$. For $A, B \in \mathcal{P}(X)$ we use the notation $A \mathbin{\dot\cup} B$ for $A \cup B$ whenever $A \cap B = \emptyset$. Denoting the set of nonnegative real numbers by $\mathbb{R}^+$, a **measure** on $\mathcal{P}(X)$ is a map $\nu\colon \mathcal{P}(X) \to \mathbb{R}^+$ satisfying the **additivity condition**

$$\nu(A \mathbin{\dot\cup} B) = \nu(A) + \nu(B) \tag{2.1}$$

for all disjoint $A, B \in \mathcal{P}(X)$. No matter what we are measuring, the reason for (2.1) is intuitively clear. We call the pair (X, ν) a **finite measure space**. It immediately follows from (2.1) that $\nu(\emptyset) = 0$ and

$$\nu\left(\dot{\bigcup_{i=1}^{m}} A_i\right) = \sum_{i=1}^{m} \nu(A_i) \tag{2.2}$$

If $\nu\colon \mathcal{P}(X) \to \mathbb{R}$ satisfies (2.1) we call ν a **signed measure** and if $\nu\colon \mathcal{P}(X) \to \mathbb{C}$ satisfies (2.1) we call ν a **complex measure**. For all types of measures we use the shorthand notation $\nu(x_i) = \nu(\{x_i\})$.

Denoting the complement of a set A by A', since

$$A = (A \cap B) \mathbin{\dot\cup} (A \cap B')$$

we have that $\nu(A \cap B') = \nu(A) - \nu(A \cap B)$ for all of the previous types of measure. Also,

$$A \cup B = (A \cap B') \mathbin{\dot\cup} (A \cap B) \mathbin{\dot\cup} (B \cap A')$$

so we obtain the inclusion-exclusion formula

$$\nu(A \cup B) = \nu(A) + \nu(B) - \nu(A \cap B) \tag{2.3}$$

A **probability measure** is a measure ν that satisfies $\nu(X) = 1$. In this case, the elements $x_i \in S$ are interpreted as sample points or elementary events and the sets $A \in \mathcal{P}(X)$ are interpreted as events. Then $\nu(A)$ is the **probability** that the event A occurs. For example, suppose we flip a fair coin twice. Denoting heads and tails by H and T, respectively, the sample space becomes

$$X = \{HH, HT, TH, TT\}$$

The probability measure ν satisfies

$$\nu(HH) = \nu(HT) = \nu(TH) = \nu(TT) = 1/4$$

The event that at least one head occurs is given by $A = \{HH, HT, TH\}$ and it follows from (2.2) that $\nu(A) = 3/4$.

We conclude from (2.2) that a measure ν is determined by its values $\nu(x_i)$, $i = 1, \ldots, n$. In fact, we have

$$\nu\left(\{x_{i_1}, \ldots, x_{i_m}\}\right) = \sum_{j=1}^{m} \nu(x_{i_j})$$

Conversely, given any nonnegative numbers p_i, $i = 1, \ldots, n$, we obtain a measure ν given by

$$\nu(A) = \sum \{p_i \colon x_i \in A\}$$

This same observation applies to signed and complex measures.

Until now, life in the classical world has been simple and intuitive. But along comes quantum mechanics and our desire to explain it mathematically. It turns out that quantum measures need not satisfy additivity (2.1) and are therefore not really measures. But if (2.1) is so intuitively clear, how can it not hold in a physical theory like quantum mechanics? The reason is because of a phenomenon called quantum interference. If the points of X represent quantum objects, they can interfere with each other both constructively and destructively. For example, suppose x_1, x_2 represent subatomic particles and μ is a measure of mass. Then we could have $\mu(x_1) > 0$ and $\mu(x_2) > 0$ but x_1, x_2 could be a particle-antiparticle pair that annihilate each other producing pure energy (fission). Taken together we would have $\mu\left(\{x_1, x_2\}\right) = 0$. Hence, $\mu\left(\{x_1, x_2\}\right) \neq \mu(x_1) + \mu(x_2)$ and additivity fails. On the other hand, two particles colliding at high kinetic energy can convert some of this energy to mass and combine (fusion) to form a single particle in which case $\mu\left(\{x_1, x_2\}\right) > \mu(x_1) + \mu(x_2)$.

For another example, suppose a beam of subatomic particles such as electrons or photons impinges on a screen containing two closely spaced narrow slits. The particles that pass through the slits hit a black target screen and produce small white dots at their points of absorption. It is well known experimentally that this results in a diffraction pattern consisting of many light and dark strips. Why do the particles accumulate in the light regions and not in the dark regions? It seems as though the particles communicate with each other to conspire to land along the white strips. Let $X = \{x_1, \ldots, x_n\}$ represent this set of particles and let R be a region of the target screen. For $A \subseteq X$, let $\mu(A)$ measure the propensity for the particles in A to hit a point in R. Now it can happen that $\mu(x_1) = \mu(x_2) = 0$ and yet $\mu(\{x_1, x_2\}) > 0$ or that $\mu(x_1), \mu(x_2) > 0$ and $\mu(\{x_1, x_2\}) = 0$. More generally, we can have $\mu(\{x_1, x_2\}) > \mu(x_1) + \mu(x)$ or $\mu(\{x_1, x_2\}) < \mu(x_1) + \mu(x_2)$. We then say that the particles interfere constructively or destructively, respectively.

In a deeper analysis, the points of X represent particle paths and it is the paths that interfere. In this case, a single particle results in two possible paths, one through each of the slits and these paths interfere. The standard explanation for this phenomenon is called **wave-particle duality**. The diffraction pattern is easily explained for waves. Two waves interfere constructively if they combine with crests close together and destructively if they combine with a crest close to a trough. In wave-particle duality, an unobserved subatomic particle behaves like a wave . When the wave impinges upon the first screen, it divides into two subwaves each going through one of the two slits. These subwaves combine, interfere and then hit the target screen. It is then observed (as a small white dot) at which point it acts like a particle. Whether you like this explanation or not (and some don't), the mathematics of quantum mechanics accurately describes the diffraction pattern.

3 Quantum Measures

We have seen in Section 2 that quantum measures need not be additive. To find the properties that they do possess, we examine some of the mathematics of quantum mechanics. Let $X = \{x_1, \ldots, x_n\}$ be a set of quantum objects. In various quantum formalisms an important role is played by a **decoherence function** $D\colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{C}$ [3, 4]. This function (or at least its real part) represents the amount of interference between pairs of subsets of X and has the following properties:

$$D(A \mathbin{\dot\cup} B, C) = D(A,C) + D(B,C) \tag{3.1}$$

$$D(A,B) = \overline{D(B,A)} \tag{3.2}$$

$$D(A,A) \geq 0 \tag{3.3}$$

$$|D(A,B)|^2 \leq D(A,A)D(B,B) \tag{3.4}$$

In (3.2), the bar is complex conjugation and (3.1), (3.2) imply that D is additive in one of the arguments when the other argument is fixed. A quantum measure is defined by $\mu(A) = D(A,A)$ and μ is a measure of the inference of A with itself. A simple example of a decoherence function is $D(A,B) = \nu(A)\overline{\nu(B)}$ where ν is a complex measure on $\mathcal{P}(X)$. In this case ν is called an **amplitude** (which comes from the analogy with waves) and we have $\mu(A) = |\nu(A)|^2$. In fact, quantum probabilities are frequently computed by taking the modulus squared of a complex amplitude. This example illustrates the nonadditivity of μ because

$$\mu(A \mathbin{\dot\cup} B) = |\nu(A \mathbin{\dot\cup} B)|^2 = |\nu(A) + \nu(B)|^2 = \mu(A) + \mu(B) + 2\mathrm{Re}\left[\nu(A)\overline{\nu(B)}\right]$$

Hence, $\mu(A \mathbin{\dot\cup} B) = \mu(A) + \mu(B)$ if and only if $\mathrm{Re}\left[\nu(A)\overline{\nu(B)}\right] = 0$. In this case we say that A and B do not interfere or A and B are compatible.

Theorem 3.1. *Let $D\colon \mathcal{P}(X) \times \mathcal{P}(X) \to \mathbb{C}$ be a decoherence function and define $\mu(A) = D(A,A)$. Then $\mu\colon \mathcal{P}(X) \to \mathbb{R}^+$ has the following properties:*

$$\mu(A \mathbin{\dot\cup} B \mathbin{\dot\cup} C) = \mu(A \mathbin{\dot\cup} B) + \mu(A \mathbin{\dot\cup} C) + \mu(B \mathbin{\dot\cup} C) - \mu(A) - \mu(B) - \mu(C) \tag{3.5}$$

If $\mu(A) = 0$, then $\mu(A \mathbin{\dot\cup} B) = \mu(B)$ for all $B \in \mathcal{P}(X)$ with $B \cap A = \emptyset$ (3.6)

If $\mu(A \mathbin{\dot\cup} B) = 0$, then $\mu(A) = \mu(B)$ (3.7)

Proof. To prove (3.5), let R be the right side of (3.5) and apply (3.1) and

(3.2) to obtain

$$\begin{aligned}R &= D(A \cup B, A \cup B) + D(A \cup C, A \cup C) + D(B \cup C, B \cup C)\\ &\quad - \mu(A) - \mu(B) - \mu(C)\\ &= 2[D(A,A)+D(B,B)+D(C,C)+\mathrm{Re}D(A,B)+\mathrm{Re}D(A,C)+\mathrm{Re}(B,C)]\\ &\quad - \mu(A) - \mu(B) - \mu(C)\\ &= D(A,A) + D(B,B) + D(C,C)\\ &\quad + 2\left[\mathrm{Re}D(A,B) + \mathrm{Re}D(A,C) + \mathrm{Re}D(B,C)\right]\\ &= D(A \cup B \cup C, A \cup B \cup C) = \mu(A \cup B \cup C)\end{aligned}$$

To prove (3.6), apply (3.1) and (3.2) to obtain

$$\mu(A \cup B) = D(A \cup B, A \cup B) = \mu(A) + \mu(B) + 2\mathrm{Re}D(A,B)$$

By (3.4) if $\mu(A) = 0$, then $D(A,B) = 0$ so that $\mu(A \cup B) = \mu(B)$. To prove (3.7), applying (3.1)–(3.4) we have

$$\begin{aligned}\mu(A \cup B) &= \mu(A) + \mu(B) + 2\mathrm{Re}D(A,B) \geq \mu(A) + \mu(B) - 2\left|D(A,B)\right|\\ &\geq \mu(A) + \mu(B) - 2\mu(A)^{1/2}\mu(B)^{1/2} = \left[\mu(A)^{1/2} - \mu(B)^{1/2}\right]^2\end{aligned}$$

Hence, $\mu(A \cup B) = 0$ implies that $\mu(A) = \mu(B)$. □

Condition (3.5) is a generalized additivity that we call **grade-2 additivity**. The usual additivity (2.1) is called **grade-1 additivity**. Of course, grade-1 additivity implies grade-2 additivity but the converse does not hold. Conditions (3.6) and (3.7) do not follow from (3.5) and a map satisfying (3.6) and (3.7) is called **regular**. A grade-2 additive map $\mu\colon \mathcal{P}(X) \to \mathbb{R}^+$ is a **grade-2 measure** and a regular grade-2 measure is a **quantum measure** (or q-**measure**, for short) [6, 7]. If μ is a q-measure we call (X, μ) a q-**measure space**. We have seen that if $\mu(A) = D(A,A)$ for a decoherence function D, then μ is a q-measure. We will later exhibit more general q-measures that do not have this form. Simple examples of q-measures are $\mu(A) = \nu(A)^2$ where ν is a signed measure. It follows from (3.5) that any q-measure μ satisfies $\mu(\emptyset) = 0$.

Example 1. *Let (X, ν) be the probability space of our fair coin example. But now we have a quantum coin with q-measure $\mu(A) = \nu(A)^2$. Then the sample points have "quantum probability" $1/16$ and the certain event X has "quantum probability" 1 as it should. The event A that at least one head appears has "quantum probability" $9/16$.*

Example 2. *Let* $X = \{x_1, x_2\}$ *and define* $\mu(x_1) = \mu(x_2) = 1$, $\mu(\emptyset) = 0$ *and* $\mu(X) = 6$. *Then* (X, μ) *is a q-measure space, but* μ *does not have the form* $\mu(A) = D(A, A)$ *for a decoherence function* D. *Indeed, if such a* D *exists we would have*

$$2D(x_1, x_2) + D(x_1, x_1) + D(x_2, x_2) = D(X, X) = \mu(X) = 6$$

Hence, $D(x_1, x_2) = 2$ *but then* (3.4) *is not satisfied which is a contradiction.*

Example 3. *Let* $X = \{x_1, x_2, x_3\}$ *with* $\mu(\emptyset) = \mu(x_1) = 0$ *and* $\mu(A) = 1$ *for all other* $A \in \mathcal{P}(X)$. *Then* (X, μ) *is a q-measure space.*

Example 4. *Let* $X = \{x_1 \ldots, x_m, y_1, \ldots, y_m, z_1, \ldots, z_n\}$ *and call* (x_i, y_i) $i = 1, \ldots, m$, ***destructive pairs*** *(or* ***particle-antiparticle pairs****. Denoting the cardinality of a set* B *by* $|B|$ *we define*

$$\mu(A) = |A| - 2\left|\{(x_i, y_i) \colon x_i, y_i \in A\}\right| \tag{3.8}$$

for every $A \in \mathcal{P}(X)$. *For instance* $\mu\left(\{x_1, y_1, z_1\}\right) = 1$ *and* $\mu\left(\{x_1, y_1, y_2, z_1\}\right) = 2$. *We now check that* μ *is a q-measure on* X. *If* $\mu(A) = 0$, *then* $A = \emptyset$ *or* A *has the form*

$$A = \left\{x_{i_1}, y_{i_1}, \ldots, x_{i_j}, y_{i_j}\right\}$$

If $B \in \mathcal{P}(X)$ *with* $A \cap B = \emptyset$, *then*

$$\begin{aligned}\mu(A \cup B) &= |A| + |B| - 2\left|\{(x_i, y_i) \colon x_i, y_i \in A\}\right| - 2\left|\{(x_i, y_i) \colon x_i, y_i \in B\}\right| \\ &= |B| - 2\left|\{(x_i, y_i) \colon x_i, y_i \in B\}\right| = \mu(B)\end{aligned}$$

Hence, (3.6) *holds. To show that* (3.7) *holds, suppose* $\mu(A \cup B) = 0$. *Then* $z_i \notin A \cup B$, $i = 1, \ldots, n$. *If* $x_i \in A$, *then* $y_i \in A \cup B$ *and if* $y_i \in A$ *then* $x_i \in A \cup B$. *Hence,*

$$\begin{aligned}\mu(A) &= \left|\{x_i \in A \colon y_i \in B\}\right| + \left|\{y_i \in A \colon x_i \in B\}\right| \\ &= \left|\{y_i \in B \colon x_i \in A\}\right| + \left|\{x_i \in B \colon y_i \in A\}\right| = \mu(B)\end{aligned}$$

We conclude that μ *is regular. To prove grade-2 additivity* (3.5), *let* $A_1, A_2, A_3 \in \mathcal{P}(X)$ *be mutually disjoint. If* $x_i \in A_r$ *and* $y_i \in A_s$, $r, s = 1, 2, 3$, *we call*

(x_i, y_i) *an* rs-***pair***. *We then have*

$$
\begin{aligned}
&\mu(A_1 \mathbin{\dot\cup} A_2) + \mu(A_1 \mathbin{\dot\cup} A_3) + \mu(A_2 \mathbin{\dot\cup} A_3) - \mu(A_1) - \mu(A_2) - \mu(A_3)\\
&\quad = |A_1| + |A_2| - 2\,|\{rs\text{-pairs}, r, s = 1, 2\}| + |A_1| + |A_3|\\
&\qquad - 2\,|\{rs\text{-pairs}, r, s = 1, 3\}| + |A_2| + |A_3| - 2\,|\{rs\text{-pairs}, r, s = 2, 3\}|\\
&\qquad - |A_1| + 2\,|\{11\text{-pairs}\}| - |A_2| + 2\,|\{22\text{-pairs}\}| - |A_3| + 2\,|\{33\text{-pairs}\}|\\
&\quad = |A_1 \mathbin{\dot\cup} A_2 \mathbin{\dot\cup} A_3| - 2\,|\{rs\text{-pairs}, r, s = 1, 2, 3\}|\\
&\quad = \mu(A_1 \mathbin{\dot\cup} A_2 \mathbin{\dot\cup} A_3)
\end{aligned}
$$

We conclude that (X, μ) *is a q-measure space.*

The next result shows that grade-2 additivity is equivalent to a generalization of (2.3). The **symmetric difference** of A and B is $A \bigtriangleup B = (A \cap B') \cup (A' \cap B)$.

Theorem 3.2. *A map* $\mu\colon \mathcal{P}(X) \to \mathbb{R}^+$ *is grade-2 additive if and only if* μ *satisfies*

$$\mu(A\cup B) = u(A)+\mu(B)-\mu(A\cap B)+\mu(A\bigtriangleup B)-\mu(A\cap B')-\mu(A'\cap B) \quad (3.9)$$

Proof. If μ is grade-2 additive, we have

$$
\begin{aligned}
\mu(A \cup B) &= \mu\left[(A \cap B') \mathbin{\dot\cup} (A' \cup B) \mathbin{\dot\cup} (A \cap B)\right]\\
&= \mu(A \bigtriangleup B) + \mu(A) + \mu(B) - \mu(A \cap B') - \mu(A' \cap B) - \mu(A \cap B)
\end{aligned}
$$

which is (3.9). Conversely, if (3.9) holds, then letting $A_1 = A \mathbin{\dot\cup} C$, $B_1 = B \mathbin{\dot\cup} C$ we have

$$
\begin{aligned}
\mu(A \mathbin{\dot\cup} B \mathbin{\dot\cup} C) &= \mu(A_1 \cup B_1)\\
&= \mu(A_1) + \mu(B_1) - \mu(A_1 \cap B_1) + \mu(A_1 \bigtriangleup B_1)\\
&\quad - \mu(A_1 \cap B_1') - \mu(A_1' \cap B_1)\\
&= \mu(A \mathbin{\dot\cup} C) + \mu(B \mathbin{\dot\cup} C) - \mu(C) + \mu(A \mathbin{\dot\cup} B) - \mu(A) - \mu(B)
\end{aligned}
$$

which is grade-2 additivity. □

We now show that grade-2 additivity can be extended to more than three mutually disjoint sets [5].

Theorem 3.3. *If $\mu\colon \mathcal{P}(X) \to \mathbb{R}^+$ is grade-2 additive, then for any $m \geq 3$ we have*

$$\mu\left(\dot{\bigcup_{i=1}^{m}} A_i\right) = \sum_{i<j=1}^{m} \mu(A_i \mathbin{\dot\cup} A_j) - (m-2)\sum_{i=1}^{m} \mu(A_i) \tag{3.10}$$

Proof. We prove the result by induction on m The result holds for $m = 3$. Assuming the result holds for $m - 1 \geq 2$ we have

$$\begin{aligned}
\mu\left(\dot{\bigcup_{i=1}^{m}} A_i\right) &= \mu\left[A_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} (A_{m-1} \mathbin{\dot\cup} A_m)\right] \\
&= \sum_{i<j=1}^{m-2} \mu(A_i \mathbin{\dot\cup} A_j) + \sum_{i=1}^{m-2} \mu\left[A_i \mathbin{\dot\cup} (A_{m-1} \mathbin{\dot\cup} A_m)\right] \\
&\quad - (m-3)\left[\sum_{i=1}^{m-2} \mu(A_i) + \mu(A_{m-1} \mathbin{\dot\cup} A_m)\right] \\
&= \sum_{i<j=1}^{m-2} \mu(A_i \mathbin{\dot\cup} A_j) + \sum_{i=1}^{m-2} \mu(A_i \mathbin{\dot\cup} A_{m-1}) \\
&\quad + \sum_{i=1}^{m-2} \mu(A_i \mathbin{\dot\cup} A_m) + (m-2)\mu(A_{m-1} \mathbin{\dot\cup} A_m) \\
&\quad - \sum_{i=1}^{m-2} \mu(A_i) - (m-2)\mu(A_{m-1}) - (m-2)\mu(A_m) \\
&\quad - (m-3)\left[\sum_{i=1}^{m-2} \mu(A_i) + \mu(A_{m-1} \mathbin{\dot\cup} A_m)\right] \\
&= \sum_{i<j=1}^{m} \mu(A_i \mathbin{\dot\cup} A_j) - (m-2)\sum_{i=1}^{m} \mu(A_i)
\end{aligned}$$

The result follows by induction. □

Notice that Theorem 3.3 also holds for signed and complex grade-2 additive measures.

4 Quantum Interference

Unlike a measure on $\mathcal{P}(X)$, a q-measure μ is not determined by its values on singleton sets. However, by Theorem 3.3, μ is determined by its values on singleton and doubleton sets. Thus, if $p_i \geq 0$ and $q_{ij} \geq 0$, $i, j = 1, \ldots, n$, satisfy $q_{ij} = q_{ji}$ and

$$\sum_{i<j=1}^{m} q_{ij} - (m-2) \sum_{i=1}^{m} p_i \geq 0$$

for $3 \leq m \leq n$, then there exists a unique q-measure μ on $X = \{x_1, \ldots, x_n\}$ such that $\mu(x_i) = p_i$ and $\mu(\{x_i, x_j\}) = q_{ij}$, $i, j = 1, \ldots, n$. Conversely, given a q-measure μ on X, then $p_i = \mu(x_i)$, $q_{ij} = \mu(\{x_i, x_j\})$ have these properties. We now introduce a physically relevant parameter call quantum interference that can also be used to determine a q-measure.

For a q-measure μ on $X = \{x_1, \ldots, x_n\}$ we define the **quantum interference function** $I_\mu \colon X \times X \to \mathbb{R}$ by

$$I_\mu(x_i, x_j) = \mu(\{x_i, x_j\}) - \mu(x_i) - \mu(x_j)$$

if $i \neq j$ and $I_\mu(x_i, x_i) = 0$, $i, j = 1, \ldots, n$. The function I_μ gives the deviation of μ from being a measure on the sets $\{x_i, x_j\}$ and hence is an indicator of the interference between x_i and x_j. Notice that I_μ can have positive or negative values. For instance, in Example 3, $I_\mu(x_2, x_3) = -1$ while in Example 2, $I_\mu(x_1, x_2) = 4$. By Theorem 3.3, μ is determined by the numbers $\mu(x_i)$ and $I_\mu(x_i, x_j)$, $i, j = 1 \ldots, n$. We extend I_μ to a signed measure λ_μ on $\mathcal{P}(X \times X)$ by defining

$$\lambda_\mu(B) = \sum \{I_\mu(x_i, x_j) \colon (x_i, x_j) \in B\}$$

Since $I_\mu(x_i, x_j) = I_\mu(x_j, x_i)$ it follows that λ_μ is **symmetric** in the sense that $\lambda_\mu(A \times B) = \lambda_\mu(B \times A)$ for all $A, B \in \mathcal{P}(X)$. The **classical part** of μ is defined to be the unique measure ν_μ on $\mathcal{P}(X)$ that satisfies $\nu_\mu(x_i) = \mu(x_i)$, $i = 1 \ldots, n$. The next result shows that we can always decompose μ into the classical part and its interference part.

Theorem 4.1. *If μ is a q-measure on $X = \{x_1, \ldots, x_n\}$, then for any $A \in \mathcal{P}(X)$ we have*

$$\mu(A) = \nu_\mu(A) + \frac{1}{2} \lambda_\mu(A \times A) \tag{4.1}$$

Proof. We first prove that $\delta(A)=\lambda_\mu(A\times A)$ is a grade-2 signed measure on $\mathcal{P}(X)$. To show this we have

$$\begin{aligned}
&\delta(A\cup B)+\delta(A\cup C)+\delta(B\cup C)-\delta(A)-\delta(B)-\delta(C)\\
&\quad=\lambda_\mu(A\cup B\times A\cup B)+\lambda_\mu(A\cup C\times A\cup C)+\lambda_\mu(B\cup C\times B\cup C)\\
&\qquad-\lambda_\mu(A\times A)-\lambda_\mu(B\times B)-\lambda_\mu(C\times C)\\
&\quad=\lambda_\mu(A\times A)+2\lambda_\mu(A\times B)+\lambda_\mu(B\times B)+\lambda_\mu(A\times A)+2\lambda_\mu(A\times C)\\
&\qquad+\lambda_\mu(C\times C)+\lambda_\mu(B\times B)+2\lambda_\mu(B\times C)+\lambda_\mu(C\times C)-\lambda_\mu(A\times A)\\
&\qquad-\lambda_\mu(B\times B)-\lambda_\mu(C\times C)\\
&\quad=\lambda_\mu(A\times A)+\lambda_\mu(B\times B)+\lambda_\mu(C\times C)\\
&\qquad+2\left[\lambda_\mu(A\times B)+\lambda_\mu(A\times C)+\lambda_\mu(B\times C)\right]\\
&\quad=\lambda_\mu(A\cup B\cup C\times A\cup B\cup C)=\delta(A\cup B\cup C)
\end{aligned}$$

Hence, $\nu_\mu(A)+\frac{1}{2}\lambda_\mu(A\times A)$ is a grade-2 signed measure. Now

$$\begin{aligned}
\nu_\mu(x_i)+\frac{1}{2}\lambda_\mu\left(\{x_i\}\times\{x_i\}\right)&=\nu_\mu(x_i)-\frac{1}{2}I_\mu(x_i,x_i)\\
&=\nu_\mu(x_i)=\mu(x_i)
\end{aligned}$$

and for $i\neq j$ we have

$$\begin{aligned}
&\nu_\mu\left(\{x_i,x_j\}\right)+\frac{1}{2}\lambda_\mu\left(\{x_i,x_j\}\times\{x_i,x_j\}\right)\\
&\quad=\nu_\mu(x_i)+\nu_\mu(x_j)+I_\mu(x_i,x_j)\\
&\quad=\mu(x_i)+\mu(x_j)+\mu\left(\{x_i,x_j\}\right)-\mu(x_i)-\mu(x_j)\\
&\quad=\mu\left(\{x_i,x_j\}\right)
\end{aligned}$$

Since μ and $A\mapsto\nu_\mu(A)+\frac{1}{2}(A\times A)$ are both grade-2 signed measures that agree on singleton and doubleton sets, it follows from Theorem 3.3 that they coincide. □

Notice that (4.1) can be written

$$\mu(A)=\nu_\mu(A)+\frac{1}{2}\sum\left\{I_\mu(x_i,x_j)\colon x_i,x_j\in A\right\}\tag{4.2}$$

We shall now illustrate (4.2) in some examples. In Example 1 we have that $\nu_\mu(x_i)=1/16$ and $I_\mu(x_i,x_j)=1/8$ for $i\neq j$. By (4.2) we have for all $A\in\mathcal{P}(X)$ that

$$\mu(X)=\frac{1}{16}\left|A\right|+\frac{1}{16}\left|A\right|\left(\left|A\right|-1\right)=\frac{1}{16}\left|A\right|^2$$

In Example 4 we have $\nu_\mu(x_i) = 1$, $I_\mu(x_i, y_i) = I_\mu(y_i, x_i) = -2$ and I_μ vanishes for all other pairs. Hence, (4.2) agrees with (3.8). We can use (4.2) to construct q-measures. For example, letting $\nu(x_i) = 0$ for all i and $I(x_i, x_j) = 1$ for $i \neq j$ we conclude from (4.2) that

$$\mu(A) = \binom{|A|}{2} = \frac{1}{2}\,|A|\,(|A| - 1)$$

For another example, let $X = \{x_1, \ldots, x_{2n+1}\}$, $\nu(x_i) = n$ for all i and $I(x_i, x_j) = -1$ for all $i \neq j$. Applying (4.2) gives

$$\mu(A) = n\,|A| - \binom{|A|}{2} = \frac{1}{2}\,|A|\,(|X| - |A|)$$

5 Compatibility and the Center

Let (X, μ) be a quantum measure space. We say that $A, B \in \mathcal{P}(X)$ are μ-**compatible** and write $A\mu B$ if

$$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

Recalling (2.3) we see that μ acts like a measure on $A \cup B$ so in some weak sense A and B do not interfere with each other. For example, $\{x\}$ and $\{y\}$ are μ-compatible if and only if $I_\mu(x, y) = 0$. This analogy is not completely accurate because $A\mu A$ for all $A \in \mathcal{P}(X)$ and certainly points of A can interfere with each other. It follows from (3.9) that $A\mu B$ if and only if

$$\mu(A \bigtriangleup B) = \mu(A \cap B') + \mu(A' \cap B) \tag{5.1}$$

The μ-**center** of $\mathcal{P}(X)$ is

$$Z_\mu = \{A \in \mathcal{P}(X) \colon A\mu B \text{ for all } B \in \mathcal{P}(X)\}$$

The elements of Z_μ are called **macroscopic sets** because they behave like large objects at the human scale [7].

Lemma 5.1. (i) *If $A \subseteq B$, then $A\mu B$.* (ii) *If $A\mu B$, then $A'\mu B'$.* (iii) $\emptyset, X \in Z_\mu$. (iv) *If $A \in Z_\mu$, then $A' \in Z_\mu$.*

Proof. (i) If $A \subseteq B$, then

$$\mu(A \cup B) = \mu(B) = \mu(A) + \mu(B) - \mu(A \cap B)$$

Hence, $A\mu B$. (ii) If $A\mu B$, then by (5.1)

$$\begin{aligned}\mu(A' \triangle B') &= \mu(A \triangle B) = \mu(A \cap B') + \mu(A' \cap B) \\ &= \mu\left[(A')' \cap B'\right] + \mu\left[A' \cap (B')'\right]\end{aligned}$$

Hence, by (5.1), $A'\mu B'$. (iii)follows from (i) and (iv) follows from (ii). □

A set $A \in \mathcal{P}(X)$ is μ-**splitting** if $\mu(B) = \mu(B \cap A) + \mu(B \cap A')$ for all $B \in \mathcal{P}(X)$.

Lemma 5.2. *A is μ-splitting if and only if $A \in Z_\mu$.*

Proof. Suppose A is μ-splitting. Then for every $B \in \mathcal{P}(X)$ we have

$$\begin{aligned}\mu(A \cup B) &= \mu\left[(A \cup B) \cap A\right] + \mu\left[(A \cup B) \cap A'\right] \\ &= \mu(A) + \mu(B \cap A') = \mu(A) + \mu(B) - \mu(A \cap B)\end{aligned}$$

Hence, $A \in Z_\mu$. Conversely, suppose $A \in Z_\mu$. Then for every $B \in \mathcal{P}(X)$ we have

$$\mu(A \cup B) = \mu\left[A \,\dot\cup\, (B \cap A')\right] = \mu(A) + \mu(B \cap A')$$

Thus,

$$\mu(B) = \mu(A \cup B) - \mu(A) + \mu(A \cap B) = \mu(B \cap A) + \mu(B \cap A')$$

so A is μ-splitting. □

A **Boolean subalgebra** of $\mathcal{P}(X)$ is a collection of sets $\mathcal{A} \subseteq \mathcal{P}(X)$ such that $X \in \mathcal{A}$, $A \in \mathcal{A}$ implies $A' \in \mathcal{A}$ and $A, B \in \mathcal{A}$ implies $A \cup B \in \mathcal{A}$. A measure on $\mathcal{A}$ is defined just as was on $\mathcal{P}(X)$.

Theorem 5.3. *Z_μ is a Boolean subalgebra of $\mathcal{P}(X)$ and the restriction $\mu \mid Z_\mu$ of μ to Z_μ is a measure. Moreover, if $A_i \in Z_\mu$ are mutually disjoint, then for every $B \in \mathcal{P}(X)$ we have*

$$\mu\left[\dot\cup (B \cap A_i)\right] = \sum \mu(B \cap A_i)$$

Proof. By Lemma 5.1, $X \in Z_\mu$ and $A' \in Z_\mu$ whenever $A \in Z_\mu$. Now suppose $A, B \in Z_\mu$ and $C \in \mathcal{P}(X)$. Since A is μ-splitting we have

$$\begin{aligned}\mu\left[C \cap (A \cup B)\right] &= \mu\left[(C \cap A) \cap (A \cup B)\right] + \mu\left[(C \cap A') \cap (A \cup B)\right] \\ &= \mu(C \cap A) + \mu(C \cap A' \cap B)\end{aligned}$$

Hence, since B is μ-splitting we conclude that

$$\begin{aligned}\mu(C) &= \mu(C \cap A) + \mu(C \cap A') = \mu(C \cap A) + \mu(C \cap A' \cap B) + \mu(C \cap A' \cap B') \\ &= \mu\left[C \cap (A \cup B)\right] + \mu\left[C \cap (A \cup B)'\right]\end{aligned}$$

It follows that $A \cup B$ is μ-splitting so $A \cup B \in Z_\mu$. Hence, Z_μ is a Boolean subalgebra of $\mathcal{P}(X)$. Moreover, $\mu \mid Z_\mu$ is a measure because if $A, B \in Z_\mu$ with $A \cap B = \emptyset$, since $A\mu B$ we have $\mu(A \mathbin{\dot\cup} B) = \mu(A) + \mu(B)$. To prove the last statement, let $A_i \in Z_\mu$ be mutually disjoint, $i = 1, \ldots, m$, and let $S_r = \dot\cup_{i=1}^r A_i$, $r \le m$. We prove by induction on r that for $B \in \mathcal{P}(X)$ we have

$$\mu(B \cap S_r) = \sum_{i=1}^r \mu(B \cap A_i)$$

The case $r = 1$ is obvious. Suppose the result is true for $r < m$. Since $S_r \in Z_\mu$ we have

$$\begin{aligned}\mu(B \cap S_{r+1}) &= \mu(B \cap S_{r+1} \cap S_r) + \mu(B \cap S_{r+1} \cap S_r') \\ &= \mu(B \cap S_r) + \mu(B \cap A_{r+1}) \\ &= \sum_{i=1}^r \mu(B \cap A_i) + \mu(B \cap A_{r+1}) = \sum_{i=1}^{r+1} \mu(B \cap A_i)\end{aligned}$$

By induction, the result holds for $r = m$ so that

$$\mu\left[\dot{\bigcup_{i=1}^m} (B \cap A_i)\right] = \mu(B \cap S_m) = \sum_{i=1}^m \mu(B \cap A_i) \qquad \square$$

We now illustrate these ideas in Example 4. All the results in the rest of this section apply to the quantum measure space (X, μ) of Example 4.

Theorem 5.4. *$A\mu B$ if and only if $x_i \in A \cap B'$ implies that $y_i \notin B \cap A'$ and $y_i \in A \cap B'$ implies that $x_i \notin B \cap A'$.*

Proof. The condition is equivalent to the following. If $\{x_i, y_i\} \subseteq A \triangle B$, then $\{x_i, y_i\} \subseteq A \cap B'$ or $\{x_i, y_i\} \subseteq B \cap A'$. Suppose the condition holds. We may assume without loss of generality that

$$\{x_1, y_1, \dots, x_r, y_r\} \subseteq A \cap B'$$
$$\{x_{r+1}, y_{r+1}, \dots, x_s, y_s\} \subseteq B \cap A'$$

and there are no other destructive pairs in $A \triangle B$. Then

$$\mu(A \triangle B) = |A \triangle B| - 2s = |A \cap B'| - 2r + |B \cap A'| - 2(s - r)$$
$$= \mu(A \cap B') + \mu(B \cap A')$$

By Theorem 3.2, $A\mu B$. Conversely, suppose $A\mu B$. Again, without loss of generality we can assume that $\{x_1, y_1, \dots, x_r, y_r\}$ are all the destructive pairs in $A \cap B'$ and $\{x_{r+1}, y_{r+1}, \dots, x_s, y_s\}$ are all the destructive pairs in $B \cap A'$. Assume that

$$S = \{x_{s+1}, y_{s+1}, \dots, x_t, y_t\} \subseteq A \triangle B$$

Then

$$|A \triangle B| - 2t = \mu(A \triangle B) = \mu(A \cap B') + \mu(B \cap A')$$
$$= |A \cap B'| - 2r + |B \cap A'| - 2(s - r)$$

It follows that $t = s$ so that $S = \emptyset$. Hence, all the destructive pairs in $A \triangle B$ are in $A \cap B'$ or $B \cap A'$. □

Corollary 5.5. *$A \in Z_\mu$ if and only if for all $i = 1, \dots, m$, either $\{x_i, y_i\} \subseteq A$ of $\{x_i, y_i\} \subseteq A'$.*

Proof. If $A \in Z_\mu$, then $A\mu A'$. By Theorem 5.4, if $x_i \in A$ then $y_i \notin A'$ so $y_i \in A$ and similarly if $y_i \in A$ then $x_i \in A$. Conversely, suppose the condition holds and $B \in \mathcal{P}(X)$. Then

$$\mu(B \cap A') + \mu(B \cap A) = |B \cap A| - 2\,|\{(x_i, y_i)\colon\ \{x_i, y_i\} \subseteq B \cap A\}|$$
$$+ |B \cap A'| - 2\,|\{(x_i, y_i)\colon\ \{x_i, y_i\} \subseteq B \cap A'\}|$$
$$= |B| - 2\,|\{(x_i, y_i)\colon\ \{x_i, y_i\} \subseteq B\}| = \mu(B)$$

By Lemma 5.2, $A \in Z_\mu$. □

Corollary 5.6. *The following statements are equivalent.* (i) *$A\mu A'$,* (ii) *$A \in Z_\mu$,* (ii) *$\mu(X) = \mu(A) + \mu(A')$.*

Proof. (i)⇒(ii) follows from Theorem 5.4 and Corollary 5.5. (ii)⇒(iii)⇒(i) are trivial. □

It follows from Theorem 5.3 that $\mu \mid Z_\mu$ is a measure. In fact, by Corollary 5.5 we have for every $B \in Z_\mu$ that $\mu(B) = |\{z_i \colon z_i \in B\}|$ and this is clearly a measure.

6 Quantum Covers

A q-measure μ on $X = \{x_1, \ldots, x_n\}$ is called a q-**probability** if $\mu(X) = 1$. Of course, a q-probability would give a very strange probability because it need not be additive and we could have $\mu(A) > 1$ for some $A \in \mathcal{P}(X)$. Nevertheless, q-probabilities have been studied and have been useful for certain applications. If μ is a q-measure on X for which $\mu(X) \neq 0$, then μ can be "normalized" by forming the q-probability $\mu_1 = \mu/\mu(X)$. Another reason for wanting to know whether $\mu(X) \neq 0$ is that this would mean that "X happens." Why can't we just check to see whether $\mu(X) = 0$ or not? This may be difficult when X is a large, complicated system. For example, in some applications of this work, X represents the entire physical universe! In this case, q-measures are used to study the evolution of the universe going back to the big bang [1, 2, 6, 7]. More specifically, X is the set of possible "histories" of the universe and for $A \in \mathcal{P}(X)$, $\mu(A)$ gives the "propensity" that the true history is an element of A and this is studied in the field of quantum gravity and cosmology. To check whether $\mu(X) = 0$ we could test simpler subsets of X to see if they have zero q-measure. If many of these sets have zero q-measure it would be an indication (but not a guarantee) that $\mu(X) = 0$. The quantum covers that we shall consider give a guarantee.

A collection of sets $A_i \in \mathcal{P}(X)$ is a **cover** for X if $\cup A_i = X$. If ν is an ordinary measure, then applying additivity we conclude that $\nu(X) \neq 0$ if and only if X does not have a cover consisting of sets with ν-measure zero. But this doesn't work for q-measures. For example, let $X = \{x_1, x_2, x_3\}$ and define $\mu(\{x_2, x_3\}) = 4$ and

$$\mu(\emptyset) = \mu(\{x_1, x_2\}) = \mu(\{x_1, x_3\}) = 0$$
$$\mu(\{x_1\}) = \mu(\{x_2\}) = \mu(\{x_3\}) = \mu(X) = 1$$

It is easy to check that μ is a q-measure on X. Now the sets $\{x_1, x_2\}$, $\{x_1, x_3\}$ cover X, have μ-measure zero but $\mu(X) \neq 0$. We call a cover $\{A_i\}$

for X a **quantum cover** if $\mu(A_i) = 0$ for all i implies that $\mu(X) = 0$ for every q-measure μ on X [8]. Notice that a quantum cover applies to all q-measures. This is because in quantum mechanics, the q-measures correspond to physical states of the system and one frequently needs to consider many states simultaneously.

A cover $\{A_i\}$ for X is a **partition** if $A_i \cap A_j = \emptyset$ for $i \neq j$. An arbitrary partition $\{A_1, \ldots, A_m\}$ for X is an example of a quantum cover. Indeed, let μ be a q-measure on X and suppose that $\mu(A_i) = 0$, $i = 1, \ldots, m$. Then by regularity we have

$$\mu(X) = \mu(A_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} A_m) = \mu(A_2 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} A_m) = \cdots = \mu(A_m) = 0$$

We now show that there are other types of quantum covers. A subset $A \subseteq X$ is a k-**set** if $|A| = k$. The k-**set cover** for X is the collection of all k sets in X. Thus, the 1-set cover is the collection of singleton sets in X and the 2-set cover is the collection of all doubleton sets in X. The next result appears in [8] and uses a nice combinatorial argument.

Theorem 6.1. *The k-set cover is a quantum cover.*

Proof. The result is true for $k = 1$ since the 1-set cover is a partition. Let $2 \le k \le n$ and assume that every k-set has μ-measure zero. By Theorem 3.3 we have

$$(2-k)\sum_{j=1}^{k}\mu(x_{i_j}) + \sum_{r<s=1}^{k}\mu\left(\{x_{i_r}, x_{i_s}\}\right) = \mu\left(\{x_{i_1}, \ldots, x_{i_k}\}\right) = 0$$

Adding up the $\binom{n}{k}$ possible k-sets, since x_{i_j} appears in $\binom{n-1}{k-1}$ k-sets and $\{x_{i_r} x_{i_s}\}$ is a subset of $\binom{n-2}{k-2}$ k-sets we have

$$\binom{n-1}{k-1}(2-k)\sum_{j=1}^{n}\mu(x_j) + \binom{n-2}{k-2}\sum_{r<s=1}^{n}\left(\{x_r, x_s\}\right) = 0$$

Since

$$(k-2)\binom{n-1}{k-1}\bigg/\binom{n-2}{k-2} = \frac{(n-1)(k-2)}{k-1}$$

we conclude that

$$\sum_{r<s=1}^{n} \mu\left(\{x_r, x_s\}\right) = \frac{(n-1)(k-2)}{k-1} \sum_{j=1}^{n} \mu(x_j)$$

Since $k \leq n$ we have by Theorem 3.3 that

$$\begin{aligned} \mu(X) = \mu\left(\{x_1, \ldots, x_n\}\right) &= \sum_{r<s=1}^{n} \mu\left(\{x_r, x_s\}\right) - (n-2) \sum_{j=1}^{n} \mu(x_j) \\ &= \frac{k-n}{k-1} \sum_{j=1}^{n} \mu(x_j) \leq 0 \end{aligned}$$

Hence, $\mu(X) = 0$ so the k-set cover is a quantum cover. □

We now briefly consider a generalization of the k-set cover that has physical significance [8]. An **antichain** in $\mathcal{P}(X)$ is a nonempty collection of sets $\{A_1, \ldots, A_m\}$ in $\mathcal{P}(X)$ that are incomparable. That is, $A_i \not\subseteq A_j$ for $i \neq j$, $i, j = 1, \ldots, m$. An antichain $\{A_1, \ldots, A_m\}$ is **maximal** if it is not contained in a strictly larger antichain. That is, for any $B \in \mathcal{P}(X)$ either $B \subseteq A_i$ or $A_i \subseteq B$ for some $i = 1, \ldots, m$. Notice that a maximal antichain $\{A_1, \ldots, A_m\}$ forms a cover for X. Indeed, for any $x \in X$ there is an A_i such that $\{x\} \subseteq A_i$. Thus $\cup A_i = X$. The k-set cover $\mathcal{C}_k$ is an example of a maximal antichain. To see this, first notice that two different k-sets are incomparable so $\mathcal{C}_k$ forms an antichain. To show maximality, let $B \in \mathcal{P}(X)$. If $|B| < k$, then there exists an $A \in \mathcal{C}_k$ such that $B \subseteq A$ while if $|B| \geq k$, then there exists an $A \in \mathcal{C}_k$ such that $A \subseteq B$. We conclude that maximal antichains generalize the k-set cover, $k = 1, \ldots, n$. It is conjectured in [8] that every maximal antichain is a quantum cover. This is an interesting unsolved problem that the reader is invited to investigate.

7 Super-Quantum Measure Spaces

We say that a map $\mu\colon \mathcal{P}(X) \to \mathbb{R}^+$ is a **grade-m measure** if μ satisfies the **grade-m additivity condition**

$$\begin{aligned}
&\mu(A_1 \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} A_{m+1}) \\
&\quad= \sum_{i_1<\cdots<i_m=1}^{m+1} \mu(A_{i_1} \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} A_{i_m}) - \sum_{i_1<\cdots<i_{m-1}=1}^{m+1} \mu(A_{i_1} \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} A_{i_{m-1}}) \\
&\qquad+ \cdots + (-1)^{m+1} \sum_{i=1}^{m+1} \mu(A_i)
\end{aligned} \tag{7.1}$$

Grade-m measures for $m \geq 3$ correspond to super-quantum measures and these may describe theories that are more general than quantum mechanics. It can be shown by induction that a grade-m measure is a grade-$(m+1)$ measure [5]. Thus, we have a hierarchy of measure grades with each grade contained in all higher grades. Instead of giving the induction proof we will just check that any grade-2 measure μ is also a grade-3 measure. Indeed by Theorem 3.3 we have

$$\begin{aligned}
&\sum_{i<j<k=1}^{4} \mu(A_i \mathbin{\dot\cup} A_j \mathbin{\dot\cup} A_k) - \sum_{i<j=1}^{4} \mu(A_i \mathbin{\dot\cup} A_j) + \sum_{i=1}^{4} \mu(A_i) \\
&\quad= 2\sum_{i<j=1}^{4} \mu(A_i \mathbin{\dot\cup} A_j) - 3\sum_{i=1}^{4} \mu(A_i) - \sum_{i<j=1}^{4} \mu(A_i \mathbin{\dot\cup} A_j) + \sum_{i=1}^{4} \mu(A_i) \\
&\quad= \sum_{i<j=1}^{4} \mu(A_i \mathbin{\dot\cup} A_j) - 2\sum_{i=1}^{4} \mu(A_i) = \mu(A_1 \mathbin{\dot\cup} A_2 \mathbin{\dot\cup} A_3 \mathbin{\dot\cup} A_4)
\end{aligned}$$

The next result whose proof we omit gives a general method of generating grade-m measures. We denote the Cartesian product of a set A with itself m times by A^m.

Theorem 7.1. *If λ is a signed measure on $\mathcal{P}(X^m)$, then $\mu(A) = \lambda(A^m)$ is a grade-m measure.*

Let μ be a grade-3 measure on $\mathcal{P}(X)$. We define the classical part $\lambda_\mu^1 = \nu_\mu$ and the two-point interference function $I_\mu^2 = I_\mu$ just as we did in Section 4.

We now define the **three-point interference function** by

$$\begin{aligned}I_\mu^3(x_i,x_j,x_k) = \mu\left(\{x_i,x_j,x_k\}\right) - \mu\left(\{x_i,x_j\}\right) - \mu\left(\{x_i,x_k\}\right) - \mu\left(\{x_j,x_k\}\right)\\ + \mu(x_i) + \mu(x_j) + \mu(x_k)\end{aligned}$$

if $i \neq j \neq k$ and $I_\mu^3 = 0$, otherwise. Of course, $I_\mu^3 = 0$ for quantum measures so this term introduces a new type of phenomenon that does not seem to occur in quantum mechanics. We next define the signed measures $\lambda_\mu^2, \lambda_\mu^3$ on $\mathcal{P}(X^2)$ and $\mathcal{P}(X^3)$, respectively by

$$\begin{aligned}\lambda_\mu^2(B) &= \sum\left\{I_\mu^2(x_i,x_j)\colon (x_i,x_j)\in B\right\}\\ \lambda_\mu^3(B) &= \sum\left\{I_\mu^3(x_i,x_j,x_k)\colon (x_i,x_j,x_k)\in B\right\}\end{aligned}$$

The next result extends Theorem 4.1.

Theorem 7.2. *If μ is a grade-3 measure, then for every $B \in \mathcal{P}(X)$ we have*

$$\mu(B) = \lambda_\mu^1(B) + \frac{1}{2!}\,\lambda_\mu^2(B^2) + \frac{1}{3!}\,\lambda_\mu^3(B^3) \tag{7.2}$$

Proof. It follows from Theorem 7.1 that the right side of (7.2) is a grade-3 measure. As in Theorem 4.1, we are finished if we show that this grade-3 measure coincides with μ for k-sets, $k = 1, 2, 3$. They clearly agree for 1-sets and they agree on 2-sets as in the proof of Theorem 4.1. Finally, if $B = \{x_1, x_2, x_3\}$, we have

$$\begin{aligned}\lambda_\mu^1(B) + \frac{1}{2!}\,\lambda_\mu^2(B^2) + \frac{1}{3!}\,\lambda_\mu^3(B^3) &= \mu(x_1) + \mu(x_2) + \mu(x_3) + I_\mu^2(x_1,x_2)\\ &\quad + I_\mu^2(x_1,x_3) + I_\mu^2(x_2,x_3) + I_\mu^3(x_1,x_2,x_3)\end{aligned}$$

Expanding I_μ^2 and I_μ^3 in terms of their definitions, the right side becomes $\mu\left(\{x_1,x_2,x_3\}\right)$. □

Of course, Theorem 7.2 can be generalized to grade-m measures to obtain

$$\mu(B) = \sum_{i=1}^{m} \frac{1}{i!}\,\lambda_\mu^i(B^i) \tag{7.3}$$

In this last paragraph, let your imagination take flight and don't worry about technicalities. If we consider the vector space V of complex-valued functions on $X = \{x_1, \ldots, x_n\}$ with the natural inner product, then V is isomorphic to $\mathbb{C}^n$. In a similar way, the vector space of complex-valued functions on $X \times X$ is isomorphic to $\mathbb{C}^n \otimes \mathbb{C}^n$. We now form the inner product space

$$H = \mathbb{C}^n \oplus \mathbb{C}^n \otimes \mathbb{C}^n \oplus \cdots \oplus \mathbb{C}^{\otimes m}$$

where the last summand is the m-fold tensor product. The measure λ_μ^i in (7.3) provides a linear functional on $\mathbb{C}^{\otimes i}$ via its integral and hence is itself a member of $\mathbb{C}^{\otimes i}$. The q-measure μ in (7.3) is given by some kind of "collapse" of these vectors. For those who know about these sorts of things, this is beginning to look like a Fock space in quantum field theory. But this is a new story that has not yet been told. In any case, the subject of quantum and super-quantum measure spaces may usher in a whole new world for mathematicians to explore.

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