

EXAMPLES OF QUANTUM INTEGRALS

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Abstract

We first consider a method of centering and a change of variable formula for a quantum integral. We then present three types of quantum integrals. The first considers the expectation of the number of heads in n flips of a “quantum coin.” The next computes quantum integrals for destructive pairs examples. The last computes quantum integrals for a (Lebesgue)² quantum measure. For this last type we prove some quantum counterparts of the fundamental theorem of calculus.

1 Introduction

Quantum measure theory was introduced by R. Sorkin in his studies of the histories approach to quantum mechanics and quantum gravity [7]. Since then, he and several other authors have continued this study [1, 2, 6, 8, 9, 10] and the author has developed a general quantum measure theory for infinite cardinality spaces [3]. Very recently the author has introduced the concept of a quantum integral [4]. Although this integral generalizes the classical Lebesgue integral, it may exhibit unusual behaviors that the Lebesgue integral does not. For example, the quantum integral may be nonlinear and nonmonotone. Because of this possible nonstandard behavior we lack intuition concerning properties of the quantum integral. To help us gain some intuition for this new integral, we present various examples of quantum integrals.

The paper begins with a method of centering and a change of variable formula for a quantum integral. Examples of centering and variable changes are given. We also consider quantum integrals over subsets of the measure spaces. We then present three types of quantum integrals. The first considers the expectation a_n of the number of heads in n flips of a “quantum coin.” We prove that a_n asymptotically approaches the classical value $n/2$ as n approaches infinity, and numerical data are given to illustrate this. The next computes quantum integrals for destructive pairs examples. The functions integrated in these examples are monomials. The last computes quantum integrals for (Lebesgue)² quantum measure. For this last type, some quantum counterparts of the fundamental theorem of calculus are proved.

2 Centering and Change of Variables

If (X, \mathcal{A}) is a measurable space, a map $\mu: \mathcal{A} \rightarrow \mathbb{R}^+$ is *grade-2 additive* [1, 2, 3, 7], if

$$\mu(A \cup B \cup C) = \mu(A \cup B) + \mu(A \cup C) + \mu(B \cup C) - \mu(A) - \mu(B) - \mu(C)$$

for any mutually disjoint $A, B, C \in \mathcal{A}$ where $A \cup B$ denotes $A \cup B$ whenever $A \cap B \neq \emptyset$. A *q-measure* is a grade-2 additive map $\mu: \mathcal{A} \rightarrow \mathbb{R}^+$ that also satisfies the following continuity conditions [3]

(C1) For any increasing sequence $A_i \in \mathcal{A}$ we have

$$\mu(\cup A_i) = \lim_{i \rightarrow \infty} \mu(A_i)$$

(C2) For any decreasing sequence $B_i \in \mathcal{A}$ we have

$$\mu(\cap A_i) = \lim_{i \rightarrow \infty} \mu(A_i)$$

A *q-measure* μ is not always additive, that is, $\mu(A \cup B) \neq \mu(A) + \mu(B)$ in general. A *q-measure space* is a triple (X, \mathcal{A}, μ) where (X, \mathcal{A}) is a measurable space and $\mu: \mathcal{A} \rightarrow \mathbb{R}^+$ is a *q-measure* [2, 3, 7]. Let (X, \mathcal{A}, μ) be a *q-measure space* and let $f: X \rightarrow \mathbb{R}$ be a measurable function. It is shown in [4] that the following real-valued functions of $\lambda \in \mathbb{R}$ are Lebesgue measurable:

$$\lambda \mapsto \mu(\{x: f(x) > \lambda\}), \quad \lambda \mapsto \mu(\{x: f(x) < -\lambda\})$$

We define the *quantum integral* of f to be

$$\int f d\mu = \int_0^\infty \mu(\{x: f(x) > \lambda\}) d\lambda - \int_0^\infty \mu(\{x: f(x) < -\lambda\}) d\lambda \quad (2.1)$$

where $d\lambda$ is Lebesgue measure on \mathbb{R} . If μ is an ordinary measure (that is; μ is additive) then $\int f d\mu$ is the usual Lebesgue integral [4]. The quantum integral need not be linear or monotone. That is, $\int (f+g) d\mu \neq \int f d\mu + \int g d\mu$ and $\int f d\mu \not\leq \int g d\mu$ whenever $f \leq g$, in general. However, the integral is homogenous in the sense that $\int \alpha f d\mu = \alpha \int f d\mu$, for $\alpha \in \mathbb{R}$.

Definition (2.1) gives the number zero a special status which is unimportant when μ is a measure, but which makes a nontrivial difference when μ is a general q -measure. It may be useful in applications to define for $a \in \mathbb{R}$ the *a-centered quantum integral*

$$\begin{aligned} \int f d\mu_a &= \int_a^\infty \mu[f^{-1}(\lambda, \infty)] d\lambda - \int_{-a}^\infty \mu[f^{-1}(-\infty, -\lambda)] d\lambda \\ &= \int_a^\infty \mu[f^{-1}(\lambda, \infty)] d\lambda - \int_{-\infty}^a \mu[f^{-1}(-\infty, \lambda)] d\lambda \end{aligned} \quad (2.2)$$

Of course, $\int f d\mu_0 = \int f d\mu$ but we shall omit the subscript 0. Our first result shows how to compute $\int f d\mu_a$ when f is a simple function.

Lemma 2.1. *Let $f = \sum_{i=1}^n \alpha_i \chi_{A_i}$ be a simple function with $A_i \cap A_j = \emptyset$, $i \neq j$, $\cup_{i=1}^n A_i = X$. If $\alpha_1 < \dots < \alpha_m \leq a < \alpha_{m+1} < \dots < \alpha_n$ then*

$$\begin{aligned} \int f d\mu_a &= (\alpha_{m+1} - a) \mu \left(\bigcup_{i=m+1}^n A_i \right) + (\alpha_{m+2} - \alpha_{m-1}) \mu \left(\bigcup_{i=m+2}^n A_i \right) \\ &\quad + \dots + (\alpha_n - \alpha_{n-1}) \mu(A_n) - \left[(a - \alpha_m) \mu \left(\bigcup_{i=1}^m A_i \right) \right. \\ &\quad \left. - (\alpha_m - \alpha_{m-1}) \mu \left(\bigcup_{i=1}^{m-1} A_i \right) + \dots + (\alpha_2 - \alpha_1) \mu(A_1) \right] \end{aligned} \quad (2.3)$$

$$\begin{aligned}
&= \alpha_1 \mu(A_1) + \alpha_2 [\mu(A_1 \cup A_2) - \mu(A_1)] + \cdots + \alpha_m [\mu(A_1 \cup A_m) \\
&\quad + \cdots + \mu(A_{m-1} \cup A_m) - \mu(A_1) - \cdots - \mu(A_{m-1}) - (m-2)\mu(A_m)] \\
&\quad + \alpha_{m+1} [\mu(A_{m+1} \cup A_{m+2}) + \cdots + \mu(A_{m+1} \cup A_n) \\
&\quad - (n-m-2)\mu(A_{m+1}) - \mu(A_{m-2}) - \cdots - \mu(A_n)] \\
&\quad + \cdots + \alpha_{n-1} [\mu(A_{n-1} \cup A_n) - \mu(A_n)] + \alpha_n \mu(A_n) \\
&\quad - a \left[\mu \left(\bigcup_{i=1}^m A_i \right) + \mu \left(\bigcup_{i=m+1}^n A_i \right) \right] \tag{2.4}
\end{aligned}$$

Proof. Equation (2.3) is a straightforward application of the definition (2.2). We can rewrite (2.3) as

$$\begin{aligned}
\int f d\mu_a &= \alpha_{m+1} \left[\mu \left(\bigcup_{i=m+1}^n A_i \right) - \mu \left(\bigcup_{i=m+2}^n A_i \right) \right] \\
&\quad + \alpha_{m+2} \left[\mu \left(\bigcup_{i=m+2}^n A_i \right) - \mu \left(\bigcup_{i=m+3}^n A_i \right) \right] \\
&\quad + \cdots + \alpha_{n-1} [\mu(A_{n-1} \cup A_n) - \mu(A_n)] + \alpha_n \mu(A_n) + \alpha_1 \mu(A_1) \\
&\quad + \alpha_2 \left[\mu \left(\bigcup_{i=1}^2 A_i \right) - \mu(A_1) \right] + \alpha_3 \left[\mu \left(\bigcup_{i=1}^3 A_i \right) - \mu \left(\bigcup_{i=1}^2 A_i \right) \right] \\
&\quad + \cdots + \alpha_m \left[\mu \left(\bigcup_{i=1}^m A_i \right) - \mu \left(\bigcup_{i=1}^{m-1} A_i \right) \right] \\
&\quad - a \left[\mu \left(\bigcup_{i=1}^m A_i \right) + \mu \left(\bigcup_{i=m+1}^n A_i \right) \right]
\end{aligned}$$

Applying Theorem 3.3 [3] the result follows. \square

Corollary 2.2. *If μ is a measure and f is integrable, then*

$$\int f d\mu_a = \int f d\mu - a\mu(X)$$

Proof. By (2.4) the formula holds for simple functions. Approximate f by an increasing sequence of simple functions and apply the monotone convergence theorem. \square

As an illustration of Lemma 2.1, let $f = a\chi_A + b\chi_B$ be a simple function with $A \cap B = \emptyset$, $A \cup B = X$, $0 \leq a < b$. By (2.4) we have

$$\int f d\mu = a [\mu(A \cup B) - \mu(B)] + b\mu(B)$$

This also shows that the quantum integral is nonlinear because if $\mu(A \cup B) \neq \mu(A) + \mu(B)$ then

$$\int (a\chi_A + b\chi_B) d\mu = \int f d\mu \neq a\mu(A) + b\mu(B) = a \int \chi_A d\mu + b \int \chi_B d\mu$$

Corollary 2.2 shows that if μ is a measure, then $\int f d\mu_a$ is just a translation of $\int f d\mu$ by the constant $a\mu(X)$ for all integrable f . We now show that this does not hold when μ is a general q -measure.

Example 1. Let $a > 0$ be a fixed constant and let $f = c\chi_A$ be a simple function with $c \neq 0$ and $A \neq \emptyset, X$. We can write f in the canonical form

$$f = 0\chi_{A'} + c\chi_A$$

where A' is the complement of A . By Lemma 2.1 we have that

$\int f d\mu = c\mu(A)$ and applying (2.4) to the various cases we obtain the following:

Case 1. If $0 < a < c$, then $\alpha_1 = 0$, $\alpha_2 = c$, $\alpha_1 < a < \alpha_2$, $A_1 = A'$ and $A_2 = A$. We compute

$$\begin{aligned} \int f d\mu_a &= a\mu(A') + c\mu(A) - a [\mu(A') + \mu(A)] \\ &= \int f d\mu - a [\mu(A') + \mu(A)] \end{aligned}$$

Case 2. If $0 < c < a$, then $\alpha_1 = 0$, $\alpha_2 = c$, $\alpha_1 < \alpha_2 < a$, $A_1 = A'$ and $A_2 = A$. We compute

$$\begin{aligned} \int f d\mu_a &= 0\mu(A') + c [\mu(X) - \mu(A')] - a\mu(X) \\ &= \int f d\mu - c [\mu(A) + \mu(A') - \mu(X)] - a\mu(X) \end{aligned}$$

Case 3. If $c < 0 < a$, then $\alpha_1 = c$, $\alpha_2 = 0$, $\alpha_1 < \alpha_2 < a$, $A_1 = A$ and $A_2 = A'$. We compute

$$\int f d\mu_a = c\mu(A) + 0 [\mu(X) - \mu(A)] - a\mu(X) = \int f d\mu - a\mu(X)$$

We now derive a change of variable formula. Suppose g is an increasing and differentiable function on \mathbb{R} and let $g^{-1}(\pm\infty) = \lim_{\lambda \rightarrow \pm\infty} g^{-1}(\lambda)$. If $f: X \rightarrow \mathbb{R}$ is measurable, then so is $g \circ f$ and we have

$$\begin{aligned} \int g \circ f d\mu_a &= \int_a^\infty \mu[\{x: g \circ f(x) > \lambda\}] d\lambda - \int_{-\infty}^a \mu[\{x: g \circ f(x) < \lambda\}] d\lambda \\ &= \int_a^\infty \mu[\{x: f(x) > g^{-1}(\lambda)\}] d\lambda - \int_{-\infty}^a \mu[\{x: f(x) < g^{-1}(\lambda)\}] d\lambda \end{aligned}$$

Letting $t = g^{-1}(\lambda)$, $g(t) = \lambda$, $g'(t)dt = d\lambda$, by the usual change of variable formula we obtain

$$\begin{aligned} \int g \circ f d\mu_a & \tag{2.5} \\ &= \int_{g^{-1}(a)}^{g^{-1}(\infty)} \mu[\{x: f(x) > t\}] g'(t)dt - \int_{g^{-1}(-\infty)}^{g^{-1}(a)} \mu[\{x: f(x) < t\}] g'(t)dt \end{aligned}$$

For example, if $f \geq 0$, letting $g(t) = t^n$ we have

$$\int f^n d\mu = \int_0^\infty \mu[\{x: f(x) > t\}] nt^{n-1} dt \tag{2.6}$$

As with the Lebesgue integral, if $A \in \mathcal{A}$ we define

$$\int_A f d\mu = \int \chi_A f d\mu$$

We then have

$$\begin{aligned} \int_A f d\mu &= \int_0^\infty \mu[\{x: \chi_A(x)f(x) > \lambda\}] d\lambda - \int_0^\infty \mu[\{x: \chi_A(x)f(x) < -\lambda\}] d\lambda \\ &= \int_0^\infty \mu[A \cap f^{-1}(\lambda, \infty)] d\lambda - \int_0^\infty \mu[A \cap f^{-1}(-\infty, -\lambda)] d\lambda \\ &= \int_0^\infty \{\mu[A \cap f^{-1}(\lambda, \infty)] - \mu[A \cap f^{-1}(-\infty, -\lambda)]\} d\lambda \end{aligned}$$

Now $(A, \mathcal{A} \cap A)$ is a measurable space and it is easy to check that $\mu_A(B) = \mu(A \cap B)$ is a q -measure on $\mathcal{A} \cap A$ so $(A, \mathcal{A} \cap A, \mu_A)$ is a q -measure space. Hence, for a measurable function $f: X \rightarrow \mathbb{R}$, the restriction $f|_A: A \rightarrow \mathbb{R}$ is measurable and

$$\int_A f d\mu = \int f|_A d\mu_A$$

Similar definitions apply to the centered integrals $\int_A f d\mu_a$.

3 A Quantum Coin

If we flip a coin n times the resulting sample space X_n consists of 2^n outcomes each being a sequence of n heads or tails. For example, a possible outcome for 3 flips is HHT and $X_2 = \{HH, HT, TH, TT\}$. If this were an ordinary fair coin then the probability of a subset $A \subseteq X_n$ would be $|A|/2^n$ where $|A|$ is the cardinality of A . However, suppose we are flipping a “quantum coin” for which the probability is replaced by the q -measure $\mu_n(A) = |A|^2/2^{2n}$. It is easy to check that μ is indeed a q -measure. In fact the square of any measure is a q -measure.

Let $f_n: X_n \rightarrow \mathbb{R}$ be the random variable that gives the number of heads in n flips. For example, $f_3(HHT) = 2$. For an ordinary coin the expectation of f_n is $n/2$. We are interested in computing the “quantum expectation” $\int f_n d\mu_n$ for a “quantum coin.” For the case $n = 1$ we have $X_1 = \{x_1, x_2\}$ with $f_1(x_1) = 1$, $f_1(x_2) = 0$. Then $f_1 = \chi_{\{x_1\}}$ and by (2.3) we have

$$\int f_1 d\mu_1 = \mu_1(\{x_1\}) = \frac{1}{4}$$

For the case $n = 2$, we have $X_2 = \{x_1, x_2, x_3, x_4\}$ with $f_2(x_1) = 2$, $f_2(x_2) = f_2(x_3) = 1$, $f_2(x_4) = 0$. Then

$$f_2 = \chi_{\{x_2, x_3\}} + 2\chi_{\{x_1\}}$$

and by (2.3) we have

$$\int f_2 d\mu_2 = \mu_2(\{x_1, x_2, x_3\}) + \mu_2(\{x_1\}) = \frac{9}{16} + \frac{1}{16} = \frac{5}{8}$$

Continuing this process, $X_3 = \{x_1, \dots, x_8\}$ and

$$f_3 = \chi_{\{x_5, x_6, x_7\}} + 2\chi_{\{x_2, x_3, x_4\}} + 3\chi_{\{x_1\}}$$

By (2.3) we obtain

$$\begin{aligned} \int f_3 d\mu_3 &= \mu_3(\{x_1, \dots, x_7\}) + \mu_3(\{x_1, \dots, x_4\}) + \mu_3(\{x_1\}) \\ &= \frac{49}{64} + \frac{16}{64} + \frac{1}{64} = \frac{33}{32} \end{aligned}$$

For 4 flips, $X_4 = \{x_1, \dots, x_{16}\}$ and

$$\begin{aligned} \int f_4 d\mu_4 &= \mu_4(\{x_1, \dots, x_{15}\}) + \mu_4(\{x_1, \dots, x_{11}\}) \\ &\quad + \mu_4(\{x_1, \dots, x_5\}) + \mu(\{x_1\}) \\ &= \frac{15^2 + 11^2 + 5^2 + 1}{16^2} = \frac{93}{64} \end{aligned}$$

For 5 flips, $X_5 = \{x_1, \dots, x_{32}\}$ and

$$\begin{aligned} \int f_5 d\mu_5 &= \mu_5(\{x_1, \dots, x_{31}\}) + \mu_5(\{x_1, \dots, x_{26}\}) + \mu_5(\{x_1, \dots, x_{16}\}) \\ &\quad + \mu_5(\{x_1, \dots, x_6\}) + \mu(\{x_1\}) \\ &= \frac{31^2 + 26^2 + 16^2 + 6^2 + 1}{32^2} = \frac{965}{512} \end{aligned}$$

Letting $a_n = \int f_n d\mu_n$ we have that

$$\begin{aligned} a_1 &= \frac{1}{2^2} \binom{1}{0}^2 \\ a_2 &= \frac{1}{2^4} \left\{ \binom{2}{0}^2 + \left[\binom{2}{0} + \binom{2}{1} \right]^2 \right\} \\ a_3 &= \frac{1}{2^6} \left\{ \binom{3}{0}^2 + \left[\binom{3}{0} + \binom{3}{1} \right]^2 + \left[\binom{3}{0} + \binom{3}{1} + \binom{3}{2} \right]^2 \right\} \\ &\quad \vdots \\ a_n &= \frac{1}{2^{2n}} \left\{ \binom{n}{0}^2 + \left[\binom{n}{0} + \binom{n}{1} \right]^2 + \dots + \left[\binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n-1} \right]^2 \right\} \end{aligned}$$

We shall show that a_n asymptotically approaches the classical value $n/2$ for large n ; that is

$$\lim_{n \rightarrow \infty} \frac{2a_n}{n} = 1 \quad (3.1)$$

As numerical evidence for this result the first seven values of $2a_n/n$ are: 0.5000, 0.6250, 0.6875, 0.7266, 0.7539, 0.7749, 0.7905 and the twentieth value is 0.8737. The next result shows that the quantum expectation does not exceed the classical expectation.

Lemma 3.1. For all $n \in \mathbb{N}$, $\int f_n d\mu_n \leq n/2$.

Proof. Letting $A_i = f_n^{-1}(\{i\})$, $i = 1, \dots, n$, applying (2.4) we obtain

$$\begin{aligned}
\int f_n d\mu_n &= [\mu_n(A_1 \cup A_2) + \dots + \mu_n(A_1 \cup A_n) - (n-2)\mu_n(A_1) \\
&\quad - \mu_n(A_2) - \dots - \mu_n(A_n)] \\
&\quad + 2[\mu_n(A_1 \cup A_3) + \dots + \mu_n(A_2 \cup A_n) - (n-3)\mu_n(A_2) \\
&\quad - \mu_n(A_3) - \dots - \mu_n(A_n)] \\
&\quad \vdots \\
&\quad + (n-1)[\mu_n(A_{n-1} \cup A_n) - \mu_n(A_n)] + n\mu_n(A_n) \\
&= \frac{1}{2^{2n}} \left\{ |A_1 \cup A_2|^2 + \dots + |A_1 \cup A_n|^2 - (n-2)|A_1|^2 - |A_2|^2 \right. \\
&\quad \left. - \dots - |A_n|^2 \right. \\
&\quad \left. + 2[|A_1 \cup A_3|^2 + \dots + |A_1 \cup A_n|^2 - (n-3)|A_2|^2 - |A_3|^2 \right. \\
&\quad \left. - \dots - |A_n|^2] \right. \\
&\quad \left. \vdots \right. \\
&\quad \left. + (n-1)[|A_{n-1} \cup A_n|^2 - |A_n|^2] + n|A_n|^2 \right\} \\
&= \frac{1}{2^{2n}} \left\{ |A_1|^2 + 2|A_1|(|A_2| + \dots + |A_n|) \right. \\
&\quad \left. + 2[|A_2|^2 + 2|A_2|(|A_3| + \dots + |A_n|)] \right. \\
&\quad \left. + \dots + (n-1)[|A_{n-1}|^2 + 2|A_{n-1}||A_n|] + n|A_n|^2 \right\}
\end{aligned}$$

By the binomial theorem we conclude that

$$\begin{aligned}
\int f_n d\mu_n &= \frac{1}{2^{2n}} \left\{ |A_1|[1 + 2(2^n - |A_1| - 1)] \right. \\
&\quad \left. + 2|A_2|[1 + 2(2^n - |A_2| - |A_1| - 1)] \right. \\
&\quad \left. + \dots + (n-1)|A_{n-1}|[1 + 2(2^n - |A_{n-1}| - \dots - |A_1| - 1)] \right. \\
&\quad \left. + n|A_n|[2^n - 1 - |A_1| - |A_2| - \dots - |A_{n-1}|] \right\} \\
&\leq \frac{1}{2^{2n}} [|A_1|2^n + 2|A_2|2^n + \dots + n|A_n|2^n] \\
&= \frac{1}{2^n} (|A_1| + 2|A_2| + \dots + n|A_n|) = \frac{n}{2}
\end{aligned}$$

where the last equality follows from the classical expectation. \square

We now give a_n in closed form and prove (3.1).

Theorem 3.2. (a) For all $n \in \mathbb{N}$ we have

$$a_n = \frac{1}{2} \left[n + 2 - \left(\frac{n \binom{2n}{n} + 2}{2^{2n}} \right) \right]$$

(b) Equation (3.1) holds.

Proof. (a) Letting

$$b_n = \binom{n}{0}^2 + \left[\binom{n}{0} + \binom{n}{1} \right]^2 + \cdots + \left[\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} \right]^2$$

it is shown in [5] that

$$b_n = (n+2)2^{2n-1} - \frac{1}{2}n \binom{2n}{n} - 1$$

Since $a_n = b_n/2^{2n}$, the result follows.

(b) By Stirling's formula we have the asymptotic approximation

$$n! \approx \sqrt{2\pi n} \frac{n^n}{e^n}$$

for large n . Hence,

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{2^{2n}} \binom{2n}{n} &= \lim_{n \rightarrow \infty} \frac{1}{2^{2n}} \frac{(2n)!}{(n!)^2} = \lim_{n \rightarrow \infty} \frac{1}{2^{2n}} \frac{\sqrt{2\pi n} (2n)^{2n}}{e^{2n}} \cdot \frac{e^{2n}}{2\pi n n^{2n}} \\ &= \lim_{n \rightarrow \infty} \frac{1}{\sqrt{2\pi n}} = 0 \end{aligned}$$

Hence,

$$\lim_{n \rightarrow \infty} \frac{2a_n}{n} = \lim_{n \rightarrow \infty} \left(\frac{n+2}{n} \right) - \lim_{n \rightarrow \infty} \frac{\binom{2n}{n} + 2}{2^{2n}} = 1 \quad \square$$

The next example illustrates the a -centered integral $\int f_2 d\mu_{2a}$ for two flips of a “quantum coin.”

Example 2. The following computations result from applying (2.3). If $a \leq 0$, then

$$\begin{aligned}\int f_2 d\mu_{2a} &= (0 - a)\mu(\{x_1, x_2, x_3, x_4\}) + \mu(\{x_1, x_2, x_3\}) - \mu(\{x_1\}) \\ &= \frac{5}{8} - a\end{aligned}$$

If $0 \leq a \leq 1$, then

$$\begin{aligned}\int f_2 d\mu_{2a} &= (1 - a)\mu(\{x_1, x_2, x_3\}) + \mu(\{x_1\}) - (a - 0)\mu(\{x_4\}) \\ &= (1 - a)\frac{9}{16} + \frac{1}{16} - \frac{1}{16}a = \frac{5}{8} - \frac{5}{8}a\end{aligned}$$

If $1 \leq a \leq 2$, then

$$\begin{aligned}\int f_2 d\mu_{2a} &= (2 - a)\mu(\{x_1\}) - (a - 1)\mu(\{x_2, x_3, x_4\}) - \mu(\{x_4\}) \\ &= (2 - a)\frac{1}{16} - (a - 1)\frac{9}{16} - \frac{1}{16} = \frac{5}{8} - \frac{5}{8}a\end{aligned}$$

If $2 \leq a$, then

$$\begin{aligned}\int f_2 d\mu_{2a} &= -[(a - 2)\mu(\{x_1, x_2, x_3, x_4\}) + \mu(\{x_2, x_3, x_4\}) + \mu(\{x_4\})] \\ &= -\left[(a - 2) + \frac{9}{16} + \frac{1}{16}\right] = \frac{11}{8} - a\end{aligned}$$

We conclude that $\int f_2 d\mu_{2a}$ is piecewise linear as a function of a .

4 Destructive Pairs Examples

Consider $X = [0, 1]$ as consisting of particles for which pairs of the form $(x, x + 3/4)$, $x \in [0, 1/4]$ are *destructive pairs* (or *particle-antiparticle pairs*). Thus, particles in $x \in [0, 1/4]$ annihilate their counterparts in $[3/4, 1]$ while particles in $(1/4, 3/4)$ do not interact with other particles. Let $\mathcal{B}(X)$ be the set of Borel subsets of X and let ν be Lebesgue measure on $\mathcal{B}(X)$. For $A \in \mathcal{B}(X)$ define

$$\mu(A) = \nu(A) - 2\nu(\{x \in A: x + 3/4 \in A\})$$

Thus, $\mu(A)$ is the Lebesgue measure of A after the destructive pairs in A annihilate each other. For example, $\mu([0, 1]) = 1/2$ and $\mu([0, 3/4]) = 3/4$. It can be shown that $(X, \mathcal{B}(X), \mu)$ is a q -measure space [3].

Letting $f(x) = x$ and $0 < b \leq 1$ we shall compute

$$\int_0^b f(x)d\mu = \int_{[0,b]} f(x)d\mu$$

We first define

$$F(\lambda) = \mu(\{x: f\chi_{[0,b]}(x) > \lambda\}) = \mu(\{x \in [0, b] : x > \lambda\}) = \mu((\lambda, b])$$

If $b \leq 3/4$ then

$$F(\lambda) = \begin{cases} b - \lambda & \text{if } \lambda \leq b \\ 0 & \text{if } \lambda > b \end{cases}$$

We obtain

$$\int_0^b x d\mu = \int_0^b F(\lambda) d\lambda = \int_0^b (b - \lambda) d\lambda = \frac{b^2}{2}$$

which is the expected classical result because there is no interference (annihilation).

Now suppose that $b \geq 3/4$ in which case there is interference. If $\lambda \geq b - 3/4$, then $F(\lambda) = b - \lambda$ as before. If $\lambda < b - 3/4$, then

$$F(\lambda) = (b - \lambda) - 2\left(b - \frac{3}{4} - \lambda\right) = \frac{3}{2} + \lambda - b$$

We then obtain

$$\begin{aligned} \int_0^b x d\mu &= \int_0^b F(\lambda) d\lambda = \int_0^{b-3/4} \left(\frac{3}{2} + \lambda - b\right) d\lambda + \int_{b-3/4}^b (b - \lambda) d\lambda \\ &= \frac{3}{2}b - \frac{9}{16} - \frac{1}{2}b^2 \end{aligned}$$

For example, $\int_0^1 x d\mu = 7/16$ which is slightly less than $\int_0^1 x d\lambda = 1/2$. Of course, interference is the cause of this difference. Also, $\int_0^{3/4} x d\mu = 9/32$ which agrees with $\int_0^{3/4} x dx$ as shown in the $b \leq 3/4$ case.

We next compute $\int_0^b x^n d\mu$. If $b \leq 3/4$, then by our change of variable formula we have

$$\int_0^b x^n d\mu = n \int_0^b (b - \lambda) \lambda^{n-1} d\lambda = \frac{b^{n+1}}{n+1}$$

in agreement with the classical result. If $b \geq 3/4$, we obtain by the change of variable formula

$$\begin{aligned} \int_0^b x^n d\mu &= n \int_0^b F(\lambda) \lambda^{n-1} d\lambda \\ &= n \left[\int_0^{b-3/4} \left(\frac{3}{2} + \lambda - b\right) \lambda^{n-1} d\lambda + \int_{b-3/4}^b (b - \lambda) \lambda^{n-1} d\lambda \right] \\ &= \frac{1}{n+1} \left[b^{n+1} - 2 \left(b - \frac{3}{4}\right)^{n+1} \right] \end{aligned}$$

As a check, if $n = 1$ we obtain our previous result. Notice that the deviation from the classical integral becomes

$$\int_0^b x^n dx - \int_0^b x^n d\mu = \frac{2}{n+1} \left(b - \frac{3}{4}\right)^{n+1}$$

which increases as b approaches 1.

We now change the previous example so that we only have destructive pairs in which case we obtain more interference. We again let $X = [0, 1]$, but now we define the q -measure

$$\mu(A) = \nu(A) - 2\nu(\{x \in A: x + \frac{1}{2} \in A\})$$

In this case, $(x, x + 1/2)$, $x \in [0, 1/2]$ are destructive pairs. For instance, $\mu(X) = 0$, $\mu([1/16, 5/6]) = 1/3$, and $\mu([0, 1/2]) = 1/2$. Letting $f(x) = x$ and $0 \leq a < b \leq 1$, we shall compute

$$\int_a^b x d\mu = \int_{(a,b)} x d\mu$$

We then have

$$\begin{aligned} F(\lambda) &= \mu(\{x: f\chi_{(a,b)}(x) > \lambda\}) = \mu(\{x \in (a, b): x > \lambda\}) \\ &= \begin{cases} \mu((a, b)) & \text{if } \lambda \leq a \\ \mu((\lambda, b)) & \text{if } a \leq \lambda \leq b \\ 0 & \text{if } \lambda \geq b \end{cases} \end{aligned}$$

Now $\{x \in (a, b) : x + \frac{1}{2} \in (a, b)\} = \emptyset$ if and only if $b - a \leq 1/2$. If $b - a \leq 1/2$ we have

$$F(\lambda) = \begin{cases} b - a & \text{if } \lambda \leq a \\ b - \lambda & \text{if } a \leq \lambda \leq b \\ 0 & \text{if } \lambda \geq b \end{cases}$$

We then obtain

$$\int_a^b x d\mu = \int_0^a (b - a) d\lambda + \int_a^b (b - \lambda) d\lambda = \frac{b^2}{2} - \frac{a^2}{2}$$

which is expected because there is not interference.

If $b - a \geq 1/2$, letting $c = b - 1/2$ we have that $c \geq a$ and

$$\mu((a, b)) = b - a - 2(c - a) = b - a - 2(b - \frac{1}{2} - a) = a - b + 1$$

If $\lambda \leq b - 1/2$, then $\mu((\lambda, b)) = \lambda - b + 1$ and if $\lambda \geq b - 1/2$, then $\mu((\lambda, b)) = b - \lambda$. Hence,

$$F(\lambda) = \begin{cases} a - b + 1 & \text{if } \lambda \leq a \\ \lambda - b + 1 & \text{if } a \leq \lambda \leq b - 1/2 \\ b - \lambda & \text{if } b - 1/2 \leq \lambda \leq b \end{cases}$$

We conclude that

$$\begin{aligned} \int_a^b x d\mu &= \int_0^a (a - b + 1) d\lambda + \int_a^{b-1/2} (\lambda - b + 1) d\lambda + \int_{b-1/2}^b (b - \lambda) d\lambda \\ &= \frac{a^2}{2} - \frac{b^2}{2} + b - \frac{1}{4} \end{aligned}$$

The deviation from the classical integral becomes

$$\Delta = \int_a^b x dx - \int_a^b x d\mu = b^2 - a^2 - b + \frac{1}{4}$$

Notice that $\Delta = 0$ if and only if $b = a + 1/2$. Special cases of the integral are

$$\begin{aligned}\int_0^b x d\mu &= -\frac{b^2}{2} + b - \frac{1}{4} \\ \int_0^{1/2} x d\mu &= \frac{1}{8} \\ \int_0^{3/4} x d\mu &= \frac{7}{32} \\ \int_0^1 x d\mu &= \frac{1}{4}\end{aligned}$$

5 (Lebesgue)² Quantum Measure

We again let $X = [0, 1]$ and let ν be Lebesgue measure on $\mathcal{B}(X)$. We define (Lebesgue)² q -measure by $\mu(A) = \nu(A)^2$ for $A \in \mathcal{B}(X)$ and consider the q -measure space $(X, \mathcal{B}(X), \mu)$. The first example in this section is the a -centered quantum integral $\int x^n d\mu_a$. Applying the change of variable formula we obtain

$$\begin{aligned}\int x^n d\mu_a &= n \int_a^1 \mu(\{x: x > t\}) t^{n-1} dt - n \int_0^a \mu(\{x: x < t\}) t^{n-1} dt \\ &= n \int_a^1 (1-t)^2 t^{n-1} dt - n \int_0^a t^2 t^{n-1} dt \\ &= \frac{2}{(n+1)(n+2)} - a^n \left(1 - \frac{2n}{n+1} a + \frac{2n}{n+2} a^2 \right)\end{aligned}$$

As special cases we have

$$\begin{aligned}\int x d\mu_a &= \frac{1}{3} - a + a^2 - \frac{2}{3} a^3 \\ \int x^n d\mu &= \frac{2}{(n+1)(n+2)}\end{aligned}$$

We now compute the quantum integral $\int_a^b x^n d\mu$ for $0 \leq a < b \leq 1$. Again

the change of variable formula gives

$$\begin{aligned}
\int_a^b x^n d\mu &= \int_0^\infty \mu((a, b) \cap \{x: x > \lambda^{1/n}\}) d\lambda \\
&= n \int_0^\infty \mu((a, b) \cap \{x: x > t\}) t^{n-1} d\lambda \\
&= n \int_a^b (b-t)^2 t^{n-1} dt + n \int_0^a (b-a)^2 t^{n-1} dt \\
&= \frac{2}{(n+1)(n+2)} (b^{n+2} - a^{n+2}) - \frac{2a^{n+1}}{n+1} (b-a)
\end{aligned}$$

As special cases we have

$$\begin{aligned}
\int_a^b x d\mu &= \frac{b^3}{3} - \frac{a^3}{3} - a^2(b-a) \\
\int_a^b d\mu &= (b-a)^2
\end{aligned}$$

We can compute $\int_a^b x^n d\mu$ another way without relying on a change of variables:

$$\begin{aligned}
\int_a^b x^n d\mu &= \int_0^\infty \mu((a, b) \cap \{x: x > \lambda^{1/n}\}) d\lambda \\
&= \int_{a^n}^{b^n} (b - \lambda^{1/n})^2 d\lambda + \int_0^{a^n} (b-a)^2 d\lambda \\
&= \int_{a^n}^{b^n} (b^2 - 2b\lambda^{1/n} + \lambda^{2/n}) d\lambda + (b-a)^2 a^n \\
&= \frac{2}{(n+1)(n+2)} b^{n+2} - 2a^{n+1} \left(\frac{b}{n+1} - \frac{a}{n+2} \right)
\end{aligned}$$

which agrees with our previous result.

Until now we have only integrated monomials. We now integrate the more complex function e^x . By the change of variable formula

$$\begin{aligned}
\int_a^b e^x d\mu &= \int_{-\infty}^\infty \mu((a, b) \cap \{x: x > t\}) e^t dt \\
&= \int_a^b (b-t)^2 e^t dt + \int_{-\infty}^a (b-a)^2 e^t dt \\
&= 2 [e^b - e^a - e^a(b-a)]
\end{aligned}$$

In particular,

$$\int_0^b e^x dx = 2(e^b - 1 - b)$$

For the Lebesgue integral we have the formula

$$\int_a^b f(x)dx = \int_0^b f(x)dx - \int_0^a f(x)dx$$

which is frequently used to simplify computations. This formula does not hold for our q -measure μ . However, we do have the following result.

Theorem 5.1. *If f is increasing, differentiable, nonnegative on $[0, 1]$ and $f^{-1}(\infty) \geq b$, $f^{-1}(0) \leq a$, then*

$$\int_a^b f d\mu = \int_0^b f d\mu - \int_0^a f d\mu - 2(b-a) \int_0^a f(t)dt$$

Proof. Employing the change of variable formula gives

$$\begin{aligned} \int_a^b f d\mu &= \int_{f^{-1}(0)}^{f^{-1}(\infty)} \mu((a, b) \cap \{x: x > t\}) f'(t) dt \\ &= \int_a^b (b-t)^2 f'(t) dt + \int_{f^{-1}(0)}^a (b-a)^2 f'(t) dt \\ &= \int_0^b (b-t)^2 f'(t) dt - \int_0^a (b-t)^2 f'(t) dt + (b-a)^2 f(a) \end{aligned}$$

On the other hand, using integration by parts we have

$$\begin{aligned} \int_a^b f d\mu - \int_0^a f d\mu &= \int_0^b (b-t)^2 f'(t) dt + b^2 f(0) - \int_0^a (a-t)^2 f'(t) dt - a^2 f(0) \\ &= \int_0^b (b-t)^2 f'(t) dt - \int_0^a (b-t)^2 f'(t) dt \\ &\quad + \int_0^a [(b-t)^2 - (a-t)^2] f'(t) dt + (b^2 - a^2) f(0) \\ &= \int_0^b (b-t)^2 f'(t) dt - \int_0^a (b-t)^2 f'(t) dt \\ &\quad + (b^2 - a^2) f(a) - 2(b-a) \int_0^a t f'(t) dt \end{aligned}$$

$$\begin{aligned}
&= \int_0^a (b-t)^2 f'(t) dt - \int_0^a (b-t)^2 f'(t) dt \\
&\quad + (b-a)^2 f(a) + 2(b-a) \int_0^a f(t) dt
\end{aligned}$$

The result now follows. \square

Example 3. In this example we use some previous computations to verify Theorem 5.1. We have shown that

$$\int_a^b x^n d\mu = \frac{2}{(n+1)(n+2)} b^{n+2}$$

Hence, by Theorem 5.1 we have

$$\begin{aligned}
\int_a^b x^n d\mu &= \int_0^b x^n d\mu - \int_0^a x^n d\mu - 2(b-a) \int_0^a t^n dt \\
&= \frac{2}{(n+1)(n+2)} (b^{n+2} - a^{n+2}) - \frac{2a^{n+1}}{n+1} (b-a)
\end{aligned}$$

which agrees with our previous result. We have shown that

$$\int_a^b e^x d\mu = 2(e^b - 1 - b)$$

Hence, by Theorem 5.1 we have

$$\begin{aligned}
\int_a^b e^x d\mu &= \int_0^b e^x d\mu - \int_0^a e^x d\mu - 2(b-a) \int_0^a e^t dt \\
&= 2(e^b - 1 - b) - 2(e^a - 1 - a) - 2(b-a)(e^a - 1) \\
&= 2[e^b - e^a - e^a(b-a)]
\end{aligned}$$

which agrees with our previous result.

Example 4. We compute the quantum integral of $f(x) = x + x^2$. By the change of variable formula we have

$$\begin{aligned}
\int_0^b (x + x^2) d\mu &= \int_0^b (b-t)^2 (1+2t) dt = \int_0^b (b-t)^2 dt + 2 \int_0^b (b-t)^2 t dt \\
&= \frac{b^3}{3} + \frac{b^4}{6}
\end{aligned}$$

This gives the surprising result that

$$\int_0^b (x + x^2) d\mu = \int_0^b x d\mu + \int_0^b x^2 d\mu$$

We shall later show that this quantum integral is always additive for sums of increasing continuous functions even if they are not differentiable. The next example shows that this result does not hold for two monomials if their sum is not increasing.

Example 5. Let $f(x) = x - x^2$ for $x \in [0, 1]$. To evaluate $\int_0^b f(x) d\mu$ we cannot use the change of variable formula because f is not increasing, so we will proceed directly. Let $1/2 \leq b \leq 1$. For $0 \leq \lambda \leq 1/4$ we have that $\lambda = x - x^2$, if and only if $x = (1 \pm \sqrt{1 - 4\lambda}) / 2$. Hence, for $\lambda \geq b - b^2$ we have

$$\nu((0, b) \cap \{x: x - x^2 > \lambda\}) = \begin{cases} \sqrt{1 - 4\lambda} & \text{if } 0 \leq \lambda \leq 1/4 \\ 0 & \text{if } 1/4 \leq \lambda \leq 1 \end{cases}$$

and for $\lambda \leq b - b^2$ we have

$$\nu((0, b) \cap \{x: x - x^2 > \lambda\}) = b - \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4\lambda}$$

Hence,

$$\begin{aligned} \int_0^b (x - x^2) d\mu &= \int_0^{b-b^2} \left(b - \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4\lambda}\right)^2 d\lambda + \int_{b-b^2}^{1/4} (1 - 4\lambda) d\lambda \\ &= -\frac{1}{24} + \frac{1}{3} b - b^2 + \frac{5}{3} b^3 - \frac{5}{6} b^4 \end{aligned}$$

Notice that this does not coincide with

$$\int_0^b x d\mu - \int_0^b x^2 d\mu = \frac{1}{3} b^3 - \frac{1}{6} b^4$$

For completeness we evaluate the integral with $0 \leq b \leq 1/2$. Since f is increasing on this interval we obtain the expected result:

$$\int_0^b (x - x^2) d\mu = \int_0^{b-b^2} \left(b - \frac{1}{2} + \frac{1}{2} \sqrt{1 - 4\lambda}\right)^2 d\lambda = \frac{1}{3} b^3 - \frac{1}{6} b^4$$

Example 6. Let f be the following piecewise linear function:

$$f(x) = \begin{cases} 2x & \text{if } 0 \leq x \leq 1/2 \\ 2 - 2x & \text{if } 1/2 \leq x \leq 1 \end{cases}$$

Let $1/2 \leq b \leq 1$. For $0 \leq \lambda \leq 2 - 2b$ we have

$$\nu((0, b) \cap \{x: f(x) > \lambda\}) = b - \frac{\lambda}{2}$$

and for $2 - 2b \leq \lambda \leq 1$ we have

$$\nu((0, b) \cap \{x: f(x) > \lambda\}) = 1 - \lambda$$

Hence

$$\int_0^b f d\mu = \int_0^{2-2b} (b - \frac{\lambda}{2})^2 d\lambda + \int_{2-2b}^1 (1 - \lambda)^2 d\lambda = \frac{1}{3} - 2b + 4b^2 - 2b^3$$

If $0 \leq b \leq 1/2$ we obtain the expected result

$$\int_0^b f d\mu = \int_0^{2b} (b - \frac{\lambda}{2})^2 d\lambda = \frac{2}{3} b^3$$

Observe that

$$\begin{aligned} \frac{1}{2} \frac{d^2}{db^2} \int_0^b x^n d\mu &= b^n \\ \frac{1}{2} \frac{d^2}{db^2} \int_0^b e^x d\mu &= e^b \end{aligned}$$

However, in Example 5 we have for $b > 1/2$ that

$$\frac{1}{2} \frac{d^2}{db^2} \int_0^b (x - x^2) d\mu = -1 + 5b - 5b^3 \neq b - b^2$$

and in Example 6 we have for $b > 1/2$ that

$$\frac{1}{2} \frac{d^2}{db^2} \int_0^b f d\mu = 4 - 6b \neq 2 - 2b = f(b)$$

These examples again illustrate the special nature of increasing functions. The next result is a quantum counterpart to the fundamental theorem of calculus.

Theorem 5.2. *If f is continuous and monotone on $[0, 1]$, then*

$$\frac{1}{2} \frac{d^2}{db^2} \int_0^b f d\mu = f(b)$$

Proof. If f is decreasing then $-f$ is increasing so we can assume f is increasing. For a positive integer n , let g be the following increasing step function on $[0, 1]$:

$$g = c_1 \chi_{[0, 1/n]} + c_2 \chi_{(1/n, 2/n]} + \cdots + c_n \chi_{((n-1)/n, 1]}$$

where $0 < c_1 < \cdots < c_n$. For $0 < b \leq 1$ we have that $(m-1)/n < b \leq m/n$ for some integer $0 < m \leq n$ and

$$g \chi_{[0, b]} = c_1 \chi_{[0, 1/n]} + c_2 \chi_{(1/n, 2/n]} + \cdots + c_{m-1} \chi_{((m-2)/n, (m-1)/n]} + c_m \chi_{((m-1)/n, b]}$$

Letting $A_i = ((i-1)/n, i/n]$, $i = 1, \dots, m-1$, $A_m = ((m-1)/n, b]$ and $\widehat{b} = b - (m-1)/n$ we have by (2.4) of Lemma 2.1 that

$$\begin{aligned} \int_0^b g d\mu &= c_1 [\mu(A_1 \cup A_2) + \cdots + \mu(A_1 \cup A_m)] \\ &\quad - (m-2)\mu(A_1) - \mu(A_2) - \cdots - \mu(A_m)] \\ &\quad + c_2 [\mu(A_2 \cup A_3) + \cdots + \mu(A_2 \cup A_m)] \\ &\quad - (m-3)\mu(A_2) - \mu(A_3) - \cdots - \mu(A_m)] \\ &\quad + \cdots + c_{m-1} [\mu(A_{m-1} \cup A_m) - \mu(A_m)] + c_m \mu(A_m) \\ &= c_1 \left[(m-2) \left(\frac{2}{n} \right)^2 + \left(\frac{1}{n} + \widehat{b} \right)^2 - (2m-4) \left(\frac{1}{n} \right)^2 - \widehat{b}^2 \right] \\ &\quad + c_2 \left[(m-3) \left(\frac{2}{n} \right)^2 + \left(\frac{1}{n} + \widehat{b} \right)^2 - (2m-6) \left(\frac{1}{n} \right)^2 - \widehat{b}^2 \right] \\ &\quad + \cdots + c_{m-1} \left[\left(\frac{1}{n} + \widehat{b} \right)^2 - \widehat{b}^2 \right] + c_m \widehat{b}^2 \\ &= c_1 \left[(2m-3) \frac{1}{n^2} + \frac{2}{n} \widehat{b} \right] + c_2 \left[(2m-5) \frac{1}{n^2} + \frac{2}{n} \widehat{b} \right] \\ &\quad + \cdots + c_{m-1} \left(\frac{1}{n^2} + \frac{2}{n} \widehat{b} \right) + c_m \widehat{b}^2 \end{aligned}$$

It follows that

$$\frac{d}{db} \int_0^b g d\mu = \frac{2}{n} (c_1 + c_2 + \cdots + c_{m-1}) + 2c_m \widehat{b} \quad (5.1)$$

and that

$$\frac{1}{2} \frac{d^2}{db^2} \int_0^b g d\mu = c_m = g(b) \quad (5.2)$$

We can assume without loss of generality that f is nonnegative. Then there exists an increasing sequence of increasing nonnegative step functions s_i converging uniformly to f . Since

$$\mu [s_{i+1}^{-1}(\lambda, \infty)] \geq \mu [s_i^{-1}(\lambda, \infty)]$$

we have by the continuity of μ that

$$\mu [f^{-1}(\lambda, \infty)] = \mu [\cup s_i^{-1}(\lambda, \infty)] = \lim \mu [s_i^{-1}(\lambda, \infty)]$$

These same formulas apply to $f\chi_{[0,b]}$ and $s_i\chi_{[0,b]}$. By the quantum bounded monotone convergence theorem [4] we conclude that

$$\int_0^b f d\mu_i = \lim \int_0^b s_i d\mu$$

Applying (5.1) with g replaced by s_i it can be checked that the sequence of functions of b given by

$$\frac{d}{db} \int_0^b s_i d\mu$$

is uniformly Cauchy so the sequence converges and hence

$$\frac{d}{db} \int_0^b f d\mu = \lim \frac{d}{db} \int_0^b s_i d\mu$$

By (5.2)

$$\frac{d^2}{db^2} \int_0^b s_i d\mu$$

converges uniformly so we have

$$\frac{1}{2} \frac{d^2}{db^2} \int_0^b f d\mu = \lim \frac{1}{2} \frac{d^2}{db^2} \int_0^b s_i d\mu = f(b) \quad \square$$

Lemma 5.3. *If f is continuous and monotone on $[0, 1]$, then*

$$\left[\frac{d}{db} \int_0^b f d\mu \right] (0) = 0$$

Proof. We can assume without loss of generality that f is increasing. Let $g = \sum c_i \chi_{A_i}$ be a step function as in the proof of Theorem 5.2. If b is sufficiently small we have

$$g\chi_{[0,b]} = c_1 \chi_{A_1 \cap [0,b]} = c_1 \chi_{[0,b]}$$

Hence, for such b we have

$$\int_0^b g d\mu = \int g \chi_{[0,b]} d\mu = c_1 \int \chi_{[0,b]} d\mu = c_1 b^2$$

Therefore,

$$\left[\frac{d}{db} \int_0^b g d\mu \right] (0) = \left(\frac{d}{db} c_1 b^2 \right) (0) = 0$$

As shown in the proof of Theorem 5.2, there exists an increasing sequence of step functions s_i such that

$$\frac{d}{db} \int_0^b f d\mu = \lim \frac{d}{db} \int_0^b s_i d\mu$$

The result follows. □

Part (b) of the next theorem is the second half of the quantum fundamental theorem of calculus.

Theorem 5.4. (a) *If f is continuous and monotone on $[0, 1]$, then*

$$\int_0^b f d\mu = 2 \int_0^b \int_0^t f(x) dx dt$$

(b) *If f'' is monotone and continuous on $[0, 1]$, then*

$$\int_0^b \frac{1}{2} f'' d\mu = f(b) - f(0) - f'(0)b$$

Proof. (a) If $g'' = f$, then integrating gives

$$\int_0^t f(x) dx = g'(t) - g'(0)$$

Integrating again we have

$$\int_0^b \int_0^t f(x) dx dt = g(b) - g(0) - g'(0)b$$

Hence, for all $b \in [0, 1]$ we have

$$g(b) = g(0) + g'(0)b + \int_0^b \int_0^t f(x) dx dt$$

Since by Theorem 5.2

$$\frac{d^2}{db^2} \int_0^b \frac{1}{2} f d\mu = f(b)$$

letting $g(b) = \int_0^b \frac{1}{2} f d\mu$ we have that $g(0) = 0$ and by Lemma 5.3 we obtain $g'(0) = 0$. Hence,

$$\int_0^b \int_0^t f(x) dx dt = g(b) = \frac{1}{2} \int_0^b f d\mu$$

(b) By Part (a) we have

$$\begin{aligned} \int_0^b \frac{1}{2} f'' d\mu &= \int_0^b \int_0^t f''(x) dx dt = \int_0^b [f'(t) - f'(0)] dt \\ &= f(b) - f(0) - f'(0)b \end{aligned} \quad \square$$

The next corollary follows from Theorem 5.4(a)

Corollary 5.5. *The quantum (Lebesgue)² integral is additive for increasing (decreasing) continuous functions.*

Example 7. We compute some quantum integrals using Theorem 5.4(a).

$$\begin{aligned} \int_0^b \cos x d\mu &= 2 \int_0^b \int_0^t \cos x dx dt = 2 \int_0^b \sin t dt = 2(1 - \cos b) \\ \int_0^b \sin x d\mu &= 2 \int_0^b \int_0^t \sin x dx dt = 2 \int_0^b (1 - \cos t) dt = 2(b - \sin b) \\ \int_0^b \cosh \sqrt{2} x d\mu &= 2 \int_0^b \int_0^t \cosh \sqrt{2} x dx dt = \sqrt{2} \int_0^b \sinh \sqrt{2} t dt \\ &= \cosh \sqrt{2} b - 1 \end{aligned}$$

The last integral shows that the quantum counterpart of e^x is $\cosh \sqrt{2} x$.

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