DIMENSION GROUPS AND INVARIANT MEASURES FOR POLYNOMIAL ODOMETERS

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1. Introduction

Adic maps on simple ordered Bratteli diagrams, called Bratteli-Vershik systems, have been proven to be extremely useful as models for minimal Cantor systems. In particular, the dimension group given by the diagram is order isomorphic to the ordered cohomology group for the system, and this along with a distinguished order unit is shown to be the complete invariant for strong orbit equivalence in [8], see also [9, 14]. A wider class of Bratteli-Vershik systems involves adic maps on potentially non-simple ordered Bratteli diagrams. These examples include the well-studied Pascal adic system [13, 16, 17, 20], the Stirling adic system [7, 10], and the Euler adic system [2, 6, 10, 18]. Although these are examples of adic maps on non-simple, non-stationary Bratteli diagrams, the diagrams are highly structured, and thus these seem natural extensions of the minimal Cantor systems to study. In addition, there are connections between these systems and reinforced random walks [7, 10]. In this paper we unify the study of some of these examples by examining a class which we call *polynomial odometers*. These are adic maps defined by a sequence of polynomials with positive integer coefficients. This class includes the Pascal and Stirling (but not the Euler) adic systems as examples as well as the classical odometers. As we have defined them here, the adic maps of this type are defined everywhere, but there are countably many points of discontinuity unless the system is a classical odometer. Except for the classical odometer, these systems are not minimal. In the non-minimal case but there is a rich supply of fully supported invariant measures.

In the next two sections of the paper we establish basic definitions and facts about polynomial odometers. In particular, we show in Theorem 5 that for a polynomial odometer (X, T), the dimension group of the diagram is order isomorphic to $C(X, \mathbb{Z})/(\partial_T C(X, \mathbb{Z}) \cap C(X, \mathbb{Z}))$ where

$$\partial_T C(X, \mathbb{Z}) = \{g \circ T - g \mid g \in C(X, \mathbb{Z})\}.$$

We show in Theorem 6 that this dimension group can be seen as being comprised of a subgroup of rational functions (quotients of polynomials over $\mathbb{Z}[t]$) with a natural order structure, whereas the dimension groups for classical odometers are subgroups of $(\mathbb{Q}, \mathbb{Q}_+)$. For example, the dimension group for the Pascal adic system is presented as

$$\left\{ \frac{a(x)}{(1+x)^n} : a \in \mathbb{Z}[t], \deg a(x) \leq n \right\}$$

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with usual addition of rational functions, and the positive cone is given by those rational functions which can be expressed with a numerator polynomial $a(x)$ with all coefficients positive. The distguished order unit is the constant function 1.

In Section 4 we use this characterization to explore the space of invariant measures for these systems, a main subject of interest in this category. We see (Proposition 7) that the these measures can be seen as states on the dimension group. For some of these polynomial odometers, we show that the space of invariant measures is affinely homeomorphic to the space of probability measures on $[0,1]$ (Theorem 10). We call these *reasonable* polynomial odometers. Reasonable polynomial odometers include those like the Pascal adic system in which the transition from level n to $n+1$ is given by the same polynomial for any n and the Stirling adic system (Remark 12).

In Section 5 we give a method for constructing some examples of unreasonable polynomial odometers. The paper [10] also produces unreasonable examples via a different approach.

Finally, in Section 6 we characterize the isomorphisms that can exist between unital ordered groups for polynomial odometers (Theorem 18). We also introduce the notion of extremal orbit equivalence which is similar to the concept of strong orbit equivalence for Cantor minimal systems but more appropriate for the setting of polynomial odometers. We show that if the dimension groups of two reasonable polynomial odometers (X,T) and (Y,S) are order isomorphic then either the systems T and S are extremely orbit equivalent or the systems T and S^{-1} are (Theorem 20).

2. Polynomial Odometers

A Bratteli diagram, denoted $(\mathcal{V},\mathcal{E})$ is an infinite directed graph with vertex set $\mathcal{V}$ and edge set $\mathcal{E}$ such that $\mathcal{V}$ and $\mathcal{E}$ are each the union of countably many pairwise disjoint finite sets, $\mathcal{V}=\cup_{i=0}^{\infty}\mathcal{V}_i$ and $\mathcal{E}=\cup_{i=0}^{\infty}\mathcal{E}_i$. The edges in $\mathcal{E}_i$ connect the vertices in $\mathcal{V}_i$ and $\mathcal{V}_{i+1}$. For notational purposes for each set $\mathcal{V}_n$ the vertices are numbered 0 through $|\mathcal{V}_n|-1=I(n)$. The vertices are drawn in ascending numerical order from left to right. We will let a specific vertex $v\in\mathcal{V}$ be denoted by (n,k) whenever $v\in\mathcal{V}_n$ and is the kth vertex. Every vertex has at least one edge connecting it to the level below, and every vertex except $(0,0)$ has at least one edge connecting it to the level above.

For $n=1,2,\ldots$ and $j=0,1,\ldots,d_n$ let $a_{n,j}\in\mathbb{N}$ and $p_n(x)=\displaystyle\sum_{j=0}^{d_n}a_{n,j}x^j$. Then $(p_1(x),p_2(x),\ldots)$ is a sequence of positive integer polynomials, where d_n is the degree of $p_n(x)$. Every such sequence determines a Bratteli-Vershik system. The Bratteli diagram associated to this sequence is determined as follows:

(1) $|\mathcal{V}_0|=1$ and $|\mathcal{V}_n|=|\mathcal{V}_{n-1}|+d_n=1+\displaystyle\sum_{j=1}^{n}d_j$.

(2) The number of edges from the vertex $(n-1,k)$ to vertex $(n,k+j)$ is $a_{n,j}$ for $0\le j\le d_n$ and 0 for $j<0$ and $j>d_n$.

If for every $n=1,2,\ldots$, we have that $p_n(x)=p(x)$ for some fixed polynomial $p(x)$, we call the system a $p(x)$-adic odometer.

Suppose we have a polynomial odometer system defined by polynomials $(p_1(x),p_2(x),\ldots)$. Let $c(n,k)$ be the number of edges from the top (level 0) to the kth vertex at level

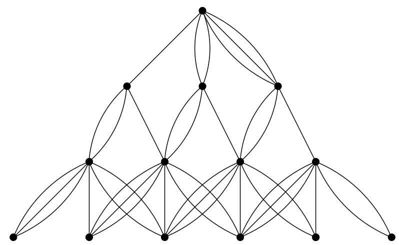

FIGURE 1. An example of a polynomial odometer associated to a sequence beginning $(1 + 2x + 3x^2, 2 + x, 3 + x + 2x^2, \dots)$.

n. Note that by the above definition,

$$c(n, k) = \sum_{i+j=k} c(n - 1, j)a_{n,i}, \tag{1}$$

see Figure 2.

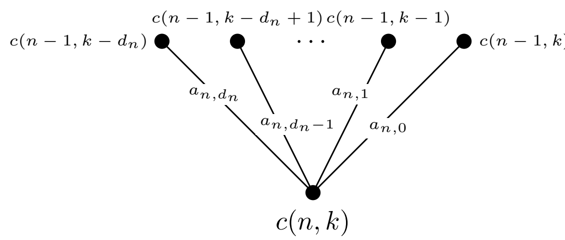

FIGURE 2. The recurrence relations of the $c(n, k)$ seen graphically.

To any Bratteli diagram associate the space $X = X(\mathcal{V}, \mathcal{E})$ of infinite edge paths on $(\mathcal{V}, \mathcal{E})$ beginning at the vertex $v_0 = (0, 0)$. If γ is a path in X, for each $n = 0, 1, \dots,$ and $k = 0, 1, \dots, I(n)$, denote by $(n, k_n(\gamma))$ the vertex through which γ passes on level n. Then denote by γ_i the edge along which γ travels between vertices $(i, k_i(\gamma))$ and $(i + 1, k_{i+1}(\gamma))$. X is a compact metric space with the metric given by, $d(\gamma, \xi) = 2^{-j}$, where $j = \inf\{i|\gamma_i \neq \xi_i\}$.

On any Bratteli diagram one can define a *partial order* on the set $\mathcal{E}$ of edges. Specifically, two edges e and $\tilde{e}$ are said to be *comparable* if both e and $\tilde{e}$ terminate into the same vertex. The edges' sources may be different. We choose and fix a total order on each set of edges with the same range. For the polynomial odometer systems, the edge ordering increases from left to right.

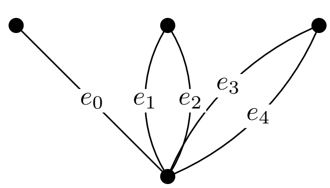

FIGURE 3. An edge ordering. $e_0 < e_1 < e_2 < e_3 < e_4$

On an ordered Bratteli diagram, the partial ordering of edges can be extended to a partial ordering of the entire path space X. Two paths γ and ξ are *comparable* if they agree after some level n ($\gamma_j = \xi_j$ for all $j \geq n$) and $\gamma_{n-1} \neq \xi_{n-1}$; then we define

$\gamma < \xi$ if and only if $\gamma_{n-1} < \xi_{n-1}$. The set of maximal paths is denoted by $X_{\max}$. For any path $\gamma \in X_{\max}$, and for all $i \in \{0, 1, \dots\}$, γ_i is a maximal edge according to the partial ordering on edges. Likewise there are minimal paths, which make up the set $X_{\min}$. Both $X_{\max}$ and $X_{\min}$ are countable. For $k = 1, 2, \dots$ we define $\gamma_{\max}^{(k)}$ to be the unique maximal path that passes through vertex (n, k) for all n for which $k \leq I(n)$. Likewise, $\gamma_{\min}^{(k)}$ is the unique minimal path that passes through vertex $(n, I(n) - k)$ for all n for which $0 \leq I(n) - k$. We also define $\gamma_{\max}^{\infty}$ $(\gamma_{\min}^{\infty})$ to be the unique maximal (minimal) path for passing through vertex $(n, I(n))$ for all n. Lastly, let $\gamma_{\max}^{(0)}$ $(\gamma_{\min}^{(0)})$ be the unique maximal (minimal) path through vertex $(n, 0)$ for all n. We then define the adic transformation $T : X \to X$ to be the map that sends $\gamma \in X \setminus X_{\max}$ to the next largest path, and $\gamma_{\max}^{(k)}$ to $\gamma_{\min}^{(k)}$ for $k \in \mathbb{Z}_+ \cup \{\infty\}$.

The basis of this topology consists of cylinder sets which specify a finite number of edges. It can be assumed that the cylinders start from the root vertex $(0, 0)$ and specify every edge until the terminal vertex of the cylinder. In order to speak about specific cylinders we introduce the following notation. Let $Y_n(k, 0)$ denote the minimal cylinder into vertex (n, k) and $Y_n(k, i) = T^i(Y_n(k, 0))$ for $i = 0, 1, \dots, c(n, k) - 1$. We will also let $Y_n = \cup_{k=0}^{I(n)} Y_n(k, 0)$ be the union of all minimal cylinder sets which terminate at level n.

Lemma 1. *For all $k, n \geq 0$, $c(n, k)$ is the coefficient of x^k in the polynomial $\prod_{i=1}^{n} p_i(x)$.*

Proof. We will prove by induction. Clearly $c(1, k) = a_{1,k}$. Assume for $k = 0, \dots, I(n-1)$, $c(n-1, k)$ is the coefficient of x^k in the polynomial $\prod_{i=1}^{n-1} p_i(x)$. Then

$$\prod_{i=1}^{n-1} p_i(x) = c(n-1, 0) + c(n, 1)x + \cdots + c(n, I(n-1))x^{I(n-1)},$$

and hence

$$\prod_{i=1}^{n} p_i(x) = \sum_{i=0}^{d_n} \sum_{j=0}^{I(n-1)} c(n-1, j) a_{n,i} x^{i+j} = \sum_{j=0}^{I(n)} c(n, j) x^j.$$

□

Lemma 2. *Fix two vertices l and k on levels m, n respectively where $m < n$. Then the number of paths from vertex l down to k is equal to the coefficient of x^{k-l} in the polynomial $\prod_{i=m+1}^{n} p_i(x)$.*

The proof is as above.

Remark 3. *Note that if (X, T) is a polynomial odometer, then so is (X, T^{-1}). That is, if (X, T) is a polynomial odometer defined by the sequence of polynomials $(p_n(x))_{n \in \mathbb{N}}$, then (X, T^{-1}) is conjugate to an adic map on the ordered Bratteli diagram defined by $q_n(x) = x^{d_n} p_n(1/x)$ where d_n is the degree of $p_n(x)$.*

3. Ordered Groups

For every Bratteli diagram $(\mathcal{V}, \mathcal{E})$ there is an associated dimension group, denoted $K_0(\mathcal{V}, \mathcal{E})$, which is the direct limit of the following directed system:

$$\mathbb{Z}^{|\mathcal{V}_0|=1} \xrightarrow{A_1} \mathbb{Z}^{|\mathcal{V}_1|} \xrightarrow{A_2} \mathbb{Z}^{|\mathcal{V}_2|} \xrightarrow{A_3} \dots$$

where for each $i = 1, 2, \ldots$ A_i is the group homomorphism determined by the incidence matrix between levels $i-1$ and i of the Bratteli diagram. The positive set consists of the equivalence classes for which there is a nonnegative vector representative. An *order isomorphism* between two such groups is an order-preserving group isomorphism. The equivalence class of $1 \in \mathbb{Z}$ is the *distinguished order unit.* It is important to note that this dimension group is not dependent on the associated dynamical system.

A dynamical system (X, T) also has an associated dimension group, let $C(X, \mathbb{Z})$ denote the additive group of continuous functions from the space X to $\mathbb{Z}$ and define

$$\partial_T C(X, \mathbb{Z}) = \{g \circ T - g \mid g \in C(X, \mathbb{Z})\}.$$

The elements of $\partial_T C(X, \mathbb{Z})$ are called the *coboundaries* of (X, T). In the case that T is a homeomorphism, $\partial_T C(X, \mathbb{Z}) \subset C(X, \mathbb{Z})$, and we define $K^0(X, T)$ to be $C(X, \mathbb{Z})/\partial_T C(X, \mathbb{Z})$.

Theorem 4 ([14], Theorem 5.4 and Corollary 6.3)**.** *Let (X, T) be a Cantor minimal system with associated Bratteli diagram $(\mathcal{V}, \mathcal{E})$ then $K_0(\mathcal{V}, \mathcal{E}) \cong K^0(X, T)$ by an order isomorphism that maps distinguished order unit to distinguished order unit.*

Except for the case of a sequence of constant functions, polynomial odometers are not continuous on $X_{\max}$ and the above notion is not well defined. Nevertheless, we can slightly adjust the definition of $K^0(X, T)$ to be $C(X, \mathbb{Z})/(\partial_T C(X, \mathbb{Z}) \cap C(X, \mathbb{Z}))$ and arrive at the next theorem. The following theorem is stated for a wider class of systems, but it is easy to see that the polynomial odometers satisfy all the necessary conditions.

Theorem 5. *Let (X, T) be a Brattei-Vershik system for which $|\mathcal{V}_n| \leq |\mathcal{V}_{n+1}|$ and there are a positive number of edges connecting vertex (n, k) and vertex $(n+1, k+j)$ where $j = 0, 1, \ldots, |\mathcal{V}_{n+1}| - |\mathcal{V}_n|$ and no edges elsewhere, and the edge ordering has the property that the left most edge is the minimal edge and the right most edge is the maximal edge. Then there is an order isomorphism*

$$K_0(\mathcal{V}, \mathcal{E}) \cong K^0(X, T)$$

which maps the distinguished order unit of $K_0(\mathcal{V}, \mathcal{E})$ to the equivalence class of the constant function 1.

Proof. This proof is an adaptation of the dynamical proof of Theorem 4 in [14] given by Glasner and Weiss in [9]. The construction of the isomorphism is the same and the adaptation comes in describing the elements of $\partial_T C(X, \mathbb{Z}) \cap C(X, \mathbb{Z})$. We will describe the elements of $\partial_T C(X, \mathbb{Z}) \cap C(X, \mathbb{Z})$ here and refer the reader to [9] for the construction of the isomorphism.

Let $f \in C(X, \mathbb{Z})$. Since X is compact, f is bounded and hence takes on only finitely many values. Let $\{l_1, \ldots l_j\}$ be the set of these values and let $U_i = f^{-1}\{l_i\}$ for each i. For $\gamma \in U_i$, let C_γ be a cylinder set containing γ and $C_\gamma \subset U_i$. Then $\{C_\gamma | \gamma \in X\}$ covers X, select a finite subcover $\{C_{\gamma^1}, C_{\gamma^2}, \ldots, C_{\gamma^r}\}$. Then for some $s \in \{1, 2, \ldots, r\}$, C_{γ^s} is of longest length, $N_1(f)$, and f is constant on any cylinder of length $n \geq N_1(f)$.

Recall that Y_n is the union of all the minimal cylinder sets into level n. Let $G = \{g \in C(X, \mathbb{Z}) | \exists N_2(g)$ such that g is constant on $Y_{N_2(g)}\}$. Note that if g is constant on $Y_{N_2(g)}$ then for any $n \geq N_2(g)$, g is constant on Y_n. Define $B = \{g \circ T - g | g \in G\}$. We will show that $B = C(X, \mathbb{Z}) \cap \partial_T C(X, \mathbb{Z})$.

Clearly $B \subset \partial_T C(X,\mathbb{Z})$ and we now show that $B \subset C(X,\mathbb{Z})$. For $f \in B$ with $f = g \circ T - g$, f is continuous on $X \setminus X_{\max}$, since $g \in C(X,\mathbb{Z})$ and T is continuous on $X \setminus X_{\max}$. Hence we only need to check continuity of f on $X_{\max}$. Let $m \geq \max\{N_1(g), N_2(g)\}$, then g is constant on each cylinder of length m and g is also constant on Y_m. For $\gamma_{\max} \in X_{\max}$ and $\xi \in X$, $d(\gamma_{\max}, \xi) < 2^{-m}$ implies that $\gamma_{\max}$ and ξ are both in the same maximal cylinder terminating at vertex $(m, k_m(\gamma_{\max}))$, and hence $g(\gamma_{\max}) = g(\xi)$. Since $T(\gamma_{\max})$ and $T(\xi)$ are both in Y_m, we have $(g \circ T)(\gamma_{\max}) = (g \circ T)(\xi)$. Hence $f(\gamma_{\max}) = f(\xi)$, and so f is continuous.

All that is left to show is that $\partial_T C(X,\mathbb{Z}) \cap C(X,\mathbb{Z}) \subset B$. Let $f \in \partial_T C(X,\mathbb{Z}) \cap C(X,\mathbb{Z})$ be given. Then $f = g \circ T - g$ for some $g \in C(X,\mathbb{Z})$, and f is continuous. We have to show that there is an $N_2(g)$ so that for each $n \geq N_2(g)$, g takes the same value on all of Y_n. Since $g \in C(X,\mathbb{Z})$, we can choose $l = N_1(g)$ such that g is constant on cylinder sets of length l. Then for every level $j \geq l$, and every $i \in \{0, 1, \ldots, d_{l+1} + d_{l+2} + \cdots + d_j\}$,

$$Y_j(i,0) \subset Y_l(0,0) \tag{2}$$

(see Figure 4). Now consider $k < I(l)$ and $\gamma^k_{\max} \in X_{\max}$. Then

$$T(\gamma^k_{\max}) = \gamma^k_{\min} \in Y_l(I(l) - k, 0) \text{ (see Figure 5).} \tag{3}$$

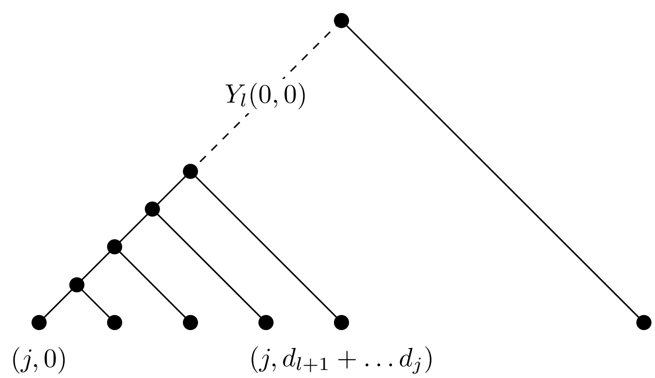

Figure 4. Connections from level l to j.

Since $f, g \in C(X,\mathbb{Z})$, there is a δ such that $d(\gamma^k_{\max}, \xi) < \delta$ implies $f(\gamma^k_{\max}) = f(\xi)$ and $g(\gamma^k_{\max}) = g(\xi)$. Let j be such that $2^{-j} < \delta$ and $d_{l+1} + d_{l+2} + \cdots + d_j > k+1$. Let xi be the path in X such that $xi_i = (\gamma^k_{\max})_i$ for each $i = 0, 1, \ldots, j-1$ and $\xi_j \neq (\gamma^k_{\max})_j$. Then $f(\gamma^k_{\max}) = f(\xi)$ and $g(\gamma^k_{\max}) = g(\xi)$ which implies $(g \circ T)(\gamma^k_{\max}) = (g \circ T)(\xi)$. Since $s(\xi_j) = (j,k)$, and ξ_j is the first non-maximal edge of ξ, $T\xi$ is in either $Y_j(k,0)$ or $Y_j(k+1,0)$ depending on the source of the successor of ξ_j (see Figure 6). Since $k+1 < d_{l+1} + d_{l+1} + \ldots d_j$, Equation 2 implies $T\xi \in Y_l(0,0)$. Then $(g \circ T)(\gamma^k_{\max}) = (g \circ T)(\xi)$, and g constant on each cylinder of length l implies $g(Y_l(I(l) - k, 0)) = g(Y_l(0,0))$. Since $k < I(l)$ was arbitrary, we have shown that g is constant on all $Y_l(k,0)$ for $k < I(l)$. It remains only to show that g takes this same value on $Y_l(I(l), 0)$. Consider $\gamma^\infty_{\max}$, and choose $j \geq l$ so that $2^{-j} < \delta$. Then $d(\gamma^\infty_{\max}, \gamma^{I(j)}_{\max}) < \delta$, which implies $\gamma^\infty_{\max}$ and $\gamma^{I(j)}_{\max}$ are both in the maximal cylinder terminating at vertex $(l, I(l))$. Thus $g(\gamma^\infty_{\max}) = g(\gamma^{I(j)}_{\max})$. Then $f(\gamma^\infty_{\max}) = f(\gamma^{I(j)}_{\max})$ and $g(\gamma^\infty_{\max}) = g(\gamma^{I(j)}_{\max})$ implies $(g \circ T)(\gamma^\infty_{\max}) = (g \circ T)(\gamma^{I(j)}_{\max})$.

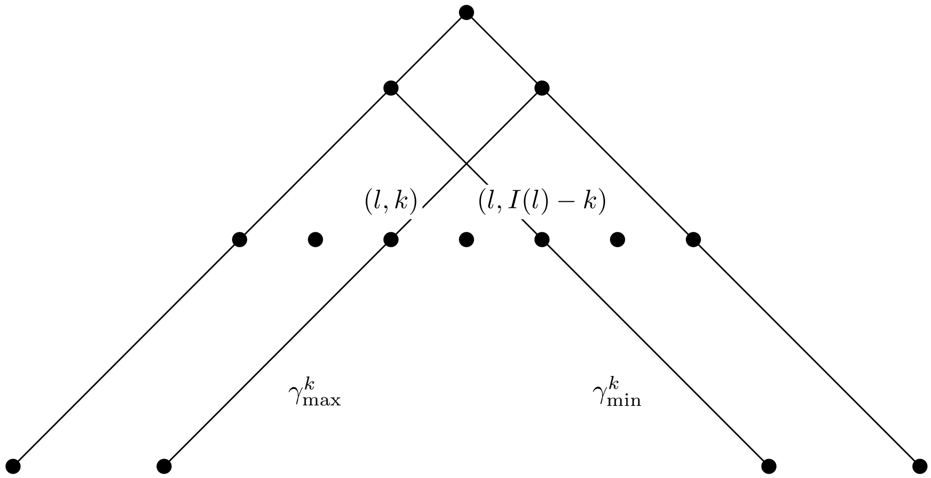

FIGURE 5. $T(\gamma^k_{\max}) = \gamma^k_{\min} \in Y_l(I(l) - k, 0)$.

Thus $T\gamma^\infty_{\max} \in Y_l(I(l), 0)$, $T\gamma^{I(j)}_{\max} \in Y_l(0, 0)$ and g constant on each cylinder of length l implies $g(Y_l(0, 0)) = g(Y_l(I(l), 0))$. Hence g is constant on Y_l, as required. □

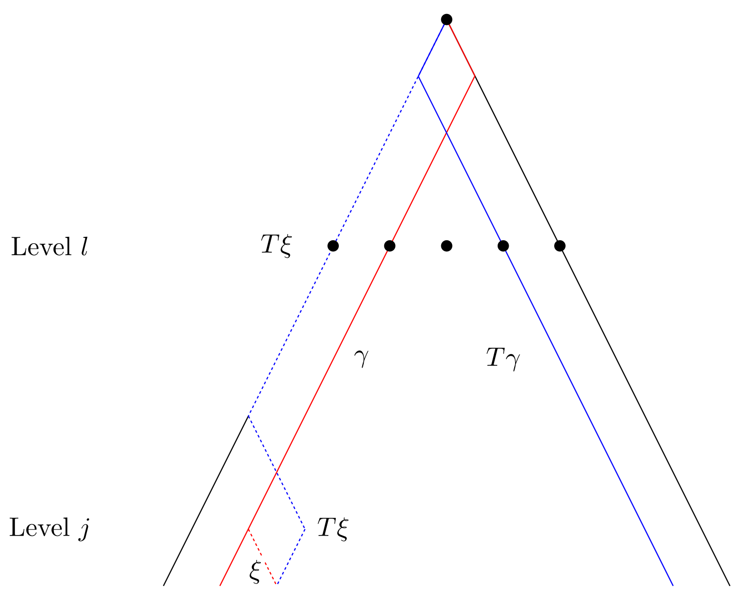

FIGURE 6. Tracking $T\gamma$ and $T\xi$.

In light of this isomorphism, we will use the two definitions interchangeably. We now compute the dimension groups of polynomial odometers. The dimension group of the diagram corresponding to the Pascal adic is computed in [19].

Theorem 6. *The dimension group $K_0(\mathcal{V}, \mathcal{E})$ associated to the Bratteli-Vershik system associated to the sequence of positive integer polynomials $(p_1(x), p_2(x) \dots)$ is* *order isomorphic to the ordered group G of rational functions of the form*

$$\frac{r(x)}{\prod_{i=1}^{m} p_i(x)},$$

where $r(x)$ is any polynomial with integer coefficients such that $\deg(r(x)) \leq \prod_{i=1}^{n} p_i(x)$. *Addition of two elements is given by normal rational function addition. The positive set consists of the elements of $K_0(\mathcal{V}, \mathcal{E})$ such that there is an l for which the numerator of*

$$\frac{r(x)\prod_{i=m+1}^{m+l} p_i(x)}{\prod_{i=1}^{m+l} p_i(x)}$$

has all positive coefficients. The distinguished order unit of $K_0(\mathcal{V}, \mathcal{E})$ is the constant polynomial 1.

Proof. We will construct an order isomorphism from $K_0(\mathcal{V}, \mathcal{E})$ into G. The incidence matrix, A_n is an $|\mathcal{V}_{n-1}| \times |\mathcal{V}_n|$ matrix with

$$(A_n)_{ij} = \begin{cases} a_{n,(j-i)} & \text{if } 0 \leq j-i \leq d_n \\ 0 & \text{otherwise} \end{cases}$$

Identify $\mathbb{Z}^i$ with the additive group of polynomials of degree at most $i-1$, $\mathbb{Z}_{i-1}[x]$ in the following manner. For $v = [v_0\ v_1 \ldots v_{i-1}]$ in $\mathbb{Z}^i$, define $v(x)$ in $\mathbb{Z}_{i-1}[x]$ by $v(x) = \sum_{j=0}^{i-1} v_j x^j$. Now if v is a vector in $\mathbb{Z}^{1+\sum_{j=1}^{n-1} d_j}$, we have $(vA_n)(x) = v(x)p_n(x)$. Under the above correspondence, A_n becomes multiplication by $p_n(x)$ for all n, and the transition functions A_{lm} correspond to multiplication by $\prod_{i=l+1}^{m} p_i(x)$.

Define $\rho_n : \mathbb{Z}_{(1+\sum_{j=1}^{n} d_j)}[x] \to G$ by $\rho_n(r(x)) = \dfrac{r(x)}{\prod_{i=1}^{n} p_i(x)}$. In order to satisfy the hypothesis of the universal mapping property of direct limits, it needs to be shown that for $l \leq m$, $\rho_l = \rho_m \circ A_{lm}$:

$$\begin{aligned} \rho_m \circ A_{lm}(r(x)) &= \rho_m(r(x)\textstyle\prod_{i=l+1}^{m} p_i(x)) \\ &= \frac{r(x)\prod_{i=l+1}^{m} p_i(x)}{\prod_{i=1}^{m} p_i(x)} \\ &= \frac{r(x)}{\prod_{i=1}^{l} p_i(x)} \\ &= \rho_l(r(x)). \end{aligned}$$

Hence the hypothesis for the universal mapping property of direct limits is satisfied, and the ρ_l are constant on equivalence classes. It follows that there is a unique homomorphism $\rho : K_0(\mathcal{V}, \mathcal{E}) \to G$, which can be defined on an equivalence class by taking any representative in $\mathbb{Z}^{1+\sum_{j=1}^{n} d_j}$ and applying ρ_n to it. An easy computation shows that ρ is an order isomorphism with appropriate positive set and distinguished order unit. □

4. States and Measures

Let (X, T) be a polynomial odometer. Consider the set of *states* on $K^0(X, T)$ i.e., the set of homomorphisms $\sigma : K^0(X, T) \to \mathbb{R}$ which send the positve set in

$K^0(X,T)$ to $\mathbb{R}_+$ and the distinguished order unit to 1. The following two propositions apply in situations more general than for polynomial odometers (E.g., see [14, 4]),

Proposition 7. *Let (X,T) be a polynomial odometer. Let μ be a T-invariant Borel probability measure. The homomorphism $\sigma_\mu : [f] \mapsto \int f d\mu$ is a state on $K^0(X,T)$. Furthermore, the map $\mu \mapsto \sigma_\mu$ is a bijection from the set of T-invariant Borel probability measures $\mathcal{M}_T$ to the set of states on $K^0(X,T)$.*

Proof. That σ_μ is a state is clear. Any state σ on $K^0(X,T)$ assigns positive values to the semi-algebra of clopen sets to a Borel probability measure μ is standard (see e.g. [23, Chapter 0]). The only thing to check is T-invariance of μ, and for this we refer the reader to Lemma 3.2.6 of [1]. □

Given the above, we will think of elements of $\mathcal{M}_T$ interchangeably as measures on X and states on $K^0(X,T)$. Below gives another characterization of the states on $K^0(X,T)$, the proof follows from the definition of a state.

Proposition 8. *Let G be a dimension group. That is,*

$$G = \mathbb{Z} \overset{A_1}{\to} \mathbb{Z}^{n(1)} \overset{A_2}{\to} \mathbb{Z}^{n(2)} \overset{A_3}{\to} \cdots$$

where A_k are nonnegative $n(k) \times n(k-1)$ matrices $(n(0)=1)$ and u is the equivalence class of the sequence $(1, A_1, A_2A_1, \ldots)$. There is a one-to-one correspondence between the set of states on G and elements of the inverse limit

$$S(G) = \{1\} \overset{A_1}{\leftarrow} (\mathbb{Z}_+)^{n(1)} \overset{A_2}{\leftarrow} (\mathbb{Z}_+)^{n(2)} \overset{A_3}{\leftarrow}$$

That is, $S(G)$ consists of sequences $\{\mu_k\} \in \mathbb{R}^{n(k)}$ of row vectors which satisfy $\mu_0 = 1$, $\mu_k = \mu_{k+1}A_k$ and $\mu_k \geq 0$ for all k.

Note that $S(G)$ is convex, and we have the natural (product) topology on $S(G)$, namely, for two sequences $\mu = [1, \mu_1, \mu_2, \mu_3, \ldots]$, $\nu = [1, \nu_1, \nu_2, \nu_3, \ldots]$ in $\mathcal{S}(G)$, define $d(\mu,\nu) = \sum_{i=0}^{\infty} \|\mu_i - \nu_i\|_1 2^{-i}$.

Proposition 9. *The spaces $\mathcal{M}_T$ (with the weak*-topology) and $\mathcal{S}(T) = \mathcal{S}\left(K^0(X,T)\right)$ (with the topology defined above) are affinely homeomorphic.*

Proof. If μ is a sequence in $S(T)$, then the corresponding measure in $\mathcal{M}_T$ is the measure defined on cylinder sets in such a way that the measure of any cylinder terminating into vertex (n,k) is $\mu_n(k)$. It is clear that the correspondence respects convex combinations. We can enumerate the paths ending at vertex $(n,k) \in \mathcal{V}_n$. The topology on $\mathcal{M}_T$ is metrizable with the metric

$$d(\mu,\nu) = \sum_{m=1}^{\infty} \sum_{i=1}^{I(m)} \left|\mu\left(C_i^{(m)}\right) - \nu\left(C_i^{(m)}\right)\right| 2^{-m}$$

which is equivalent to the aforementioned metric on $\mathcal{S}(T)$. □

Now we specialize to the polynomial odometer setting, and construct some elements of $\mathcal{S}(T)$. Let $\alpha \in [0,\infty)$. Consider the map $m_\alpha : \frac{q(x)}{p(x)} \mapsto \frac{q(\alpha)}{p(\alpha)}$ which takes $K^0(X,T)$ into $\mathbb{R}$. It is easy to check that $m_\alpha : K^0(X,T) \to \mathbb{R}$ is a homomorphism, that m_α of a positive element is in $\mathbb{R}_+$ and $m_\alpha(1) = 1$, thus $m_\alpha \in \mathcal{S}(T)$. Note that since the elements of $K^0(X,T)$ correspond to rational functions $\frac{q(x)}{p(x)}$

where $\deg q \leq \deg p$, and as such it makes sense to talk about the evaluation map $m_\infty : \frac{q(x)}{p(x)} \mapsto \lim_{n\to\infty} \frac{q(x)}{p(x)}$ as well.

By taking convex combinations of these *point evaluation measures*, we obtain more states on $K^0(X,T)$, and given a Borel probability measure m on $[0,\infty]$, we can extend this idea to find more elements of $\mathcal{S}(T)$ of the form $m : \frac{q(x)}{p(x)} \mapsto \int_0^\infty \frac{q(x)}{p(x)} dm(x)$. The measures m_α are the extreme points of the set of Borel measures on $[0,\infty]$.

Theorem 10 (Theorem 2.1 of [5]). *If T is a $p(x)$-adic odometer, states of the form m_α for $\alpha \in [0,\infty]$ are exactly the ergodic T-invariant measures, and the fully supported ergodic measures correspond to the point evaluation measures m_α where $\alpha \in (0,\infty)$.*

In [1, 5], these measures are given by a parameter q which corresponds to a m_α where α is the unique solution in $[0,\infty]$ to the equation $q = 1/p(x)$.

Remark 11. *The case for which $p(x) = x + 1$ is commonly known as the Pascal adic and many proofs exist, see* [13, 15, 17, 20] *and the references they contain. The case for which $p(x)$ has all ones for coefficients was proven by Méla in* [15].

We will call a polynomial odometer *reasonable* if the states of the form m_α for $\alpha \in [0,\infty]$ are exactly the ergodic invariant measures. With this terminology, Theorem 10 above states that all $p(x)$-adic polynomial odometers are reasonable. The *Stirling system* is an example of a non $p(x)$-adic polynomial odometer which is also reasonable. The Stirling system, is the polynomial odometer (X,T) defined by $p_n(x) = (n+x)$ (see figure 7).

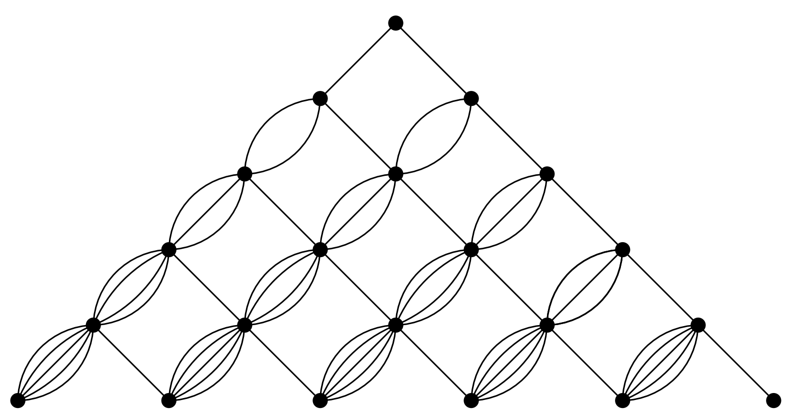

Figure 7. The Stirling adic

Remark 12. *The proof that the Stirling system is reasonable can be found in* [10, 11]. *In the next section, we give a way to construct examples of polynomial odometers which are not reasonable. Examples can also be found in* [10].

5. Unreasonable Examples

We wish to define a collection of polynomials $p_n(x) = a_n + x$ for the counterexample. When $p_n(x) = a_n + x$, we have that A_n is the $(n+1) \times n$ matrix

$$A_n = \begin{bmatrix} a_n & 0 & 0 & 0 & 0 \\ 1 & a_n & 0 & \ddots & 0 \\ 0 & 1 & a_n & \ddots & 0 \\ 0 & 0 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & 1 & a_n \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Our counterexample will be constructed by recursively defining positive integers a_n, and vectors μ_n such that $\mu_0 = 1$, $\mu_n = \mu_{n+1} A_n$ and $\mu_n \geq 0$ for all n. First we prove the key lemma.

Lemma 13. *Given any set of $n+1$ positive numbers $\mu_n = (\mu_n(0), \mu_n(1), \ldots, \mu_n(n))$, there is an integer N such that for any integer $a_{n+1} > N$ there exists a set of $n+2$ positive numbers $\mu_{n+1} = (\mu_{n+1}(0), \mu_{n+1}(1), \ldots, \mu_{n+1}(n+1))$ such that $\mu_n = \mu_{n+1} A_{n+1}$.*

Proof. Performing column reduction on the above matrix A_{n+1} and solving, we obtain the following.

$$\mu_{n+1}(k) = \sum_{i=0}^{n-k} \frac{(-1)^i \mu_n(k+i)}{(a_{n+1})^{i+1}} + \frac{(-1)^{n-k+1} \mu_{n+1}(n+1)}{(a_{n+1})^{n-k+1}}$$

where $\mu_{n+1}(n+1)$ is a free variable. Fix a number s between 0 and $\mu_n(n)$ and set $\mu_{n+1}(n+1) = s$. We have

$$\begin{aligned} \mu_{n+1}(n) &= \frac{\mu_n(n)}{a_{n+1}} - \frac{\mu_{n+1}(n+1)}{a_{n+1}} \\ &= \frac{1}{a_{n+1}} (\mu_n(n) - s) > 0 \end{aligned}$$

Next we see

$$\begin{aligned} \mu_{n+1}(n-1) &= \frac{\mu_n(n-1)}{a_{n+1}} - \frac{\mu_n(n)}{(a_{n+1})^2} + \frac{\mu_{n+1}(n+1)}{(a_{n+1})^2} \\ &= \frac{1}{a_{n+1}} \left(\mu_n(n-1) - \frac{(\mu_n(n) - s)}{a_{n+1}} \right) \end{aligned}$$

so if a_{n+1} is sufficiently large, we have $\mu_{n+1}(n-1) > 0$. Similarly,

$$\begin{aligned} \mu_{n+1}(n-2) &= \sum_{i=0}^{2} \frac{(-1)^i \mu_n(n-2+i)}{(a_{n+1})^{i+1}} - \frac{\mu_{n+1}(n+1)}{(a_{n+1})^3} \\ &= \frac{1}{a_{n+1}} \left(\mu_n(n-2) - \frac{\mu_n(n-1)}{a_{n+1}} + \frac{(\mu_n(n) - s)}{(a_{n+1})^2} \right) \end{aligned}$$

So $\mu_{n+1}(n-1) > 0$ if a_{n+1} is sufficiently large. In like fashion, if a_{n+1} is sufficiently large, then $\mu_{n+1}(k) > 0$ for $0 \leq k \leq n+1$. □

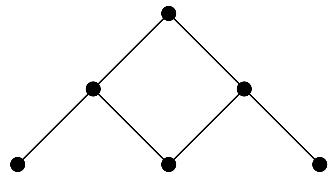

FIGURE 8. The first two levels when $a_1 = a_2 = 1$

We may set $a_1 = a_2 = 1$ and set $\mu_0 = 1$, $\mu_1 = \left(\frac{1}{2}, \frac{1}{2}\right)$, $\mu_2 = \left(\frac{1}{8}, \frac{3}{8}, \frac{1}{8}\right)$.

Then by the above lemma we may recursively choose $\{a_n : n \geq 0\}$ and $\{\mu_n : n \geq 0\}$ such that $\mu_0 = 1$, $\mu_n = \mu_{n+1} A_{n+1}$ and $\mu_n > 0$ for all $n \geq 0$. Consider the diagram corresponding to this system with $p_n(x) = a_n + x$ and the invariant measure μ corresponding to the sequence $[1, \mu_1, \mu_2, \mu_3, \ldots]$.

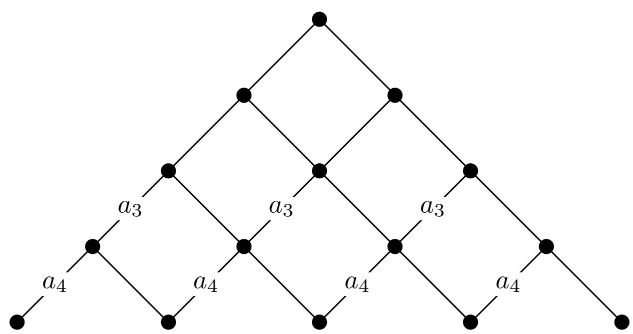

FIGURE 9. The first four levels where a_3 and a_4 indicate the number of edges connecting the two vertices.

We would like to show that the point evaluations are not the entire set of ergodic measures for the above system. The ergodic measures correspond to the extreme points of $\mathcal{M}_T$ and therefore by the Krein-Milman Theorem convex combinations of ergodic measures are dense in $\mathcal{M}_T$.

Proposition 14. *There is an $\epsilon > 0$ such that the measure μ constructed above is not within ϵ of a convex combination of point evaluation states.*

Proof. Suppose not, then for any $n > 0$ there is an integer $M(n) > 0$ and a choice of $\alpha_i^{(n)} \in [0, \infty]$ and $\lambda_i^{(n)} \in [0, 1]$ for $i = 1, 2, \ldots, M(n)$ such that

$$\sum_{i=1}^{M(n)} \lambda_i^{(n)} = 1$$

$$\left| \sum_{i=1}^{M(n)} \lambda_i^{(n)} \frac{1}{\left(1+\alpha_i^{(n)}\right)^2} - \frac{1}{8} \right| < \frac{1}{n}$$

$$\left| \sum_{i=1}^{M(n)} \lambda_i^{(n)} \frac{\alpha_i^{(n)}}{\left(1+\alpha_i^{(n)}\right)^2} - \frac{3}{8} \right| < \frac{1}{n}$$

$$\left| \sum_{i=1}^{M(n)} \lambda_i^{(n)} \frac{\left(\alpha_i^{(n)}\right)^2}{\left(1+\alpha_i^{(n)}\right)^2} - \frac{1}{8} \right| < \frac{1}{n}$$

Now consider the same sequence of combinations of point evaluations in the Pascal example, namely $\{\mu_n\}$ where $\mu_n = \sum_{i=1}^{M(n)} \lambda_i^{(n)} m_{\left(\alpha_i^{(n)}\right)}$.

Since the set of T-invariant measures is compact (one can see from the state characterization that the set of solutions is closed), a subsequence of these measures μ_{n_k} must converge to an invariant measure μ. Further, the vector which gives the value of the measure μ on the clopen sets corresponding to the three edges ending at level 2 is $\left(\frac{1}{8}, \frac{3}{8}, \frac{1}{8}\right)$. This is a contradiction since there is no such invariant measure for the Pascal example since there is no solution to

$$\left(\frac{1}{8}, \frac{3}{8}, \frac{1}{8}\right) = (w, x, y, z) \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

with $w, x, y, z \geq 0$. □

By the above, the collection of point evaluation measures $\{m_\alpha : \alpha \in [0, \infty]\}$ for the given example cannot be the set of all ergodic measures. Let us also show that there is a choice of $a_n \in \mathbb{N}$ such that for the polynomial odometer with $p_n(x) = a_n + x$, not all point evaluation measures are ergodic.

We know that no matter what the values of a_n, there is a point evaluation measure m_α with $\alpha = 1$. Let us call this measure ρ. Our goal is to create two distinct invariant measures μ and ν with $\rho = \frac{1}{2}(\mu + \nu)$, thus showing that ρ is not an extreme point of $\mathcal{M}_T$.

Let $a_1 = a_2 = 1$, $\mu_1 = (\frac{1}{2}, \frac{1}{2})$, and $\mu_2 = (\frac{1}{8}, \frac{3}{8}, \frac{1}{8})$. We first show that we can choose a_n in such a way that $0 < \mu_n < 2\rho_n$ for all $n \in \mathbb{N}$. This is satisfied for $n = 1, 2$ since $\rho_1 = (\frac{1}{2}, \frac{1}{2})$ and $\rho_2 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{4})$. Assume that $\mu_n < 2\rho_n$. Then we can choose a_{n+1} large enough so that Lemma 13 is satisfied and

$$\mu_n < 2\rho_n \left(\frac{a_{n+1}}{1 + a_{n+1}}\right). \tag{4}$$

Since $\mu_n = \mu_{n+1}A_{n+1}$ we have for $k = 0, 1, \ldots, n$ that $\mu_n(k) = \mu_{n+1}(k) + \mu_{n+1}(k+1)$, and hence for $k = 0, 1, \ldots, n$

$$\mu_{n+1}(k) = \frac{\mu_n(k) - \mu_{n+1}(k)}{a_{n+1}} < \frac{\mu_n(k)}{a_{n+1}}. \tag{5}$$

Also, for $k = 0, \ldots, n+1$

$$\rho_{n+1}(k) = \frac{\rho_n(k)}{1 + a_{n+1}}. \tag{6}$$

Combining 4, 5, and 6 we have for $k = 0, 1, \ldots, n$

$$\mu_{n+1}(k) < \frac{\mu_n(k)}{a_{n+1}} < \frac{2\rho_n(k)}{1 + a_{n+1}} = 2\rho_{n+1}(k). \tag{7}$$

For $k = n+1$, note from the proof of Lemma 13 that $\mu_{n+1}(n+1)$ is a free variable and hence can be chosen so that $0 < \mu_{n+1}(n+1) < 2\rho_{n+1}(n+1)$. Therefore $\mu_n < 2\rho_n$ for all $n = 0, 1, \ldots$. We can now define $\nu_n = 2\rho_n - \mu_n$. Then $0 < \nu_n(k) < 1$ for all $n = 0, 1, dots$ and $k = 0, 1, \ldots, n$ and $\frac{1}{2}(\mu_n + \nu_n) = \rho_n$ for $n = 0, 1, \ldots$. Therefore the measures $\frac{1}{2}(\mu + \nu)$ and ρ coincide on all clopen sets, which implies that they are equal and proves that the measure ρ is not ergodic.

6. Isomorphism Theorem

In this section we characterize the kinds of unital ordered group isomorphisms that can exist from $K^0(X,T) \to K^0(Y,S)$ for two reasonable polynomial odometer systems (X,T) and (Y,S). With this characterization, we are able to address the question of what sort of dynamical relation exists between two such systems when the associated unital ordered groups are isomorphic.

First let us consider some possibilities for the isomorphism. As shown in Theorem 6 there is a sequence of polynomials $\{p_n(x)\}$ such that the group $K^0(X,T)$ can be presented in the form

$$K^0(X,T) = \left\{ \frac{q(x)}{p_1(x)\, p_2(x) \cdots p_n(x)} : q(x) \in \mathbb{Z}[x], \deg q(x) \leq \sum_{i=1}^{n} \deg p_i(x) \right\}.$$

One possibility is that $K^0(Y,S)$ is presented by the exact same set of polynomials as $K^0(X,T)$. Even this possibility is non-trivial as seen in the following example.

Example 15. *Let $(\mathcal{V}, \mathcal{E}, \leq)$ be the ordered Bratteli diagram defined by telescoping the $(1+x)$-diagram to every third level, and let (X,T) be the associated polynomial odometer system. Let $\leq'$be the left-right ordering on $(\mathcal{V}, \mathcal{E})$, and (X,S) the associated polynomial odometer system. One can show that there is no possible continuous conjugacy between these two polynomial odometers. Whether there is a weaker kind of conjugacy between them is an interesting problem, e.g. where continuity on all of X is not required.*

A second possibility is that $K^0(Y,S)$ is presented by replacing x by $\frac{1}{x}$ in set of polynomials for $K^0(X,T)$.

Example 16. *Let T be the $p(x)$-adic odometer with $p(x) = 1 + 2x$, and let S be the $q(x)$-adic odometer with $q(x) = 2 + x$. Then we have a unital ordered group*

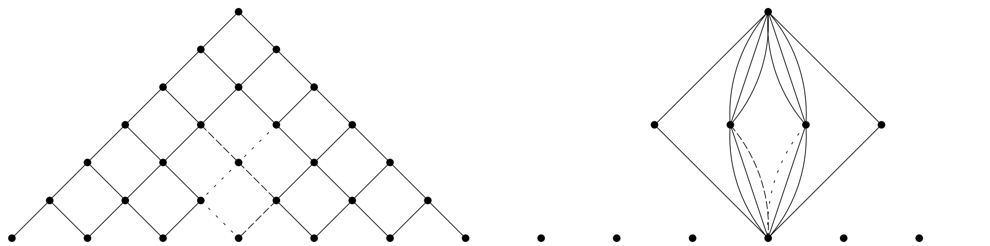

FIGURE 10. In the telescoping the dotted and the dashed paths become edges and the dotted path is less than the dashed path. Therefore the edge ordering is no longer left to right and hence the identity map is not a conjugacy between (X,T) and (X,S).

isomorphism $h : K^0(X,T) \to K^0(X,S)$ *given by*

$$h : \frac{r(x)}{(1+2x)^n} \mapsto \frac{r\left(\frac{1}{x}\right)}{\left(1+\frac{2}{x}\right)^n} = \frac{x^n r\left(\frac{1}{x}\right)}{(2+x)^n} = \frac{x^n r\left(\frac{1}{x}\right)}{q(x)^n} \in K^0(X,S)$$

noting that since $\deg(r) \leq n$, $x^n r\left(\frac{1}{x}\right) \in \mathbb{Z}[x]$ *with degree at most* n.

A third, more subtle possibility is that an isomorphism is presented by replacing x by $\frac{a}{b}x$ where $a, b \in \mathbb{N}$.

Example 17. *Let* T *be the* $p(x)$*-adic odometer with* $p(x) = 6(1+x)$*, and let* S *be the* $q(x)$*-adic odometer with* $q(x) = 6(2+3x)$*. Then we have an isomorphism* $h : K^0(X,T) \to K^0(X,S)$ *given by* $x \mapsto \frac{3}{2}x$*. More specifically,*

$$h : \frac{r(x)}{6^n(1+x)^n} \mapsto \frac{r\left(\frac{3}{2}x\right)}{6^n\left(1+\frac{3}{2}x\right)^n} = \frac{r\left(\frac{3}{2}x\right)}{3^n(2+3x)^n} = \frac{2^n r\left(\frac{3}{2}x\right)}{q(x)^n} \in K^0(X,S)$$

$$h^{-1} : \frac{r(x)}{6^n(2+3x)^n} \mapsto \frac{r\left(\frac{2}{3}x\right)}{6^n\left(2+3\left(\frac{2}{3}x\right)\right)^n} = \frac{r\left(\frac{2}{3}x\right)}{6^n(2+2x)^n} = \frac{3^n r\left(\frac{2}{3}x\right)(1+x)^n}{p(x)^{2n}} \in K^0(X,T)$$

We show that in the case where the T and S are reasonable and $h : K^0(X,T) \to K^0(Y,S)$ is a unital ordered group isomorphism, then it is a composition of isomorphisms of the above types.

Theorem 18. *If* (X,T) *and* (Y,S) *are two reasonable polynomial odometer systems and* $K^0(X,T)$ *is order isomorphic to* $K^0(Y,S)$ *by an isomorphism which identifies distinguished order units. Then the isomorphism is given by* $x \mapsto \frac{a}{b}x$ *for some* $a, b \in \mathbb{N}$ *or by* $x \mapsto \frac{a}{b}\frac{1}{x}$ *for some* $a, b \in \mathbb{N}$.

Proof. Since the unital ordered groups are isomorphic there is an affine homeomorphism σ from $\mathcal{M}_S$ to $\mathcal{M}_T$. Since the homeomorphism σ respects extreme points, if T and S are reasonable polynomial odometers and $\mathcal{E}_S$ and $\mathcal{E}_T$ are the extreme points of $\mathcal{M}_S$ and $\mathcal{M}_T$ respectively, then $\sigma : \mathcal{E}_S \to \mathcal{E}_T$ is an affine homeomorphism. Since $\mathcal{E}_S = \mathcal{E}_T = \{m_\alpha : \alpha \in [0,\infty]\}$ we have $\sigma : m_\alpha \mapsto m_{f(\alpha)}$ where $f : [0,\infty] \to [0,\infty]$ is a bijection. Because σ is weak*-continuous, $f : [0,\infty] \to [0,\infty]$ is continuous in the usual topology on $[0,\infty]$. Since f must also be a bijection, f is a homeomorphism.

Suppose that $\zeta : K^0(X,T) \to K^0(Y,S)$ is the isomorphism. Since for each μ in $\mathcal{M}_T$, $\sigma(\mu) = \mu \circ \zeta$, we have $\sigma(m_\alpha) = m_\alpha \circ \zeta = m_{f(\alpha)}$.

Let $\frac{1}{p(t)}, \frac{t}{p(t)} \in K^0(X,T)$, and let $\frac{q_0(t)}{r_0(t)} = \zeta\left(\frac{1}{p(t)}\right) \in K^0(Y,S)$ and $\frac{q_1(t)}{r_1(t)} = \zeta\left(\frac{t}{p(t)}\right) \in K^0(Y,S)$. We have for any $\alpha \in [0,\infty]$, and $i = 0, 1$,

$$\begin{aligned}\frac{q_i(\alpha)}{r_i(\alpha)} &= m_\alpha\left(\frac{q_i(t)}{r_i(t)}\right)\\ &= m_\alpha \circ \zeta\left(\frac{t^i}{p(t)}\right)\\ &= m_{f(\alpha)}\left(\frac{t^i}{p(t)}\right)\\ &= \frac{(f(\alpha))^i}{p(f(\alpha))}\end{aligned}$$

In particular, it follows from this that

$$f(\alpha) = \frac{q_1(\alpha)\, r_0(\alpha)}{r_1(\alpha)\, q_0(\alpha)}$$

for all $\alpha \in [0,\infty]$. That is to say, $f(t)$ coincides with the rational function $\frac{q_1(t)r_0(t)}{r_1(t)q_0(t)}$ on $[0,\infty]$. Since f is only defined on $[0,\infty]$ we can say without conflict of notation that $f(t)$ is a rational function with coefficients in $\mathbb{Z}$. This rational function has the following properties: it gives a homeomorphism from $[0,\infty]$ to itself and since the same argument as above can be applied to $f^{-1}(t)$, $f^{-1}(t)$ is also a rational function. The only way all of these conditions hold is if $f(t)$ is $\frac{a}{b}t$ or $\frac{a}{bt}$ where $a, b \in \mathbb{N}$.

Now take an arbitrary element $\frac{s(t)}{p(t)} \in K^0(X,T)$, and let $\zeta\left(\frac{s(t)}{p(t)}\right) = \frac{r(t)}{q(t)}$. For every $\alpha \in [0,\infty]$, we have

$$\begin{aligned}\frac{r(\alpha)}{q(\alpha)} &= m_\alpha\left(\frac{r(t)}{q(t)}\right)\\ &= m_\alpha \circ \zeta\left(\frac{s(t)}{p(t)}\right)\\ &= m_{f(x)}\left(\frac{s(t)}{p(t)}\right)\\ &= \frac{s(f(\alpha))}{p(f(\alpha))}\end{aligned}$$

which implies that ζ is given by the identity map or replacement of t by $f(t)$ which is $\frac{a}{b}t$ or $\frac{a}{bt}$. □

Let us consider implications of the above theorem in the case where (X,T) is the Pascal adic system. Suppose (Y,S) is some other reasonable polynomial odometer system such that $K^0(Y,S)$ is order isomorphic to $K^0(X,T)$ with the distinguished order units identified. Theorem 18 gives us very strict conditions on the order isomorphism. In fact, in this case the conditions are strong enough to imply that (Y,S) must be a polynomial odometer associated to a Bratteli diagram that is just a telescoping (and possibly some trivial microscoping) of the Pascal diagram. In the case of the Pascal, $x \mapsto 1/x$ is just the identity map, so we restrict our explanation to the case where elements of $K^0(Y,S)$ gets sent to elements of $K^0(X,T)$ via the map $x \mapsto (a/b)x$. Let $r(x)$ be a positive integer polynomial such that the two functions $1/r(x)$ and $x/r(x)$ are elements of $K^0(Y,S)$. Then

$$\frac{1}{r(x)} \mapsto \frac{1}{r\left(\frac{a}{b}x\right)} = \frac{s(x)}{(x+1)^m} \in K^0(X,T),$$

for some integer polynomial $s(x)$. Then, $r\left(\frac{a}{b}x\right)s(x) = (x+1)^m$ implies $s(x) = (x+1)^j$ and $r\left(\frac{a}{b}x\right) = (x+1)^k$. We also have

$$\frac{x}{r(x)} \mapsto \frac{\frac{a}{b}x}{r\left(\frac{a}{b}x\right)} = \frac{\frac{a}{b}x}{(x+1)^k} \in K^0(X,T).$$

Therefore $b=1$ and so $r(ax) = (x+1)^k$ which implies that $a=1$.

We now wish to apply the above theorem to the following question. Given two reasonable polynomial odometers and a unital order isomorphism from $K^0(X,T)$ to $K^0(Y,S)$ what then can we say about the dynamical relation between (X,T) and (Y,S)? Certainly there is a homeomorphism from X to Y which carries cofinal points in X to cofinal points in Y, as this would be true for any Bratteli-Vershik systems with isomorphic unital ordered groups via a homeomorphism like the one constructed in the proof of Theorem 2.1 of [8]. As in that situation, however, we would like to go further, and establish that the systems in question are actually orbit equivalent - that the homeomorphism can be constructed in such a way that the infinite minimal points and maximal points are identified. The additional tool that is used in [8] is that the diagrams were simple. In our case, this is not true, however, we are examining ordered Bratteli diagrams with a very specific structures, and this we are able to use to prove a similar theorem.

In the below definition, we extend the notion of strong orbit equivalence as defined by Giordano-Putnam-Skau. The term "extremely" is used here as a reference to the fact that minimal and maximal points in the ordered diagrams are preserved in the orbit equivalence in a particular way.

Definition 19. *Let $(\mathcal{V}, E, \leq)$ and $(\mathcal{V}', \mathcal{E}', \leq')$ be ordered Bratteli diagrams, and $T : X \to X$ and $S : Y \to Y$ the associated adic maps. Let $X_{\max}$ and $Y_{\max}$ be the set of infinite maximal points in X, Y, respectively. We say that $(\mathcal{V}, \mathcal{E}, \leq)$ and $(\mathcal{V}', \mathcal{E}', \leq')$ are* extremely orbit equivalent *if there is a homeomorphism $h : X \to Y$, and functions $m, n : X \to \mathbb{Z}$ such that*

1. *$hT(x) = S^{m(x)}h(x)$ for all $x \in X$,*
2. *$hT^{n(x)}(x) = Sh(x)$ for all $x \in X$,*
3. *the $h(X_{\max}) = Y_{\max}$,*
4. *$m(x) = n(x) = 1$ for all $x \in X_{\max}$,*
5. *m, n are continuous on $X \setminus X_{\max}$.*

Theorem 20. *Suppose (X,T) and (Y,S) are two reasonable polynomial odometer systems. $K^0(X,T)$ and $K^0(Y,S)$ are order isomorphic with distinguished order units identified if and only if either the systems T and S are extremely orbit equivalent or the systems T and S^{-1} are.*

Proof. First suppose that either T and S are extremely orbit equivalent or T and S^{-1} are extremely orbit equivalent. We will use the characterization of $\partial_S C(Y,\mathbb{Z}) \cap C(Y,\mathbb{Z})$ used in the proof of Theorem 5. This states that $f \in \partial_S C(Y,\mathbb{Z}) \cap C(Y,\mathbb{Z})$ if and only if $f = g \circ S - g$ where $g \in C(Y,\mathbb{Z})$ is constant on the union of minimal cylinder sets into level n for some n. Note that without loss of generality, we may assume $g \equiv 0$ on the minimal cylinder sets into level n because for any constant k, $(g+k) \circ S - (g+k) = g \circ S - g$.

Note that $K^0(Y,S) = K^0(Y,S^{-1})$. This follows because if $g \in C(Y,\mathbb{Z})$ then

$$g \circ S - g = g' \circ S^{-1} - g'$$

where $g' = -g \circ S$. Further, if g is zero on minimal cylinder sets into level n in the diagram for S then $g \circ S$ is zero on maximal cylinder sets into level n. But this means that g' is zero on minimal cylinder sets into level n in the diagram for S^{-1}.

By the above paragraph, we may assume that T and S are extremely orbit equivalent by a homeomorphism $h : X \to Y$. Consider the isomorphism from $C(Y,\mathbb{Z})$ to $C(X,\mathbb{Z})$ given by $f \mapsto f \circ h$.

Suppose $f \in \partial_S C(Y,\mathbb{Z}) \cap C(Y,\mathbb{Z})$. Then $f = g \circ S - g$ where $g \in C(Y,\mathbb{Z})$ and g is zero on the minimal cylinder sets into level n for some n. Therefore the function g can be written in the form $\sum_{k=1}^{K} c_k 1_{A_k}$ where $c_k \in \mathbb{Z}$ and $A_k \subset Y$ are cylinder sets of nonminimal paths. We have

$$f = g \circ S - g \tag{8}$$

$$= \sum_{k=1}^{K} c_k 1_{A_k} \circ S - \sum_{k=1}^{K} c_k 1_{A_k} \tag{9}$$

$$= \sum_{k=1}^{K} c_k \left(1_{S^{-1}A_k} - 1_{A_k}\right) \tag{10}$$

Now the set $S^{-1}A_k$ contains no maximal points so the function $n(\cdot)$ with $Sh = hT^{n(x)}$ is continuous on $S^{-1}A_k$. In particular, this means $S^{-1}A_k = \cup_{i=-I}^{I} B_k(i)$ where $B_k(i)$ is the clopen set of $S^{-1}A_k$ upon which $Sh = hT^i$. Now

$$fh = \sum_{k=1}^{K} c_k \left(1_{S^{-1}A_k} h - 1_{A_k} h\right) \tag{11}$$

$$= \sum_{k=1}^{K} \sum_{i=-I}^{I} c_k \left(1_{h^{-1}B_k(i)} - 1_{h^{-1}SB_k(i)}\right) \tag{12}$$

$$= \sum_{k=1}^{K} \sum_{i=-I}^{I} c_k \left(1_{h^{-1}B_k(i)} - 1_{T^i h^{-1}B_k(i)}\right) \tag{13}$$

$$= \sum_{k=1}^{K} \sum_{i=-I}^{I} c_k \left(1_{T^i h^{-1}B_k(i)} \circ T^i - 1_{T^i h^{-1}B_k(i)}\right) \tag{14}$$

Now we have expressed fh as a sum of functions of the form $a \circ T^i - a$ where $a \in C(X,\mathbb{Z})$. Note that for $i > 0$,

$$a \circ T^i - a = \sum_{j=0}^{i-1} \left[\left(aT^j\right) \circ T - \left(aT^j\right)\right] = b \circ T - b$$

where $b = \sum_{j=0}^{i-1} aT^j$. For $i < 0$,

$$a \circ T^i - a = -\sum_{j=i}^{-1} \left[\left(aT^j\right) \circ T - \left(aT^j\right)\right] = b \circ T - b$$

where $b = -\sum_{j=i}^{-1} aT^j$. This shows that $fh \in \partial_T C(X,\mathbb{Z})$, and finishes the proof of the first direction.

Now for the other direction, let $\left(\mathcal{V}^1, \mathcal{E}^1, \leq^1\right)$ and $\left(\mathcal{V}^2, \mathcal{E}^2, \leq^2\right)$ be ordered Bratteli diagrams for (X,T) and (Y,S), respectively. The isomorphism of unital ordered groups gives the existence of a Bratteli diagram $(\mathcal{V}, \mathcal{E})$ which when telescoped to odd levels gives $\left(\mathcal{V}^1, \mathcal{E}^1\right)$ and to even levels gives $\left(\mathcal{V}^2, \mathcal{E}^2\right)$, e.g. see [8]. There, one uses this fact to construct a strong orbit equivalence, but this uses the fact that the diagrams are simple, which is not the case for us. However, because of the special form of the isomorphisms in our case we will be able to construct $(\mathcal{V}, \mathcal{E})$ in a particular way, so that an orbit equivalence can be achieved.

Suppose $\{p_k(x)\}$ are the polynomials for (X,T) and $\{q_k(x)\}$ the polynomials for (Y,S). Let us assume that the isomorphism ζ is given by $\zeta : x \mapsto \frac{a}{b}x$ where $a, b \in \mathbb{N}$. If this is not the case, then replace T by T^{-1}.

Let $\mathcal{V}_0 = \{v_0\}$. Let $\mathcal{E}_1$ and V_1 be as determined by the polynomial $p_1(t)$. That is, $\mathcal{V}_1 = \{(1,k) \mid 0 \leq k \leq I(n)\}$ and the number of edges from $\mathcal{V}_0 = \{(0,0)\}$ to $(1,j)$ is equal to the coefficient of t^j in $p_1(t)$.

To form $\mathcal{V}_2$, note that there is a denominator polynomial, call it $q_{n(1)}(t)$, for $K^0(Y,S)$, and polynomials $r_i \in \mathbb{Z}[t]$ such that

$$\zeta\left(\frac{t^i}{p_1(t)}\right) = \frac{\left(\frac{a}{b}t\right)^i}{p_1\left(\frac{a}{b}t\right)} = \frac{r_i(t)}{q_{n(1)}(t)}$$

for $i = 0, 1, \ldots, d(1)$. Note that it follows from the above that $r_i(t) = \left(\frac{a}{b}t\right)^i r_0(t)$ for $0 \leq i < d(1)$ and that $\left(\frac{a}{b}\right)^i r_0(t) \in \mathbb{Z}[t]$.

Set $\mathcal{V}_2 = \left\{1, t, t^2, \ldots, t^{d(2)}\right\}$ where $d(2) = \deg\left(q_{n(1)}(t)\right)$. Let the number of edges from vertex t^j in $\mathcal{V}_1$ to vertex t^k in $\mathcal{V}_2$ be equal to the coefficient of t^{k-j} in $\left(\frac{a}{b}\right)^j r_0(t)$. Then one can show that the number of edges from v_0 to $t^k \in \mathcal{V}_2$ is the coefficient of t^k in $r_0(t)\, p_1\left(\frac{a}{b}t\right)$. But since

$$\zeta\left(\frac{1}{p_1(t)}\right) = \frac{1}{p_1\left(\frac{a}{b}t\right)} = \frac{r_0(t)}{q_{n(1)}(t)}$$

we have $q_{n(1)}(t) = r_0(t)\, p_1\left(\frac{a}{b}t\right)$.

To form $\mathcal{V}_3$, note that there is a denominator polynomial, call it $p_{m(2)}(t)$, for $G(T)$, and polynomials $s_i \in \mathbb{Z}[t]$ such that

$$\zeta^{-1}\left(\frac{t^i}{q_{n(1)}(t)}\right) = \frac{\left(\frac{b}{a}t\right)^i}{p_{m(2)}\left(\frac{b}{a}t\right)} = \frac{s_i(t)}{p_{m(2)}(t)}$$

for $i = 0, 1, \ldots, d(2)$. Set $\mathcal{V}_3 = \left\{1, t, t^2, \ldots, t^{d(3)}\right\}$ where $d(3) = \deg\left(p_{m(2)}(t)\right)$. Let the number of edges from vertex t^j in $\mathcal{V}_2$ to vertex t^k in $\mathcal{V}_3$ be equal to the coefficient of t^{k-j} in $\left(\frac{b}{a}\right)^j s_0(t)$. Again one can show that the number of edges from a vertex $t^j \in \mathcal{V}_1$ to $t^k \in \mathcal{V}_3$ is the coefficient of t^{k-j} in $s_0(t)\, q_{n(1)}\left(\frac{b}{a}t\right) = p_{m(2)}(t)$.

Continuing in this way, we construct a Bratteli diagram $(\mathcal{V}, \mathcal{E})$ which when telescoped to odd levels gives a telescoping of $\left(\mathcal{V}^1, \mathcal{E}^1\right)$, and when telescoped to odd levels gives a telescoping of $\left(\mathcal{V}^2, \mathcal{E}^2\right)$. For every vertex $t^k \in \mathcal{V}_n \subset \mathcal{V}$, among all edges terminating at t^k we can identify a *leftmost* edge $l(k,n)$ and a *rightmost* edge $r(k,n)$. The leftmost edge $l(k,n)$ should have the property that it originates at the vertex $t^i \in \mathcal{V}_{n-1}$ where i is the minimum exponent of a vertex connected to

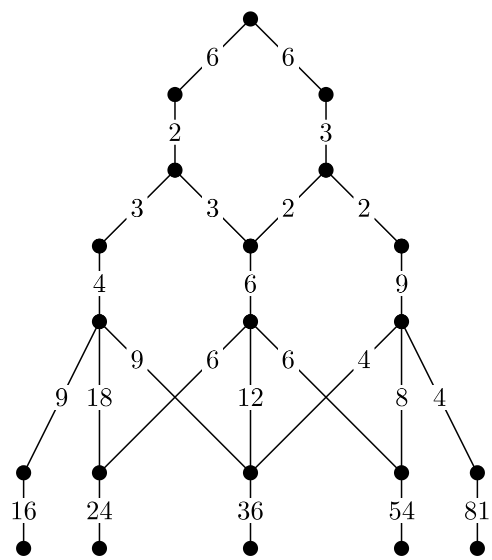

FIGURE 11. The first six levels of (V, E) corresponding to Example 17.

t^k. The rightmost edge $r(k, n)$ should have the property that it originates at the vertex $t^j \in \mathcal{V}_{n-1}$ with the maximum exponent of a vertex connected to t^k. Because telescoping $(\mathcal{V}, \mathcal{E})$ to odd levels gives a telescoping of $(\mathcal{V}^1, \mathcal{E}^1)$, there is a bijection ϕ_k between paths connecting individual vertices on levels $m(k)$ and $m(k+1)$ in $(\mathcal{V}^1, \mathcal{E}^1)$ to paths connecting the corresponding vertices on levels $2k$ and $2k+2$ in $(\mathcal{V}, \mathcal{E})$. Further, by the way the ordering on $(\mathcal{V}^1, \mathcal{E}^1)$ is defined, we can define ϕ_k in such a way that for any minimal path P, the image $\phi_k(P)$ is comprised of two leftmost edges in $(\mathcal{V}, \mathcal{E})$, and similarly for maximal paths and rightmost edges. At the same time, we may define maps ψ_k between paths between levels $n(k)$ and $n(k+1)$ in $(\mathcal{V}^2, \mathcal{E}^2)$ to paths connecting the corresponding vertices on levels $2k-1$ and $2k+1$ in $(\mathcal{V}, \mathcal{E})$. Like the maps ϕ_k, these can be defined so that the image of minimal/maximal paths are leftmost/rightmost paths.

Every point $x \in X$ is a concatenation of paths $P_1 P_2 P_3 \ldots$ where P_k originates at a vertex at level $m(k)$ and terminates at a vertex at level $m(k+1)$. Thus we can define $\phi(x) = \phi_1(P_1)\, \phi_2(P_2)\, \phi_3(P_3) \ldots$. Similarly, we can define $\psi(y)$ for $y \in Y$. Finally, we can define the extremal orbit equivalence as the map $\psi^{-1}\phi$. $\square$

One consequence of Theorem 18 and Theorem 20 is that the Pascal adic system is extremely orbit equivalent to another reasonable system (Y, S), if and only if (Y, S) is generated by a diagram that is a telescoping of the Pascal diagram.

REFERENCES

[1] Bailey (Frick), S., Dynamical properties of some non-stationary, non-simple Bratteli-Vershik systems. *Ph.D. thesis*, University of North Carolina, Chapel Hill.

[2] Bailey (Frick), S., Keane, M.; Petersen, K.; Salama, I., Ergodicity of the Euler adic transformation on the Euler graph, *Math. Proc. Camb. Phil. Soc.* **141** Issue 2, (2006), 231–238.

[3] Durand, F.; Host, B.; Skau, C., Substitutional dynamical systems, Bratteli diagrams and dimension groups. *Ergodic Theory Dynam. Systems* **19** (1999), no. 4, 953–993.

[4] Effros, Edward G., Dimensions and C^*-algebras, CBMS Regional Conference Series in Mathematics, **46**. *Conference Board of the Mathematical Sciences, Washington, D.C.*, 1981. v+74 pp. ISBN: 0-8218-1697-7
[5] Frick, Sarah Bailey, Limited scope adic transformations. *Discrete Contin. Dyn. Syst. Ser. S* **2** (2009), no. 2, 269–285.
[6] Frick, Sarah Bailey; Petersen, Karl, Random permutations and unique fully supported ergodicity for the Euler adic transformation. *Ann. Inst. Henri Poincar Probab. Stat.* **44** (2008), no. 5, 876–885.
[7] Frick, Sarah Bailey; Petersen, Karl, Reinforced random walks and adic transformations (preprint).
[8] Giordano, Thierry; Putnam, Ian F.; Skau, Christian F., Topological orbit equivalence and C^*-crossed products. *J. Reine Angew. Math.* **469** (1995), 51–111.
[9] Glasner, Eli; Weiss, Benjamin, Weak orbit equivalence of Cantor minimal systems, *Internat. J. Math.* **6** (1995), no. 4, 559–579.
[10] Gnedin, A., Boundaries from inhomogeneous Bernoulli trials (preprint - arXiv:0909.4933).
[11] Gnedin, A.; Pitman, J., Exchangeable Gibbs partitions and Stirling triangles. Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) **325** (2005), Teor. Predst. Din. Sist. Komb. i Algoritm. Metody. **12**, 83–102, 244–245; translation in J. Math. Sci. (N. Y.) **138** (2006), no. 3, 5674–5685
[12] Kerov, S. V. Combinatorial examples in the theory of AF-algebras. (Russian) *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **172** (1989), Differentsialnaya Geom. Gruppy Li i Mekh. Vol. 10, 55–67, 169–170; *translation in J. Soviet Math.* **59** (1992), no. 5, 1063–1071.
[13] Hajian, Arshag; Ito, Yuji; Kakutani, Shizuo, *Invariant measures and orbits of dissipative transformations. Advances in Math.* **9**, 52–65. (1972).
[14] Herman, Richard H., Putnam, Ian F., and Skau, Christian F., Ordered Bratteli diagrams, dimension groups and topological dynamics. *Internat. J. Math.* **3** (1992), no. 6, 827–864.
[15] Méla, Xavier, A class of nonstationary adic transformations. *Ann. Inst. H. Poincaré Probab. Statist.* **42** (2006), no. 1, 103–123.
[16] Méla, Xavier; Petersen, Karl, Dynamical properties of the Pascal adic transformation *Ergodic Theory Dynam. Systems* **25** (2005), no. 1, 227–256.
[17] Petersen, Karl; Schmidt, Klaus Symmetric Gibbs measures. *Trans. Amer. Math. Soc.* **349** (1997), no. 7, 2775–2811.
[18] Petersen, Karl; Varchenko, Alexander, The Euler adic dynamical system and path counts in the Euler graph, (to appear *Tokyo J. Math.*).
[19] Renault, Jean, A groupoid approach to C^*-algebras. Lecture Notes in Mathematics, 793. *Springer, Berlin*, 1980. ii+160 pp. ISBN: 3-540-09977-8
[20] Veršik, A. M., A description of invariant measures for actions of certain infinite-dimensional groups, *Dokl. Akad. Nauk SSSR* **218**, (1974), 749–752.
[21] Vershik, A. M., Livshits, A. N., Adic models of ergodic transformations, spectral theory, substitutions, and related topics, Representation theory and dynamical systems, Adv. Soviet Math. **9**, *Amer. Math. Soc., Providence, RI* (1992), 185–204.
[22] Vershik, A. M.; Kerov, S. V., Asymptotic theory of the characters of a symmetric group (Russian), *Funktsional. Anal. i Prilozhen.* **15** (1981), no. 4, 15–27, 96.
[23] Peter Walters, An introduction to ergodic theory, *Springer, New York,* 1982, ISBN 0-387-95152-0.

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