QUANTUM INTEGRALS AND ANHOMOMORPHIC LOGICS

Stan Gudder
Department of Mathematics
University of Denver
Denver, Colorado 80208
sgudder@math.du.edu

Abstract

The full anhomomorphic logic of coevents $\mathcal{A}^*$ is introduced. Atoms of $\mathcal{A}^*$ and embeddings of the event set $\mathcal{A}$ into $\mathcal{A}^*$ are discussed. The quantum integral over an event A with respect to a coevent ϕ is defined and its properties are treated. Integrals with respect to various coevents are computed. Reality filters such as preclusivity and regularity of coevents are considered. A quantum measure μ that can be represented as a quantum integral with respect to a coevent ϕ is said to 1-generate ϕ. This gives a stronger reality filter that may produce a unique coevent called the "actual reality" for a physical system. What we believe to be a more general filter is defined in terms of a double quantum integral and is called 2-generation. It is shown that ordinary measures do not 1 or 2-generate coevents except in a few simple cases. Examples are given which show that there are quantum measures that 2-generate but do not 1-generate coevents. Examples also show that there are coevents that are 2-generated but not 1-generated. For simplicity only finite systems are considered.

1 Introduction

Quantum measure theory and anhomomorphic logics have been studied for the past 16 years [1, 2, 3, 5, 6, 7, 8, 9, 10]. The main motivations for these studies have been investigations into the histories approach to quantum mechanics and quantum gravity and cosmology. The author has recently introduced a quantum integration theory [4] and the present article presents some connections between this theory in a slightly different setting and anhomomorphic logics. It turns out that classical logic, in which truth functions are given by homomorphisms, is not adequate for quantum mechanical studies. instead, one must employ truth functions that are not homomorphism and this is the origin of the term anhomomorphic logic. These more general truth functions are Boolean-valued functions on the set $\mathcal{A}$ of quantum events (or propositions) and are called coevents. We denote the set of coevents with reasonable properties by $\mathcal{A}^*$ and call $\mathcal{A}^*$ the full anhomomorphic logic. The elements of $\mathcal{A}^*$ correspond to potential realities for a quantum system. For $A \in \mathcal{A}$, $\phi \in \mathcal{A}^*$, $\phi(A) = 1$ if and only if the event A occurs (or the proposition A is true) in the reality described by ϕ. The main goal of the theory is to find the "actual reality" ϕ_a for the system.

Even for a system with a small finite sample space Ω of cardinality n, the cardinality of $\mathcal{A}^*$ is $2^{(2^n-1)}$ which is huge. Thus, finding ϕ_a in this huge space may not be an easy task. It is thus important to develop filters or criteria for reducing this number of potential realities to make the selection of ϕ_a more manageable. One of the main filters used in the past has been preclusivity. Nature has provided us with an underlying quantum measure μ on $\mathcal{A}$. The measure $\mu(A)$ is sometimes interpreted as the propensity of occurrence for the event A. If $\mu(A) = 0$, then A does not occur and we say that A is *precluded*. If $\phi(A) = 0$ for every precluded event A, then A is *preclusive*. It is postulated that ϕ_a must be preclusive [1, 7, 8]. Unfortunately, there may still be many preclusive coevents so this criteria does not specify ϕ_a uniquely. In this article we propose a stronger filter that may uniquely determine ϕ_a.

Motivated by previous work on quantum integration [4], we define an integral $\int_A f d\phi$ over $A \in \mathcal{A}$ of a real-valued function on Ω with respect to a coevent ϕ. If μ is a quantum measure on $\mathcal{A}$ and if $\mu(A) = \int_A f d\phi$ for all $A \in \mathcal{A}$ where f is a strictly positive function, we say that μ 1-*generates* ϕ. If μ 1-generates ϕ, then ϕ is automatically preclusive (relative to μ). We do not know whether a 1-generated coevent ϕ is unique but we have a partial result in that direction. In any case, 1-generated coevents provide a much stronger filter than preclusivity. Unfortunately, as we discuss in more detail later, one cannot expect an arbitrary quantum measure μ to 1-generate a coevent and we shall show that this only holds for a very restricted set of quantum measures. For this reason we introduce what we believer is a more general method that holds for a much larger set of (but not all) quantum measures. We say that μ 2-*generates* ϕ if

$$\mu(A) = \int_A \left[\int_A f(\omega, \omega') d\phi(\omega) \right] d\phi(\omega')$$

for all $A \in \mathcal{A}$ where f is a symmetric, strictly positive function on $\Omega \times \Omega$. Again, if μ 2-generates ϕ, then ϕ is preclusive (relative to μ). A result which we find interesting is that except for a few simple cases, no ordinary measure 1- or 2-generates a coevent. Thus, the concept of generating coevents is essentially purely quantum mechanical. This article includes many examples that illustrate various concepts. For simplicity, we only consider finite sample spaces.

We now briefly summarize the contents of the paper. In Section 2, we define the full anhomomorphic logic $\mathcal{A}^*$. We briefly discuss the additive, multiplicative and quadratic sublogics of $\mathcal{A}^*$. These sublogics have been considered in the past and it has not yet been settled which is the most suitable or whether some other sublogic is preferable. For this reason and for generality, we do not commit to a particular sublogic here. We point out that $\mathcal{A}^*$ is a Boolean algebra and we discuss the atoms of $\mathcal{A}^*$. Two embeddings of $\mathcal{A}$ into $\mathcal{A}^*$ denoted by $A \mapsto A_*$ and $A \mapsto A^*$ are treated.

Section 3 introduces the quantum integral $\int f d\phi$ with respect to the coevent ϕ. Properties of this integral and the more general integral $\int_A f d\phi$ are discussed. Integrals with respect to various coevents such as A_* and A^* are computed. Reality filters are considered in Section 4. Preclusive and regular coevents are discussed. Most of the section is devoted to the study of 1- and 2-generated coevents. Section 5 presents some general theorems. A uniqueness result shows that if $\phi, \psi \in \mathcal{A}^*$ are regular and μ 1-generates both ϕ and ψ, then $\phi = \psi$. Expansions of quantum measures and coevents are defined. It is shown that μ 1- or 2-generates ϕ if and only if expansions of μ 1- or 2-generate corresponding expansions of ϕ. It is shown that if ϕ is 1-generated by μ and $\phi(A) \neq 0$ whenever $\mu(A) = 0$, then ϕ is 2-generated by μ. Sections 6 and 7 are devoted to examples of 1- and 2-generated coevents. For instance, it is shown that there are quantum measures that 2-generate coevents but do not 1-generate coevents. Examples of coevents that are 2-generated but not 1-generated are given. Also, examples of coevents that are not 1- or 2-generated are presented.

2 Full Anhomomorphic Logics

Let Ω be a finite nonempty set with cardinality $|\Omega| < \infty$. We call Ω a *sample space*. The elements of Ω correspond to outcomes or trajectories of an experiment or physical system and the collection of subsets 2^Ω of Ω correspond to possible events. We can also think of the sets in 2^Ω as propositions concerning the system. Contact with reality is given by a truth function $\phi\colon 2^\Omega \to \{0,1\}$. The function ϕ specifies what actually happens where we interpret $\phi(A) = 1$ to mean that A is true or occurs and $\phi(A) = 0$ means that A is false or does not occur. It is convenient to view $\{0,1\}$ as the two element Boolean algebra $\mathbb{Z}_2$ with the usual multiplication and addition given by $0 \oplus 0 = 1 \oplus 1 = 0$ and $0 \oplus 1 = 1 \oplus 0 = 1$.

For $\omega \in \Omega$ we define the *evaluation map* $\omega^*\colon 2^\Omega \to \mathbb{Z}_2$ by

$$\omega(A) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

For classical systems, it is assumed that a truth function ϕ is a homomorphism; that is, ϕ satisfies:

(H1) $\phi(\Omega) = 1$ (unital)

(H2) $\phi(A \,\dot\cup\, B) = \phi(A) \oplus \phi(B)$ whenever $A \cap B = \emptyset$ (additive)

(H3) $\phi(A \cap B) = \phi(A)\phi(B)$ (multiplicative)

In (H2) $A \,\dot\cup\, B$ denotes $A \cup B$ whenever $A \cap B = \emptyset$. It is well-known that ϕ is a homomorphism if and only if $\phi = \omega^*$ for some $\omega \in \Omega$. Thus, there are $|\Omega|$ truth functions for classical systems.

As discussed in [7, 8, 9], for a quantum system a truth function need not be a homomorphism. However, a quantum truth function should satisfy some requirements or else there would be no theory at all. Various proposals have been presented concerning what these requirements should be [1, 7, 8, 9]. In [7] it is assumed that quantum truth functions satisfy (H2) and these are called *additive* truth functions, while in [1, 9] it is assumed that quantum truth functions satisfy (H3) and these are called *multiplicative* truth functions. In [5] it is argued that quantum truth functions need not satisfy (H1), (H2) or (H3) but should be *quadratic* or *grade*-2 *additive* in the sense that

(H4) $\phi(A \cup B \cup C) = \phi(A \cup B) \oplus \phi(A \cup C) \oplus \phi(B \cup C) \oplus \phi(A) \oplus \phi(B) \oplus \phi(C)$

If $\phi, \psi \colon 2^\Omega \to \mathbb{Z}_2$ are truth functions, we define $\phi\psi$ by $(\phi\psi)(A) = \phi(A)\psi(A)$ and $\phi \oplus \psi$ by $(\phi \oplus \psi)(A) = \phi(A) \oplus \psi(A)$ for all $A \in 2^\Omega$. We make the standing assumption that $\phi(\emptyset) = 0$ for any truth function and we do admit the constant 0 function as a truth function and denote it by 0. We also define 1 as the truth function given by $1(A) = 1$ for all $A \neq \emptyset$. It can be shown [1, 5, 9] that ϕ is additive if and only if ϕ is 0 or a degree-1 polynomial

$$\phi = \omega_1^* \oplus \cdots \oplus \omega_n^*$$

and that ϕ is multiplicative if and only if ϕ is a monomial

$$\phi = \omega_1^* \omega_2^* \cdots \omega_n^*$$

Moreover, one can show [1, 5] that ϕ is quadratic if and only if ϕ is a degree-1 polynomial or ϕ is a degree-2 polynomial of the form

$$\phi = \omega_1^* \oplus \cdots \oplus \omega_n^* \oplus \omega_i^* \omega_j^* \oplus \cdots \oplus \omega_n^* \omega_s^*$$

Notice that we do not allow a constant term in our polynomials.

Since it is not completely clear what conditions a quantum truth function should satisfy and for the sake of generality, we shall not commit to any particular type of truth function here. We use the notation $\Omega_n = \{\omega_1, \ldots, \omega_n\}$, $\mathcal{A}_n = 2^{\Omega_n}$ for an n-element sample space. Leaving out the truth functions satisfying $\phi(\emptyset) = 1$ we have $2^{(2^n - 1)}$ admissible truth functions. We denote this set of truth functions by $\mathcal{A}_n^*$ or $\mathcal{A}^*$ when no confusion arises. Similarly, we sometimes denote the set of events $\mathcal{A}_n$ by $\mathcal{A}$. We call $\mathcal{A}^*$ the *full anhomomorphic logic* and the elements of $\mathcal{A}^*$ are called *coevents*. In $\mathcal{A}^*$ there are n classical, 2^n additive, 2^n multiplicative and $2^{n(n+1)/2}$ quadratic coevents. All nonzero multiplicative coevents are unital and half of the additive and quadratic coevents are unital, namely those with an odd number of summands. It can be shown that any nonzero coevent ϕ can be uniquely written as a polynomial in the evaluation maps. We call this the *evaluation map representation* of ϕ. Notice that $\mathcal{A}_1^*$ consists of the 0 coevent and the single classical coevent ω_1^*. The following examples discuss $\mathcal{A}_2^*$ and $\mathcal{A}_3^*$.

Example 1. The full anhomomorphic logic $\mathcal{A}_2^*$ contain $2^3 = 8$ elements. We list them according to types. The 0 coevent is type (0) and the two classical coevents ω_1^*, ω_2^* are type (1). The additive coevent $\omega_1^* \oplus \omega_2^*$ is type

$(1,2)$. The multiplicative coevent $\omega_1^*\omega_2^*$ is type (12). The two quadratic coevents $\omega_1^*\oplus\omega_1^*\omega_2^*$, $\omega_2^*\oplus\omega_1^*\omega_2^*$ are type $(1,12)$. Finally, the quadratic coevent $1=\omega_1^*\oplus\omega_2^*\oplus\omega_1^*\omega_2$ is type $(1,2,12)$. $\square$

Example 2. The full anhomomorphic logic $\mathcal{A}_3^*$ contains $2^7=128$ elements. There are too many to list so we shall just consider some of them. The coevent ω_1^* is one of the three classical type (1) coevents and $\omega_1^*\oplus\omega_2^*$ is one of the three type $(1,2)$ coevents. The coevent $\omega_1^*\omega_2^*$ is one of the three type (12) coevents and $\omega_1^*\oplus\omega_2^*\oplus\omega_2^*$ is the only type $(1,2,3)$ coevent. There is one type (123) coevent $\omega_1^*\omega_2^*\omega_3^*$ and there are six type $(1,12)$ coevents, one being $\omega_1^*\oplus\omega_1^*\omega_2^*$. There are three type $(1,23)$ coevents, one being $\omega_1^*\oplus\omega_2^*\omega_3^*$ and three type $(1,2,12)$ coevents, one being $\omega_1^*\oplus\omega_2^*\oplus\omega_1^*\omega_2^*$. The coevent $\omega_1^*\oplus\omega_2^*\oplus\omega_1^*\omega_3^*$ is one of the six type $(1,2,13)$ coevents and $\omega_1^*\oplus\omega_2^*\oplus\omega_3^*\oplus\omega_1^*\omega_2^*$ is one of the three type $(1,2,3,12)$ coevents. Our last example is the type $(1,2,3,12,13,23,123)$ coevent

$$1=\omega_1^*\oplus\omega_2^*\oplus\omega_3^*\oplus\omega_1^*\omega_2^*\oplus\omega_1^*\omega_3^*\oplus\omega_2^*\omega_3^*\oplus\omega_1^*\omega_2^*\omega_3^* \qquad \square$$

It is of interest to note that the coevent 1 in $\mathcal{A}_n^*$ has the form

$$1=\bigoplus_{i=1}^n\omega_i^*\oplus\bigoplus_{i<j=1}^n\omega_i^*\omega_j^*\oplus\cdots\oplus\omega_1^*\omega_2^*\cdots\omega_n^* \tag{2.1}$$

To show this, let $A\in\mathcal{A}_n$ and assume without loss of generality that $A=\{\omega_1,\ldots,\omega_m\}$. Denoting the right side of (2.1) by ϕ, the number of terms of ϕ that are 1 on A becomes

$$\binom{m}{1}+\binom{m}{2}+\cdots+\binom{m}{m}=2^m-1$$

Since 2^m-1 is odd, we conclude that $\phi(A)=1$.

For $\phi,\psi\in\mathcal{A}^*$ we define $\phi\le\psi$ if $\phi(A)\le\psi(A)$ for all $A\in\mathcal{A}$. It is clear that $\le$ is a partial order on $\mathcal{A}^*$ and that $0\le\phi\le 1$ for all $\phi\in\mathcal{A}^*$. For $\phi\in\mathcal{A}^*$ we define the *complement* ϕ' of ϕ by $\phi'=1\oplus\phi$. It is easy to check that $(\mathcal{A}^*,0,1,\le,')$ is a Boolean algebra in which the meet and join are given by

$$(\phi\wedge\psi)(A)=\min\left(\phi(A),\psi(A)\right)=\phi(A)\psi(A)$$
$$(\phi\vee\psi)(A)=\max\left(\phi(A),\psi(A)\right)=\phi(A)\oplus\psi(A)\oplus\phi(A)\psi(A)$$

Examples of meets and joins are:

$$\begin{aligned}
&\omega_1^* \wedge \omega_2^* \wedge \cdots \wedge \omega_m^* = \omega_1^*\omega_2^* \cdots \omega_m^* \\
&\omega_1^* \vee \omega_2^* = \omega_1^* \oplus \omega_2^* \oplus \omega_1^*\omega_2^* \\
&(\omega_1^* \oplus \omega_1^*\omega_2^*) \vee (\omega_2^* \oplus \omega_1^*\omega_2^*) = \omega_1^* \oplus \omega_2^* \\
&(\omega_1^* \oplus \omega_1^*\omega_2^* \oplus \omega_1^*\omega_3^* \oplus \omega_1^*\omega_2^*\omega_3^*) \vee (\omega_2^* \oplus \omega_1^*\omega_2^* \oplus \omega_2^*\omega_3^* \oplus \omega_1^*\omega_2^*\omega_3^*) \\
&\quad = \omega_1^* \oplus \omega_2^* \oplus \omega_1^*\omega_3^* \oplus \omega_2^*\omega_3^*
\end{aligned}$$

The *atoms* of $\mathcal{A}^*$ are the minimal nonzero elements of $\mathcal{A}^*$ and every nonzero coevent ϕ is the unique join of the atoms below ϕ. Thus, every atom has the form ϕ_A for $\emptyset \neq A \in \mathcal{A}$ where $\phi_A(B) = 1$ if and only if $B = A$. The atoms of $\mathcal{A}_2^*$ are $\phi_{\{\omega_1\}} = \omega_1^* \oplus \omega_1^*\omega_2^*$, $\phi_{\{\omega_2\}} = \omega_2^* \oplus \omega_1^*\omega_2^*$, and $\phi_\Omega = \omega_1^*\omega_2^*$. We then have $\omega_1^* = \phi_{\{\omega_1\}} \vee \phi_\Omega$, $\omega_2^* = \phi_{\{\omega_2\}} \vee \phi_\Omega$, $\omega_1^* \oplus \omega_2^* = \phi_{\{\omega_1\}} \vee \phi_{\{\omega_2\}}$ and $1 = \phi_{\{\omega_1\}} \vee \phi_{\{\omega_2\}} \vee \phi_\Omega$. The seven atoms of $\mathcal{A}_3^*$ are the type (123), the three type $(1, 12, 13, 123)$ and the three type $(12, 123)$ coevents.

We now describe the atoms of $\mathcal{A}_n^*$ in terms of their evaluation map representations. Without loss of generality suppose $A \in \mathcal{A}_n$ has the form $A = \{\omega_1, \ldots, \omega_m\}$ and let $\phi = \omega_1^*\omega_2^* \cdots \omega_m^*$. We claim that

$$\phi_A = \phi \oplus \bigoplus_{i=m+1}^{n} \phi\omega_i^* \oplus \bigoplus_{i<j=m+1}^{n} \phi\omega_i^*\omega_j^* \oplus \cdots \oplus \phi\omega_{m+1}^* \cdots \phi_n^* \tag{2.2}$$

To show this, let ψ be the right side of (2.2). It is clear that $\psi(A) = \phi(A) = 1$. Now suppose that $B \in \mathcal{A}_n$ with $\emptyset \neq B \neq A$. If $A \not\subseteq B$, then $\psi(B) = 0$. If $A \subseteq B$, then we can assume without loss of generality that $B = \{\omega_1, \ldots, \omega_m, \omega_{m+1}, \ldots, \omega_{m+r}\}$. The number of terms on the right side of (2.2) that are 1 on B is

$$1 + \binom{r}{1} + \binom{r}{2} + \cdots + \binom{r}{r} = 2^r$$

Since 2^r is even, we conclude that $\psi(B) = 0$ and this proves (2.2).

We next give a natural embedding of the Boolean algebra $\mathcal{A}$ into $\mathcal{A}^*$. For $A \in \mathcal{A}$, we define $A_* \in \mathcal{A}$ by $A_*(B) = 1$ if and only if $\emptyset \neq B \subseteq A$. Notice that $\emptyset_* = 0$ and $\Omega_* = 1$. Also, $\{\omega\}_* = \phi_{\{\omega\}}$ so $\{\omega\}_*$ has the form (2.2) with $m = 1$ and $\phi = \omega^* = \omega_1^*$. The evaluation map representation of A_* is a generalization of (2.2) that we shall discuss later. We denote the complement of a set A by A'.

Theorem 2.1. (a) *$A \subseteq B$ if and only if $A_* \leq B_*$.* (b) $(A \cap B)_* = A_* \wedge B_*$.

Proof. (a) Suppose that $A \subseteq B$. If $A_*(C) = 1$, then $\emptyset \neq C \subseteq A$. Hence, $C \subseteq B$ so $B_*(C) = 1$. It follows that $A_* \leq B_*$. Conversely, suppose that $A_* \leq B_*$. If $A \not\subseteq B$, then $A \cap B' \neq \emptyset$ and $A \cap B' \subseteq A$. Hence, $A_*(A \cap B') = 1$ so $B_*(A \cap B') = 1$. We conclude that $A \cap B' \subseteq B$ which is a contradiction. Therefore, $A \subseteq B$. (b) If $A_* \wedge B_*(C) = 1$, then $\min(A_*(C), B_*(C)) = 1$. Hence, $A_*(C) = B_*(C) = 1$ so $C \subseteq A \cap B$. It follows that $(A \cap B)_*(C) = 1$. Conversely, if $(A \cap B)_*(C) = 1$, then $C \subseteq A \cap B$. Hence, $A_*(C) = B_*(C) = 1$ so $(A_* \wedge B_*)(C) = 1$. Therefore, $(A \cap B)_* = A_* \wedge B_*$. □

In general, $A \mapsto A_*$ is not a Boolean homomorphism. One reason is that $(A')_* \neq (A_*)'$. Indeed, if $B \cap A \neq \emptyset$ and $B \cap A' \neq \emptyset$, then $(A')_*(B) = A_*(B) = 0$. Another reason is that $(A \cup B)_* \neq A_* \vee B_*$. To see this, suppose that $C \not\subseteq A$, $C \not\subseteq B$ and $C \subseteq A \cup B$. Then $(A \cup B)_*(C) = 1$ while $A_*(C) = B_*(C) = 0$. To describe the evaluation map representation of A_* we can assume without loss of generality that $A \subseteq \Omega_n$ and $A = \{\omega_1, \ldots, \omega_m\}$, $m \leq n$. We claim that A_* is the sum of all monomials that contain at least one of the elements ω_i^*, $i = 1, \ldots, m$ as a factor. For instance, in Ω_3 we have

$$\{\omega_1, \omega_2\}_* = \omega_1^* \oplus \omega_2^* \oplus \omega_1^*\omega_2^* \oplus \omega_1^*\omega_3^* \oplus \omega_2^*\omega_3^* \oplus \omega_1^*\omega_2^*\omega_3^*$$

as is easy to check. Instead of giving the general proof of this claim, which is tedious, we present some other examples.

Example 3. In Ω_4 with $A = \{\omega_1, \omega_2\}$ our claim states that

$$\begin{aligned} A_* = \omega_1^* \oplus \omega_2^* \oplus \omega_1^*\omega_2^* \oplus \omega_1^*\omega_3^* \oplus \omega_1^*\omega_4^* \oplus \omega_2^*\omega_3^* \oplus \omega_2^*\omega_4^* \\ \oplus \omega_1^*\omega_2^*\omega_3^* \oplus \omega_1^*\omega_2^*\omega_4^* \oplus \omega_1^*\omega_3^*\omega_4^* \oplus \omega_2^*\omega_3^*\omega_4^* \oplus \omega_1^*\omega_2^*\omega_3^*\omega_4^* \end{aligned} \tag{2.3}$$

Let ψ be the right side of (2.3). Clearly $\psi(B) = 1$ for every $\emptyset \neq B \subseteq A$ and $\psi(B) = 0$ for every $B \subseteq A'$. Next we have

$$\psi(\{\omega_1, \omega_3\}) = \psi(\{\omega_1, \omega_4\}) = \psi(\{\omega_2, \omega_3\}) = \psi(\{\omega_2, \omega_4\}) = 0$$

Also, $\psi(\Omega_4) = 0$ and

$$\psi(\{\omega_1, \omega_2, \omega_3\}) = \psi(\{\omega_1, \omega_2, \omega_4\}) = 0$$

It follows that $\psi = A_*$. As another example, let $B = \{\omega_1, \omega_2, \omega_3\}$. Our claim states that

$$B_* = A_* \oplus \omega_3^* \oplus \omega_3^*\omega_4^*$$

and this is easily verified. $\square$

We now present another embedding of $\mathcal{A}$ into $\mathcal{A}^*$. For $A = \{\omega_1, \ldots, \omega_m\}$, let A^* be the sum of all monomials that contain only elements $\omega_1^*, \ldots, \omega_n^*$ as factors. Thus,

$$A^* = \bigoplus_{i=1}^{m} \omega_i^* \oplus \bigoplus_{i<j=1}^{m} \omega_i^* \omega_j^* \oplus \cdots \oplus \omega_1^* \omega_2^* \cdots \omega_m^* \tag{2.4}$$

Notice that $\emptyset^* = 0$, $\Omega^* = 1$ and $\{\omega\}^* = \omega^*$.

Theorem 2.2. (a) $A^*(B) = 1$ *if and only if* $B \cap A \neq \emptyset$. (b) $A \subseteq B$ *if and only if* $A^* \leq B^*$. (c) $(A \cup B)^* = A^* \vee B^*$.

Proof. (a) If $B \cap A = \emptyset$, then clearly $A^*(B) = 0$. Suppose $B \cap A \neq \emptyset$ and $|B \cap A| = r$. Representing A^* as in (2.4) we have

$$A^*(B) = \bigoplus_{i=1}^{m} \omega_i^*(B) \oplus \bigoplus_{i<j=1}^{m} \omega_i^* \omega_j^*(B) \oplus \cdots \oplus \omega_1^* \omega_2^* \cdots \omega_m^*(B) \tag{2.5}$$

The number of terms on the right side of (2.5) that equal 1 are

$$\binom{r}{1} + \binom{r}{2} + \cdots + \binom{r}{r} = 2^r - 1$$

Since $2^r - 1$ is odd, $A^*(B) = 1$. (b) Suppose that $A \subseteq B$. If $A^*(C) = 1$, then by (a), $C \cap A \neq \emptyset$. Hence, $C \cap B \neq \emptyset$ so by (a) $B^*(C) = 1$. Therefore, $A^* \leq B^*$. Conversely, suppose that $A^* \leq B^*$. If $A \not\subseteq B$, then $A \cap B' \neq \emptyset$. Since $A \cap B' \subseteq A$, by (a) $A^*(A \cap B') = 1$. Hence, $B^*(A \cap B') = 1$ so by (a) $(A \cap B') \cap B \neq \emptyset$ which is a contradiction. Hence, $A \cap B' = \emptyset$ so $A \subseteq B$. (c) If $(A \cup B)^*(C) = 1$, then $C \cap (A \cup B) \neq \emptyset$. Hence, $C \cap A \neq \emptyset$ or $C \cap B \neq \emptyset$ so $A^*(C) = 1$ or $B^*(C) = 1$. Hence,

$$(A^* \vee B^*)(C) = \max\left(A^*(C), B^*(C)\right) = 1$$

Conversely, if $(A^* \vee B^*)(C) = 1$ then $A^*(C) = 1$ or $B^*(C) = 1$. Hence, $C \cap A \neq \emptyset$ or $C \cap B \neq \emptyset$. It follows that $C \cap (A \cup B) \neq \emptyset$ so $(A \cup B)^*(C) = 1$. We conclude that $(A \cup B)^* = A^* \vee B^*$. $\square$

As before $A \mapsto A^*$ is not a Boolean homomorphism because $(A')^* \neq (A^*)'$ and $(A \cap B)^* \neq A^* \wedge B^*$ in general. For the first case, let B satisfy $B \cap A \neq \emptyset$ and $B \cap A' \neq \emptyset$. Then

$$A^*(B) = (A')^*(B) = 1$$

Hence, $(A')^* \neq (A^*)'$. For the second case, let C satisfy $C \cap A$, $C \cap B \neq \emptyset$ but $C \cap (A \cap B) = \emptyset$. Then $(A \cap B)^*(C) = 0$ but

$$(A^* \wedge B^*)(C) = \min\left(A^*(C), B^*(C)\right) = 1$$

Hence, $(A \cap B)^* \neq A^* \wedge B^*$. It is interesting to note that $A_* \leq A^*$.

We close this section by mentioning that the coevent $\psi_A(B) = 1$ if and only if $A \subseteq B$ is much simpler than A_* or A^*. If $A = \{\omega_1, \ldots, \omega_m\}$, then $\psi_A = \omega_1^* \omega_2^* \cdots \omega_m^*$ which is multiplicative.

3 Quantum Integrals

As in Section 2, Ω is a finite set, $\mathcal{A} = 2^\Omega$ and $\mathcal{A}^*$ is the full anhomomorphic logic. Following [4], for $f \colon \Omega \to \mathbb{R}$ and $\phi \in \mathcal{A}^*$, we define the *quantum integral* (*q-integral*, for short)

$$\int f d\phi = \int_0^\infty \phi\left(\{\omega \colon f(\omega) > \lambda\}\right) d\lambda - \int_0^\infty \phi\left(\{\omega \colon f(\omega) < -\lambda\}\right) d\lambda \qquad (3.1)$$

where $d\lambda$ denotes Lebesgue measure on $\mathbb{R}$. Any $f \colon \Omega \to \mathbb{R}$ has a unique representation $f = f_1 - f_2$ where $f_1, f_2 \geq 0$ and $f_1 f_2 = 0$. It follows from (3.1) that

$$\int f d\phi = \int f_1 d\phi - \int f_2 d\phi$$

We conclude that q-integrals are determined by q-integrals of nonnegative functions. In fact, all the q-integrals that we consider will be for nonnegative functions.

Denoting the characteristic function of a set A by χ_A, any nonnegative function $f \colon \Omega \to \mathbb{R}^+$ has the canonical representation

$$f = \sum_{i=1}^n \alpha_i \chi_{A_i} \qquad (3.2)$$

where $0 < \alpha_1 < \cdots < \alpha_n$ and $A_i \cap A_j = \emptyset$, $i \neq j = 1, \ldots, n$. Thus, α_i are the nonzero values of f and $A_i = f^{-1}(\alpha_i)$, $i = 1, \ldots, n$. Since $f \geq 0$ we can write (3.1) as

$$\int f d\phi = \int_0^\infty \phi\left(\{\omega \colon f(\omega) > \lambda\}\right) d\lambda \tag{3.3}$$

and it follows from (3.2) and (3.3) that

$$\int f d\phi = \alpha_1 \phi\left(\dot{\bigcup_{i=1}^n} A_i\right) + (\alpha_2 - \alpha_1)\phi\left(\dot{\bigcup_{i=2}^n} a_i\right) + \cdots + (\alpha_n - \alpha_{n-1})\phi(A_n) \tag{3.4}$$

$$= \sum_{j=1}^n \alpha_j \left[\phi\left(\dot{\bigcup_{i=j}^n} A_i\right) - \phi\left(\dot{\bigcup_{i=j+1}^n} A_i\right)\right] \tag{3.5}$$

In (3.5) we use the convention that $\dot{\cup}_{i=n+1}^n A_i = \emptyset$.

It is clear from (3.4) that $\int f d\phi \geq 0$ and $\int \alpha \chi_A d\phi = \alpha \phi(A)$ for all $\alpha \geq 0$. In particular, $\phi(A) = \int \chi_A d\phi$ so the q-integral generalizes the coevent ϕ. Also it is easy to check that $\int \alpha f d\phi = \alpha \int f d\phi$ and that

$$\int (\alpha + f) d\phi = \int \alpha d\phi + \int f d\phi = \alpha\phi(A) + \int f d\phi$$

for all $\alpha \geq 0$. The q-integral is nonlinear, in general. For example, suppose that $A, B \in \mathcal{A}$ are disjoint nonempty sets, $0 < \alpha < \beta$ and $\phi \in \mathcal{A}^*$ satisfies $\phi(A \dot{\cup} B) \neq \phi(A) + \phi(B)$. Then by (3.5) we have

$$\int (\alpha \chi_A + \beta \chi_B) d\phi = \alpha \left[\phi(A \dot{\cup} B) - \phi(B)\right] + \beta \phi(B)$$

$$\neq \alpha\phi(A) + \beta\phi(B) = \int \alpha \chi_A d\phi + \int \beta \chi_B d\phi$$

Also, we do not have $\int f d(\phi \oplus \psi) = \int f d\phi + \int f d\psi$. For example, suppose that $\omega_1, \omega_2 \in A$. Then

$$\int \chi_A d(\omega_1^* \oplus \omega_2^*) = (\omega_1^* \oplus \omega_2^*)(A) = 0 \neq \omega_1^*(A) + \omega_2^*(A)$$

$$= \int \chi_A d\omega_1^* + \int \chi_A d\omega_2^*$$

It is frequently convenient to write (3.4) and (3.5) in a different form. For $f \colon \Omega \to \mathbb{R}^+$, let $\omega_1, \ldots, \omega_r$ be the points of Ω such that $f(\omega_i) > 0$,

$i = 1, \ldots, r$, with $f(\omega_1) \le \cdots \le f(\omega_r)$. It easily follows from (3.4) that

$$\int f d\phi = f(\omega_1)\phi(\{\omega_1, \ldots, \omega_r\}) + [f(\omega_2) - f(\omega_1)]\,\phi(\{\omega_2, \ldots, \omega_r\}) + \cdots + [f(\omega_r) - f(\omega_{r-1})]\,\phi(\omega_r) \tag{3.6}$$

$$= \sum_{i=1}^{r} f(\omega_i)\,[\phi(\{\omega_i, \ldots, \omega_r\}) - \phi(\{\omega_{i+1}, \ldots, \omega_r\})] \tag{3.7}$$

In (3.6) we have used the shorthand notation $\phi(\omega_r) = \phi(\{\omega_r\})$.

We now compute q-integrals relative to some of the common coevents. These formulas follow easily from (3.4) or (3.6) and their verification is left to the reader. First, it is not surprising that $\int f d\omega^* = f(\omega)$. If $0 \le f(\omega_1) \le \cdots \le f(\omega_r)$ then

$$\int f d(\omega_1^* \oplus \cdots \oplus \omega_r^*) = f(\omega_r) - f(\omega_{r-1}) + \cdots + (-1)^{r+1} f(\omega_1) \tag{3.8}$$

For example, if $f \colon \Omega \to \mathbb{R}^+$ is arbitrary then

$$\int f d(\omega_1^* \oplus \omega_2^*) = \max(f(\omega_1), f(\omega_2)) - \min(f(\omega_1), f(\omega_2))$$

For $f \colon \Omega \to \mathbb{R}^+$ we have

$$\int f d(\omega_1^* \cdots \omega_r^*) = \min(f(\omega_1), \ldots, f(\omega_r)) \tag{3.9}$$

$$\int f dA^* = \max\{f(\omega) \colon \omega \in A\} \tag{3.10}$$

For $f \colon \Omega \to \mathbb{R}^+$, letting $\alpha = \max\{f(\omega) \colon \omega \in \Omega\}$, $\beta = \min\{f(\omega) \colon \omega \in A\}$ we have

$$\int f dA_* = \begin{cases} \alpha - \beta & \text{if } f^{-1}(\alpha) \subseteq A \\ 0 & \text{if } f^{-1}(\alpha) \not\subseteq A \end{cases} \tag{3.11}$$

Example 4. We illustrate (3.8)–(3.11) by computing these q-integrals for specific examples. On Ω_5 suppose $0 < f(\omega_1) < \cdots < f(\omega_5)$. Applying (3.6) gives

$$\begin{aligned} \int f d(\omega_2^* \oplus \omega_3^* \oplus \omega_4^*) &= f(\omega_1) + [f(\omega_2) - f(\omega_1)] + [f(\omega_3) - f(\omega_2)] \cdot 0 \\ &\quad + [f(\omega_4) - f(\omega_3)] + [f(\omega_5) - f(\omega_4)] \cdot 0 \\ &= f(\omega_4) - f(\omega_3) + f(\omega_2) \end{aligned}$$

$$\int f d(\omega_2^*\omega_3^*\omega_4^*) = f(\omega_1) + [f(\omega_2) - f(\omega_1)] + [f(\omega_3) - f(\omega_2)] \cdot 0$$
$$+ [f(\omega_4) - f(\omega_3)] \cdot 0 + [f(\omega_5) - f(\omega_4)] \cdot 0$$
$$= f(\omega_2) = \min\left(f(\omega_2), f(\omega_3), f(\omega_4)\right)$$

which are (3.8) and (3.9), respectively. Letting $A = \{\omega_2, \omega_3, \omega_4\}$ we have

$$\int f dA^* = f(\omega_1) + [f(\omega_2) - f(\omega_1)] + [f(\omega_3) - f(\omega_2)] + [f(\omega_4) - f(\omega_3)]$$
$$+ [f(\omega_5) - f(\omega_4)] \cdot 0 = f(\omega_4) = \max\{f(\omega) \colon \omega \in A\}$$

$$\int f dA_* = f(\omega_1) \cdot 0 + [f(\omega_2) - f(\omega_1)] \cdot 0 + [f(\omega_3) - f(\omega_2)] \cdot 0$$
$$+ [f(\omega_4) - f(\omega_3)] \cdot 0 + [f(\omega_5) - f(\omega - 4)] \cdot 0 = 0$$

which are (3.10) and (3.11), respectively. This last example is more interesting if we let $B = \{\omega_3, \omega_4, \omega_5\}$ in which case we have

$$\int f dB_* = f(\omega_1) \cdot 0 + [f(\omega_2) - f(\omega_1)] \cdot 0 + [f(\omega_3) - f(\omega_2)] + [f(\omega_4) - f(\omega_3)]$$
$$+ [f(\omega_5) - f(\omega_4)] = f(\omega_5) - f(\omega_2)$$

which are (3.11). □

Let f have the representation (3.2) and let ϕ_A be an atomic coevent for $A \in \mathcal{A}$. A straightforward application of (3.4) shows that

$$\int f d\phi_A = \begin{cases} \alpha_m - \alpha_{m-1} & \text{if } A = \dot{\bigcup}_{i=m}^{n} A_i \text{ for some } m > 1 \\ \alpha_1 & \text{if } A = \dot{\bigcup}_{i=1}^{n} A_i \\ 0 & \text{otherwise} \end{cases} \tag{3.12}$$

We mentioned previously that any $\phi \in \mathcal{A}^*$ has a unique representation $\phi = \phi_1 \vee \cdots \vee \phi_m$ where $\phi_1, \ldots, \phi_m$ are the distinct atoms below ϕ. The next result shows that the q-integral with respect to ϕ is just the sum of the q-integrals with respect to the ϕ_i, $i = 1, \ldots, m$.

Theorem 3.1. *If $\phi_1, \ldots, \phi_m$ are distinct atoms in $\mathcal{A}^*$ and $f\colon \Omega \to \mathbb{R}^+$, then*

$$\int f d(\phi_1 \vee \cdots \vee \phi_m) = \int f d\phi_1 + \cdots + \int f d\phi_m$$

Proof. We give the proof for $m = 2$ and the general case is similar. Letting f have the representation (3.2) and $\phi_1 = \phi_A$, $\phi_2 = \phi_B$ for $A \neq B$, then (3.4) gives

$$\int f d(\phi_A \vee \phi_B) = \alpha_1 (\phi_A \vee \phi_B) \left(\dot{\bigcup_{i=1}^{n}} A_i \right) + (\alpha_2 - \alpha_1)(\phi_A \vee \phi_B) \left(\dot{\bigcup_{i=2}^{n}} A_i \right) + \cdots + (\alpha_n - \alpha_{n-1})(\phi_A \vee \phi_B)(A_n) \tag{3.13}$$

If the right side of (3.13) is zero, then it is clear that

$$\int f d\phi_A = \int f d\phi_B = 0$$

and we are finished. Otherwise, at most two of the terms on the right side of (3.13) are nonzero. Suppose the jth term is the only nonzero term. We can assume without loss of generality that

$$\phi_A \left(\dot{\bigcup_{i=j}^{n}} A_i \right) = 1 \quad \text{and} \quad \phi_B \left(\dot{\bigcup_{i=j}^{n}} A_i \right) = 0$$

Then $\int f d\phi_B = 0$ and we have

$$\int f d(\phi_A \vee \phi_B) = \alpha_j - \alpha_{j-1} = \int f d\phi_A + \int f d\phi_B$$

Next suppose the jth term and the kth term are nonzero. There are three cases one of which being

$$\phi_A \left(\dot{\bigcup_{i=j}^{n}} A_i \right) = 1,\ \phi_B \left(\dot{\bigcup_{i=j}^{n}} A_i \right) = 0 \ \text{ and } \ \phi_B \left(\dot{\bigcup_{i=k}^{n}} A_i \right) = 1,\ \phi_A \left(\dot{\bigcup_{i=k}^{n}} A_i \right) = 0$$

In this case we have

$$\int f d(\phi_A \vee \phi_B) = \alpha_j - \alpha_{j-1} + \alpha_k - \alpha_{k-1} = \int f d\phi_A + \int f d\phi_B$$

The other cases are similar. □

As usual in integration theory, for $A \in \mathcal{A}$ and $\phi \in \mathcal{A}^*$ we define

$$\int_A f d\phi = \int f \chi_A d\phi$$

Simple integrals of this type are $\int_{\{\omega\}} f d\phi = f(\omega)\phi(\omega)$ and $\int_A 1 d\phi = \phi(A)$. Other examples are

$$\int_A f d1 = \max\{f(\omega) \colon \omega \in A\}$$
$$\int_A f dB^* = \max\{f(\omega) \colon \omega \in A \cap B\}$$

Moreover, if $0 \leq f(\omega_1) \leq f(\omega_2)$, then

$$\int_{\{\omega_1,\omega_2\}} f d\phi = f(\omega_1)\phi\left(\{\omega_1,\omega_2\}\right) + \left[f(\omega_2) - f(\omega_1)\right]\phi(\omega_2)$$

Example 5. This example shows that the q-integral is not additive, even if ϕ is additive, in the sense that in general

$$\int_{A \cup B} f d\phi \neq \int_A f d\phi + \int_B f d\phi$$

Let $A \cap B = \emptyset$, $\omega_1 \in A$, $\omega_2 \in B$ and $f \colon \Omega \to \mathbb{R}^+$ with $0 < f(\omega_1) < f(\omega_2)$. We then have

$$\int_{A \cup B} f d(\omega_1^* \oplus \omega_2^*) = \int f \chi_{A \cup B} d(\omega_1^* \oplus \omega_2^*) = f(\omega_2) - f(\omega_1)$$
$$\neq f(\omega_1) + f(\omega_2) = \int_A f d(\omega_1^* \oplus \omega_2^*) + \int_B f d(\omega_1^* \oplus \omega_2^*) \qquad \square$$

Example 6. This example shows that the q-integral is not grade-2 additive, even if ϕ is additive, in the sense that in general

$$\int_{A \cup B \cup C} f d\phi \neq \int_{A \cup B} f d\phi + \int_{A \cup C} f d\phi + \int_{B \cup C} f d\phi - \int_A f d\phi - \int_B f d\phi - \int_C f d\phi$$

Let $A, B, C \in \mathcal{A}$ be mutually disjoint, $\omega_1 \in A$, $\omega_2 \in B$, $\omega_3 \in C$ and $f \colon \Omega \to \mathbb{R}^+$ with $0 < f(\omega_1) < f(\omega_2) < f(\omega_3)$. For $\phi = \omega_1^* \oplus \omega_2^* \oplus \omega_3^*$ we have

$$\int_{A \cup B \cup C} f d\phi = f(\omega_3) - f(\omega_2) + f(\omega_1)$$

However,

$$\int_{A\cup B} f d\phi + \int_{A\cup C} f d\phi + \int_{B\cup C} f d\phi - \int_A f d\phi - \int_B f d\phi - \int_C f d\phi$$
$$= f(\omega_3) - f(\omega_2) - 3f(\omega_1) \qquad \square$$

4 Reality Filters

We interpret $\mathcal{A}^*$ as the set of coevents that correspond to possible realities of our physical system. Presumably, there is only one "actual reality" ϕ_a. But how do we find ϕ_a? We need methods for filtering out the unwanted potential realities until we are left only with ϕ_a. We have seen that $|\mathcal{A}^*| = 2^{(2^n-1)}$ which can be an inconceivably large number. Thus, finding ϕ_a is like finding a needle in a double exponential haystack. For example, for a small system with $n = |\Omega| = 6$ sample points, we have

$$|\mathcal{A}^*| = 2^{63} = 9,223,372,036,854,775,808$$

If this were a classical system, we only have to sort through 6 potential realities and this could be achieved by a simple observation or measurement. Filters that have been used in the past have been to assume that the potential realities must be additive or that they must be multiplicative. In these two cases there are $2^6 = 64$ potential realities which is quite manageable. Another filter that has been considered is the assumption that the potential realities must be quadratic in which case we have $2^{21} = 1,097,152$ potential realities. Although large, this is a lot better than 2^{63}.

We now discuss another method for eliminating unwanted coevents. Suppose there is an experimental or theoretical reason for assuming that $A \in \mathcal{A}$ does not occur (or that A is false). We then say that A is *precluded* [7, 8, 9]. A trivial example is the empty set $\emptyset$ which by definition is precluded. Denoting the set of precluded events by $\mathcal{A}_0$ we call $\mathcal{A}_p = \mathcal{A} \smallsetminus \mathcal{A}_0$ the set of permitted events. A coevent ϕ is *preclusive* if $\phi(A) = 0$ for all $A \in \mathcal{A}_0$ [7, 8, 9] and the set of preclusive coevents

$$\mathcal{A}_p^* = \{\phi \in \mathcal{A}^* \colon \phi(A) = 0 \text{ for all } A \in \mathcal{A}_0\}$$

is the *preclusive anhomomorphic logic*. Although $\mathcal{A}_p$ is not a Boolean algebra, $\mathcal{A}_p^*$ is a Boolean algebra and its atoms are the preclusive atoms of $\mathcal{A}^*$. Hence,

$\left|\mathcal{A}_p^*\right| = 2^{|\mathcal{A}_p|}$. For example, in Ω_2 if $\mathcal{A}_0 = \{\{\omega_1\}\}$, then $\mathcal{A}_p = \{\{\omega_2\}, \Omega_2\}$ and $\left|\mathcal{A}_p^*\right| = 2^2 = 4$. We then have

$$\mathcal{A}_p^* = \{0, \omega_2^*, \omega_1^*\omega_2^*, \omega_2^* \oplus \omega_1^*\omega_2^*\}$$

If $\mathcal{A}_0 = \{\Omega_2\}$, then $\mathcal{A}_p = \{\{\omega_1\}, \{\omega_2\}\}$ and $\left|\mathcal{A}_p^*\right| = 4$. We then have

$$\mathcal{A}_p^* = \{0, \omega_1^* \oplus \omega_1^*\omega_2^*, \omega_2^* \oplus \omega_1^*\omega_2^*, \omega_1^* \oplus \omega_2^*\}$$

If $\mathcal{A}_0 = \{\{\omega_1\}, \{\omega_2\}\}$, then $\mathcal{A}_p = \{\Omega_2\}$, $\left|\mathcal{A}_p^*\right| = 2$ and $\mathcal{A}_p^* = \{0, \omega_1^*\omega_2^*\}$.

A coevent ϕ is *regular* if

(R1) $\phi(A) = 0$ implies $\phi(A \mathbin{\dot\cup} B) = \phi(B)$ for all $B \in \mathcal{A}$

(R2) $\phi(A \mathbin{\dot\cup} B) = 0$ implies $\phi(A) = \phi(B)$

Condition (R1) is a reasonable condition and although (R2) is not as clear, it may have merit. (We will mention later that quantum measures are usually assumed to be regular.) In any case, there may be fundamental reasons for assuming that the actual reality is regular which gives another method for filtering out unwanted potential realities. For example the regular coevents on Ω_2 are $0, 1, \omega_1^*, \omega_2^*$ and $\omega_1^* \oplus \omega_2^*$.

The filter F that we now discuss is much stronger than the previous ones. In fact, it is so strong that it may filter out all of $\mathcal{A}_p^*$ in which case we say it is not successful. In all the examples we have considered, when F is successful then F admits a unique reality. We believe that if F does not produce a unique coevent, then the number of coevents it does produce is very small. To describe F we shall need the concept of a quantum measure (q-measure, for short) [2, 3, 7, 8]. A *q-measure* is a set function $\mu\colon \mathcal{A} \to \mathbb{R}^+$ that satisfies the *grade*-2 *additivity condition*

$$\mu(A \mathbin{\dot\cup} B \mathbin{\dot\cup} C) = \mu(A \mathbin{\dot\cup} B) + \mu(A \mathbin{\dot\cup} C) + \mu(B \mathbin{\dot\cup} C) - \mu(A) - \mu(B) - \mu(C) \quad (4.1)$$

Because of quantum interference, μ may not satisfy the *grade*-1 *additivity condition* $\mu(A \mathbin{\dot\cup} B) = \mu(A) + \mu(B)$ that holds for ordinary measures. Of course, grade-1 additivity implies grade-2 additivity but the converse does not hold [1, 2, 6]. A q-measure μ is *regular* if μ satisfies (R1) and (R2) (with ϕ replaced by μ) and it is usually assumed that q-measures are regular. For generality, we do not make that assumption here. Since an ordinary measure is grade-1 additive, it is determined by its values on singleton sets.

In a similar way, a q-measure is determined by its values on singleton and doubleton sets. In fact, by (4.1) we have

$$\mu(\{\omega_1,\omega_2,\omega_3\}) \\ = \mu(\{\omega_1,\omega_2\}) + \mu(\{\omega_1,\omega_3\}) + \mu(\{\omega_2,\omega_3\}) - \mu(\omega_1) - \mu(\omega_2) - \mu(\omega_3)$$

and continuing by induction we obtain

$$\mu(\{\omega_1,\ldots,\omega_m\}) = \sum_{i<j=1}^{m} \mu(\{\omega_i,\omega_j\}) - (m-2)\sum_{i=1}^{m}\mu(\omega_i) \tag{4.2}$$

There are reasons to believe that a (finite) quantum system can be described by a q-measure space $(\Omega,\mathcal{A},\mu)$ where $|\Omega|<\infty$, $\mathcal{A}=2^\Omega$ and $\mu\colon\mathcal{A}\to\mathbb{R}^+$ is a fixed q-measure that is specified by nature [1, 2, 7, 8]. A q-measure μ on $\mathcal{A}$ 1-*generates* a coevent $\phi\in\mathcal{A}^*$ if there exists a strictly positive function $f\colon\Omega\to\mathbb{R}$ such that $\mu(A)=\int_A f d\mu$ for all $A\in\mathcal{A}$. We call f a ϕ-*density* forμ. In a sense, μ is an "average" of the density f with respect to the potential reality ϕ. Put another way, μ is an "average" of the truth values of ϕ. There are some immediate questions that one might ask.

(Q1) Does every q-measure 1-generate at least one coevent?

(Q2) Is every coevent 1-generated by at least one q-measure?

(Q3) If μ 1-generates ϕ, is ϕ unique?

(Q4) If ϕ is 1-generated by μ, is μ unique?

(Q5) If f is a ϕ-density for μ, is f unique?

We shall show that the answers to (Q1), (Q2), (Q4) and (Q5) are no. We do not know the answer to (Q3) although we have a partial result.

The definition of μ 1-generating a coevent ϕ is quite simple and if ϕ is unique that's great, but we shall give examples of q-measures that do not 1-generate any coevent. One reason for this is that a function $f\colon\Omega_n\to\mathbb{R}$ gives at most n pieces of information while a q-measure is determined by its values on singleton and doubleton sets so $n(n+1)/2$ pieces of information may be needed. For this reason, we introduce a more complicated (and presumably more general) definition. A function $f\colon\Omega\times\Omega\to\mathbb{R}$ is *symmetric* if $f(\omega,\omega')=f(\omega',\omega)$ for all $\omega,\omega'\in\Omega$. Notice that a symmetric function on $\Omega_n\times\Omega_n$ has

$n(n+1)/2$ possible values. A q-measure μ on $\mathcal{A}$ 2-*generates* a coevent $\phi\in\mathcal{A}^*$ if there exists a strictly positive symmetric function $f\colon\Omega\times\Omega\to\mathbb{R}$ such that

$$\mu(A)=\int_A\left[\int_A f(\omega,\omega')d\phi(\omega)\right]d\phi(\omega')$$

for every $A\in\mathcal{A}$. We again call f a ϕ-*density* for μ. It is interesting to note that a ϕ-density f determines a symmetric matrix $[f(\omega_i,\omega_j)]$ with positive entries which reminds us of a density matrix or state but we have not found this observation useful. The 2-generation of coevents is what we referred to previously as the strong reality filter F. In Section 5 we shall present some general results involving 1 and 2-generation of coevents and in Sections 6 and 7 we discuss specific examples. We also consider questions (Q1)–(Q5) for 2-generation of coevents.

The reason we require the density f to be strictly positive is that if f is allowed to be zero, then ϕ may be generated by a q-measure in a trivial way. For example, suppose that $\phi(\omega_0)=1$ and let $f=\chi_{\{\omega_0\}}$. Then for every $A\in\mathcal{A}$ we have $\int_A fd\phi=\delta_{\omega_0}(A)$. Thus, the Dirac measure δ_{ω_0} 1-generates ϕ in a trivial manner. In this sense, δ_{ω_0} "generates" any $\phi\in\mathcal{A}^*$ satisfying $\phi(\omega_0)=1$ so ϕ is highly nonunique. However, suppose that f is strictly positive and $\delta_{\omega_0}(A)=\int_A fd\phi$ for every $A\in\mathcal{A}$. We then have

$$1=\int_{\{\omega_0\}}fd\phi=f(\omega_0)\phi(\omega_0)$$

so that $f(\omega_0)=\phi(\omega_0)=1$. Let $A=\{\omega_1,\ldots,\omega_r\}$ with $0<f(\omega_1)\le\cdots\le f(\omega_r)$. If $\omega_0\notin A$ then by (3.6) we have $f(\omega_1)\phi(A)=0$ so $\phi(A)=0$. If $\omega_0\in A$, then $\omega_0=\omega_i$ for some $i\in\{1,\ldots,r\}$. Hence, by (3.6) we have

$$\begin{aligned}1=\int_A fd\phi&=f(\omega_1)\phi(A)+[f(\omega_2)-f(\omega_1)]\,\phi\left(\{\omega_2,\ldots,\omega_r\}\right)\\&\quad+\cdots+[f(\omega_i)-f(\omega_{i-1})]\,\phi\left(\{\omega_i,\ldots,\omega_r\}\right)\end{aligned}$$

If $\phi(A)=0$, then

$$\begin{aligned}1&\le f(\omega_2)-f(\omega_1)+f(\omega_3)-f(\omega_2)+\cdots+f(\omega_0)-f(\omega_{i-1})\\&=f(\omega_0)-f(\omega_1)=1-f(\omega_1)<1\end{aligned}$$

which is a contradiction. Hence, $\phi(A)=1$. It follows that $\phi=\omega_0^*$ is the unique coevent 1-generated by δ_{ω_0}.

We now show that if ϕ is 1-generated by μ and $\phi(A) \neq 0$ whenever $\mu(A) \neq 0$, then ϕ is 2-generated by μ. Indeed, suppose that $\mu(A) = \int_A f d\phi$ for a strictly positive function $f \colon \Omega \to \mathbb{R}$. Then $g(\omega,\omega') = \frac{1}{2}\left[f(\omega) + f(\omega')\right]$ is a strictly positive symmetric function on $\Omega \times \Omega$ and we have

$$\begin{aligned}\int_A \left[\int_A g(\omega,\omega')d\phi(\omega)\right] d\phi(\omega') &= \frac{1}{2}\int_A \left[\int_A \left(f(\omega) + f(\omega')\right) d\phi(\omega)\right] d\phi(\omega')\\ &= \frac{1}{2}\int_A \left[\mu(A) + f(\omega')\phi(A)\right] d\phi(\omega')\\ &= \mu(A)\phi(A) = \mu(A)\end{aligned}$$

Hence, ϕ is 2-generated by μ. We do not know if ϕ 1-generated implies ϕ 2-generated, in general.

5 General Results

If μ is a q-measure on $\mathcal{A}$ and $\phi \in \mathcal{A}^*$ we say that ϕ is *μ-preclusive* if $\phi(A) = 0$ whenever $\mu(A) = 0$. The following result shows that generation is stronger than preclusivity.

Theorem 5.1. *If ϕ is 1-generated or 2-generated by μ, then ϕ is μ-preclusive.*

Proof. Suppose that ϕ is 1-generated by μ and $\mu(A) = \int_A f d\phi$ for all $A \in \mathcal{A}$. If $A = \{\omega_1, \ldots, \omega_r\}$ with $\mu(A) = 0$, then by (3.6) we have

$$f(\omega_1)\phi(A) + \left[f(\omega_2) - f(\omega_1)\right]\phi\left(\{\omega_2, \ldots, \omega_r\}\right) + \cdots + \left[f(\omega_r) - f(\omega_{r-1})\right]\mu(\omega_r) = 0$$

Since $f(\omega_1) > 0$ we conclude that $\phi(A) = 0$. Next suppose that ϕ is 2-generated by μ and

$$\mu(A) = \int_A \left[\int_A f(\omega,\omega')d\phi(\omega)\right] d\phi(\omega')$$

for all $A \in \mathcal{A}$. Assume that $\mu(A) = 0$ but $\phi(A) = 1$. Let

$$g(\omega') = \int_A f(\omega,\omega')d\phi(\omega)$$

If $m(\omega') = \min\{f(\omega,\omega')\colon \omega \in A\}$, then $m(\omega') > 0$ and by (3.6) $g(\omega') \geq m(\omega')$ for all $\omega' \in A$. Letting $m = \min\{m(\omega')\colon \omega' \in A\}$ we have that $g(\omega') \geq m > 0$ for all $\omega' \in A$. But then

$$\mu(A) = \int_A g(\omega')d\phi(\omega') > 0$$

This is a contradiction so $\phi(A) = 0$. □

The next theorem is a partial uniqueness result which shows that within the set of regular coevents 1-generated coevents are unique.

Theorem 5.2. *If $\phi, \psi \in \mathcal{A}^*$ are regular and μ 1-generates both ϕ and ψ, then $\phi = \psi$.*

Proof. We have that $\int_A f d\phi = \int_A g d\psi$ for all $A \in \mathcal{A}$ where f and g are strictly positive. If $\phi(\omega) = 0$, then

$$\int_{\{\omega\}} f d\phi = 0 = \int_{\{\omega\}} g d\psi = g(\omega)\psi(\omega)$$

so $\psi(\omega) = 0$. By symmetry $\phi(\omega) = 0$ if and only if $\psi(\omega) = 0$ so ϕ and ψ agree on all singleton sets. If $\phi(\omega) = 1$ so $\psi(\omega) = 1$, then

$$f(\omega) = \int_{\{\omega\}} f d\phi = \int_{\{\omega\}} g d\phi = g(\omega) \tag{5.1}$$

Suppose $\phi(\{\omega_1\omega_2\}) = 0$. By regularity $\phi(\omega_1) = \phi(\omega_2) = 1$ or $\phi(\omega_1) = \phi(\omega_2) = 0$. In the second case $\psi(\omega_1) = \psi(\omega_2) = 0$ so by regularity

$$\psi(\{\omega_1, \omega_2\}) = \phi(\{\omega_1, \omega_2\}) = 0$$

Suppose $\phi(\omega_1) = \phi(\omega_2) = 1$. Then $\psi(\omega_1) = \psi(\omega_2) = 1$ and by (5.1) we have $f(\omega_1) = g(\omega_1)$ and $f(\omega_2) = g(\omega_2)$. Suppose that $f(\omega_1) \leq f(\omega_2)$. Then if $\psi(\{\omega_1, \omega_2\}) = 1$ we have

$$\int_{\{\omega_1,\omega_2\}} f d\phi = f(\omega_2) - f(\omega_1) = \int_{\{\omega_1,\omega_2\}} g d\psi = g(\omega_1) + g(\omega_2) - g(\omega_1) = g(\omega_2) = f(\omega_2)$$

which implies $f(\omega_1) = 0$, a contradiction. Hence, ϕ and ψ agree on all doubleton sets. Suppose $\phi(\{\omega_1, \omega_2, \omega_3\}) = 0$ and $\psi(\{\omega_1, \omega_2, \omega_3\}) = 1$. If

$\phi(\{\omega_2,\omega_3\}) = \phi(\omega_1) = 0$, then $\psi(\{\omega_2,\omega_3\}) = \psi(\omega_1) = 1$. If $\phi(\omega_3) = 0$ we obtain by regularity that

$$0 = \phi(\{\omega_1,\omega_2,\omega_3\}) = \phi(\{\omega_2,\omega_3\}) = 1$$

which is a contradiction. Hence, $\phi(\omega_3) = 1$. Suppose that $f(\omega_1) \le f(\omega_2) \le f(\omega_3)$. Then

$$\begin{aligned} f(\omega_3) - f(\omega_1) &= f(\omega_2) - f(\omega_1) + f(\omega_3) - f(\omega_2) \\ &= \int_{\{\omega_1,\omega_2,\omega_3\}} f d\phi = \int_{\{\omega_1,\omega_2,\omega_3\}} g d\psi \\ &= g(\omega_1) + g(\omega_2) - g(\omega_1) + g(\omega_3) - g(\omega_2) \\ &= g(\omega_3) = f(\omega_3) \end{aligned}$$

Hence, $f(\omega_1) = 0$ which is a contradiction. We conclude that ϕ and ψ agree on all tripleton sets. Continue this process by induction. □

For a set function $\nu\colon \mathcal{A}_m \to \mathbb{R}^+$ we define the *expansion* $\widehat{\nu}\colon \mathcal{A}_n \to \mathbb{R}^+$, $n \ge m$, of ν by $\widehat{\nu}(A) = \nu(A \cap \Omega_m)$. As a special case, if $\phi\colon \mathcal{A}_m \to \mathbb{Z}_2$, the expansion $\widehat{\phi}\colon \mathcal{A}_n \to \mathbb{Z}_2$ of ϕ is given by $\widehat{\phi}(A) = \phi(A \cap \Omega_m)$. It is clear that if $\phi \in \mathcal{A}_m^*$, then $\widehat{\phi} \in \mathcal{A}_n^*$. Also, if ϕ has the evaluation map representation

$$\phi = \omega_1^* \oplus \cdots \oplus \omega_r^* \oplus \omega_s^*\omega_t^* \oplus \cdots \oplus \omega_u^*\omega_v^*\omega_w^* \oplus \cdots$$

on $\mathcal{A}_m$ then $\widehat{\phi}$ has the same representation on $\mathcal{A}_n$. It is also clear that $\nu = \widehat{\nu} \mid \mathcal{A}_m$ where $\widehat{\nu} \mid \mathcal{A}_m$ is the restriction of $\widehat{\nu}$ to $\mathcal{A}_m$. The proof of the next lemma is straightforward.

Lemma 5.3. (a) *μ is a q-measure on $\mathcal{A}_m$ if and only if $\widehat{\mu}$ is a q-measure on $\mathcal{A}_n$.* (b) *μ is a regular q-measure on $\mathcal{A}_m$ if and only if $\widehat{\mu}$ is a regular q-measure on $\mathcal{A}_n$.*

We call the next result the expansion theorem.

Theorem 5.4. (a) *μ 1-generates ϕ if and only if $\widehat{\mu}$ 1-generates $\widehat{\phi}$* (b) *μ 2-generates ϕ if and only if $\widehat{\mu}$ 2-generates $\widehat{\phi}$.*

Proof. (a) Suppose $\mu(A) = \int_A d\phi$ for all $A \in \mathcal{A}$, where $f\colon \Omega_m \to \mathbb{R}$ is strictly positive. For $n > m$, define $\widehat{f}\colon \Omega_n \to \mathbb{R}$ by

$$\widehat{f}(\omega_i) = \begin{cases} f(\omega_i) & \text{if } \omega_i \in \Omega_m \\ M & \text{if } \omega_i \in \Omega_n \smallsetminus \Omega_m \end{cases}$$

where $M = \max\{f(\omega_i)\colon \omega_i \in \Omega_m\}$. Then $\widehat{f}\colon \Omega_n \to \mathbb{R}$ is strictly positive. Let $A = \{\omega_1, \ldots, \omega_r, \omega_{r+1}, \ldots, \omega_s\}$ where $\omega_1, \ldots, \omega_r \in \Omega_m$ and $\omega_{r+1}, \ldots, \omega_s \in \Omega_n \smallsetminus \Omega_m$. We can assume without loss of generality that

$$\widehat{f}(\omega_1) \le \widehat{f}(\omega_2) \le \cdots \le \widehat{f}(\omega_r) \le M = \widehat{f}(\omega_{r+1}) = \cdots = \widehat{f}(\omega_s)$$

Then by (3.6) we have

$$\begin{aligned}\int_A \widehat{f} d\widehat{\phi} &= \widehat{f}(\omega_1)\widehat{\phi}(A) + \left[\widehat{f}(\omega_2) - \widehat{f}(\omega_1)\right] \widehat{\phi}\left(\{\omega_2, \ldots, \omega_s\}\right) \\ &\quad + \cdots + \left[\widehat{f}(\omega_r) - \widehat{f}(\omega_{r-1})\right] \widehat{\phi}\left(\{\omega_r, \ldots, \omega_s\}\right) \\ &= f(\omega_1)\phi(A \cap \Omega_m) + \left[f(\omega_2) - f(\omega_1)\right] \phi\left(\{\omega_2, \ldots, \omega_r\}\right) \\ &\quad + \cdots + \left[f(\omega_r) - f(\omega_{r-1})\right] \phi(\omega_r) \\ &= \int_{A \cap \Omega_m} f d\phi = \mu(A \cap \Omega_m) = \widehat{\mu}(A)\end{aligned}$$

Hence, $\widehat{\mu}$ 1-generates $\widehat{\phi}$. The converse is straightforward.
(b) Suppose

$$\mu(A) = \int_A \left[\int_A f(\omega, \omega') d\phi(\omega)\right] d\phi(\omega')$$

for all $A \in \mathcal{A}_m$ where $f\colon \Omega_m \times \Omega_m \to \mathbb{R}$ is strictly positive and symmetric. For $n > m$ define $\widehat{f}\colon \Omega_n \times \Omega_n \to \mathbb{R}$ by

$$\widehat{f}(\omega_i, \omega_j) = \begin{cases} f(\omega_i, \omega_j) & \text{if } (\omega_i, \omega_j) \in \Omega_m \times \Omega_m \\ M & \text{if } (\omega_i, \omega_j) \in \Omega_n \times \Omega_n \smallsetminus \Omega_m \times \Omega_m \end{cases}$$

where $M = \max(M_1, M_2)$ and

$$\begin{aligned} M_1 &= \max\{f(\omega_i, \omega_j)\colon (\omega_i, \omega_j) \in \Omega_m \times \Omega_m\} \\ M_2 &= \max\left\{\int_A f(\omega, \omega_i) d\phi(\omega)\colon \omega_i \in \Omega_m, A \in \mathcal{A}_m\right\}\end{aligned}$$

Then $\widehat{f}$ is strictly positive and symmetric on $\Omega_n \times \Omega_n$. Again, let $A = \{\omega_1, \ldots, \omega_r, \omega_{r+1}, \ldots, \omega_s\}$ where $\omega_1, \ldots, \omega_r \in \Omega_m$ and $\omega_{r+1}, \ldots, \omega_s \in \Omega_n \smallsetminus \Omega_m$. Assume for simplicity that

$$\widehat{f}(\omega_1, \omega_1) \le \cdots \le \widehat{f}(\omega_1, \omega_r) \le \widehat{f}(\omega_2, \omega_3) \le \cdots \le \widehat{f}(\omega_2, \omega_r) \le \cdots \le \widehat{f}(\omega_r, \omega_r)$$

and the other cases will be similar. Letting

$$g(\omega') = \int_A \widehat{f}(\omega, \omega') d\widehat{\phi}(\omega)$$

we have as before that

$$g(\omega_i) = \int_{A \cap \Omega_m} f(\omega, \omega_i) d\phi(\omega)$$

$i = 1, \ldots, r$, and

$$g(\omega_{r+1}) = \cdots = g(\omega_s) = M\phi(A \cap \Omega_m)$$

As in (a) we obtain

$$\begin{aligned}
\int_A \left[\int_A \widehat{f}(\omega, \omega') d\widehat{\phi}(\omega)\right] d\widehat{\phi}(\omega') &= \int_A g(\omega') d\widehat{\phi}(\omega') = \int_{A \cap \Omega_m} g(\omega') d\phi(\omega') \\
&= \int_{A \cap \Omega_m} \left[\int_{A \cap \Omega_m} f(\omega, \omega') d\phi(\omega)\right] d\phi(\omega') = \mu(A \cap \Omega_m) = \widehat{\mu}(A)
\end{aligned}$$

Hence, $\widehat{\mu}$ 2-generates $\widehat{\phi}$. Again, the converse is straightforward. □

Notice that in the proof of Theorem 5.4 we could replace M by any larger number and the result would be the same. This is one of many examples which show that the density function need not be unique. The expansion theorem can be quite useful. For example, in Section 7 we shall show that $\phi = \omega_1^* \oplus \omega_2^* \oplus \omega_3^* \oplus \omega_1^*\omega_2^*$ is 2-generated by a q-measure on $\mathcal{A}_3$. It follows from the expansion theorem that on $\mathcal{A}_n$ for $n > 3$, ϕ is 2-generated by $\widehat{\mu}$. Moreover, the restriction of ϕ to $\mathcal{A}_2$, namely $\omega_1^* \oplus \omega_2^* \oplus \omega_1^*\omega_2^*$ is 2-generated.

We now present a useful lemma that determines values of a density function when $\phi(\omega) \neq 0$.

Lemma 5.5. (a) *Suppose μ 1-generates ϕ with ϕ-density f. Then $\phi(\omega) = 0$ if and only if $\mu(\omega) = 0$ and $\phi(\omega) \neq 0$ if and only if $f(\omega) = \mu(\omega)$.* (b) *Suppose μ 2-generates ϕ with ϕ-density f. Then $\phi(\omega) = 0$ if and only if $\mu(\omega) = 0$ and $\phi(\omega) \neq 0$ if and only if $f(\omega, \omega) = \mu(\omega)$.*

Proof. (a) Since μ 1-generates ϕ, we have

$$\mu(\omega) = \int_{\{\omega\}} f d\phi = f(\omega)\phi(\omega)$$

Hence, $\mu(\omega) = 0$ if and only if $\phi(\omega) = 0$. Moreover, if $\phi(\omega) \neq 0$, then $f(\omega) = \mu(\omega)$. Conversely, if $f(\omega) = \mu(\omega)$, then

$$f(\omega) = \int_{\{\omega\}} f d\phi = f(\omega)\phi(\omega)$$

Since $f(\omega) \neq 0$, $\phi(\omega) = 1$.
(b) Since μ 2-generates ϕ, we have

$$\begin{aligned}\mu(\omega_0) &= \int_{\{\omega_0\}} \left[\int_{\{\omega_0\}} f(\omega,\omega')d\phi(\omega)\right] d\phi(\omega') = \int_{\{\omega_0\}} f(\omega_0,\omega')\phi(\omega_0)d\phi(\omega') \\ &= f(\omega_0,\omega_0)\phi(\omega_0)\end{aligned}$$

Hence $\mu(\omega_0) = 0$ if and only if $\phi(\omega_0) = 0$. Moreover, if $\phi(\omega_0) \neq 0$, then $f(\omega_0,\omega_0) = \mu(\omega_0)$. Conversely, if $f(\omega_0,\omega_0) = \mu(\omega_0)$ then

$$f(\omega_0,\omega_0) = \int_{\{\omega_0\}} \left[\int_{\{\omega_0\}} f(\omega_0,\omega')d\phi(\omega)\right] d\phi(\omega') = f(\omega_0,\omega_0)\phi(\omega_0)$$

Since $f(\omega_0,\omega_0) \neq 0$, $\phi(\omega_0) = 1$. □

The next two results show that 1 and 2-generation are strictly quantum phenomena except for a few simple cases.

Theorem 5.6. *If μ is a (grade-1) measure on $\mathcal{A}$ that is not a Dirac measure $c\delta_\omega$, then μ is not 1-generating.*

Proof. Since μ is not a Dirac measure, there exist $\omega_1, \omega_2 \in \Omega$ such that $\mu(\omega_1), \mu(\omega_2) > 0$. Suppose μ 1-generates $\phi \in \mathcal{A}^*$ so that $\mu(A) = \int_A f d\phi$ for all $A \in \mathcal{A}$ where $f\colon \Omega \to \mathbb{R}$ is strictly positive. It follows from Lemma 5.5 that $f(\omega_i) = \mu(\omega_i)$ and $\phi(\omega_i) = 1$, $i = 1,2$. We can assume without loss of generality that $f(\omega_1) \leq f(\omega_2)$. Since μ is a measure, we have

$$\begin{aligned}\mu(\omega_1) + \mu(\omega_2) &= \mu\left(\{\omega_1,\omega_2\}\right) \\ &= \int_{\{\omega_1,\omega_2\}} f d\phi = f(\omega_1)\phi\left(\{\omega_1,\omega_2\}\right) + \left[f(\omega_2) - f(\omega_1)\right]\phi(\omega_2) \\ &= \mu(\omega_1)\phi\left(\{\omega_1,\omega_2\}\right) + \mu(\omega_2) - \mu(\omega_1)\end{aligned}$$

Hence, $\phi\left(\{\omega_1,\omega_2\}\right) = 2$ which is a contradiction. □

In the sequel, we shall use the notation

$$g_A(\omega') = \int_A f(\omega,\omega')d\phi(\omega)$$

Of course, g_A depends on f and ϕ but these will be known by context.

Theorem 5.7. (a) *Any measure of the form $\mu = a_1\delta_{\omega_1} + a_2\delta_{\omega_2}$, $a_1, a_2 > 0$, 2-generates the coevent $\phi = \omega_1^* \oplus \omega_2^* \oplus \omega_1^*\omega_2^*$. In particular, any measure on $\mathcal{A}_2$ 2-generates $\omega^* \in \mathcal{A}_2^*$ or $1 \in \mathcal{A}_2^*$.* (b) *If μ is a measure on $\mathcal{A}$ that is not of the form $a_1\delta_{\omega_1} + a_2\delta_{\omega_2}$, then μ is not 2-generating.*

Proof. (a) Define the strictly positive, symmetric function f on $\Omega \times \Omega$ by $f(\omega_i,\omega_i) = a_i$, $i = 1, 2$,

$$f(\omega_1,\omega_2) = f(\omega_2,\omega_1) = a_1 + a_2$$

and $f(\omega_i,\omega_j) = M$ otherwise, where $M > a_1 + a_2$. For $A \in \mathcal{A}$, if $\omega_1, \omega_2 \notin A$, then $g_A(\omega') = M\phi(A) = 0$. Hence,

$$\mu'(A) = \int_A g(\omega')d\phi(\omega') = 0$$

If $\omega_1 \in A$, $\omega_2 \notin A$, then

$$g_A(\omega_1) = \int_A f(\omega,\omega_1)d\phi(\omega) = a_1$$

and for $\omega' \neq \omega_1$ we have

$$g_A(\omega') = \int_A f(\omega,\omega')d\phi(\omega) = 0$$

Hence,

$$\mu'(A) = \int_A g_A(\omega')d\phi(\omega') = a_1$$

Similarly, if $\omega_2 \in A$, $\omega_1 \notin A$, then $\mu'(A) = a_2$. If $\omega_1, \omega_2 \in A$, then

$$g_A(\omega_1) = \int_A f(\omega,\omega_1)d\phi(\omega) = a_1 + a_2$$

$$g_A(\omega_2) = \int_A f(\omega,\omega_2)d\phi(\omega) = a_2 + a_1$$

If $\omega' \neq \omega_1, \omega_2$, then

$$g_A(\omega') = \int_A f(\omega, \omega') d\phi(\omega') = M$$

Hence,

$$\mu'(A) = \int_A g_A(\omega') d\phi(\omega') = a_1 + a_2$$

We conclude that

$$\mu(A) = \mu(A') = \int_A \left[\int_A f(\omega, \omega') d\phi(\omega) \right] d\phi(\omega')$$

so μ 2-generates ϕ.

(b) Since μ is not of the form $a_1\delta_{\omega_1} + a_2\delta_{\omega_2}$, there exists $\omega_1, \omega_2, \omega_3 \in \Omega$ such that $\mu(\omega_1), \mu(\omega_2), \mu(\omega_3) > 0$. We can assume without loss of generality that $\mu(\omega_1) \leq \mu(\omega_2) \leq \mu(\omega_3)$. Suppose μ 2-generates $\phi \in \mathcal{A}^*$ with density f. It follows from Lemma 5.5 that $f(\omega_i, \omega_i) = \mu(\omega_i)$ and that $\phi(\omega_i) = 1$, $i = 1, 2, 3$. We now have three cases.

Case 1. $\mu(\omega_1) \leq \omega_2) \leq f(\omega_1, \omega_2)$. Letting $A = \{\omega_1, \omega_2\}$, $\phi(\{\omega_1, \omega_2\}) = a$ we have that

$$g_A(\omega_1) = \int_A f(\omega, \omega_1) d\phi(\omega) = \mu(\omega_1)a + f(\omega_1, \omega_2) - \mu(\omega_1)$$

$$g_A(\omega_2) = \int_A f(\omega, \omega_2) d\phi(\omega) = \mu(\omega_2)a + f(\omega_1, \omega_2) - \mu(\omega_2)$$

Since μ is a measure, we have

$$\begin{aligned}\mu(\omega_1) + \mu(\omega_2) = \mu(A) &= \int_A g_A(\omega') d\phi(\omega') \\ &= [\mu(\omega_2)(a-1) + f(\omega_1, \omega_2)]\, a + [\mu(\omega_1) - \mu(\omega_2)]\, a + \mu(\omega_2) - \mu(\omega_1)\end{aligned}$$

If $a = 0$, then

$$\mu(\omega_1) + \mu(\omega_2) = \mu(\omega_2) - \mu(\omega_1)$$

which is a contradiction. If $a = 1$, then $f(\omega_1, \omega_2) = \mu(\omega_1) + \mu(\omega_2)$.

Case 2. $f(\omega_1, \omega_2) \leq \mu(\omega_1) \leq \mu(\omega_2)$. With the same terminology as in Case 1, we have

$$g_A(\omega_1) = \int_A f(\omega, \omega_1) d\phi(\omega) = f(\omega_1, \omega_2)a + \mu(\omega_1) - f(\omega_1, \omega_2)$$

$$g_A(\omega_2) = \int_A f(\omega, \omega_2) d\phi(\omega) = f(\omega_1, \omega_2)a + \mu(\omega_2) - f(\omega_1, \omega_2)$$

Since μ is a measure, we have

$$\begin{aligned}\mu(\omega_1) + \mu(\omega_2) = \mu(A) &= \int_A g_A(\omega')d\phi(\omega') \\ &= [f(\omega_1, \omega_2)(a-1) + \mu(\omega_1)]\, a + \mu(\omega_2) - \mu(\omega_1) \\ &= \mu(\omega_2) + (a-1)\mu(\omega_1)\end{aligned}$$

If $a = 0$, then $\mu(\omega_1) + \mu(\omega_2) = \mu(\omega_2) - \mu(\omega_1)$ which is a contradiction. If $a = 1$, then $\mu(\omega_1) + \mu(\omega_2) = \mu(\omega_2)$ which is a contradiction.

Case 3. $\mu(\omega_1) \leq f(\omega_1, \omega_2) \leq \mu(\omega_2)$. Again, with the same notation as before, we have

$$\begin{aligned}g_A(\omega_1) &= \mu(\omega_1)a + f(\omega_1, \omega_2) - \mu(\omega_1) \\ g_A(\omega_2) &= f(\omega_1, \omega_2)a + \mu(\omega_2) - f(\omega_1, \omega_2)\end{aligned}$$

If $a = 1$, since μ is a measure we have

$$\begin{aligned}\mu(\omega_1) + \mu(\omega_2) = \mu(A) &= \int_A g_A(\omega')d\phi(\omega') \\ &= f(\omega_1, \omega_2) + \mu(\omega_2) - f(\omega_1, \omega_2) = \mu(\omega_2)\end{aligned}$$

which is a contradiction. If $a = 0$, we have two subcases. If

$$f(\omega_1, \omega_2) - \mu(\omega_1) \leq \mu(\omega_2) - f(\omega_1, \omega_2)$$

then

$$\mu(\omega_1) + \mu(\omega_2) = \mu(A) = \int_A g_A(\omega')d\phi(\omega') = \mu(\omega_2) - f(\omega_1, \omega_2)$$

which is a contradiction. If

$$\mu(\omega_2) - f(\omega_1, \omega_2) \leq f(\omega_1, \omega_2) - \mu(\omega_1)$$

then

$$\begin{aligned}\mu(\omega_1) + \mu(\omega_2) = \mu(A) &= \int_A g_A(\omega')d\phi(\omega') = f(\omega_1, \omega_2) - \mu(\omega_1) \\ &\leq \mu(\omega_2) - \mu(\omega_1)\end{aligned}$$

which is a contradiction.

Since Case 1 with $a = 1$ is the only noncontradiction, we conclude that $\phi(\omega_i) = 1$, $f(\omega_i, \omega_i) = \mu(\omega_i)$, $i = 1, 2, 3$, $\phi(\{\omega_i, \omega_j\}) = 1$, $f(\omega_i, \omega_j) = \mu(\omega_i) + \mu(\omega_j)$, $i < j = 1, 2, 3$. Letting $A = \{\omega_1, \omega_2, \omega_3\}$ and $a = \phi(\{\omega_1, \omega_2, \omega_3\})$ we obtain

$$\begin{aligned} g_A(\omega_1) &= \int_A f(\omega, \omega_1) d\phi(\omega) = \mu(\omega_1)a + \mu(\omega_2) \\ g_A(\omega_2) &= \int_A f(\omega, \omega_2) d\phi(\omega) = \mu(\omega_2)a + \mu(\omega_3) \\ g_A(\omega_3) &= \int_A f(\omega, \omega_3) d\phi(\omega) = \mu(\omega_3)a + \mu(\omega_2) \end{aligned}$$

If $a = 0$, since μ is a measure we have

$$\mu(\omega_1) + \mu(\omega_2) + \mu(\omega_3) = \mu(A) = \int_A g_A(\omega') d\phi(\omega') = \mu(\omega_3) - \mu(\omega_2)$$

which is a contradiction. If $a = 1$, we obtain

$$\mu(\omega_1) + \mu(\omega_2) + \mu(\omega_3) = \int_A g_A(\omega') d\phi(\omega') = \mu(\omega_3) + \mu(\omega_2)$$

which is again a contradiction. Since every case leads to a contradiction, μ is not 2-generating. □

6 1-Generation

This section mainly considers q-measures and their 1-generated coevents in $\mathcal{A}_2$ and $\mathcal{A}_3$. We begin by showing that only a very restricted set of q-measures 1-generate coevents.

Lemma 6.1. *Let μ be a q-measure on $\mathcal{A}$ that 1-generates a coevent $\phi \in \mathcal{A}^*$. If $\omega_1, \omega_2 \in \Omega$ with $0 < \mu(\omega_1) \leq \mu(\omega_2)$, then $\mu(\{\omega_1, \omega_2\}) = \mu(\omega_2) - \mu(\omega_1)$ or $\mu(\{\omega_1, \omega_2\}) = \mu(\omega_2)$. Moreover, $\mu(\{\omega_1, \omega_2\}) = \mu(\omega_2) - \mu(\omega_1)$ if and only if $\phi(\{\omega_1, \omega_2\}) = 0$ and $\mu(\{\omega_1, \omega_2\}) = \mu(\omega_2)$ if and only if $\phi(\{\omega_1, \omega_2\}) = 1$.*

Proof. Let f be a ϕ-density for μ. Since $0 < \mu(\omega_1) \leq \mu(\omega_2)$, by Lemma 5.5 we have that

$$f(\omega_1) = \mu(\omega_1) \leq \mu(\omega_2) = f(\omega_2)$$

and $\phi(\omega_2) = 1$. Hence,

$$\mu(\{\omega_1, \omega_2\}) = \int_{\{\omega_1,\omega_2\}} f d\phi = f(\omega_1)\phi(\{\omega_1, \omega_2\}) + [f(\omega_2) - f(\omega_1)]\, \phi(\{\omega_2\})$$
$$= \mu(\omega_1)\phi(\{\omega_1, \omega_2\}) + \mu(\omega_2) - \mu(\omega_1)$$

It follows that $\mu(\{\omega_1, \omega_2\}) = \mu(\omega_2) - \mu(\omega_1)$ if and only if $\phi(\{\omega_1\omega_2\}) = 0$ and $\mu(\{\omega_1, \omega_2\}) = \mu(\omega_2)$ if and only if $\phi(\{\omega_1, \omega_2\}) = 1$. □

Example 7. This example considers q-measures and 1-generated coevents on $\Omega_2 = \{\omega_1, \omega_2\}$. The zero measure 1-generates $0 \in \mathcal{A}_2^*$ and Dirac measures $c\delta_{\omega_i}$ 1-generate ω_i^*, $i = 1, 2$. For the type $(1, 2)$ coevent $\phi = \omega_1^* \oplus \omega_2^*$, let $f\colon \Omega_2 \to \mathbb{R}$ be strictly positive and define $\mu(A) = \int_A f d\phi$, $A \in \mathcal{A}_2$. Assuming that $f(\omega_1) \leq f(\omega_2)$ we have that $\mu(\omega_1) = f(\omega_1)$, $\mu(\omega_2) = f(\omega_2)$ and $\mu(\Omega_2) = f(\omega_2) - f(\omega_1)$. Hence, any q-measure on $\mathcal{A}_2$ that satisfies $0 < \mu(\omega_1) \leq \mu(\omega_2)$, $\mu(\Omega_2) = \mu(\omega_2) - \mu(\omega_1)$ 1-generates the coevent $\omega_1^* \oplus \omega_2^*$. The density is given by $f(\omega_1) = \mu(\omega_1)$, $f(\omega_2) = \mu(\omega_2)$ and is unique.

For the type (12) coevent $\phi = \omega_1^*\omega_2^*$, let $f\colon \Omega_2 \to \mathbb{R}$, $\mu(A) = \int_A f d\phi$ as before. Assuming $f(\omega_1) \leq f(\omega_2)$, we have that $\mu(\omega_1) = \mu(\omega_2) = 0$ and $\mu(\Omega_2) = f(\omega_1)$. Notice that μ is not regular. Hence, any q-measure μ on $\mathcal{A}_2$ that satisfies $\mu(\omega_1) = \mu(\omega_2) = 0$ 1-generates the coevent $\omega_1^*\omega_2^*$. The density is given by $f(\omega_2) \geq f(\omega_1) = \mu(\Omega_2)$ but otherwise is arbitrary. In this case, the density is not unique.

For the type $(1, 12)$ coevent $\phi = \omega_1^* \oplus \omega_1^*\omega_2^*$, let $f\colon \Omega_2 \to \mathbb{R}$, $\mu(A) = \int_A d\phi$ as before. Assuming $f(\omega_2) \leq f(\omega_1)$ we have that $\mu(\omega_1) = f(\omega_1)$, $\mu(\omega_2) = 0$ and $\mu(\Omega_2) = \mu(\omega_1) - f(\omega_2)$. Notice that μ is not regular. Hence, any q-measure on $\mathcal{A}_2$ that satisfies $\mu(\omega_1) > 0$, $\mu(\omega_2) = 0$, $\mu(\Omega_2) \leq \mu(\omega_1)$ 1-generates the coevent $\omega_1^* \oplus \omega_1^*\omega_2^*$. The density is given by $f(\omega_1) = \mu(\omega_1)$, $f(\omega_2) = \mu(\omega_1) - \mu(\Omega_2)$ and is unique.

For the type $(1, 2, 12)$ coevent

$$\Omega_2^* = 1 = \omega_1^* \oplus \omega_2^* \oplus \omega_1^*\omega_2^*$$

let $f\colon \Omega_2 \to \mathbb{R}$, $\mu(A) = \int_A f d\phi$ as before. Assuming $f(\omega_1) \leq f(\omega_2)$ we have that $\mu(\omega_1) = f(\omega_1)$, $\mu(\omega_2) = f(\omega_2)$ and $\mu(\Omega_2) = f(\omega_2)$. Hence, any q-measure μ on $\mathcal{A}_2$ that satisfies

$$0 < \mu(\omega_1) \leq \mu(\omega_2) = \mu(\Omega_2)$$

1-generates the coevent 1 and the unique density is $f(\omega_1)=\mu(\omega_1)$, $f(\omega_2)=\mu(\omega_2)$. $\square$

Examining all the cases in Example 7 shows that every coevent $\phi\in\mathcal{A}_2^*$ is 1-generated and when a q-measure on $\mathcal{A}_2$ 1-generates a coevent ϕ, then ϕ is unique.

Example 8. There are too many coevents in $\mathcal{A}_3^*$ to consider them all so we give some examples in $\mathcal{A}_3^*$ that are 1-generated and some that are not 1-generated. It follows from Theorem 5.4 that ω_i^*, $i=1,2,3$, $\omega_i^*\oplus\omega_j^*$ and $\omega_i^*\oplus\omega_j^*\oplus\omega_i^*\omega_j^*$, $i<j=1,2,3$, are 1-generated. We now describe a set of 1-generated coevents in $\mathcal{A}_3^*$ that include these nine coevents. Since $\phi(A)=\int_A d\phi$, if $\phi\colon\mathcal{A}\to\{0,1\}$ happens to be q-measure, then ϕ 1-generates itself with density $f=1$. In general, ϕ may not be regular. We now describe the 34 q-measures in $\mathcal{A}_3^*$. List the nonempty subsets of Ω_3 in the order A_i, $i=1,\ldots,7$ as follows

$$\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_1,\omega_2\},\{\omega_1,\omega_3\}\{\omega_2,\omega_3\},\Omega_3$$

We can represent a coevent by $\phi_{a_1\cdots a_7}$ where $a_i\in\{0,1\}$ are not all zero and $a_i=1$ if and only if $\phi(A_i)=1$. Now $\phi_{a_1\cdots a_7}$ is a q-measure if and only if $a_7=a_4+a_5+a_6-a_1-a_2-a_3$. If we count these according to the number of ones in $\{a_1,a_2,a_3\}$ we obtain $3\cdot 4+3\cdot 6+4=34$ q-measures. Examples are

$$\omega_1^*=\phi_{1001101},\quad \omega_1^*\oplus\omega_2^*=\phi_{1100110},\quad (\omega_1^*\omega_2^*\omega_3^*)'=\phi_{1111110}$$
$$\omega_2^*\oplus\omega_1^*\omega_2^*\oplus\omega_1^*\omega_3^*=\phi_{0100111}$$

For instance, $\phi=(\omega_1^*\omega_2^*\omega_3^*)'$ is 1-generated by itself. Moreover, ϕ is 1-generated by any q-measure μ satisfying $\mu(\omega_i)>0$, $i=1,2,3$, $\mu(\{\omega_i,\omega_j\})=\max(\mu(\omega_i),\mu(\omega_j))$, $i<j=1,2,3$. The density is $f(\omega_i)=\mu(\omega_i)$.

We now give examples of coevents in $\mathcal{A}_3^*$ that are not 1-generated. First $\phi=\omega_1^*\omega_2^*\omega_3^*$ is not 1-generated. Suppose μ is a q-measure on $\mathcal{A}_3$ and $\mu(A)=\int_A fd\phi$ where $f\colon\Omega_3\to\mathbb{R}$ is strictly positive. We can assume without loss of generality that $0<f(\omega_1)\le f(\omega_2)\le f(\omega_3)$. By Lemma 5.5 we have that $\mu(\omega_i)=0$, $i=1,2,3$. Moreover, for $i<j=1,2,3$ we obtain

$$\mu(\{\omega_i,\omega_j\})=\int_{\{\omega_i,\omega_j\}}fd\phi=0$$

and

$$\mu(\Omega_3) = \int_{\Omega_3} f d\phi = f(\omega_1) > 0$$

Since μ is a q-measure we conclude that

$$\begin{aligned}\mu(\Omega_3) &= \mu\left(\{\omega_1,\omega_2\}\right) + \mu\left(\{\omega_1,\omega_3\}\right) + \mu\left(\{\omega_2,\omega_3\}\right) - \mu(\omega_1) - \mu(\omega_2) - \mu(\omega_3)\\ &= 0\end{aligned}$$

which is a contradiction.

We next show that $\phi = \omega_1^* \oplus \omega_2^* \oplus \omega_3^*$ is not 1-generated. Suppose μ is a q-measure on $\mathcal{A}_3$ and $\mu(A) = \int_A f d\phi$ where $f\colon \Omega_3 \to \mathbb{R}$ is strictly positive. We can assume that $0 < f(\omega_1) \leq f(\omega_2) \leq f(\omega_3)$. By Lemma 5.5 we have $f(\omega_i) = \mu(\omega_i)$, $i = 1,2,3$. Now

$$\mu\left(\{\omega_1,\omega_2\}\right) = \int_{\{\omega_1,\omega_2\}} f d\phi = f(\omega_2) - f(\omega_1) = \mu(\omega_2) - \mu(\omega_1)$$

and similarly, $\mu\left(\{\omega_1,\omega_3\}\right) = \mu(\omega_3) - \mu(\omega_1)$, $\mu\left(\{\omega_2,\omega_3\}\right) = \mu(\omega_3) - \mu(\omega_2)$. We also have that

$$\mu(\Omega_3) = \int_{\Omega_3} f d\phi = f(\omega_1) + f(\omega_3) - f(\omega_2) = \mu(\omega_3) - \mu(\omega_2) + \mu(\omega_1)$$

Since μ is a q-measure, we obtain

$$\begin{aligned}\mu(\omega_3) - \mu(\omega_2) + \mu(\omega_1) &= \mu(\Omega_3)\\ &= \mu\left(\{\omega_1,\omega_2\}\right) + \mu\left(\{\omega_1,\omega_3\}\right) + \mu\left(\{\omega_2,\omega_3\}\right) - \mu(\omega_1) - \mu(\omega_2) - \mu(\omega_3)\\ &= \mu(\omega_3) - \mu(\omega_2) - 3\mu(\omega_1)\end{aligned}$$

Since this gives a contradiction, ϕ is not 1-generated.

Finally, we show that $\phi = \omega_1^* \oplus \omega_2^* \oplus \omega_3^* \oplus \omega_1^*\omega_2^*$ is not 1-generated. Suppose μ is a q-measure $\mathcal{A}_3$ and $\mu(A) = \int_A f d\mu$ where $f\colon \Omega_3 \to \mathbb{R}$ is strictly positive. By Lemma 5.5, $f(\omega_i) = \mu(\omega_i)$, $i = 1,2,3$. We now have three cases.

Case 1. $0 < f(\omega_1) \leq f(\omega_2) \leq f(\omega_3)$. We obtain

$$\mu\left(\{\omega_1,\omega_2\}\right) = \int_{\{\omega_1,\omega_2\}} f d\phi = \mu(\omega_2)$$
$$\mu\left(\{\omega_1,\omega_3\}\right) = \int_{\{\omega_1,\omega_3\}} f d\phi = \mu(\omega_3) - \mu(\omega_1)$$
$$\mu\left(\{\omega_2,\omega_3\}\right) = \int_{\{\omega_2,\omega_3\}} f d\phi = \mu(\omega_3) - \mu(\omega_2)$$
$$\mu(\Omega_3) = \int_{\Omega_3} f d\phi = \mu(\omega_3) - \mu(\omega_2)$$

Since μ is a q-measure, we have

$$\mu(\omega_3) - \mu(\omega_2) = \mu(\Omega_3) = \mu(\omega_3) - \mu(\omega_2) - 2\mu(\omega_1)$$

which is a contradiction.
Case 2. $0 < f(\omega_3) \leq f(\omega_1) \leq f(\omega_2)$. We obtain

$$\mu\left(\{\omega_1,\omega_2\}\right) = \int_{\{\omega_1,\omega_2\}} f d\phi = \mu(\omega_2)$$
$$\mu\left(\{\omega_1,\omega_3\}\right) = \int_{\{\omega_2,\omega_3\}} f d\phi = \mu(\omega_1) - \mu(\omega_3)$$
$$\mu\left(\{\omega_2,\omega_3\}\right) = \int_{\{\omega_2,\omega_3\}} f d\phi = \mu(\omega_2) - \mu(\omega_3)$$
$$\mu(\Omega_3) = \int_{\Omega_3} f d\phi = \mu(\omega_2) - \mu(\omega_3)$$

Since μ is a q-measure, we have

$$\mu(\omega_2) - \mu(\omega_3) = \mu(\Omega_3) = \mu(\omega_2) - 3\mu(\omega_3)$$

which is a contradiction.
Case 3. $0 < f(\omega_1) \leq f(\omega_3) \leq f(\omega_2)$. In a similar way as before, we obtain $\mu\left(\{\omega_1,\omega_2\}\right) = \mu(\omega_2)$, $\mu\left(\{\omega_1,\omega_3\}\right) = \mu(\omega_3) - \mu(\omega_1)$, $\mu\left(\{\omega_2,\omega_3\}\right) = \mu(\omega_2) - \mu(\omega_3)$ and $\mu(\Omega_3) = \mu(\omega_2) - \mu(\omega_3)$. Since μ is a q-measure, we have

$$\mu(\omega_2) - \mu(\omega_3) = \mu(\Omega_3) = \mu(\omega_2) - 2\mu(\omega_1) - \mu(\omega_3)$$

which is a contradiction. □

7 2-Generation

This section illustrates by examples that more q-measures are 2-generating than 1-generating and more coevents are 2-generated than are 1-generated.

Example 9. We have seen in Example 7 that only q-measures on $\mathcal{A}_2$ that satisfy $\mu(\omega_1), \mu(\omega_2) > 0$ and

$$\mu(\Omega_2) = \max\left(\mu(\omega_1), \mu(\omega_2)\right) - \min\left(\mu(\omega_1), \mu(\omega_2)\right)$$

1-generate $\phi = \omega_1^* \oplus \omega_2^*$. We now show that if a q-measure μ on $\mathcal{A}_2$ satisfies $\mu(\omega_1), \mu(\omega_2) > 0$ and

$$\mu(\Omega_2) \leq \max\left(\mu(\omega_1), \mu(\omega_2)\right) - \min\left(\mu(\omega_1), \mu(\omega_2)\right)$$

then μ 2-generates $\phi = \omega_1^* \oplus \omega_2^*$. First assume without loss of generality that $0 < \mu(\omega_1) \leq \mu(\omega_2)$ and $\mu(\Omega_2) \leq \mu(\omega_2) - \mu(\omega_1)$. Let $f\colon \Omega_2 \times \Omega_2 \to \mathbb{R}$ be the strictly positive, symmetric function defined by $f(\omega_i, \omega_i) = \mu(\omega_i)$, $i = 1, 2$ and

$$f(\omega_1, \omega_2) = f(\omega_2, \omega_1) = \frac{\mu(\Omega_2) + \mu(\omega_1) + \mu(\omega_2)}{2}$$

We then have

$$\mu(\omega_1) \leq \frac{\mu(\omega_1) + \mu(\omega_2)}{2} \leq \frac{\mu(\Omega_2) + \mu(\omega_1) + \mu(\omega_2)}{2} \leq \mu(\omega_2)$$

Hence, $f(\omega_1, \omega_1) \leq f(\omega_1, \omega_2) \leq f(\omega_2, \omega_2)$. Define the set function $\mu'\colon \mathcal{A}_2 \to \mathbb{R}^+$ by

$$\mu'(A) = \int_A \left[\int_A f(\omega, \omega') d\phi(\omega)\right] d\phi(\omega')$$

Then

$$\mu'(\omega_1) = \int_{\{\omega_1\}} f(\omega_1, \omega') d\phi(\omega') = f(\omega_1, \omega_1) = \mu(\omega_1)$$

and similarly, $\mu'(\omega_2) = \mu(\omega_2)$. Defining $g(\omega') = \int f(\omega, \omega') d\phi(\omega)$ we have

$$g(\omega_1) = \int f(\omega, \omega_1) d\phi(\omega) = f(\omega_1, \omega_2) - f(\omega_1, \omega_1) = f(\omega_1, \omega_2) - \mu(\omega_1)$$

$$g(\omega_2) = \int f(\omega, \omega_2) d\phi(\omega) = f(\omega_2, \omega_2) - f(\omega_1, \omega_2) = \mu(\omega_2) - f(\omega_1, \omega_2)$$

Since

$$f(\omega_1,\omega_2)-\mu(\omega_1)-\mu(\omega_2)+f(\omega_1,\omega_2) = 2f(\omega_1,\omega_2)-\mu(\omega_1)-\mu(\omega_2) = \mu(\Omega_2) \geq 0$$

we have that

$$\mu'(\Omega_2) = \int g(\omega')d\phi(\omega') = \mu(\Omega_2)$$

Hence, $\mu(A) = \mu'(A)$ for all $A \in \mathcal{A}_2$ so μ 2-generates ϕ. □

Just as every coevent in $\mathcal{A}_2^*$ is 1-generated, it is not hard to show that every coevent in $\mathcal{A}_2^*$ is 2-generated. For the same reason that $\omega_1^*\omega_2^*\omega_3^*$ is not 1-generated in $\mathcal{A}_3$ we have that $\omega_1^*\omega_2^*\omega_3^*$ is not 2-generated in $\mathcal{A}_3$. In fact, for $m \geq 3$, $\phi = \omega_1^* \cdots \omega_m^*$ is not 1 or 2-generated in $\mathcal{A}_n$, $n \geq m$. This is because $\phi(A) = 0$ for all $A \in \mathcal{A}_n$ such that $\{\omega_1,\ldots,\omega_m\} \not\subseteq A$. Hence, if a q-measure μ 1 or 2-generates ϕ, then

$$\mu(\omega_i) = \mu\left(\{\omega_i,\omega_j\}\right) = 0$$

for all $i,j \leq n$. However, $\mu\left(\{\omega_1,\ldots,\omega_m\}\right) > 0$ and this contradicts (4.2). The same reasoning shows that any coevent whose evaluation map representation has all terms of degree larger than 2 is not 1 or 2-generated.

Example 10. We have seen in Example 8, that

$$\phi = \omega_1^* \oplus \omega_2^* \oplus \omega_3^* \oplus \omega_1^*\omega_2^*$$

is not 1-generated in $\mathcal{A}_3$. We now show that ϕ is 2-generated in $\mathcal{A}_3$. Place a q-measure μ on $\mathcal{A}_3$ satisfying $0 < \mu(\omega_1) \leq \mu(\omega_2)$, $\mu(\omega_3) = \mu(\omega_1) + \mu(\omega_2)$, $\mu\left(\{\omega_1,\omega_2\}\right) \geq \mu(\omega_3)$, $\mu\left(\{\omega_1,\omega_3\}\right) = \mu(\omega_2)$, $\mu\left(\{\omega_2,\omega_3\}\right) = \mu(\omega_1)$ and $\mu(\Omega_3) = \mu\left(\{\omega_1,\omega_2\}\right) - \mu(\omega_3)$. To show that μ is indeed a q-measure we have

$$\sum_{i<j=1}^{3} \mu\left(\{\omega_i,\omega_j\}\right) - \sum_{i=1}^{3} \mu(\omega_i) = \mu\left(\{\omega_1,\omega_2\}\right) - \mu(\omega_3) = \mu(\Omega_3)$$

Let $f\colon \Omega_3 \times \Omega_3 \to \mathbb{R}$ be the strictly positive, symmetric function satisfying $f(\omega_i,\omega_i) = \mu(\omega_i)$, $i = 1,2,3$ and

$$f(\omega_1,\omega_2) = f(\omega_1,\omega_3) = f(\omega_2,\omega_3) = \mu\left(\{\omega_1,\omega_2\}\right)$$

Letting $\mu'(A) = \int_A \left[\int_A f(\omega,\omega')d\phi(\omega)\right] d\phi(\omega')$ for all $A \in \mathcal{A}_3$ we have $\mu'(\omega_i) = \mu(\omega_i)$, $i = 1,2,3$. Moreover,

$$g_{\{\omega_1,\omega_2\}}(\omega_1) = \int_{\{\omega_1,\omega_2\}} f(\omega,\omega_1)d\phi(\omega) = f(\omega_1,\omega_2) = \mu\left(\{\omega_1,\omega_2\}\right)$$

Similarly, $g_{\{\omega_1,\omega_2\}}(\omega_2) = \mu(\{\omega_1,\omega_2\})$ and for the other doubleton sets we have

$$\begin{aligned}
g_{\{\omega_1,\omega_3\}}(\omega_1) &= \mu(\{\omega_1,\omega_2\}) - \mu(\omega_1) \\
g_{\{\omega_1,\omega_3\}}(\omega_3) &= \mu(\{\omega_1,\omega_2\}) - \mu(\omega_3) \\
g_{\{\omega_2,\omega_3\}}(\omega_2) &= \mu(\{\omega_1,\omega_2\}) - \mu(\omega_2) \\
g_{\{\omega_2,\omega_3\}}(\omega_3) &= \mu(\{\omega_1,\omega_2\}) - \mu(\omega_3)
\end{aligned}$$

Hence,

$$\mu'(\{\omega_1,\omega_2\}) = \int_{\{\omega_1,\omega_2\}} g_{\{\omega_1,\omega_2\}}(\omega')d\phi(\omega') = \mu(\{\omega_1,\omega_2\})$$

and similarly, $\mu'(\{\omega_1,\omega_3\}) = \mu(\{\omega_1,\omega_3\})$, $\mu'(\{\omega_2,\omega_3\}) = \mu(\{\omega_2,\omega_3\})$. Finally,

$$g_\Omega(\omega_1) = \int f(\omega,\omega_1)d\phi(\omega) = f(\omega_1,\omega_3) - f(\omega_1,\omega_2) = 0$$

and similarly, $g_\Omega(\omega_2) = 0$, $g_\Omega(\omega_3) = \mu(\Omega_3)$. We conclude that

$$\mu'(\Omega_3) = \int g_\Omega(\omega')d\phi(\omega') = \mu(\Omega_3)$$

Hence, $\mu(A) = \mu'(A)$ for all $A \in \mathcal{A}_3$ so μ 2-generates ϕ. □

We do not know whether $\omega_1^* \oplus \omega_2^* \oplus \omega_3^*$ is 2-generated in $\mathcal{A}_3$.

References

[1] Y. Ghazi-Tabatabai, Quantum measure theory: a new interpretation, arXiv: quant-ph (0906.0294), 2009.

[2] S. Gudder, Finite quantum measure spaces, *Amer. Math. Monthly* (to appear).

[3] S. Gudder, Quantum measure theory, *Math. Slovaca* (to appear).

[4] S. Gudder, Quantum measure and integration theory, arXiv: quant-ph (0909.2203), 2009 and *J. Math. Phys.* (to appear).

[5] S. Gudder, An anhomomorphic logic for quantum mechanics, arXiv: quant-ph (0910.3253), 2009.

[6] R. Salgado, Some identities for the quantum measure and its generalizations, *Mod. Phys. Letts. A* **17** (2002), 711–728.

[7] R. Sorkin, Quantum mechanics as quantum measure theory, *Mod. Phys. Letts. A* **9** (1994), 3119–3127.

[8] R. Sorkin, Quantum mechanics without the wave function, *J. Phys. A* **40** (2007), 3207–3231.

[9] R. Sorkin, An exercise in "anhomomorphic logic," *J. Phys.* (to appear).

[10] S. Surya and P. Wallden, Quantum covers in quantum measure theory, arXiv, quant-ph (0809.1951), 2008.