Finite retracts of Priestley spaces and sectional coproductivity

Richard N. Ball, Aleš Pultr, and Jiří Sichler

Abstract. Let Y and P be posets, let P be finite and connected, and let $f : Y \to P$ be a surjective monotone map. The map f can be naturally extended to a Priestley surjection $\widehat{f} : \widehat{Y} \to P$ which can turn out to be a retraction even if f is not. We characterize those maps f whose Priestley extensions $\widehat{f}$ are retractions.

We then use this characterization to contrast, yet again, the behavior of cyclic and acyclic posets P insofar as their appearances in Priestley spaces are concerned. We say that P is *sectionally coproductive* if the Priestley surjection $f : \coprod X_i \to P$, induced by Priestley surjections $f_i : X_i \to P$, is a retraction only when at least one of f_i's is a retraction. We then prove that P is sectionally coproductive exactly when it is acyclic.

1. Introduction

This paper deals with two topics concerning Priestley spaces. (A Priestley space is a compact partially ordered topological spaces having a specific separation property which we recall in 2.2 below.) The first topic is finite retracts, and the second is another aspect of the contrasting behaviour of cyclic and acyclic connected finite posets; we call such posets *configurations*.

A continuous function $f : X \to Y$ onto a finite discrete space Y is trivially a retraction. If X is completely regular, the Čech-Stone extension $\beta f : \beta X \to Y$ of f is surjective, and hence a retraction, exactly when f is surjective. The situation is different for (not necessarily compact) partially ordered topological spaces with the Priestley separation property. In this case, a continuous monotone map $f : X \to P$ onto a configuration P need not have a monotone section g, that is, a monotone map g satisfying $fg = 1_P$. Furthermore, if we consider the natural Priestley extension $\widehat{f} : \widehat{X} \to P$ given in 2.5 below, it is by no means clear whether the fact that $\widehat{f}$ is a retraction implies that f is a

---

2000 *Mathematics Subject Classification*: Primary 06D55, 06A11, 54F05; Secondary 06D20, 03C05.

Key words and phrases: distributive 01-lattice, Priestley duality, sums (coproducts) of Priestley spaces, ultraproduct.

The first author would like to express his thanks for support from project LN 1M0545ITI of the Ministry of Education of the Czech Republic. The second author would like to express his thanks for support from projects 1M0545ITI and MSM 0021620838 of the Ministry of Education of the Czech Republic, from the NSERC of Canada and from a PROF grant from the University of Denver. The third author would like to express his thanks for support from the NSERC of Canada and from project MSM 0021620838 of the Ministry of Education of the Czech Republic.

retraction. It does not, and an analysis of the situation constitutes the first part of this paper.

Acyclic and cyclic configurations exhibit markedly different behaviors insofar as their appearances in Priestley coproducts are concerned; see [1, 2, 3, 4]. While an acyclic configuration P can occur in a coproduct $\coprod X_i$ only when it has a copy in some summand X_i, this is not true when P is cyclic.

Applying the result of the first part, we prove a similar result about monotone sections. Specifically, if $f : \coprod X_i \to P$ is a retraction onto an acyclic P then at least one of the associated maps $f_i : X_i \to P$ is a retraction. On the other hand, for any cyclic configuration P we construct Priestley surjections $f_i : X_i \to P$, none of which is a retraction, such that their induced coproduct map $f : \coprod X_i \to P$ is a retraction.

It should be noted that, unlike other results separating the presence of cyclic and acyclic configurations in coproducts of Priestley spaces, here we do not use Łoś's Theorem on first order properties [9]. Both implications are based on concrete constructions using only the axiom of choice.

While the interested reader will be acquainted with the basic facts about Priestley duality ([11, 12, 6]), we explicitly recall those we need. For partial order, one can consult [6] if necessary, and the category theory facts can be found in [10], for instance.

2. Preliminaries

2.1. In a partially ordered set (poset) $(X\ \leq)$ we will write, for a subset $M \subseteq X$,

$$\downarrow M = \{x \in X \mid \exists m \in M,\ x \leq m\}, \quad \uparrow M = \{x \in X \mid \exists m \in M,\ x \geq m\}.$$

We abbreviate $\downarrow\{x\}$ to $\downarrow x$ and similarly $\uparrow\{x\}$ to $\uparrow x$. The subsets U with $\uparrow U = U$ ($\downarrow U = U$) are referred to as *up-sets* (*down-sets*).

We write

$$x \prec y$$

to denote a *cover pair*, that is, a pair $x < y$ such that $x \leq z \leq y$ only when $x = z$ or $z = y$. Further,

$$x \succ\!\!\prec y$$

indicates that either $x \prec y$ or $y \prec x$. The resulting (symmetric) graph $(X, \succ\!\!\prec)$ is called the Hasse graph of $(X, \leq)$. This terminology will be used typically for finite X.

A *configuration* P is a finite poset such that its Hasse graph is connected. A configuration is *acyclic*, or a *tree*, if $(P, \succ\!\!\prec)$ is a tree; otherwise we speak of a *cyclic configuration.*.

2.2. A *Priestley space* is a compact partially ordered topological space $(X, \leq, \tau)$ such that for any $x \not\leq y$ in X there is a clopen up-set U such that $x \in U$ and $y \notin U$. The category of Priestley spaces and monotone continuous maps (the *Priestley maps*) will be denoted by

$$\mathbf{PSp}.$$

2.3. Denote by

$$\mathbf{DLat}$$

the category of bounded distributive lattices. Recall the famous *Priestley duality* ([11, 12]) constituted by the contravariant functors

$$\mathcal{U} : \mathbf{PSp} \to \mathbf{DLat}, \quad \mathcal{P} : \mathbf{DLat} \to \mathbf{PSp},$$

defined by

$$\mathcal{U}(X) = (\{U \mid U \subseteq X \text{ clopen up-set } \}, \subseteq), \quad \mathcal{U}(f)(U) = f^{-1}[U],$$
$$\mathcal{P}(L) = (\{x \mid x \text{ prime filter in } L\}, \subseteq, \tau), \quad \mathcal{P}(h)(x) = h^{-1}[x],$$

where τ is the topology of 2^L, with the natural equivalences

$$\rho_X = (x \mapsto \{U \mid x \in U\}) : X \to \mathcal{P}\mathcal{U}(X),$$
$$\lambda_L = (a \mapsto \{x \mid a \in x\}) : L \to \mathcal{U}\mathcal{P}(L).$$

2.3.1. *Notation.* We will sometimes represent a Priestley space by the isomorphic $\mathcal{P}\mathcal{U}(X)$. We will use, instead of $\rho_X(x)$, a simpler symbol

$$\mathring{x}$$

for the principal prime filter $\{U \subseteq X \mid x \in U\}$.

2.4. We will also work with the category

$$\mathbf{potop}$$

of partially ordered topological spaces and monotone continuous maps, and consider the contravariant functors

$$\mathcal{U}^\circ : \mathbf{potop} \to \mathbf{DLat}, \quad \mathcal{P}^\circ : \mathbf{DLat} \to \mathbf{potop}$$

defined by the same formulas as the $\mathcal{U}$ and $\mathcal{P}$ above. They are adjoint on the right, with the adjunction units

$$\rho^\circ_X = (x \mapsto \{U \mid x \in U\}) : X \to \mathcal{P}^\circ\mathcal{U}^\circ(X),$$
$$\lambda^\circ_L = \lambda_L : L \to \mathcal{U}^\circ\mathcal{P}^\circ(L).$$

Note that we have here again a natural equivalence $\lambda^\circ : \mathrm{Id} \cong \mathcal{U}^\circ\mathcal{P}^\circ$, while ρ° is a more general natural transformation.

2.4.1. We also have a contravariant adjunction

$$\mathfrak{U} : \mathbf{poset} \to \mathbf{DLat}, \quad \mathsf{Pf} : \mathbf{DLat} \to \mathbf{poset},$$

with

$$\mathfrak{U}(X) = \{U \mid U \subseteq X \text{ an up-set}\} \quad \text{and}$$
$$\mathsf{Pf}(L) = (\{x \mid x \text{ a prime filter in } L\}, \subseteq)$$

(with no topology). Here, of course, neither of the units is a natural equivalence.

2.5. Denote by $\mathcal{J} : \mathbf{PSp} \to \mathbf{potop}$ the embedding functor. We obviously have

$$\mathcal{J}\mathcal{P} = \mathcal{P}^\circ \quad \text{and} \quad \mathcal{U}^\circ\mathcal{J} = \mathcal{U}.$$

Thus, $\rho^\circ : \mathrm{Id} \to \mathcal{J}(\mathcal{P}\mathcal{U}^\circ)$ and the natural equivalence $(\mathcal{P}\mathcal{U}^\circ)\mathcal{J} = \mathcal{P}\mathcal{U} \cong \mathrm{Id}$ constitute an adjunction (of covariant functors) with $\mathcal{J}$ on the right and $\mathcal{P}\mathcal{U}^\circ$ on the left, and we see that **PSp** is a reflective subcategory of **potop**, with the reflection

$$\rho^\circ_X : X \to \widehat{X} = \mathcal{P}\mathcal{U}^\circ(X).$$

The notation

$$\widehat{X}$$

will be retained, and we will refer to $\widehat{X}$ as the *Priestley reflection* of X. Furthermore, for any map $f : X \to Y$ to a Priestley space Y we will write

$$\widehat{f} : \widehat{X} \to Y$$

for the Priestley map defined by $\widehat{f}\rho^\circ = f$.

2.5.1. For plain posets X we will speak of the $\mathcal{P}\mathfrak{U}(X)$ as of the *Priestley modification*. In the sequel we will often work with discrete ordered topological spaces X. Then, of course, X is only a poset, and hence its Priestley reflection and modification coincide.

2.6. By the standard properties of adjoints we have colimits in **PSp** (resp. **potop**, resp. **poset**) sent to limits in **DLat**. In particular, the order independent disjoint union

$$\bigcup_{i \in I} X_i$$

in **potop** resp. **poset** is sent to $\prod \mathcal{U}^\circ(X_i)$ resp. $\prod \mathfrak{U}(X_i)$.

Thus we can represent coproducts $\coprod X_i$ in **PSp** (the X_i assumed disjoint) by first forming the order independent union $X = \bigcup X_i$ to obtain

$$\coprod X_i = \mathcal{P}\mathcal{U}^\circ(X) = \widehat{X}.$$

That is, we have the coproduct represented as the set of prime filters on the set of (clopen, but we will typically have the summands discrete) up-sets in $\bigcup X_i$, endowed with the inclusion order and a suitable topology.

2.7. Priestley duals of finite discrete spaces are both Heyting and co-Heyting algebras and therefore so are their products. Consequently, arbitrary coproducts of systems of finite Priestley spaces are duals of double Heyting algebras, and hence (an old result, likely folklore, see also [5]) have the property that

for each open U in X, both $\uparrow U$ and $\downarrow U$ are open.

Since $\uparrow U$ and $\downarrow U$ of a closed set U are always closed, we have, then, that *for each clopen U in X, both $\uparrow U$ and $\downarrow U$ are clopen.* This will be used in Section 4.

3. Small sets and finite retracts

3.1. In this section $f : Y \to P$ will be a surjective monotone map. The posets Y and P will be later interpreted as discrete partially ordered topological spaces. We will assume that

- P is finite, and hence Priestley when interpreted as a space. In the application in the next section it will be assumed tthat it is connected, that is, a configuration, and that
- for each $p \in P$, $f^{-1}[p]$ is trivially ordered.

Remark 3.1.1. Following 2.5 we will write

$$\widehat{Y} = \mathcal{P}(B) \quad \text{where} \quad B = \mathfrak{U}(Y)$$

and

$$\widehat{f} = \mathcal{P}\mathfrak{U}(f) : \widehat{Y} \to \mathcal{P}\mathfrak{U}(P) \cong P,$$

Recall the notation $\mathring{p}$ from 2.3.1.

Lemma 3.1.2. $\widehat{f}(y) = \mathring{p}$ *iff* $\bigcup_{q \not\leq p} f^{-1}[\uparrow q] \notin y$ and $y \ni f^{-1}[\uparrow p]$.

Proof. We have $\widehat{f} = \mathcal{P}\mathfrak{U}(f)$; that is, $\widehat{f}(y) = h^{-1}[y]$ where h is defined by $h(U) = f^{-1}[U]$. Thus,

$$\begin{aligned} \widehat{f}(y) = \{U \mid p \in U\} \quad &\text{iff} \quad (\forall U \in \mathfrak{U}(P),\ p \in U \text{ iff } U \in h^{-1}[y]) \\ &\text{iff} \quad (\forall U \in \mathfrak{U}(P),\ p \in U \text{ iff } f^{-1}[U] \in y). \end{aligned}$$

Since the up-sets in P are finite unions of the $\uparrow q$'s, this amounts to

$$\ldots \quad \text{iff} \quad (\forall q \in P,\ p \in \uparrow q \text{ iff } f^{-1}[\uparrow q] \in y).$$

That is,

- if $q \leq p$ then $f^{-1}[\uparrow q] \in y$, which is equivalent to saying that $f^{-1}[\uparrow p] \in y$, $f^{-1}[\uparrow p]$ being the smallest such, and
- if $q \not\leq p$ then $f^{-1}[\uparrow q] \notin y$; since y is a prime filter this can be expressed by $\bigcup_{q \not\leq p} f^{-1}[\uparrow q] \notin y$.

∎

3.2. We will write

$$Y(p) = f^{-1}[p].$$

Recall from 3.1 that $Y(p)$ is ordered trivially.

For a subset $R \subseteq Y$ write, more generally,

$$R(p) = R \cap Y(p).$$

A subset $R \subseteq Y$ is *small at a point* $p \in P$ if

$$Y(p) \cap \bigcap_{q \le p} \uparrow R(q) = \emptyset.$$

It is *small* if it is small at some point.

3.3. Further notation. Set

$$X(p) = \beta(Y(p)),$$

that is, the set of all ultrafilters on $Y(p)$. Note that since the order in $Y(p)$ is trivial, $\mathfrak{U}(Y(p))$ is the set of *all* subsets of $Y(p)$ and hence $\mathcal{P}\mathfrak{U}(Y(p))$ is the set of all ultrafilters. Thus,

$$X(p) = \mathcal{P}\mathfrak{U}(Y(p)) = \widehat{Y(p)}.$$

Consider the product in **PSp**

$$X = \prod_{p \in P} X(p).$$

Write $A(p) = \mathfrak{U}(Y(p))$ $(= \{a \mid a \subseteq Y(p)\})$. The basic sets of $X(p)$ are the clopen sets $\{x \in X(p) \mid a \in x\}$, and hence those of X are the clopen sets

$$[a] = \{x \in X \mid \forall p \in P,\ x(p) \ni a(p)\}, \quad a \in \prod_p A(p).$$

We will be particularly interested in the clopen sets of the form

$$[R] = \{x \in X \mid \forall p \in P,\ x(p) \ni R(p)\}$$

for R a small subset of Y and $R(p)$ as in 2.3.

3.4. Set

$$Z = Z(f) = \{x \in X \mid \forall q \le p\ (U \in x(q) \ \Rightarrow\ (\uparrow U \cap Y(p)) \in x(p))\}.$$

Lemma 3.4.1. $Z(f) = X \setminus \bigcup \{[R] \mid R \subseteq Y \text{ small }\}$.

Proof. I. Let $x \notin Z(f)$. Then there exist $q < p$ such that for some $U \in x(q)$ we have $Y(p) \cap \uparrow U \notin x(p)$. Set

$$R = U \cup (Y(p) \setminus \uparrow U) \cup \bigcup_{r \neq p,q} Y(r).$$

Then R is small at p because

$$R(q) = U \in x(q), \quad R(p) = (Y(p) \setminus (\uparrow U \cap Y(p)) \in x(p),$$
$$\text{and } R(r) = Y(r) \in x(r) \text{ for } r \neq p, q$$

so that $x \in [R]$.

II. On the other hand, let $x \in [R]$ for an R small at p. We have

$$\bigcap_{q \le p} (Y(p) \cap \uparrow R(q)) = \emptyset$$

and hence there is some $q \le p$ for which $Y(p) \cap \uparrow R(q)$ is not in $x(p)$ while $R(p)$ is in $x(q)$. Thus, $U = R(q)$ violates the condition for Z, and $x \notin Z$. □

3.5. A P-*choice* for $f : Y \to P$ is a subset $S \subseteq Y$ such that for each $p \in P$ there is an element $s(p)$ such that

$$S \cap Y(p) = \{s(p)\}.$$

Note that $f(s(p)) = p$ but that, generally, the resulting map $s : P \to Y$ does not have to be a section of the surjection f because we have not made any assumption making s monotone.

Proposition 3.5.1. *For the map $f : Y \to P$ from 3.1, the following statements are equivalent:*

(1) $Z(f) = \emptyset$.
(2) There is a finite cover of X by a family of sets of the form $[R]$ with R small,
(3) There is a finite family $\mathcal{R}$ of small sets such that each P-choice S for f is a subset of some $R \in \mathcal{R}$.

Proof. (1)⇔(2) by 3.4.1: the set $\bigcup\{[R] \mid R \text{ small}\}$ is open because all $[R]$ are clopen and Z is its complement in a compact space. Thus Z is compact.

(3)⇒(1): Let $Z \neq \emptyset$ and choose any $x \in Z$. Let $\mathcal{R}$ be a finite family of small subsets of Y. Since $x \notin [R]$ for all small R, there exists $p(R) \in P$ such that

$$R(p(R)) \notin x(p(R)).$$

Choose elements

$$s(q) \in Y(q) \setminus \bigcup\{R(q) \mid R \in \mathcal{R},\ p(R) = q\}.$$

The displayed union, which is empty if q is not $p(R)$ for any $R \in \mathcal{R}$, is not a member of $x(p)$ in any case, hence is not $Y(p)$. Then $S = \{s(q) \mid q \in P\}$ is a P-choice, and for all $R \in \mathcal{R}$, $s(q(R)) \notin R$; hence $S \nsubseteq R$.

(2)⇒(3): Let $X = \bigcup\{[R] \mid R \in \mathcal{R}\}$ for a finite family $\mathcal{R}$ of small sets. Let $\{s(p) \mid p \in P\}$ be a P-choice. Consider a point $s = (\mathring{s}(p))_{p \in P} \in X$, and a small set $R \in \mathcal{R}$ such that $x \in [R]$. Then $\mathring{s}(p) \ni R(p)$ for every $p \in P$, and hence $s(p) \in R(p)$. Thus $S \subseteq R$. □

3.6. A morphism $f : A \to B$ in a category $\mathcal{A}$ is a *retraction* if there is a $g : B \to A$ in $\mathcal{A}$ such that $f \cdot g = \mathrm{id}_B$. Any such g is called a *section* of f.

Note that a surjection $f : A \to B$ in a concrete category always (that is, assuming the Axiom of Choice) admits an unstructured mapping σ from the carrier of B to the carrier of A such that $f(\sigma(x)) = x$. Such σ is what we call a P-choice. The question of whether f is a retraction can therefore be reformulated as the question of whether there is a P-choice that is a section.

Theorem 3.6.1. *Let $f : Y \to P$ be a monotone surjection onto a configuration P such that $Y(p) = f^{-1}[p]$ is trivially ordered for all $p \in P$. Then $\widehat{f}$ is a retraction iff for every finite family $\mathcal{R}$ of small subsets of Y there is a P-choice S contained in no $R \in \mathcal{R}$.*

Proof. Suppose that for every finite family $\mathcal{R}$ of small subsets of Y there is a P-choice S contained in no $R \in \mathcal{R}$. Then $Z(f) \neq \emptyset$ by 3.5, and we can choose a point $x \in Z(f)$. For $p \in P$ set

$$g(p) = \{U \in \mathfrak{U}(Y) \mid U \cap Y(p) \in x(p)\}.$$

First, each $g(p)$ is clearly a proper filter. It is also prime, for if $U_1 \cup U_2$ is in $g(p)$ then

$$(U_1 \cap Y(p)) \cup (U_2 \cap Y(p)) = (U_1 \cup U_2) \cap Y(p) \in x\,(p)\,,$$

hence $U_i \cap Y(p) \in x(p)$ for some i, and $U_i \in g(p)$.

Second, g is monotone. To verify this, consider $U \in g(q)$ and $q \leq p$. Since $x \in Z$ we have

$$x\,(p) \ni \uparrow(U \cap Y(q)) \cap Y(p) \subseteq \uparrow U \cap Y(p) = U \cap Y(p),$$

hence $U \cap Y(p) \in x(p)$ and $U \in g(p)$.

Finally, $\widehat{f}(g(p)) = \mathring{p}$. Using Lemma 3.1.2, we have

$$f^{-1}[\uparrow p] \cap Y(p) = Y(p) \in g(p), \text{ and, for } q \nleq p,$$
$$f^{-1}[\uparrow q] \cap Y(p) = f^{-1}[\uparrow q \cap \{p\}] = \emptyset \notin g(p).$$

Now suppose $\widehat{f}$ is a retraction with section g. By Lemma 3.1.2 that means that, for $p \in P$, $\bigcup_{q \nleq p} f^{-1}[\uparrow q] \notin g\,(p)$ and $g(p) \ni f^{-1}[\uparrow p]$. Define

$$x(p) = \{V \cap Y(p) \mid V \in g(p)\}.$$

Note that, for $U \subseteq Y(p)$,

$$U \in x\,(p) \text{ iff } \uparrow U = \{y' \in Y \mid y' \geq y\} \in g(p).$$

Clearly $\uparrow U \in g\,(p)$ implies $U \in x(p)$, since $(\uparrow U) \cap Y(p) = U$. On the other hand, consider $U \in x(p)$, say $U = V \cap Y(p)$ for $V \in g\,(p)$, and for the sake of argument suppose that $\uparrow U \notin g\,(p)$. Since

$$(\uparrow U) \cup (\uparrow(Y(p) \smallsetminus U)) = f^{-1}[\uparrow p] \in g\,(p)\,,$$

it follows that $\uparrow(Y\,(p) \smallsetminus U) \in g\,(p)$. But that leads to the contradiction

$$g\,(p) \ni V \cap (\uparrow(Y(p) \smallsetminus U)) \subseteq \bigcup_{q \nleq p} f^{-1}[\uparrow q].$$

We conclude that $\uparrow U \in g\,(p)$.) Therefore

$$x\,(p) = \{U \subseteq Y(p) :\ \uparrow U \in g\,(p)\}\,.$$

This description makes it clear that $x(p)$ is a prime filter on $Y(p)$, and that the point $x = (p \longmapsto x\,(p))$ lies in X.

We claim that x actually lies in Z. To check this claim, consider points $q \leq p$ in P and subset $U \in x(q)$. Since g is monotone, we have $\uparrow U \in g(q) \subseteq g(p)$, hence $\uparrow U \cap Y(p) \in x\,(p)$. Combined with Lemma 2.6, the claim shows that for every finite family $\mathcal{R}$ of small subsets of Y there is a P-choice S contained in no $R \in \mathcal{R}$. □

Remark 3.6.2. In 4.4 of the subsequent section we present an example of a surjection $f : Y \to P$ that is not a retraction while $\widehat{f} : \widehat{Y} \to P$ is a retraction.

4. Sectional coproductivity

4.1. In a category let

$$j_i : A_i \to A = \coprod_{i \in I} A_i, \quad i \in I,$$

be a coproduct, let $f_i : A_i \to B$ be morphisms, and let $f : A \to B$ be the resulting unique morphism satisfying $fj_i = f_i$ for all i. If some f_i is a retraction then so is f; indeed, if $g : B \to A_i$ is a section for f_i, that is, if $f_i g = \mathrm{id}$, then we have $f \cdot (j_i g) = \mathrm{id}$.

Conversely, if $f : A \to B$ is a retraction we can ask whether, under some natural conditions on B, there is an index i such that $f_i : A_i \to B$ is a retraction. In **poset** or **potop**, where the coproduct is the disjoint union, this is true for any order-connected B, in particular for any configuration. Indeed, if $g : B \to \bigcup A_i$ is a section then, because of the connectedness of $g\,(B)$, $g = j_i g'$ for some index i, and $f_i g' = fj_i g' = fg = \mathrm{id}$.

The situation in **PSp** is more complicated and will be dealt with in this section. Let us call a configuration P *sectionally coproductive* if for every coproduct $j_i : Y_i \to \widehat{Y} = \coprod_{i \in I} Y_i$ with Y_i finite (I arbitrary), whenever $\widehat{f} : \widehat{Y} \to P$ is a retraction then at least one of the maps $f_i = \widehat{f} j_i$ is a retraction as well. (We will continue to use the notation from previous sections. In particular, for a family $\{f_i : Y_i \to P\}$ of monotone maps with finite domains and codomains, $\widehat{f}$ designates their **PSp** sum, whose domain is the Priestley sum $\widehat{Y} = \coprod_{i \in I} Y_i$, while f is reserved for their **poset** or **potop** sum, whose domain is the order disjoint union $\bigcup_{i \in I} Y_i$.)

Remark 4.1.1. The summands Y_i are restricted because we are unable to cope with general coproducts, whose topology is not very well understood. We need the property that the up- and down-sets of open sets are open. By 2.7, this is the characteristic feature of the Priestley duals of double Heyting algebras. Finite Priestley spaces constitute a transparent subclass of the latter class.

4.2. Recall from 2.1 that a configuration P is acyclic, or a tree, if the corresponding graph $(P, \prec\!\succ)$ is a tree. In particular, for any two points p and q in such a configuration P, there is a unique path connecting p to q in the $\prec\!\succ$ relation. One speaks of the length of such path as of the *distance* from p to q in P.

Proposition 4.2.1. *Any acyclic configuration is sectionally coproductive.*

Proof. Let $j_i : Y_i \to \widehat{Y} = \coprod Y_k$, $i \in I$, be a coproduct in $\mathbf{PSp}$, let P be a tree, and let $\widehat{f} : \widehat{Y} \to P$ and $g : P \to \widehat{Y}$ be Priestley maps such that $\widehat{f}g = \mathrm{id}_P$. Let $f_i = \widehat{f}j_i$.

Fix a reference point $r \in P$, and for each $p \in P$ let d_p be the distance from r to p. We define a pairwise disjoint family $\{V_p \mid p \in P\}$ of clopen subsets of $\widehat{Y}$; the definition is inductive downwards on the distance d_p.

- If p is a point at maximal distance from r, i.e., if p has no successor or predecessor q such that $d_q > d_p$, define $V_p = \widehat{f}^{-1}[p]$.
- Suppose V_q has been defined for all points q which are predecessors or successors of p with $d_q > d_p$. Define

$$V_p = \widehat{f}^{-1}[p] \cap \Big(\bigcap_{\substack{q \prec p \\ d_q > d_p}} \uparrow V_q \Big) \cap \Big(\bigcap_{\substack{q \succ p \\ d_q > d_p}} \downarrow V_q \Big).$$

Using the property from 2.7, we find that all sets V_p are clopen.

Let Y be the union $\bigcup_{i \in I} j_i[Y_i]$. It is dense in $\widehat{Y}$ (see [8]) and hence the nonempty open set $V_r \subseteq \widehat{Y}$ must contain a point y_r of Y, say $y_r \in j_{i_0}[Y_{i_0}]$ for some $i_0 \in I$. Since, for each predecessor $p \prec r$ (successor $p \succ r$), we have $V_r \subseteq \uparrow V_p$ ($V_r \subseteq \downarrow V_p$), there is a point $y_p \in V_p$ such that $y_p < y_r$ ($y_p > y_r$). By continuing this procedure, this time inducting upward on the distance from r, we get a collection of points $\{y_p \mid p \in P\} \subseteq \widehat{Y}$ such that the map $p \mapsto y_p$ is another section of $\widehat{f}$. But the $j_i[Y_i]$'s are pairwise disjoint and order independent, and Y is order independent from $\widehat{Y} \setminus Y$ (see [8]). Since the points y_p corresponding to related indices are related, and since P is connected, it follows that, for all $p \in P$, $y_p \in j_{i_0}[Y_{i_0}]$ and y_p has the form $j_{i_0}(y'_p)$ for a unique $y'_p \in Y_{i_0}$. The map $(p \mapsto y'_p) : P \to Y_{i_0}$ is the desired section for f_{i_0}. □

4.3. Construction. Let P be a cyclic configuration. Then there is a pair $r \prec s$ such that the poset which results from the removal of the rs edge from

$(P, \rightarrowtail\!\!\!\!\prec)$ is still connected. In the set

$$Y_n = \{n\} \times P \times \{0, 1, \ldots, n\}$$

we write (n, p, i) simply as npi, and we define an order $\leq$ by specifying predecessors as follows:

$$npi \prec mqj \;\text{ iff }\; n = m,\; p \prec q \text{ and } \begin{cases} i \neq j \text{ if } (p, q) = (r, s) \\ i = j \text{ otherwise.} \end{cases}$$

Set $Y = \bigcup_{n=1}^{\infty} Y_n$. Note that the role of the n in npi is just to make the union disjoint. The subsets of Y_n of the form

$$\{npi \mid p \in P\}, \quad i = 0, 1, \ldots, n,$$

are called the *pages* of Y_n.

Denote the embeddings of Y_n into Y by $j'_n : Y_n \to Y$; note that they constitute a coproduct in **poset** or **potop**. Consider the mappings

$$f = (npi \mapsto p) : Y \to P \quad \text{and} \quad f_n = (npi \mapsto p) : Y_n \to P;$$

we have $f \cdot j'_n = f_n$.

Lemma 4.3.1. *None of the maps $f_n : Y_n \to P$ is a retraction.*

Proof. Let $g : P \to Y_n$ be a section of f_n. Then $g(p) = npi_p$ for suitable i_p. Since g is monotone we have $i_p = i_q$ for $p \prec q$ unless $(p, q) = (r, s)$. Since $(P, \rightarrowtail\!\!\!\!\prec)$ remains connected after removing the rs edge, it follows that all the i_p's coincide, say, $i_p = i$. But then g is not monotone since $r \prec s$ and $g(r) \not\leq g(s)$. □

Lemma 4.3.2. *No $R \subseteq Y$ containing at least two pages of any single Y_n is small.*

Proof. This is straightforward to verify. Check that the intersection $Y(p) \cap \bigcap_{q \leq p} \uparrow R(q)$ from 3.2 is nonempty for each p. □

Theorem 4.3.3. *A configuration P is sectionally coproductive iff it is acyclic.*

Proof. By Proposition 4.2.1, if P is acyclic then it is sectionally coproductive.

Now let P be cyclic. Consider the posets Y_n and Y from 4.3 and the coproduct

$$j_n : Y_n \to Y$$

from 2.6 (see also Remark 3.1.1), and the $\widehat{f} : \widehat{Y} \to P$ determined by $\widehat{f} \cdot j_n = f_n$. Note that this notation is in agreement with determining $\widehat{f}$ for the f we already have, as in 3.1.1.

Let $\mathcal{R}$ be a finite family of small subsets of Y, and set $k = |\mathcal{R}|$. Obviously each page is a P-choice; by Lemma 4.3.2, at least one of the pages of Y_{k+1} is not covered by an $R \in \mathcal{R}$ and hence, by Theorem 3.6.1, $\widehat{f}$ is a retraction. By Lemma 4.3.1, however, none of the f_n is. Thus, P is not sectionally coproductive. □

4.4. The construction in 4.3 provides an example of a retraction $\widehat{f} : \widehat{Y} \to P$ extending a non-retraction $f : Y \to P$. Indeed, the map f given by $f(npi) = p$ is not a retraction, for otherwise at least one of the maps f_n would then be a retraction, see 4.1.

References

[1] Ball, R.N, Pultr, A.: Forbidden Forests in Priestley Spaces. Cahiers Top. Géom. Diff. Cat. **45** (2004), 2–22.

[2] Ball, R.N., Pultr A., Sichler, J.: Configurations in Coproducts of Priestley Spaces. Appl. Cat. Struct. **13** (2005), 121–130.

[3] Ball, R.N., Pultr, A., Sichler, J.: Combinatorial trees in Priestley spaces. Comment. Math. Univ. Carolin. **46** (2005), 217–234.

[4] Ball, R.N., Pultr, A., Sichler, J.: The Mysterious 2-Crown. Algebra Universalis **55** (2006), 213–226.

[5] Ball, R.N., Pultr, A., Sichler, J.: Priestley configurations and Heyting varieties. Algebra Universalis **59** (2008), 31–47.

[6] Davey, B.A., Priestley, H.A.: Introduction to Lattices and Order, Second Edition. Cambridge University Press (2001).

[7] Frayne, T., Morel, A.C., Scott, D.S.: Reduced direct products. Fund. Math. **51** (1962/63), 195–228.

[8] Koubek, V., Sichler, J.: On Priestley duals of products. Cahiers Top. Géom. Diff. Cat. **32** (1991), 243–256.

[9] Łoś, J.: Quelques remarques, théorèmes et problèmes sur les classes définisables d'algèbres. Mathematical interpretation of formal systems 98–113 North-Holland (1955).

[10] Mac Lane, S.: Categories for the Working Mathematician. Graduate Texts in Mathematics 5, Springer-Verlag (1971).

[11] Priestley, H.A.: Representation of distributive lattices by means of ordered Stone spaces. Bull. London Math. Soc. **2** (1970), 186–190.

[12] Priestley, H.A.: Ordered topological spaces and the representation of distributive lattices. Proc. London Math. Soc. **324** (1972), 507–530.

Richard N. Ball

Department of Mathematics, University of Denver, Denver, CO 80208, U.S.A.
e-mail: rball@du.edu

Aleš Pultr

Department of Applied Mathematics and ITI, MFF Charles University, CZ 11800 Praha 1, Malostranské nám. 25
e-mail: pultr@kam.ms.mff.cuni.cz

Jiří Sichler

Department of Mathematics, University of Manitoba Winnipeg, Canada R3T 2N2
e-mail: sichler@cc.umanitoba.ca