# Minimal Paths in the Commuting Graphs of Semigroups

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**Abstract**

Let $S$ be a finite non-commutative semigroup. The commuting graph of $S$, denoted $\mathcal{G}(S)$, is the graph whose vertices are the non-central elements of $S$ and whose edges are the sets $\{a, b\}$ of vertices such that $a \neq b$ and $ab = ba$. Denote by $T(X)$ the semigroup of full transformations on a finite set $X$. Let $J$ be any ideal of $T(X)$ such that $J$ is different from the ideal of constant transformations on $X$. We prove that if $|X| \geq 4$, then, with a few exceptions, the diameter of $\mathcal{G}(J)$ is 5. On the other hand, we prove that for every positive integer $n$, there exists a semigroup $S$ such that the diameter of $\mathcal{G}(S)$ is $n$.

We also study the left paths in $\mathcal{G}(S)$, that is, paths $a_1 - a_2 - \cdots - a_m$ such that $a_1 \neq a_m$ and $a_1 a_i = a_m a_i$ for all $i \in \{1, \ldots, m\}$. We prove that for every positive integer $n \geq 2$, except $n = 3$, there exists a semigroup whose shortest left path has length $n$. As a corollary, we use the previous results to solve a purely algebraic old problem posed by B.M. Schein.

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## 1 Introduction

The commuting graph of a finite non-abelian group $G$ is a simple graph whose vertices are all non-central elements of $G$ and two distinct vertices $x, y$ are adjacent if $xy = yx$. Commuting graphs of various groups have been studied in terms of their properties (such as connectivity or diameter), for example in [4], [6], [9], and [15]. They have also been used as a tool to prove group theoretic results, for example in [5], [12], and [13].

The concept of the commuting graph carries over to semigroups. Let $S$ be a finite non-commutative semigroup with center $Z(S) = \{a \in S : ab = ba \text{ for all } b \in S\}$. The *commuting graph* of $S$, denoted $\mathcal{G}(S)$, is the simple graph (that is, an undirected graph with no multiple

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edges or loops) whose vertices are the elements of $S - Z(S)$ and whose edges are the sets $\{a, b\}$ such that $a$ and $b$ are distinct vertices with $ab = ba$.

This paper initiates the study of commuting graphs of semigroups. Our main goal is to study the lengths of minimal paths. We shall consider two types of paths: ordinary paths from graph theory and so called left paths.

We first investigate the semigroup $T(X)$ of full transformations on a finite set $X$, and determine the diameter of the commuting graph of every ideal of $T(X)$ (Section 2). We find that, with a few exceptions, the diameter of $\mathcal{G}(J)$, where $J$ is an ideal of $T(X)$, is 5. This small diameter does not extend to semigroups in general. We prove that for every $n \geq 2$, there is a finite semigroup $S$ whose commuting graph has diameter $n$ (Theorem 4.1). To prove the existence of such a semigroup, we use our work on the *left paths* in the commuting graph of a semigroup.

Let $S$ be a semigroup. A path $a_1 - a_2 - \cdots - a_m$ in $\mathcal{G}(S)$ is called a *left path* (or $l$-path) if $a_1 \neq a_m$ and $a_1 a_i = a_m a_i$ for every $i \in \{1, \ldots, m\}$. If there is any $l$-path in $\mathcal{G}(S)$, we define the *knit degree* of $S$, denoted $\mathrm{kd}(S)$, to be the length of a shortest $l$-path in $\mathcal{G}(S)$.

For every $n \geq 2$ with $n \neq 3$, we construct a band (semigroup of idempotents) of knit degree $n$ (Section 3). It is an open problem if there is a semigroup of knit degree 3. The constructions presented in Section 3 also give a band $S$ whose commuting graph has diameter $n$ (for every $n \geq 4$). As another application of our work on the left paths, we settle a conjecture on bands formulated by B.M. Schein in 1978 (Section 5). Finally, we present some problems regarding the commuting graphs of semigroups (Section 6).

## 2 Commuting Graphs of Ideals of $T(X)$

Let $T(X)$ be the semigroup of full transformations on a finite set $X$, that is, the set of all functions from $X$ to $X$ with composition as the operation. We will write functions on the right and compose from left to right, that is, for $a, b \in T(X)$ and $x \in X$, we will write $xa$ (not $a(x)$) and $x(ab) = (xa)b$ (not $(ba)(x) = b(a(x))$). In this section, we determine the diameter of the commuting graph of every ideal of $T(X)$. Throughout this section, we assume that $X = \{1, \ldots, n\}$.

Let $\Gamma$ be a simple graph, that is, $\Gamma = (V, E)$, where $V$ is a finite non-empty set of vertices and $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$ is a set of edges. We will write $u - v$ to mean that $\{u, v\} \in E$. Let $u, w \in V$. A *path* in $\Gamma$ from $u$ to $w$ is a sequence of pairwise distinct vertices $u = v_1, v_2, \ldots, v_m = w$ ($m \geq 1$) such that $v_i - v_{i+1}$ for every $i \in \{1, \ldots, m-1\}$. If $\lambda$ is a path $v_1, v_2, \ldots, v_m$, we will write $\lambda = v_1 - v_2 - \cdots - v_m$ and say that $\lambda$ has *length* $m - 1$. We say that a path $\lambda$ from $u$ to $w$ is a *minimal path* if there is no path from $u$ to $w$ that is shorter than $\lambda$.

We say that the *distance* between vertices $u$ and $w$ is $k$, and write $d(u, w) = k$, if a minimal path from $u$ to $w$ has length $k$. If there is no path from $u$ to $w$, we say that the distance between $u$ and $w$ is infinity, and write $d(u, w) = \infty$. The maximum distance $\max\{d(u, w) : u, w \in V\}$ between vertices of $\Gamma$ is called the *diameter* of $\Gamma$. Note that the diameter of $\Gamma$ is finite if and only if $\Gamma$ is connected.

If $S$ is a finite non-commutative semigroup, then the commuting graph $\mathcal{G}(S)$ is a simple graph with $V = S - Z(S)$ and, for $a, b \in V$, $a - b$ if and only if $a \neq b$ and $ab = ba$.

For $a \in T(X)$, we denote by $\mathrm{im}(a)$ the image of $a$, by $\ker(a) = \{(x, y) \in X \times X : xa = ya\}$ the kernel of $a$, and by $\mathrm{rank}(a) = |\mathrm{im}(a)|$ the rank of $a$. It is well known (see [7, Section 2.2]) that in $T(X)$ the only element of $Z(T(X))$ is the identity transformation on $X$, and that $T(X)$ has exactly $n$ ideals: $J_1, J_2, \ldots, J_n$, where, for $1 \leq r \leq n$,

$$J_r = \{a \in T(X) : \mathrm{rank}(a) \leq r\}.$$

Each ideal $J_r$ is principal and any $a \in T(X)$ of rank $r$ generates $J_r$. The ideal $J_1$ consists of the transformations of rank 1 (that is, constant transformations), and it is clear that $\mathcal{G}(J_1)$ is the graph with $n$ isolated vertices.

Let $S$ be a semigroup. We denote by $\mathcal{G}_E(S)$ the subgraph of $\mathcal{G}(S)$ induced by the non-central idempotents of $S$. The graph $\mathcal{G}_E(S)$ is said to be the *idempotent commuting graph* of $S$. We first determine the diameter of $\mathcal{G}_E(J_r)$. This approach is justified by the following lemma.

**Lemma 2.1.** *Let $a, b \in J_r$ be such that $ab \neq ba$. Then there are idempotents $e_1, e_2, \ldots, e_k \in J_r$ ($k \geq 1$) such that $a - e_1 - e_2 - \cdots - e_k - b$ is a minimal path in $\mathcal{G}(J_r)$ from $a$ to $b$.*

*Proof.* Let $a - a_1 - a_2 - \cdots - a_k - b$ be a minimal path in $\mathcal{G}(J_r)$ from $a$ to $b$. Then $k \geq 1$ since $ab \neq ba$. Since $J_r$ is finite, there is an integer $p \geq 1$ such that $a_1^p$ is an idempotent in $J_r$. Since $a_1$ commutes with $a$ and $a_2$, the idempotent $e_1 = a_1^p$ also commutes with $a$ and $a_2$, and so $a - e_1 - a_2 - \cdots - a_k - b$. Repeating the foregoing argument for $a_2, \ldots, a_k$, we obtain idempotents $e_2, \ldots, e_k$ in $J_r$ such that $a - e_1 - e_2 - \cdots - e_k - b$. Since the path $a - a_1 - a_2 - \cdots - a_k - b$ is minimal, it follows that $a, e_1, e_2, \ldots, e_k, b$ are pairwise distinct and the path $a - e_1 - e_2 - \cdots - e_k - b$ is minimal. $\square$

It follows from Lemma 2.1 that if $d$ is the diameter of $\mathcal{G}_E(J_r)$, then the diameter of $\mathcal{G}(J_r)$ is at most $d + 2$.

## 2.1 Idempotent Commuting Graphs

In this subsection, we assume that $n \geq 3$ and $2 \leq r < n$. We will show that, with some exceptions, the diameter of $\mathcal{G}_E(J_r)$ is 3 (Theorem 2.8).

Let $e \in T(X)$ be an idempotent. Then there is a unique partition $\{A_1, A_2, \ldots, A_k\}$ of $X$ and unique elements $x_1 \in A_1, x_2 \in A_2, \ldots, x_k \in A_k$ such that for every $i$, $A_i e = \{x_i\}$. The partition $\{A_1 \ldots, A_k\}$ is induced by the kernel of $e$, and $\{x_1, \ldots, x_k\}$ is the image of $e$. We will use the following notation for $e$:

$$e = (A_1, x_1\rangle (A_2, x_2\rangle \ldots (A_k, x_k\rangle. \tag{2.1}$$

Note that $(X, x\rangle$ is the constant idempotent with image $\{x\}$. The following result has been obtained in [1] and [10] (see also [2]).

**Lemma 2.2.** *Let $e = (A_1, x_1\rangle (A_2, x_2\rangle \ldots (A_k, x_k\rangle$ be an idempotent in $T(X)$ and let $b \in T(X)$. Then $b$ commutes with $e$ if and only if for every $i \in \{1, \ldots, k\}$, there is $j \in \{1, \ldots, k\}$ such that $x_i b = x_j$ and $A_i b \subseteq A_j$.*

We will use Lemma 2.2 frequently, not always mentioning it explicitly. The following lemma is an immediate consequence of Lemma 2.2.

**Lemma 2.3.** *Let $e, f \in J_r$ be idempotents and suppose there is $x \in X$ such that $x \in \text{im}(e) \cap \text{im}(f)$. Then $e - (X, x\rangle - f$.*

**Lemma 2.4.** *Let $e, f \in J_r$ be idempotents such that $\text{im}(e) \cap \text{im}(f) = \emptyset$. Suppose there is $(x, y) \in \text{im}(e) \times \text{im}(f)$ such that $(x, y) \in \ker(e) \cap \ker(f)$. Then there is an idempotent $g \in J_r$ such that $e - g - f$.*

*Proof.* Let $e = (A_1, x_1\rangle \ldots (A_k, x_k\rangle$ and $f = (B_1, y_1\rangle \ldots (B_m, y_m\rangle$. We may assume that $x = x_1$ and $y = y_1$. Since $(x, y) \in \ker(e) \cap \ker(f)$, we have $y \in A_1$ and $x \in B_1$. Let $g = (\text{im}(e), x\rangle (X - \text{im}(e), y\rangle$. Then $g$ is in $J_r$ since $\text{rank}(g) = 2$ and $r \geq 2$. By Lemma 2.2, we have $eg = ge$ (since $y \in A_1$) and $fg = gf$ (since $\text{im}(f) \subseteq X - \text{im}(e)$ and $x \in B_1$). Hence $e - g - f$. $\square$

**Lemma 2.5.** *Let $e, f \in J_r$ be idempotents such that $\text{im}(e) \cap \text{im}(f) = \emptyset$. Then there are idempotents $g, h \in J_r$ such that $e - g - h - f$.*

*Proof.* Let $e = (A_1, x_1\rangle \ldots (A_k, x_k\rangle$ and $f = (B_1, y_1\rangle \ldots (B_m, y_m\rangle$. Since $\text{im}(e) \cap \text{im}(f) = \emptyset$, there is $i$ such that $y_1 \in A_i$. We may assume that $y_1 \in A_1$. Let $g = (X - \{y_1\}, x_1\rangle (\{y_1\}, y_1\rangle$ and $h = (X, y_1\rangle$. Then $g$ and $h$ are in $J_r$ (since $r \geq 2$). By Lemma 2.2, $eg = ge$, $gh = hg$, and $hf = fh$. Thus $e - g - h - f$. $\square$

**Lemma 2.6.** *Let $m$ be a positive integer such that $2m \le n$, $\sigma$ be an $m$-cycle on $\{1, \dots, m\}$, and*

$$e = (A_1, x_1\rangle(A_2, x_2\rangle \dots (A_m, x_m\rangle \text{ and } f = (B_1, y_1\rangle(B_2, y_2\rangle \dots (B_m, y_m\rangle$$

*be idempotents in $T(X)$ such that $x_1, \dots, x_m, y_1, \dots, y_m$ are pairwise distinct, $y_i \in A_i$, and $x_{i\sigma} \in B_i$ $(1 \le i \le m)$. Suppose that $g$ is an idempotent in $T(X)$ such that $e - g - f$. Then:*

1. *$x_j g = x_j$ and $y_j g = y_j$ for every $j \in \{1, \dots, m\}$.*
2. *If $1 \le i, j \le m$ are such that $A_i = \{x_i, y_i, z\}$, $B_j = \{y_j, x_{j\sigma}, z\}$ and $A_i \cap B_j = \{z\}$, then $zg = z$.*

*Proof.* Since $eg = ge$, $x_1 g = x_i$ for some $i$. Then $x_i g = x_i$ (since $g$ is an idempotent). Thus, $e - g - f$ and Lemma 2.2 imply that $y_i g = y_i$. Since $x_i = x_{(i\sigma^{-1})\sigma} \in B_{i\sigma^{-1}}$ and $g$ commutes with $f$, we have $y_{i\sigma^{-1}} g = y_{i\sigma^{-1}}$. But now, since $y_{i\sigma^{-1}} \in A_{i\sigma^{-1}}$ and $g$ commutes with $e$, we have $x_{i\sigma^{-1}} g = x_{i\sigma^{-1}}$. Continuing this way, we obtain $x_{i\sigma^{-k}} g = x_{i\sigma^{-k}}$ and $y_{i\sigma^{-k}} g = y_{i\sigma^{-k}}$ for every $k \in \{1, \dots, m-1\}$. Since $\sigma$ is an $m$-cycle, it follows that $x_j g = x_j$ and $y_j g = y_j$ for every $j \in \{1, \dots, m\}$. We have proved (1).

Suppose $A_i = \{x_i, y_i, z\}$, $B_j = \{y_j, x_{j\sigma}, z\}$, and $A_i \cap B_j = \{z\}$. Then $zg \in \{x_i, y_i, z\}$ (since $x_i g = x_i$ and $eg = ge$) and $zg \in \{y_j, x_{j\sigma}, z\}$ (since $y_j g = y_j$ and $fg = gf$). Since $A_i \cap B_j = \{z\}$, we have $zg = z$, which proves (2). ☐

**Lemma 2.7.** *Let $n \ge 4$. If $n \ne 5$ or $r \ne 4$, then for some idempotents $e, f \in J_r$, there is no idempotent $g \in J_r$ such that $e - g - f$.*

*Proof.* Let $n \ne 5$ or $r \ne 4$. Suppose that $r < n - 1$ or $n$ is even. Then there is an integer $m$ such that $m \le r$ and $r < 2m \le n$. Let $e$ and $f$ be idempotents from Lemma 2.6. Then $e, f \in J_r$ since $m \le r$. But every idempotent $g \in T(X)$ such that $e - g - f$ fixes at least $2m$ elements, and so $g \notin J_r$ since $r < 2m$.

Suppose that $r = n - 1$ and $n = 2m + 1$ is odd. Then $n \ge 7$ since we are working under the assumption that $n \ne 5$ or $r \ne 4$. We again consider idempotents $e$ and $f$ from Lemma 2.6, which belong to $J_r$ since $m < n - 1 = r$. Note that $X = \{x_1, \dots, x_m, y_1, \dots, y_m, z\}$. We may assume that $z \in A_m$ and $z \in B_1$. Since $n \ge 7$, we have $m \ge 3$. Thus, the intersection of $A_m = \{x_m, y_m, z\}$ and $B_1 = \{y_1, x_2, z\}$ is $\{z\}$, and so $zg = z$ by Lemma 2.6. Hence $g = \mathrm{id}_X \notin J_r$, which concludes the proof. ☐

**Theorem 2.8.** *Let $n \ge 3$ and let $J_r$ be an ideal in $T(X)$ such that $2 \le r < n$. Then:*

1. *If $n = 3$ or $n = 5$ and $r = 4$, then the diameter of $\mathcal{G}_E(J_r)$ is 2.*
2. *In all other cases, the diameter of $\mathcal{G}_E(J_r)$ is 3.*

*Proof.* Suppose $n = 3$ or $n = 5$ and $r = 4$. In these special cases, we obtained the desired result using GRAPE [16], which is a package for GAP [8].

Let $n \ge 4$ and suppose that $n \ne 5$ or $r \ne 4$. By Lemmas 2.3 and 2.5, the diameter of $\mathcal{G}_E(J_r)$ is at most 3. By Lemma 2.7, the diameter of $\mathcal{G}_E(J_r)$ is at least 3. Thus the diameter of $\mathcal{G}_E(J_r)$ is 3, which concludes the proof of (2). ☐

## 2.2 Commuting Graphs of Proper Ideals of $T(X)$

In this subsection, we determine the diameter of every proper ideal of $T(X)$. The ideal $J_1$ consists of the constant transformations, so $\mathcal{G}(J_1)$ is the graph with $n$ isolated vertices. Thus $J_1$ is not connected and its diameter is $\infty$. Therefore, for the remainder of this subsection, we assume that $n \ge 3$ and $2 \le r < n$.

It follows from Lemma 2.1 and Theorem 2.8 that the diameter of $\mathcal{G}(J_r)$ is at most 5. We will prove that this diameter is in fact 5 except when $n = 3$ or $n \in \{5, 6, 7\}$ and $r = 4$. It also follows from Lemma 2.1 that if $e$ and $f$ are idempotents in $J_r$, then the distance between $e$ and $f$ in $\mathcal{G}(J_r)$ is the same as the distance between $e$ and $f$ in $\mathcal{G}_E(J_r)$. So no ambiguity will arise when we talk about the distance between idempotents in $J_r$.

For $a \in T(X)$ and $x, y \in X$, we will write $x \xrightarrow{a} y$ when $xa = y$.

**Lemma 2.9.** *Let $a, b \in T(X)$. Then $ab = ba$ if and only if for all $x, y \in X$, $x \xrightarrow{a} y$ implies $xb \xrightarrow{a} yb$.*

*Proof.* Suppose $ab = ba$. Let $x, y \in X$ with $x \xrightarrow{a} y$, that is, $y = xa$. Then, since $ab = ba$, we have $yb = (xa)b = x(ab) = x(ba) = (xb)a$, and so $xb \xrightarrow{a} yb$.

Conversely, suppose $x \xrightarrow{a} y$ implies $xb \xrightarrow{a} yb$ for all $x, y \in X$. Let $x \in X$. Since $x \xrightarrow{a} xa$, we have $xb \xrightarrow{a} (xa)b$. But this means that $(xb)a = (xa)b$, which implies $ab = ba$. □

Let $a \in T(X)$. Suppose $x_1, \ldots, x_m$ are pairwise distinct elements of $X$ such that $x_i a = x_{i+1}$ $(1 \le i < m)$ and $x_m a = x_1$. We will then say that $a$ contains a cycle $(x_1\, x_2 \ldots x_m)$.

**Lemma 2.10.** *Let $a \in J_r$ be a transformation containing a unique cycle $(x_1\, x_2 \ldots x_m)$. Let $e \in J_r$ be an idempotent such that $ae = ea$. Then $x_i e = x_i$ for every $i \in \{1, \ldots, m\}$.*

*Proof.* Since $a$ contains $(x_1\, x_2 \ldots x_m)$, we have $x_1 \xrightarrow{a} x_2 \xrightarrow{a} \cdots \xrightarrow{a} x_m \xrightarrow{a} x_1$. Thus, by Lemma 2.9,

$$x_1 e \xrightarrow{a} x_2 e \xrightarrow{a} \cdots \xrightarrow{a} x_m e \xrightarrow{a} x_1 e.$$

Thus $(x_1 e\, x_2 e \ldots x_m e)$ is a cycle in $a$, and is therefore equal to $(x_1\, x_2 \ldots x_m)$. Hence, for every $i \in \{1, \ldots, m\}$, there exists $j \in \{1, \ldots, m\}$ such that $x_i = x_j e$, and so $x_i e = (x_j e)e = x_j(ee) = x_j e = x_i$. □

To construct transformations $a, b \in J_r$ such that the distance between $a$ and $b$ is 5, it will be convenient to introduce the following notation.

**Notation 2.11.** Let $x_1, \ldots, x_m, z_1, \ldots, z_p$ be pairwise distinct elements of $X$, and let $s$ be fixed such that $1 \le s < p$. We will denote by

$$a = (*\, z_s\rangle(z_p\, z_{p-1} \ldots z_1\, x_1\rangle(x_1\, x_2 \ldots x_m) \tag{2.2}$$

the transformation $a \in T(X)$ such that

$$z_p a = z_{p-1},\ z_{p-1} a = z_{p-2}, \ldots, z_2 a = z_1,\ z_1 a = x_1,$$
$$x_1 a = x_2,\ x_2 a = x_3, \ldots, x_{m-1} a = x_m,\ x_m a = x_1,$$

and $ya = z_s$ for all other $y \in X$. Suppose $w \in X$ such that $w \notin \{x_1, \ldots, x_m, z_1, \ldots, z_p\}$ and $1 \le t < p$ with $t \ne s$. We will denote by

$$b = (*\, z_s\rangle(w\, z_t\rangle(z_p\, z_{p-1} \ldots z_1\, x_1\rangle(x_1\, x_2 \ldots x_m) \tag{2.3}$$

the transformation $b \in T(X)$ that is defined as $a$ in (2.2) except that $wb = z_t$.

**Lemma 2.12.** *Let $a \in J_r$ be the transformation defined in* (2.2) *such that $m + p > r$. Let $e \in J_r$ be an idempotent such that $ae = ea$. Then:*

(1) *$x_i e = x_i$ for every $i \in \{1, \ldots, m\}$.*

(2) *$z_j e = x_{m-j+1}$ for every $j \in \{1, \ldots, p\}$.*

(3) *$ye = x_{m-s}$ for every $y \in X - \{x_1, \ldots, x_m, z_1, \ldots, z_p\}$.*

*(We assume that for every integer $u$, $x_u = x_v$, where $v \in \{1, \ldots, m\}$ and $u \equiv v \pmod{m}$.)*

*Proof.* Statement (1) follows from Lemma 2.10. By the definition of $a$, we have

$$z_p \xrightarrow{a} z_{p-1} \xrightarrow{a} \cdots \xrightarrow{a} z_1 \xrightarrow{a} x_1.$$

Thus, by Lemma 2.9,

$$z_p e \xrightarrow{a} z_{p-1} e \xrightarrow{a} \cdots \xrightarrow{a} z_1 e \xrightarrow{a} x_1 e = x_1.$$

Since $z_1 e \xrightarrow{a} x_1$, either $z_1 e = x_m$ or $z_1 e \notin \{x_1, \dots, x_m\}$. We claim that the latter is impossible. Indeed, suppose $z_1 e \notin \{x_1, \dots, x_m\}$. Then $z_j e \notin \{x_1, \dots, x_m\}$ for every $j \in \{1, \dots, p\}$. Thus the set $\{x_1, \dots, x_m, z_1 e, \dots, z_p e\}$ is a subset of $\operatorname{im}(e)$ with $m + p$ elements. But this implies that $e \notin J_r$ (since $m + p > r$), which is a contradiction. We proved the claim. Thus $z_1 e = x_m$. Now, $z_2 e \xrightarrow{a} z_1 e = x_m$, which implies $z_2 e = x_{m-1}$. Continuing this way, we obtain $z_3 e = x_{m-2}$, $z_4 e = x_{m-3}, \dots$. (A special argument is required when $j = qm + 1$ for some $q \geq 1$. Suppose $q = 1$, that is, $j = m + 1$. Then $z_j e \xrightarrow{a} z_{j-1} e = z_m e = x_1$, and so either $z_j e = x_m$ or $z_j e = z_1$. But the latter is impossible since we would have $x_m = z_1 e = z_j(ee) = z_j e = z_1$, which is a contradiction. Hence, for $j = m + 1$, we have $z_j e = x_m$. Assuming, inductively, that $z_j e = x_m$ for $j = qm + 1$, we prove by a similar argument that $z_j e = x_m$ for $j = (q + 1)m + 1$.) This concludes the proof of (2).

Let $y \in X - \{x_1, \dots, x_m, z_1, \dots, z_p\}$. Then $y \xrightarrow{a} z_s$, and so $ye \xrightarrow{a} z_s e = x_{m-s+1}$. Suppose $s$ is not a multiple of $m$. Then $x_{m-s+1} \neq x_1$, and so $ye \xrightarrow{a} x_{m-s+1}$ implies $ye = x_{m-s}$. Suppose $s$ is a multiple of $m$. Then $ye \xrightarrow{a} x_{m-s+1} = x_1$, and so either $ye = x_m$ or $ye = z_1$. But the latter is impossible since we would have $x_m = z_1 e = y(ee) = ye = z_1$, which is a contradiction. Hence, for $s$ that is a multiple of $m$, we have $ye = x_m$, which concludes the proof of (3). □

The proof of the following lemma is almost identical to the proof of Lemma 2.12.

**Lemma 2.13.** *Let $b \in J_r$ be the transformation defined in* (2.3) *such that $m + p > r$. Let $e \in J_r$ be an idempotent such that $be = eb$. Then:*

(1) $x_i e = x_i$ *for every* $i \in \{1, \dots, m\}$.

(2) $z_j e = x_{m-j+1}$ *for every* $j \in \{1, \dots, p\}$.

(3) $we = x_{m-t}$.

(4) $ye = x_{m-s}$ *for every* $y \in X - \{x_1, \dots, x_m, z_1, \dots, z_p, w\}$.

**Lemma 2.14.** *Let $n \in \{5, 6, 7\}$ and $r = 4$. Then there are $a, b \in J_4$ such that the distance between $a$ and $b$ in $\mathcal{G}(J_4)$ is at least 4.*

*Proof.* Let $a = (*\,4\rangle(3\,4\,1\rangle(1\,2)$ and $b = (*\,1\rangle(2\,1\,3\rangle(3\,4)$ (see Notation 2.11). Suppose $e$ and $f$ are idempotents in $J_4$ such that $a - e$ and $f - b$. Then, by Lemma 2.12, $e = (\{\dots, 3, 1\}, 1\rangle(\{4, 2\}, 2\rangle$ and $f = (\{\dots, 2, 3\}, 3\rangle(\{1, 4\}, 4\rangle$, where "…" denotes "5" (if $n = 5$), "5, 6" (if $n = 6$), and "5, 6, 7" (if $n = 7$). Then $e$ and $f$ do not commute, and so $d(e, f) \geq 2$. Thus $d(a, b) \geq 4$ by Lemma 2.1. □

**Lemma 2.15.** *Let $n \in \{6, 7\}$ and $r = 4$. Let $a \in J_4$ be a transformation that is not an idempotent. Then there is an idempotent $e \in J_4$ commuting with $a$ such that* $\operatorname{rank}(e) \neq 3$ *or* $\operatorname{rank}(e) = 3$ *and* $ye^{-1} = \{y\}$ *for some* $y \in \operatorname{im}(e)$.

*Proof.* If $a$ fixes some $x \in X$, then $a$ commutes with $e = (X, x\rangle$ of rank 1. Suppose $a$ has no fixed points. Let $p$ be a positive integer such that $a^p$ is an idempotent. If $a$ contains a unique cycle $(x_1\, x_2)$, then $e = a^p$ has rank 2. If $a$ contains a unique cycle $(x_1\, x_2\, x_3\, x_4)$ or two cycles $(x_1\, x_2)$ and $(y_1\, y_2)$ with $\{x_1, x_2\} \cap \{y_1, y_2\} = \emptyset$, then $e = a^p$ has rank 4.

Suppose $a$ contains a unique cycle $(x_1\, x_2\, x_3)$. Define $e \in T(X)$ as follows. Set $x_i e = x_i$, $1 \leq i \leq 3$.

Suppose there are $y, z \in X - \{x_1, x_2, x_3\}$ such that $ya = z$ and $za = x_i$ for some $i$. We may assume that $za = x_1$. Define $ze = x_3$ and $ye = x_2$. Let $u$ and $w$ be the two remaining elements in $X$ (only $u$ remains when $n = 6$). Since $\mathrm{rank}(a) \leq 4$, we have $\{u, w\}a \subseteq \{z, x_1, x_2, x_3\}$. Suppose $ua = wa = z$. Define $ue = x_2$ and $we = x_2$. Then $e$ is an idempotent of rank 3 such that $ae = ea$ and $x_1e^{-1} = \{x_1\}$. Suppose $ua$ or $wa$ is in $\{x_1, x_2, x_3\}$, say $ua \in \{x_1, x_2, x_3\}$. Define $ue = u$, and $we = x_{i-1}$ (if $wa = x_i$), where $x_{i-1} = x_3$ if $i = 1$, or $we = x_2$ (if $wa = z$). Then $e$ is an idempotent of rank 4 such that $ae = ea$.

Suppose that for every $y \in X - \{x_1, x_2, x_3\}$, $ya \in \{x_1, x_2, x_3\}$. Select $z \in X - \{x_1, x_2, x_3\}$ and define $ze = z$. For every $y \in X - \{z, x_1, x_2, x_3\}$, define $ye = x_{i-1}$ if $ya = x_i$. Then $e$ is an idempotent of rank 4 such that $ae = ea$.

Since $a \in J_4$, we have exhausted all possibilities, and the result follows. □

**Lemma 2.16.** *Let $n \in \{6, 7\}$ and $r = 4$. Then for all $a, b \in J_4$, the distance between $a$ and $b$ in $\mathcal{G}(J_4)$ is at most 4.*

*Proof.* Let $a, b \in J_4$. If $a$ or $b$ is an idempotent, then $d(a, b) \leq 4$ by Lemma 2.1 and Theorem 2.8. Suppose $a$ and $b$ are not idempotents. By lemma 2.15, there are idempotents $e, f \in J_4$ such that $ae = ea$, $bf = fb$, if $\mathrm{rank}(e) = 3$, then $ye^{-1} = \{y\}$ for some $y \in \mathrm{im}(e)$, and if $\mathrm{rank}(f) = 3$, then $yf^{-1} = \{y\}$ for some $y \in \mathrm{im}(f)$. We claim that there is an idempotent $g \in J_4$ such that $e - g - f$. If $\mathrm{im}(e) \cap \mathrm{im}(f) \neq \emptyset$, then such an idempotent $g$ exists by Lemma 2.3. Suppose $\mathrm{im}(e) \cap \mathrm{im}(f) = \emptyset$. Then, since $n \in \{6, 7\}$, both $\mathrm{rank}(e) + \mathrm{rank}(f) \leq 7$. We may assume that $\mathrm{rank}(e) \leq \mathrm{rank}(f)$. There are six possible cases.

**Case 1.** $\mathrm{rank}(e) = 1$.

Then $e = (X, x\rangle$ for some $x \in X$. Let $y = xf$. Then $(x, y) \in \mathrm{im}(e) \times \mathrm{im}(f)$ and $(x, y) \in \ker(e) \cap \ker(f)$. Thus, by Lemma 2.4, there is an idempotent $g \in J_4$ such that $e - g - f$.

**Case 2.** $\mathrm{rank}(e) = 2$ and $\mathrm{rank}(f) = 2$.

We may assume that $e = (A_1, 1\rangle(A_2, 2\rangle$ and $f = (B_1, 3\rangle(B_2, 4\rangle$. If $\{1, 2\} \subseteq B_i$ or $\{3, 4\} \subseteq A_i$ for some $i$, then we can find $(x, y) \in \mathrm{im}(e) \times \mathrm{im}(f)$ such that $(x, y) \in \ker(e) \cap \ker(f)$, and so a desired idempotent $g$ exists by Lemma 2.4. Otherwise, we may assume that $3 \in A_1$ and $4 \in A_2$. If $1 \in B_1$ or $2 \in B_2$, then Lemma 2.4 can be applied again. So suppose $1 \in B_2$ and $2 \in B_1$. Now we have

$$e = (\{\dots, 3, 1\}, 1\rangle(\{\dots, 4, 2\}, 2\rangle \text{ and } f = (\{\dots, 2, 3\}, 3\rangle(\{\dots, 1, 4\}, 4\rangle.$$

We define $g \in T(X)$ as follows. Set $xg = x$ for every $x \in \{1, 2, 3, 4\}$. Let $x \in \{5, 6, 7\}$ ($x \in \{5, 6\}$ if $n = 6$). If $x \in A_1 \cap B_1$, define $xg = 3$; if $x \in A_1 \cap B_2$, define $xg = 1$; if $x \in A_2 \cap B_1$, define $xg = 2$; finally, if $x \in A_2 \cap B_2$, define $xg = 4$. Then $g$ is an idempotent of rank 4 and $e - g - f$.

**Case 3.** $\mathrm{rank}(e) = 2$ and $\mathrm{rank}(f) = 3$.

We may assume that $e = (A_1, 1\rangle(A_2, 2\rangle$ and $f = (B_1, 3\rangle(B_2, 4\rangle(B_3, 5\rangle$. If $\{3, 4, 5\} \subseteq A_1$ or $\{3, 4, 5\} \subseteq A_2$, then Lemma 2.4 applies. Otherwise, we may assume that $3, 4 \in A_1$ and $5 \in A_2$. If $1 \in B_1 \cup B_2$ or $2 \in B_3$, then Lemma 2.4 applies again. So suppose $1 \in B_3$ and $2 \in B_1 \cup B_2$. We may assume that $2 \in B_1$. Note that if $z \in \{6, 7\}$, then $z$ cannot be in $B_2$ since $z \in B_2$ would imply that there is no $y \in \mathrm{im}(f)$ such that $yf^{-1} = \{y\}$. So now

$$e = (\{\dots, 3, 4, 1\}, 1\rangle(\{\dots, 5, 2\}, 2\rangle \text{ and } f = (\{\dots, 2, 3\}, 3\rangle(\{4\}, 4\rangle(\{\dots, 1, 5\}, 5\rangle.$$

We define $g \in T(X)$ as follows. Set $xg = x$ for every $x \in \{1, 2, 3, 5\}$ and $4g = 3$. Let $z \in \{6, 7\}$. If $z \in A_1 \cap B_1$, define $zg = 3$; if $z \in A_1 \cap B_3$, define $zg = 1$; if $z \in A_2 \cap B_1$, define $zg = 2$; finally, if $z \in A_2 \cap B_3$, define $zg = 5$. Then $g$ is an idempotent of rank 4 and $e - g - f$.

**Case 4.** $\mathrm{rank}(e) = 2$ and $\mathrm{rank}(f) = 4$.

We may assume that $e = (A_1, 1\rangle(A_2, 2\rangle$ and $f = (B_1, 3\rangle(B_2, 4\rangle(B_3, 5\rangle(B_4, 6\rangle$. If $\{3, 4, 5, 6\} \subseteq A_1$ or $\{3, 4, 5, 6\} \subseteq A_2$, then Lemma 2.4 applies. Otherwise, we may assume that $3, 4, 5 \in A_1$ and $6 \in A_2$ or $3, 4 \in A_1$ and $5, 6 \in A_2$.

Suppose $3,4,5 \in A_1$ and $6 \in A_2$. If $1 \in B_1 \cup B_2 \cup B_3$ or $2 \in B_4$, then Lemma 2.4 applies. So suppose $1 \in B_4$, and we may assume that $2 \in B_1$. Now we have

$$e = (\{\dots,3,4,5,1\},1\rangle(\{\dots,6,2\},2\rangle,$$
$$f = (\{\dots,2,3\},3\rangle(\{\dots,4\},4\rangle(\{\dots,5\},5\rangle(\{\dots,1,6\},6\rangle.$$

We define $g \in T(X)$ as follows. Set $xg = x$ for every $x \in \{1,2,3,6\}$, $4g = 3$, and $5g = 3$. Define $7g = 3$ if $7 \in A_1$ and $7 \in B_1 \cup B_2 \cup B_3$; $7g = 1$ if $7 \in A_1$ and $7 \in B_4$; $7g = 2$ if $7 \in A_2$ and $7 \in B_1 \cup B_2 \cup B_3$; and $7g = 6$ if $7 \in A_2$ and $7 \in B_4$. Then $g$ is an idempotent of rank 4 and $e - g - f$. The argument in the case when $3,4 \in A_1$ and $5,6 \in A_2$ is similar.

**Case 5.** $\text{rank}(e) = 3$ and $\text{rank}(f) = 3$.

Since both $e$ and $f$ have an element in their range whose preimage is the singleton, we may assume that $e = (A_1,1\rangle(A_2,2\rangle(\{3\},3\rangle$ and $f = (B_1,4\rangle(B_2,5\rangle(\{6\},6\rangle$. If $\{1,2\} \subseteq B_i$ or $\{4,5\} \subseteq A_i$ for some $i$, then Lemma 2.4 applies. Otherwise, we may assume that $4 \in A_1$ and $5 \in A_2$. If $1 \in B_1$ or $2 \in B_2$, then Lemma 2.4 applies again. So suppose $1 \in B_2$ and $2 \in B_1$. So now

$$e = (\{\dots,4,1\},1\rangle(\{\dots,5,2\},2\rangle(\{3\},3\rangle \text{ and } f = (\{\dots,2,4\},4\rangle(\{\dots,1,5\},5\rangle(\{6\},6\rangle.$$

We define $g \in T(X)$ as follows. Set $xg = x$ for every $x \in \{1,2,4,5\}$, $3g = 1$, and $6g = 4$. Define $7g = 4$ if $7 \in A_1$ and $7 \in B_1$; $7g = 1$ if $7 \in A_1$ and $7 \in B_2$; $7g = 2$ if $7 \in A_2$ and $7 \in B_1$; and $7g = 5$ if $7 \in A_2$ and $7 \in B_2$. Then $g$ is an idempotent of rank 4 and $e - g - f$.

**Case 6.** $\text{rank}(e) = 3$ and $\text{rank}(f) = 4$.

We may assume that $e = (A_1,1\rangle(A_2,2\rangle(\{3\},3\rangle$ and $f = (B_1,4\rangle(B_2,5\rangle(B_3,6\rangle(\{7\},7\rangle$. If $\{4,5,6\} \subseteq A_1$ or $\{4,5,6\} \subseteq A_2$, then Lemma 2.4 applies. So we may assume that $4,5 \in A_1$ and $6 \in A_2$. If $1 \in B_1 \cup B_2$ or $2 \in B_3$, then Lemma 2.4 applies again. So we may assume that $1 \in B_3$ and $2 \in B_1$. So now

$$e = (\{\dots,4,5,1\},1\rangle(\{\dots,6,2\},2\rangle(\{3\},3\rangle,$$
$$f = (\{\dots,2,4\},4\rangle(\{\dots,5\},5\rangle(\{\dots,1,6\},6\rangle(\{7\},7\rangle.$$

We define $g \in T(X)$ as follows. Set $xg = x$ for every $x \in \{1,2,4,6\}$ and $5g = 4$. Define $7g = 4$ if $7 \in A_1$; $7g = 6$ if $7 \in A_2$; $3g = 3$ if $3 \in B_1 \cup A_2$; and $3g = 1$ if $3 \in B_3$. Then $g$ is an idempotent of rank 4 and $e - g - f$.

□

**Theorem 2.17.** *Let $n \geq 3$ and let $J_r$ be an ideal in $T(X)$ such that $2 \leq r < n$. Then:*

(1) *If $n = 3$ or $n \in \{5,6,7\}$ and $r = 4$, then the diameter of $\mathcal{G}(J_r)$ is 4.*

(2) *In all other cases, the diameter of $\mathcal{G}(J_r)$ is 5.*

*Proof.* Let $n = 3$. Then the diameter of $\mathcal{G}(J_2)$ is at most 4 by Lemma 2.1 and Theorem 2.8. On the other hand, consider $a = (3\,1\rangle(1\,2)$ and $b = (2\,1\rangle(1\,3)$ in $J_2$. Suppose $e$ and $f$ are idempotents in $J_2$ such that $a - e$ and $f - b$. By Lemma 2.12, $e = (\{1\},1\rangle(\{3,2\},2\rangle$ and $f = (\{1\},1\rangle(\{2,3\},3\rangle$. Then $e$ and $f$ do not commute, and so $d(e,f) \geq 2$. Thus $d(a,b) \geq 4$ by Lemma 2.1, and so the diameter of $\mathcal{G}(J_2)$ is at least 4.

Let $n \in \{5,6,7\}$ and $r = 4$. If $n = 5$, then the diameter of $\mathcal{G}(J_4)$ is at least 4 (by Lemma 2.14) and at most 4 (by Lemma 2.1 and Theorem 2.8). If $n \in \{6,7\}$, then the diameter of $\mathcal{G}(J_4)$ is at least 4 (by Lemma 2.14) and at most 4 (by Lemma 2.16). We have proved (1).

Let $n \geq 4$ and suppose that $n \notin \{5,6,7\}$ or $r \neq 4$. Then the diameter of $\mathcal{G}(J_r)$ is at most 5 by Lemma 2.1 and Theorem 2.8. It remains to find $a, b \in J_r$ such that the distance between $a$ and $b$ in $\mathcal{G}(J_r)$ is at least 5. We consider four possible cases.

**Case 1.** $r = 2m - 1$ for some $m \geq 2$.

Then $2 \leq m < r < 2m \leq n$. Let $x_1, \ldots, x_m, y_1, \ldots, y_m$ be pairwise distinct elements of $X$. Let

$$a = (* \, y_2\rangle(y_1 \, y_2 \ldots y_m \, x_1\rangle(x_1 \, x_2 \ldots x_m) \text{ and } b = (* \, x_3\rangle(x_2 \, x_3 \ldots x_{m-1} \, x_1 \, y_1\rangle(y_1 \, y_2 \ldots y_m)$$

(see Notation 2.11) and note that $a, b \in J_r$ and $ab \neq ba$. Then, by Lemma 2.1, there are idempotents $e_1, \ldots, e_k \in J_r$ ($k \geq 1$) such that $a - e_1 - \cdots - e_k - b$ is a minimal path in $\mathcal{G}(J_r)$ from $a$ to $b$. By Lemma 2.12,

$$e_1 = (A_1, x_1\rangle(A_2, x_2\rangle \ldots (A_m, x_m\rangle \text{ and } e_k = (B_1, y_1\rangle(B_2, y_2\rangle \ldots (B_m, y_m\rangle,$$

where $y_i \in A_i$ ($1 \leq i \leq m$), $x_{i+1} \in B_i$ ($1 \leq i < m$), and $x_1 \in B_m$. Let $g \in T(X)$ be an idempotent such that $e_1 - g - e_k$. By Lemma 2.6, $x_j g = x_j$ and $y_j g = y_j$ for every $j \in \{1, \ldots, m\}$. Hence $\text{rank}(g) \geq 2m > r$, and so $g \notin J_r$. It follows that the distance between $e_1$ and $e_k$ is at least 3, and so the distance between $a$ and $b$ is at least 5.

**Case 2.** $r = 2m$ for some $m \geq 3$.

Then $3 \leq m < r = 2m < n$. Let $x_1, \ldots, x_m, y_1, \ldots, y_m, z$ be pairwise distinct elements of $X$. Let

$$\begin{aligned} a &= (* \, y_2\rangle(z \, y_1 \, y_2 \ldots y_m \, x_1\rangle(x_1 \, x_2 \ldots x_m), \\ b &= (* \, x_1\rangle(z \, x_3\rangle(x_2 \, x_3 \ldots x_m \, x_1 \, y_1\rangle(y_1 \, y_2 \ldots y_m) \end{aligned}$$

(see Notation 2.11) and note that $a, b \in J_r$ and $ab \neq ba$. Then, by Lemma 2.1, there are idempotents $e_1, \ldots, e_k \in J_r$ ($k \geq 1$) such that $a - e_1 - \cdots - e_k - b$ is a minimal path in $\mathcal{G}(J_r)$ from $a$ to $b$. By Lemma 2.12,

$$e_1 = (A_1, x_1\rangle(A_2, x_2\rangle \ldots (A_m, x_m\rangle \text{ and } e_k = (B_1, y_1\rangle(B_2, y_2\rangle \ldots (B_m, y_m\rangle,$$

where $y_i \in A_i$ ($1 \leq i \leq m$), $x_{i+1} \in B_i$ ($1 \leq i < m$), $x_1 \in B_m$, $A_m = \{x_m, y_m, z\}$, and $B_1 = \{y_1, x_2, z\}$. Let $g \in T(X)$ be an idempotent such that $e_1 - g - e_k$. By Lemma 2.6, $x_j g = x_j$ and $y_j g = y_j$ for every $j \in \{1, \ldots, m\}$, and $zg = z$. Hence $\text{rank}(g) \geq 2m + 1 > r$, and so $g \notin J_r$. It follows that the distance between $e_1$ and $e_k$ is at least 3, and so the distance between $a$ and $b$ is at least 5.

**Case 3.** $r = 4$.

Since we are working under the assumption that $n \notin \{5, 6, 7\}$ or $r \neq 4$, we have $n \notin \{5, 6, 7\}$. Thus $n \geq 8$ (since $r \leq n - 1$). Let

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \ldots n \\ 2 & 3 & 4 & 1 & 2 & 3 & 4 & 1 & 1 \ldots 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \ldots n \\ 5 & 6 & 7 & 8 & 6 & 7 & 8 & 5 & 1 \ldots 1 \end{pmatrix}.$$

Note that $a, b \in J_4$, $ab \neq ba$, $(1\,2\,3\,4)$ is a unique cycle in $a$, and $(5\,6\,7\,8)$ is a unique cycle in $b$. By Lemma 2.1, there are idempotents $e_1, \ldots, e_k \in J_4$ ($k \geq 1$) such that $a - e_1 - \cdots - e_k - b$ is a minimal path in $\mathcal{G}(J_4)$ from $a$ to $b$. By Lemma 2.10, $ie_1 = i$ and $(4 + i)e_k = 4 + i$ for every $i \in \{1, 2, 3, 4\}$. By Lemma 2.9, $5e_1 = 1$ or $5e_1 = 5$. But the latter is impossible since with $5e_1 = 5$ we would have $\text{rank}(e_1) \geq 5$. Similarly, we obtain $6e_1 = 2$, $7e_1 = 3$, $8e_1 = 4$, $2e_k = 5$, $3e_k = 6$, $4e_k = 7$, and $1e_k = 8$. Let $g \in T(X)$ be an idempotent such that $e_1 - g - e_k$. By Lemma 2.6, $jg = j$ for every $j \in \{1, \ldots, 8\}$. Hence $\text{rank}(g) \geq 8 > r$, and so $g \notin J_4$. It follows that the distance between $e_1$ and $e_k$ is at least 3, and so the distance between $a$ and $b$ is at least 5.

**Case 4.** $r = 2$.

In this case we let

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \ldots n \\ 2 & 1 & 2 & 1 & 1 \ldots 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \ldots n \\ 3 & 4 & 4 & 3 & 3 \ldots 3 \end{pmatrix}.$$

Note that $a, b \in J_2$, $ab \neq ba$, $(1\,2)$ is a unique cycle in $a$, and $(3\,4)$ is a unique cycle in $b$. By Lemma 2.1, there are idempotents $e_1, \dots, e_k \in J_2$ $(k \geq 1)$ such that $a - e_1 - \cdots - e_k - b$ is a minimal path in $\mathcal{G}(J_2)$ from $a$ to $b$. By Lemma 2.10, $1e_1 = 1$, $2e_1 = 2$, $3e_k = 3$, and $4e_k = 4$. By Lemma 2.9, $3e_1 = 1$ or $3e_1 = 3$. But the latter is impossible since with $3e_1 = 3$ we would have $\operatorname{rank}(e_1) \geq 3$. Again By Lemma 2.9, $4e_1 = 2$ or $4e_1 = y$ for some $y \in \{4, 5, \dots, n\}$. But the latter is impossible since we would have $ye_1 = y$ and again $\operatorname{rank}(e_1)$ would be at least 3. Similarly, we obtain $2e_k = 3$, and $1e_k = 4$. Let $g \in T(X)$ be an idempotent such that $e_1 - g - e_k$. By Lemma 2.6, $jg = j$ for every $j \in \{1, \dots, 4\}$. Hence $\operatorname{rank}(g) \geq 4 > r$, and so $g \notin J_2$. It follows that the distance between $e_1$ and $e_k$ is at least 3, and so the distance between $a$ and $b$ is at least 5.

Thus the diameter of $\mathcal{G}(J_r)$ is at least 5, which concludes the proof of (2). □

## 2.3 The Commuting Graph of $T(X)$

Let $X$ be a finite set with $|X| = n$. It has been proved in [9, Theorem 3.1] that if $n$ and $n-1$ are not prime, then the diameter of the commuting graph of $\operatorname{Sym}(X)$ is at most 5, and that the bound is sharp since the diameter of $\mathcal{G}(\operatorname{Sym}(X))$ is 5 when $n = 9$. In this subsection, we determine the exact value of the diameter of the commuting graph of $T(X)$ for every $n \geq 2$.

Throughout this subsection, we assume that $X$ is a finite set with $n \geq 2$ elements.

**Lemma 2.18.** *Let $n \geq 4$ be composite. Let $a, f \in T(X)$ such that $a, f \neq \mathrm{id}_X$, $a \in \operatorname{Sym}(X)$, and $f$ is an idempotent. Then $d(a, f) \leq 4$.*

*Proof.* Fix $x \in \operatorname{im}(f)$ and a cycle $(x_1 \dots x_m)$ of $a$ such that $x \in \{x_1, \dots, x_m\}$. Consider three cases.

**Case 1.** $a$ has a cycle $(y_1 \dots y_k)$ such that $k$ does not divide $m$.

Then $a^m$ is different from $\mathrm{id}_X$ and it fixes $x$. Thus $a - a^m - (X, x\rangle - f$, and so $d(a, f) \leq 3$.

**Case 2.** $a$ has at least two cycles and for every cycle $(y_1 \dots y_k)$ of $a$, $k$ divides $m$.

Suppose there is $z \in \operatorname{im}(f)$ such that $z \in \{y_1, \dots, y_k\}$ for some cycle $(y_1 \dots y_k)$ of $a$ different from $(x_1 \dots x_m)$. Since $k$ divides $m$, there is a positive integer $t$ such that $m = tk$. Define $e \in T(X)$ by:

$$x_1 e = y_1, \dots, x_k e = y_k,\ x_{k+1} e = y_1, \dots, x_{2k} e = y_k, \dots, x_{(t-1)k+1} e = y_1, \dots, x_{tk} e = y_k, \qquad (2.4)$$

and $ye = y$ for all other $y \in X$. Then $e$ is an idempotent such that $ae = ea$ and $z \in \operatorname{im}(e)$. Thus, by Lemma 2.3, $a - e - (X, z\rangle - f$, and so $d(a, f) \leq 3$.

Suppose that $\operatorname{im}(f) \subseteq \{x_1, \dots, x_m\}$. Consider any cycle $(y_1 \dots y_k)$ of $a$ different from $(x_1 \dots x_m)$. Since $\operatorname{im}(f) \subseteq \{x_1, \dots, x_m\}$, $y_1 f = x_i$ for some $i$. We may assume that $y_1 f = x_1$. Define an idempotent $e$ exactly as in (2.4). Then $\operatorname{im}(e) \cap \operatorname{im}(f) = \emptyset$, $(y_1, x_1) \in \operatorname{im}(e) \times \operatorname{im}(f)$, and $(y_1, x_1) \in \ker(e) \cap \ker(f)$. Thus, by Lemma 2.4, there is an idempotent $g \in T(X) - \{\mathrm{id}_X\}$ such that $e - g - f$. Hence $a - e - g - f$, and so $d(a, f) \leq 3$.

**Case 3.** $a$ is an $n$-cycle.

Since $n$ is composite, there is a divisor $k$ of $n$ such that $1 < k < n$. Then $a^k \neq \mathrm{id}_X$ is a permutation with $k \geq 2$ cycles, each of length $m = n/k$. By Case 2, $d(a^k, f) \leq 3$, and so $d(a, f) \leq 4$. □

**Lemma 2.19.** *Let $n \geq 4$ be composite. Let $a, b \in T(X)$ such that $a, b \neq \mathrm{id}_X$ and $a \in \operatorname{Sym}(X)$. Then $d(a, b) \leq 5$.*

*Proof.* Suppose $b \notin \operatorname{Sym}(X)$. Then $b^k$ is an idempotent different from $\mathrm{id}_X$ for some $k \geq 1$. By Lemma 2.18, $d(a, b^k) \leq 4$, and so $d(a, b) \leq 5$.

Suppose $b \in \operatorname{Sym}(X)$. Suppose $n-1$ is not prime. Then, by [9, Theorem 3.1], there is a path from $a$ to $b$ in $\mathcal{G}(\operatorname{Sym}(X))$ of length at most 5. Such a path is also a path in $\mathcal{G}(T(X))$, and so

$d(a,b) \le 5$. Suppose $p = n-1$ is prime. Then the proof of [9, Theorem 3.1] still works for $a$ and $b$ unless $a^p = \mathrm{id}_X$ or $b^p = \mathrm{id}_X$. (See also [9, Lemma 3.3] and its proof.) Thus, if $a^p \ne \mathrm{id}_X$ and $b^p \ne \mathrm{id}_X$, then there is a path from $a$ to $b$ in $\mathcal{G}(\mathrm{Sym}(X))$ of length at most 5, and so $d(a,b) \le 5$. Suppose $a^p = \mathrm{id}_X$ or $b^p = \mathrm{id}_X$. We may assume that $b^p = \mathrm{id}_X$. Then $b$ is a cycle of length $p$, that is, $b = (x_1 \dots x_p)(x)$. Thus $b$ commutes with the constant idempotent $f = (X, x\rangle$. By Lemma 2.18, $d(a,f) \le 4$, and so $d(a,b) \le 5$. ☐

**Lemma 2.20.** *Let $X = \{x_1, \dots, x_m, y_1, \dots, y_k\}$, $a \in \mathrm{Sym}(X)$, and $b = (y_1 \dots y_k\, x_1\rangle(x_1 \dots x_m)$. If $ab = ba$ then $a = \mathrm{id}_X$.*

*Proof.* Suppose $ab = ba$. By Lemma 2.9,

$$x_1a \xrightarrow{b} x_2a \xrightarrow{b} \cdots \xrightarrow{b} x_ma \xrightarrow{b} x_1a \quad \text{and} \quad y_1a \xrightarrow{b} y_2a \xrightarrow{b} \cdots \xrightarrow{b} y_ka \xrightarrow{b} x_1a. \tag{2.5}$$

Since $(x_1\, x_2 \dots x_m)$ is a unique cycle in $b$, (2.5) implies that

$$x_1a = x_q,\ x_2a = x_{q+1}, \dots, x_ma = x_{q+m-1}, \tag{2.6}$$

where $q \in \{1, \dots, m\}$ ($x_{q+i} = x_{q+i-m}$ if $q+i > m$). Thus $x_1a = x_j$ for some $j$. Since $y_k \xrightarrow{b} x_1$ and $x_m \xrightarrow{b} x_1$, we have $y_ka \xrightarrow{b} x_1a = x_j$ and $x_ma \xrightarrow{b} x_1a = x_j$. Suppose $j \ge 2$. Then $x_jb^{-1} = \{x_{j-1}\}$, and so $y_ka = x_{j-1} = x_ma$. But this implies $y_k = x_m$ (since $a$ is injective), which is a contradiction. Hence $j = 1$, and so $x_1a = x_1$. But then $x_ia = x_i$ for all $i$ by (2.6).

Since $y_ka \xrightarrow{b} x_1a = x_1$, we have $y_ka = y_k$ since $x_1b^{-1} = \{y_k, x_m\}$. Let $i \in \{1, \dots, k-1\}$ and suppose $y_{i+1}a = y_{i+1}$. Then $y_ia = y_i$ since $y_ia \xrightarrow{b} y_{i+1}a = y_{i+1}$ and $y_{i+1}b^{-1} = \{y_{i+1}\}$. It follows that $y_ia = y_i$ for all $i \in \{1, \dots, k\}$. ☐

**Lemma 2.21.** *Let $m$ be a positive integer such that $2m \le n$, $\sigma$ be an $m$-cycle on $\{1, \dots, m\}$, $a \in \mathrm{Sym}(X)$, and*

$$e = (A_1, x_1\rangle(A_2, x_2\rangle \dots (A_m, x_m\rangle \text{ and } f = (B_1, y_1\rangle(B_2, y_2\rangle \dots (B_m, y_m\rangle$$

*be idempotents in $T(X)$ such that $x_1, \dots, x_m, y_1, \dots, y_m$ are pairwise distinct, $y_i \in A_i$, and $x_{i\sigma} \in B_i$ ($1 \le i \le m$). Then:*

1. *Suppose $X = \{x_1, \dots, x_m, y_1, \dots, y_m, z\}$ and $z \in A_i \cap B_j$ such that $A_i \cap B_j = \{z\}$. If $e - a - f$, then $a = \mathrm{id}_X$.*
2. *Suppose $X = \{x_1, \dots, x_m, y_1, \dots, y_m, z, w\}$, $z \in A_i \cap B_j$ such that $A_i \cap B_j = \{z\}$, and $w \in A_s \cap B_t$ such that $A_s \cap B_t = \{w\}$, where $s \ne i$ and $t \ne j$. If $e - a - f$, then $a = \mathrm{id}_X$.*

*Proof.* To prove (1), suppose $e - a - f$ and note that $A_i = \{x_i, y_i, z\}$ and $B_j = \{y_j, x_{j\sigma}, z\}$. By Lemma 2.2, there is $p \in \{1, \dots, m\}$ such that $x_ia = x_p$ and $A_ia \subseteq A_p$. Suppose $p \ne i$. Then $A_p = \{x_p, y_p\}$, and so $A_ia$ cannot be a subset of $A_p$ since $a$ is injective. It follows that $p = i$, that is, $x_ia = x_i$ and $A_ia \subseteq A_i$. Similarly, $y_ja = y_j$ and $B_ja \subseteq B_j$. Thus $za \in A_i \cap B_j = \{z\}$, and so $za = z$. Hence, since $a$ is injective, $y_ia = y_i$.

We have proved that $x_ia = x_i$, $y_ia = y_i$, and $za = z$. We have $B_i = \{y_i, x_{i\sigma}\}$ or $B_i = \{y_i, x_{i\sigma}, z\}$ Since $y_ia = y_i$, we have $B_ia \subseteq B_i$ by Lemma 2.2. Since $za = z$ and $a$ is injective, it follows that $x_{i\sigma}a = x_{i\sigma}$. By the foregoing argument applied to $A_{i\sigma} = \{x_{i\sigma}, y_{i\sigma}\}$, we obtain $y_{i\sigma}a = y_{i\sigma}$. Continuing this way, we obtain $x_{i\sigma^k}a = x_{i\sigma^k}$ and $y_{i\sigma^k}a = y_{i\sigma^k}$ for every $k \in \{1, \dots, m-1\}$. Since $\sigma$ is an $m$-cycle, it follows that $x_ja = x_j$ and $y_jg = y_j$ for every $j \in \{1, \dots, m\}$. Hence $a = \mathrm{id}_X$. We have proved (1). The proof of (2) is similar. ☐

**Theorem 2.22.** *Let $X$ be a finite set with $n \ge 2$ elements. Then:*

1. *If $n$ is prime, then $\mathcal{G}(T(X))$ is not connected.*

(2) *If* $n = 4$, *then the diameter of* $\mathcal{G}(T(X))$ *is* 4.

(3) *If* $n \geq 6$ *is composite, then the diameter of* $\mathcal{G}(T(X))$ *is* 5.

*Proof.* Suppose $n = p$ is prime. Consider a $p$-cycle $a = (x_1\, x_2 \dots x_p)$ and let $b \in T(X)$ be such that $b \neq \mathrm{id}_X$ and $ab = ba$. Let $x_q = x_1 b$. Then, by Lemma 2.9, $x_i b = x_{q+i}$ for every $i \in \{1, \dots, p\}$ (where $x_{q+i} = x_{q+i-m}$ if $q + i > m$). Thus $b = a^q$, and so, since $p$ is prime, $b$ is also a $p$-cycle. It follows that if $c$ is a vertex of $\mathcal{G}(T(X))$ that is not a $p$-cycle, then there is no path in $\mathcal{G}(T(X))$ from $a$ to $c$. Hence $\mathcal{G}(T(X))$ is not connected. We have proved (1).

We checked the case $n = 4$ directly using GRAPE [16] through GAP [8]. We found that, when $|X| = 4$, the diameter of $\mathcal{G}(T(X))$ is 4.

Suppose $n \geq 6$ is composite. Let $a, b \in T(X)$ such that $a, b \neq \mathrm{id}_X$. If $a \in \mathrm{Sym}(X)$ or $b \in \mathrm{Sym}(X)$, then $d(a, b) \leq 5$ by Lemma 2.19. If $a, b \notin \mathrm{Sym}(X)$, then $a, b \in J_{n-1}$, and so $d(a, b) \leq 5$ by Theorem 2.17. Hence the diameter of $\mathcal{G}(T(X))$ is at most 5. It remains to find $a, b \in T(X) - \{\mathrm{id}_X\}$ such that $d(a, b) \geq 5$.

For $n \in \{6, 8\}$, we employed GAP [8]. When $n = 6$, we found that the distance between the 6-cycle $a = (1\,2\,3\,4\,5\,6)$ and $b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 5 & 1 & 2 & 4 \end{pmatrix}$ in $\mathcal{G}(T(X))$ is at least 5. And when $n = 8$, the distance between the 8-cycle $a = (1\,2\,3\,4\,5\,6\,7\,8)$ and $b = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 1 & 1 & 4 & 8 & 6 & 5 \end{pmatrix}$ in $\mathcal{G}(T(X))$ is at least 5.

To verify this with GAP, we used the following sequence of arguments and computer calculations:

1. By Lemma 2.1, if there exists a path $a - c_1 - c_2 - \dots - c_k - b$, then there exists a path $a - e_1 - e_2 - \dots - e_k - b$, where each $e_i$ is either an idempotent or a permutation;
2. Let $E$ be the set idempotents of $T(X) - \{\mathrm{id}_X\}$ and let $G = \mathrm{Sym}(X) - \{\mathrm{id}_X\}$. For $A \subseteq T(X)$, let $C(A) = \{f \in E \cup G : (\exists_{a \in A}) af = fa\}$;
3. Calculate $C(C(\{a\}))$ and $C(\{b\})$;
4. Verify that for all $c \in C(C(\{a\}))$ and all $d \in C(\{b\})$, $cd \neq dc$;
5. If there were a path $a - c_1 - c_2 - c_3 - b$ from $a$ to $b$, then we would have $c_2 \in C(C(\{a\}))$, $c_3 \in C(\{b\})$, and $c_2 c_3 = c_3 c_2$. But, by 4., there are no such $c_2$ and $c_3$, and it follows that the distance between $a$ and $b$ is at least 5.

Let $n \geq 9$ be composite. We consider two cases.

**Case 1.** $n = 2m + 1$ is odd ($m \geq 4$).

Let $X = \{x_1, \dots, x_m, y_1, \dots, y_m, z\}$. Consider

$$a = (z\, y_1\, y_2 \dots\, y_m\, x_1\rangle(x_1\, x_2 \dots\, x_m) \text{ and } b = (x_2\, x_3 \dots\, x_m\, x_1\, z\, y_2\rangle(y_1\, y_2 \dots\, y_m).$$

Let $\lambda$ be a minimal path in $\mathcal{G}(T(X))$ from $a$ to $b$. By Lemma 2.20, there is no $g \in \mathrm{Sym}(X)$ such that $g \neq \mathrm{id}_X$ and $ag = ga$ or $bg = gb$. Thus, by the proof of Lemma 2.1, $\lambda = a - e_1 - \cdots - e_k - b$, where $e_1$ and $e_k$ are idempotents. By Lemma 2.12,

$$e_1 = (A_1, x_1\rangle(A_2, x_2\rangle \dots (A_m, x_m\rangle \text{ and } e_k = (B_1, y_1\rangle(B_2, y_2\rangle \dots (B_m, y_m\rangle,$$

where $y_i \in A_i$ ($1 \leq i \leq m$), $x_{i+1} \in B_i$ ($1 \leq i < m$), $x_1 \in B_m$, $A_m = \{x_m, y_m, z\}$, and $B_1 = \{y_1, x_2, z\}$. Since $m \geq 4$, $A_m \cap B_1 = \{z\}$. Thus, by Lemma 2.21, there is no $g \in \mathrm{Sym}(X)$ such that $g \neq \mathrm{id}_X$ and $e_1 - g - e_2$. Hence, if $\lambda$ contains an element $g \in \mathrm{Sym}(X)$, then the length of $\lambda$ is at least 5. Suppose $\lambda$ does not contain any permutations. Then $\lambda$ is a path in $J_{n-1}$ and we may assume that all vertices in $\lambda$ except $a$ and $b$ are idempotents (by Lemma 2.12). By

Lemma 2.6, there is no idempotent $f \in J_{n-1}$ such that $e_1 - f - e_k$. (Here, the $m$-cycle that occurs in Lemmas 2.6 and 2.21 is $\sigma = (1\,2\ldots m)$.) Hence the length of $\lambda$ is at least 5.

**Case 2.** $n = 2m+2$ is even ($m \geq 4$).

Let $X = \{x_1, \ldots, x_m, y_1, \ldots, y_m, z, w\}$. Consider

$$a = (z\,y_1\,y_2 \ldots\, y_m\,w\,x_2\rangle(x_1\,x_2 \ldots\, x_m) \text{ and } b = (w\,x_2\,x_3 \ldots\, x_{m-2}\,x_m\,x_1\,x_{m-1}\,y_2\rangle(y_1\,y_2 \ldots\, y_m).$$

Let $\lambda$ be a minimal path in $\mathcal{G}(T(X))$ from $a$ to $b$. By Lemma 2.20, there is no $g \in \mathrm{Sym}(X)$ such that $g \neq \mathrm{id}_X$ and $ag = ga$ or $bg = gb$. Thus, by the proof of Lemma 2.1, $\lambda = a - e_1 - \cdots - e_k - b$, where $e_1$ and $e_k$ are idempotents. By Lemma 2.12,

$$e_1 = (A_1, x_1\rangle(A_2, x_2\rangle \ldots (A_m, x_m\rangle \text{ and } e_k = (B_1, y_1\rangle(B_2, y_2\rangle \ldots (B_m, y_m\rangle,$$

where $y_i \in A_i$ ($1 \leq i \leq m$), $x_{i+1} \in B_i$ ($1 \leq i \leq m-3$), $x_m \in B_{m-2}$, $x_1 \in B_{m-1}$, $x_{m-1} \in B_m$, $A_1 = \{x_1, y_1, w\}$, $A_m = \{x_m, y_m, z\}$, $B_1 = \{y_1, x_2, z\}$, and $B_m = \{y_m, x_{m-1}, w\}$. Since $m \geq 4$, $A_m \cap B_1 = \{z\}$ and $A_1 \cap B_m = \{w\}$. Thus, by Lemma 2.21, there is no $g \in \mathrm{Sym}(X)$ such that $g \neq \mathrm{id}_X$ and $e_1 - g - e_2$. Hence, as in Case 1, the length of $\lambda$ is at least 5. (Here, the $m$-cycle that occurs in Lemmas 2.6 and 2.21 is $\sigma = (1, 2\ldots, m-3, m-2, m, m-1)$.)

Hence, if $n \geq 6$ is composite, then the diameter of $\mathcal{G}(T(X))$ is 5. This concludes the proof. □

# 3 Minimal Left Paths

In this section, we prove that for every integer $n \geq 4$, there is a band $S$ with knit degree $n$. We will show how to construct such an $S$ as a subsemigroup of $T(X)$ for some finite set $X$.

Let $S$ be a finite non-commutative semigroup. Recall that a path $a_1 - a_2 - \cdots - a_m$ in $\mathcal{G}(S)$ is called a *left path* (or $l$-path) if $a_1 \neq a_m$ and $a_1 a_i = a_m a_i$ for every $i \in \{1, \ldots, m\}$. If there is any $l$-path in $\mathcal{G}(S)$, we define the *knit degree* of $S$, denoted $\mathrm{kd}(S)$, to be the length of a shortest $l$-path in $\mathcal{G}(S)$. We say that an $l$-path $\lambda$ from $a$ to $b$ in $\mathcal{G}(S)$ is a *minimal $l$-path* if there is no $l$-path from $a$ to $b$ that is shorter than $\lambda$.

## 3.1 The Even Case

In this subsection, we will construct a band of knit degree $n$ where $n \geq 4$ is even. The following lemma is obvious.

**Lemma 3.1.** *Let $c_x, c_y, e \in T(X)$ such that $e$ is an idempotent. Then:*

(1) *$c_x e = e c_x$ if and only if $x \in \mathrm{im}(e)$.*

(2) *$c_x e = c_y e$ if and only if $(x, y) \in \ker(e)$.*

Now, given an even $n \geq 4$, we will construct a band $S$ such that $\mathrm{kd}(S) = n$. We will explain the construction using $n = 8$ as an example. The band $S$ will be a subsemigroups of $T(X)$, where

$$X = \{y_0, y_1, y_2, y_3, y_4 = v_0, v_1, v_2, v_3, v_4, x_1, x_2, x_3, x_4, u_1, u_2, u_3, u_4, r, s\},$$

and it will be generated by idempotent transformations $a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, e_1$, whose images of the generators are defined by Table 1.

We will define the kernels in such a way that the generators with the same index will have the same kernel. For example, $\ker(a_1) = \ker(b_1) = \ker(e_1)$ and $\ker(a_2) = \ker(b_2)$. Let $i \in \{2, 3, 4\}$. The kernel of $a_i$ will have the following three classes:

$$\begin{aligned} &\text{Class-1} = \mathrm{im}(a_{i+1}) \cup \ldots \cup \mathrm{im}(a_4) \cup \mathrm{im}(b_1) \cup \ldots \cup \mathrm{im}(b_{i-1}),\\ &\text{Class-2} = \mathrm{im}(b_{i+1}) \cup \ldots \cup \mathrm{im}(b_4) \cup \mathrm{im}(e_1) \cup \mathrm{im}(a_1) \cup \ldots \cup \mathrm{im}(a_{i-1}),\\ &\text{Class-3} = \{x_i, u_i\}. \end{aligned}$$

| | | | |
|---|---|---|---|
| $\text{im}(a_1)$ | $y_0$ | $x_1$ | $y_1$ |
| $\text{im}(a_2)$ | $y_1$ | $x_2$ | $y_2$ |
| $\text{im}(a_3)$ | $y_2$ | $x_3$ | $y_3$ |
| $\text{im}(a_4)$ | $y_3$ | $x_4$ | $y_4$ |
| $\text{im}(b_1)$ | $y_4$ | $u_1$ | $v_1$ |
| $\text{im}(b_2)$ | $v_1$ | $u_2$ | $v_2$ |
| $\text{im}(b_3)$ | $v_2$ | $u_3$ | $v_3$ |
| $\text{im}(b_4)$ | $v_3$ | $u_4$ | $v_4$ |
| $\text{im}(e_1)$ | $v_4$ | $r$ | $s$ |

Table 1: Images of the generators.

For example, $\ker(a_2)$ has the following classes:

$$\begin{aligned}
\text{Class-1} &= \{y_2, x_3, y_3, x_4, y_4, u_1, v_1\},\\
\text{Class-2} &= \{v_2, u_3, v_3, u_4, v_4, r, s, y_0, x_1, y_1\},\\
\text{Class-3} &= \{x_2, u_2\}.
\end{aligned}$$

We define the kernel of $a_1$ as follows:

$$\begin{aligned}
\text{Class-1} &= \text{im}(a_2) \cup \text{im}(a_3) \cup \text{im}(a_4) \cup \{s\} = \{y_1, x_2, y_2, x_3, y_3, x_4, y_4, s\},\\
\text{Class-2} &= \text{im}(b_2) \cup \text{im}(b_3) \cup \text{im}(b_4) \cup \{y_0\} = \{v_1, u_2, v_2, u_3, v_3, u_4, v_4, y_0\},\\
\text{Class-3} &= \{x_1, u_1, r\}.
\end{aligned}$$

Now the generators are completely defined since $\ker(b_i) = \ker(a_i)$, $1 \le i \le 4$, and $\ker(e_1) = \ker(a_1)$. Order the generators as follows:

$$a_1,\ a_2,\ a_3,\ a_4,\ b_1,\ b_2,\ b_3,\ b_4,\ e_1. \tag{3.1}$$

Let $S$ be the semigroup generated by the idempotents listed in (3.1). Since the idempotents with the same index have the same kernel, they form a right-zero subsemigroup of $S$. For example, $\{a_1, b_1, e_1\}$ is a right-zero semigroup: $a_1a_1 = b_1a_1 = e_1a_1 = a_1$, $a_1b_1 = b_1b_1 = e_1b_1 = b_1$, and $a_1e_1 = b_1e_1 = e_1e_1 = e_1$. The product of any two generators with different indices is a constant transformation. For example, $a_2a_4 = c_{y_3}$, $a_4a_2 = c_{y_2}$, and $a_1b_3 = c_{v_3}$. The semigroup $S$ consists of the nine generators listed in (3.1) and 10 constants:

$$S = \{a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, e_1, c_{y_0}, c_{y_1}, c_{y_2}, c_{y_3}, c_{y_4}, c_{v_1}, c_{v_2}, c_{v_3}, c_{v_4}, c_s\},$$

so $S$ is a band. Note that $Z(S) = \emptyset$. Each idempotent in (3.1) commutes with the next idempotent, so $a_1 - a_2 - a_3 - a_4 - b_1 - b_2 - b_3 - b_4 - e_1$ is a path in $\mathcal{G}(S)$. Moreover, it is a unique $l$-path in $\mathcal{G}(S)$, so $\text{kd}(S) = 8$.

We will now provide a general construction of a band $S$ such that $\text{kd}(S) = n$, where $n$ is even.

**Definition 3.2.** Let $k \ge 2$ be an integer. Let

$$X = \{y_0, y_1, \ldots, y_k = v_0, v_1, \ldots, v_k, x_1, \ldots, x_k, u_1, \ldots, u_k, r, s\}.$$

We will define idempotents $a_1, \ldots, a_k, b_1, \ldots, b_k, e_1$ as follows. For $i \in \{1, \ldots, k\}$, let

$$\begin{aligned}
\text{im}(a_i) &= \{y_{i-1}, x_i, y_i\},\\
\text{im}(b_i) &= \{v_{i-1}, u_i, v_i\},\\
\text{im}(e_1) &= \{v_k, r, s\}.
\end{aligned}$$

For $i \in \{2, \dots, k\}$, define the $\ker(a_i)$-classes by:

$$\begin{aligned}
\text{Class-1} &= \operatorname{im}(a_{i+1}) \cup \dots \cup \operatorname{im}(a_k) \cup \operatorname{im}(b_1) \cup \dots \cup \operatorname{im}(b_{i-1}),\\
\text{Class-2} &= \operatorname{im}(b_{i+1}) \cup \dots \cup \operatorname{im}(b_k) \cup \operatorname{im}(e_1) \cup \operatorname{im}(a_1) \cup \dots \cup \operatorname{im}(a_{i-1}),\\
\text{Class-3} &= \{x_i, u_i\}.
\end{aligned}$$

(Note that for $i = k$, Class-1 $= \operatorname{im}(b_1) \cup \dots \cup \operatorname{im}(b_{k-1})$ and Class-2 $= \operatorname{im}(e_1) \cup \operatorname{im}(a_1) \cup \dots \cup \operatorname{im}(a_{i-1})$.)

Define the $\ker(a_1)$-classes by:

$$\begin{aligned}
\text{Class-1} &= \operatorname{im}(a_2) \cup \dots \cup \operatorname{im}(a_k) \cup \{s\},\\
\text{Class-2} &= \operatorname{im}(b_2) \cup \dots \cup \operatorname{im}(b_k) \cup \{y_0\},\\
\text{Class-3} &= \{x_1, u_1, r\}.
\end{aligned}$$

Let $\ker(b_i) = \ker(a_i)$ for every $i \in \{1, \dots, k\}$, and $\ker(e_1) = \ker(a_1)$. Now, define the subsemigroup $S_0^k$ of $T(X)$ by:

$$S_0^k = \text{the semigroup generated by } \{a_1, \dots, a_k, b_1, \dots, b_k, e_1\}. \tag{3.2}$$

We must argue that the idempotents $a_1, \dots, a_k, b_1, \dots, b_k, e_1$ are well defined, that is, for each of them, different elements of the image lie in different kernel classes. Consider $a_i$, where $i \in \{2, \dots, k\}$. Then $\operatorname{im}(a_i) = \{y_{i-1}, x_i, y_i\}$. Then $y_i$ lies in Class-1 (see Definition 3.2) since $y_i \in \operatorname{im}(a_{i+1})$ (or $y_i \in \operatorname{im}(b_1)$ if $i = k$), $y_{i-1}$ lies in Class-2 since $y_{i-1} \in \operatorname{im}(a_{i-1})$, and $x_i$ lies in Class-3. Arguments for the remaining idempotents are similar.

For the remainder of this subsection, $S_0^k$ will be the semigroup (3.2). Our objective is to prove that $S_0^k$ is a band such that $\pi = a_1 - \dots - a_k - b_1 - \dots - b_k - e_1$ is a shortest $l$-path in $S_0^k$. Since $\pi$ has length $2k = n$, it will follow that $S_0^k$ is a band with knit degree $n$.

We first analyze products of the generators of $S_0^k$.

**Lemma 3.3.** *Let $1 \le i < j \le k$. Then:*

(1) $a_i b_i = b_i$, $b_i a_i = a_i$, $a_1 e_1 = b_1 e_1 = e_1$, $e_1 a_1 = b_1 a_1 = a_1$, *and* $e_1 b_1 = a_1 b_1 = b_1$.

(2) $a_i a_j = c_{y_{j-1}}$ *and* $a_j a_i = c_{y_i}$.

(3) $a_i b_j = c_{v_j}$ *and* $a_j b_i = c_{v_{i-1}}$.

(4) $b_i a_j = c_{y_j}$ *and* $b_j a_i = c_{y_{i-1}}$.

(5) $b_i b_j = c_{v_{j-1}}$ *and* $b_j b_i = c_{v_i}$.

(6) $e_1 a_j = c_{y_{j-1}}$ *and* $a_j e_1 = c_s$.

(7) $e_1 b_j = c_{v_j}$ *and* $b_j e_1 = c_{v_k}$.

*Proof.* Statement (1) is true because the generators of $S_0^k$ are idempotents and the ones with the same index have the same kernel. By Definition 3.2, Class-2 of $\ker(a_j)$ contains both $\operatorname{im}(a_{j-1}) = \{y_{j-2}, x_{j-1}, y_{j-1}\}$ and $\operatorname{im}(a_i)$ (since $i < j$). Since $y_{j-1} \in \operatorname{im}(a_j) = \{y_{j-1}, x_j, y_j\}$, $a_j$ maps all elements of Class-2 to $y_{j-1}$. Hence $a_i a_j = c_{y_{j-1}}$. Similarly, since $i < j$, Class-1 of $\ker(a_i)$ contains both $\operatorname{im}(a_{i+1}) = \{y_i, x_{i+1}, y_{i+1}\}$ and $\operatorname{im}(a_j)$. Since $y_i \in \operatorname{im}(a_i) = \{y_{i-1}, x_i, y_i\}$, $a_i$ maps all elements of Class-1 to $y_i$. Hence $a_j a_i = c_{y_i}$. We have proved (2). Proofs of (3)-(7) are similar. For example, $b_j e_1 = c_{v_k}$ because Class-2 of $\ker(e_1) = \ker(a_1)$ contains both $\operatorname{im}(b_j)$ and $\operatorname{im}(b_k) = \{v_{k-1}, u_k, v_k\}$, and $v_k \in \operatorname{im}(e_1)$. $\square$

The following corollaries are immediate consequences of Lemma 3.3.

**Corollary 3.4.** *The semigroup $S_0^k$ is a band. It consists of $2k+1$ generators from Definition 3.2 and $2k+2$ constant transformations:*

$$S = \{a_1, \dots, a_k, b_1, \dots, b_k, e_1, c_{y_0}, c_{y_1}, \dots, c_{y_k}, c_{v_1}, \dots, c_{v_k}, c_s\}.$$

**Corollary 3.5.** *Let $g, h \in S_0^k$ be generators from the list*

$$a_1, \dots, a_k, b_1, \dots, b_k, e_1. \tag{3.3}$$

*Then $gh = hg$ if and only if $g$ and $h$ are consecutive elements in the list.*

Lemma 3.3 gives a partial multiplication table for $S_0^k$. The following lemma completes the table.

**Lemma 3.6.** *Let $1 \le p \le k$ and $1 \le i < j \le k$. Then:*

(1) $c_{y_p}a_p = c_{y_p}$, $c_{y_p}b_p = c_{v_{p-1}}$, $c_{y_i}a_j = c_{y_{j-1}}$, $c_{y_j}a_i = c_{y_i}$, $c_{y_i}b_j = c_{v_j}$, $c_{y_j}b_i = c_{v_{i-1}}$, $c_{y_p}e_1 = c_s$, $c_{y_0}a_p = c_{y_{p-1}}$, $c_{y_0}b_p = c_{v_p}$, *and* $c_{y_0}e_1 = c_{v_k}$.

(2) $c_{v_p}a_p = c_{y_{p-1}}$, $c_{v_p}b_p = c_{v_p}$, $c_{v_i}a_j = c_{y_j}$, $c_{v_j}a_i = c_{y_{i-1}}$, $c_{v_i}b_j = c_{v_{j-1}}$, $c_{v_j}b_i = c_{v_i}$, *and* $c_{v_p}e_1 = c_{v_k}$.

(3) $c_s a_j = c_{y_{j-1}}$, $c_s b_j = c_{v_j}$, $c_s a_1 = c_{y_1}$, $c_s b_1 = c_{v_0}$, *and* $c_s e_1 = c_s$.

*Proof.* We have $c_{y_p}a_p = c_{y_p}$ since $y_p \in \mathrm{im}(a_p)$. By Definition 3.2, Class-1 of $\ker(b_p)$ contains both $\mathrm{im}(a_{p+1})$ and $\mathrm{im}(b_{p-1})$. Since $y_p \in \mathrm{im}(a_{p+1})$ and $v_{p-1} \in \mathrm{im}(b_{p-1})$, both $y_p$ and $v_{p-1}$ are in Class-1. Hence $y_p b_p = v_{p-1} b_p = v_{p-1}$, where the last equality is true because $v_{p-1} \in \mathrm{im}(b_p)$. Thus $c_{y_p}b_p = c_{v_{p-1}}$. By Definition 3.2, $y_p$ and $s$ belong to Class-1 of $\ker(e_1)$, and $s \in \mathrm{im}(e_1)$. It follows that $c_{y_p}e_1 = c_s$. Again by Definition 3.2, $y_0$ and $y_{p-1}$ belong to Class-2 of $\ker(a_p)$, and $y_{p-1} \in \mathrm{im}(a_p)$. Hence $c_{y_0}a_p = c_{y_{p-1}}$. Similarly, $c_{y_0}b_p = c_{v_p}$ and $c_{y_0}e_1 = c_{v_k}$. By Lemma 3.3,

$$\begin{aligned}
c_{y_i}a_j &= (c_{y_i}a_i)a_j = c_{y_i}(a_i a_j) = c_{y_i}c_{y_{j-1}} = c_{y_{j-1}},\\
c_{y_j}a_i &= (c_{y_j}a_j)a_i = c_{y_j}(a_j a_i) = c_{y_j}c_{y_i} = c_{y_i},\\
c_{y_i}b_j &= (c_{y_i}a_i)b_j = c_{y_i}(a_i b_j) = c_{y_i}c_{v_j} = c_{v_j},\\
c_{y_j}b_i &= (c_{y_j}a_j)b_i = c_{y_j}(a_j b_i) = c_{y_j}c_{v_{i-1}} = c_{v_{i-1}}.
\end{aligned}$$

We have proved (1). Proofs of (2) and (3) are similar. □

Table 2 presents the Cayley table for $S_0^2$.

| | $a_1$ | $a_2$ | $b_1$ | $b_2$ | $e_1$ | $c_{y_0}$ | $c_{y_1}$ | $c_{y_2}$ | $c_{v_1}$ | $c_{v_2}$ | $c_s$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $a_1$ | $a_1$ | $c_{y_1}$ | $b_1$ | $c_{v_2}$ | $e_1$ | $c_{y_0}$ | $c_{y_1}$ | $c_{y_2}$ | $c_{v_1}$ | $c_{v_2}$ | $c_s$ |
| $a_2$ | $c_{y_1}$ | $a_2$ | $c_{y_2}$ | $b_2$ | $c_s$ | $c_{y_0}$ | $c_{y_1}$ | $c_{y_2}$ | $c_{v_1}$ | $c_{v_2}$ | $c_s$ |
| $b_1$ | $a_1$ | $c_{y_2}$ | $b_1$ | $c_{v_1}$ | $e_1$ | $c_{y_0}$ | $c_{y_1}$ | $c_{y_2}$ | $c_{v_1}$ | $c_{v_2}$ | $c_s$ |
| $b_2$ | $c_{y_0}$ | $a_2$ | $c_{v_1}$ | $b_2$ | $c_{v_2}$ | $c_{y_0}$ | $c_{y_1}$ | $c_{y_2}$ | $c_{v_1}$ | $c_{v_2}$ | $c_s$ |
| $e_1$ | $a_1$ | $c_{y_1}$ | $b_1$ | $c_{v_2}$ | $e_1$ | $c_{y_0}$ | $c_{y_1}$ | $c_{y_2}$ | $c_{v_1}$ | $c_{v_2}$ | $c_s$ |
| $c_{y_0}$ | $c_{y_0}$ | $c_{y_1}$ | $c_{v_1}$ | $c_{v_2}$ | $c_{v_2}$ | $c_{y_0}$ | $c_{y_1}$ | $c_{y_2}$ | $c_{v_1}$ | $c_{v_2}$ | $c_s$ |
| $c_{y_1}$ | $c_{y_1}$ | $c_{y_1}$ | $c_{y_2}$ | $c_{v_2}$ | $c_s$ | $c_{y_0}$ | $c_{y_1}$ | $c_{y_2}$ | $c_{v_1}$ | $c_{v_2}$ | $c_s$ |
| $c_{y_2}$ | $c_{y_1}$ | $c_{y_2}$ | $c_{y_2}$ | $c_{v_1}$ | $c_s$ | $c_{y_0}$ | $c_{y_1}$ | $c_{y_2}$ | $c_{v_1}$ | $c_{v_2}$ | $c_s$ |
| $c_{v_1}$ | $c_{y_0}$ | $c_{y_2}$ | $c_{v_1}$ | $c_{v_1}$ | $c_{v_2}$ | $c_{y_0}$ | $c_{y_1}$ | $c_{y_2}$ | $c_{v_1}$ | $c_{v_2}$ | $c_s$ |
| $c_{v_2}$ | $c_{y_0}$ | $c_{y_1}$ | $c_{v_1}$ | $c_{v_2}$ | $c_{v_2}$ | $c_{y_0}$ | $c_{y_1}$ | $c_{y_2}$ | $c_{v_1}$ | $c_{v_2}$ | $c_s$ |
| $c_s$ | $c_{y_1}$ | $c_{y_1}$ | $c_{y_2}$ | $c_{v_2}$ | $c_s$ | $c_{y_0}$ | $c_{y_1}$ | $c_{y_2}$ | $c_{v_1}$ | $c_{v_2}$ | $c_s$ |

Table 2: Cayley table for $S_0^2$.

**Lemma 3.7.** *Let $g, h, c_z \in S_0^k$ such that $c_z$ is a constant and $g - c_z - h$ is a path in $\mathcal{G}(S_0^k)$. Then $gh = hg$.*

*Proof.* Note that $g, h$ are not constants since different constants do not commute. Thus $g$ and $h$ are generators from list (3.3). We may assume that $g$ is to the left of $h$ in the list. Since $c_z$ commutes with both $g$ and $h$, $z \in \mathrm{im}(g) \cap \mathrm{im}(h)$ by Lemma 3.1. Suppose $g = a_i$, where $1 \le i \le k-1$. Then $h = a_{i+1}$ since $a_{i+1}$ is the only generator to the right of $a_i$ whose image is not disjoint from $\mathrm{im}(a_i)$. Similarly, if $g = a_k$ then $h = b_1$; if $g = b_i$ $(1 \le i \le k-1)$ then $h = b_{i+1}$; and if $g = b_k$ then $h = e_1$. Hence $gh = hg$ by Corollary 3.5. □

**Lemma 3.8.** *The paths*

(i) $\tau_1 = c_{y_0} - a_1 - \cdots - a_k - b_1 - \cdots - b_k - c_{v_k}$,

(ii) $\tau_2 = c_{y_1} - a_2 - \cdots - a_k - b_1 - \cdots - b_k - e_1 - c_s$

*are the only minimal $l$-paths in $\mathcal{G}(S_0^k)$ with constants as the endpoints.*

*Proof.* We have that $\tau_1$ and $\tau_2$ are $l$-paths by Lemmas 3.3 and 3.6. Suppose that $\lambda = c_z - \cdots - c_w$ is a minimal $l$-path in $\mathcal{G}(S_0^k)$ with constants $c_z$ and $c_w$ as the endpoints. Recall that $z, w \in \{y_0, y_1, \dots, y_k, v_1, \dots, v_k, s\}$. We may assume that $z$ is to the left of $w$ in the list $y_0, y_1, \dots, y_k, v_1, \dots, v_k, s$. Since $\lambda$ is minimal, Lemma 3.7 implies that $\lambda$ does not contain any constants except $c_z$ and $c_w$. There are five cases to consider.

(a) $\lambda = c_{y_i} - \cdots - c_{y_j}$, where $0 \le i < j \le k$.

(b) $\lambda = c_{y_i} - \cdots - c_{v_j}$, where $0 \le i \le k$, $1 \le j \le k$.

(c) $\lambda = c_{y_i} - \cdots - c_s$, where $0 \le i \le k$.

(d) $\lambda = c_{v_i} - \cdots - c_{v_j}$, where $1 \le i < j \le k$.

(e) $\lambda = c_{v_i} - \cdots - c_s$, where $1 \le i \le k$.

Suppose (a) holds, that is, $\lambda = c_{y_i} - \cdots - h - c_{y_j}$, $0 \le i < j \le k$. Since $hc_{y_j} = c_{y_j}h$, either $h = a_j$ or $h = a_{j+1}$ (where $a_{k+1} = b_1$) (since $a_j$ and $a_{j+1}$ are the only generators that have $y_j$ in their image). Suppose $h = a_{j+1}$. Then, by Corollary 3.5, either $\lambda = c_{y_i} - \cdots - a_j - a_{j+1} - c_{y_j}$ or $\lambda = c_{y_i} - \cdots - a_{j+2} - a_{j+1} - c_{y_j}$ (where $a_{j+2} = b_1$ if $j = k-1$, and $a_{j+2} = b_2$ if $j = k$). In the latter case,

$$\lambda = c_{y_i} - \cdots - a_1 - e_1 - b_k - \cdots - b_1 - a_k - \cdots - a_{j+2} - a_{j+1} - c_{y_j},$$

which is a contradiction since $a_1$ and $e_1$ do not commute. Thus either $\lambda = c_{y_i} - \cdots - a_j - c_{y_j}$ or $\lambda = c_{y_i} - \cdots - a_j - a_{j+1} - c_{y_j}$. In either case, $\lambda$ contains $a_j$, and so $c_{y_i}a_j = c_{y_j}a_j$ (since $\lambda$ is an $l$-path). But, by Lemma 3.6, $c_{y_i}a_j = c_{y_{j-1}}$ and $c_{y_j}a_j = c_{y_j}$. Hence $c_{y_{j-1}} = c_{y_j}$, which is a contradiction.

Suppose (b) holds, that is, $\lambda = c_{y_i} - g - \cdots - h - c_{v_j}$, $0 \le i \le k$ and $1 \le j \le k$. Then $g$ is either $a_i$ or $a_{i+1}$ ($g = a_{i+1}$ if $i = 0$) and $h$ is either $b_j$ or $b_{j+1}$ (where $b_{k+1} = e_1$). In any case, $\lambda = c_{y_i} - g - \cdots - a_k - b_1 - \cdots - h - c_{v_j}$. Suppose $i \ge 1$. Then, by Lemma 3.6 and the fact that $\lambda$ is an $l$-path, $c_{v_0} = c_{y_i}b_1 = c_{v_j}b_1 = c_{v_1}$, which is a contradiction. If $i = 0$ and $j < k$, then $c_{y_{k-1}} = c_{y_0}a_k = c_{v_j}a_k = c_{y_k}$, which is again a contradiction. If $i = 0$ and $j = k$, then $g = a_1$, and so $\lambda = \tau_1$.

Suppose (c) holds, that is, $\lambda = c_{y_i} - g - \cdots - a_k - b_1 - \cdots - b_k - e_1 - c_s$, $0 \le i \le k$, where $g$ is either $a_i$ or $a_{i+1}$ ($g = a_{i+1}$ if $i = 0$). If $i > 1$, then $c_{v_{i-1}} = c_{y_i}b_i = c_sb_i = c_{v_i}$, which is a contradiction. If $i = 0$, then $c_{v_k} = c_{y_0}e_1 = c_se_1 = c_s$, which is a contradiction. If $i = 1$ and $g = a_1$, then $\lambda$ is not minimal since $c_{y_1} - a_2$, so $a_1$ can be removed. Finally, if $i = 1$ and $g = a_2$, then $\lambda = \tau_2$.

Suppose (d) holds, that is, $\lambda = c_{v_i} - g - \cdots - h - c_{v_j}$, $1 \le i < j \le k$, where $g$ is either $b_i$ or $b_{i+1}$ and $h$ is either $b_j$ or $b_{j+1}$ (where $b_{k+1} = e_1$). In any case, $\lambda$ contains $b_j$, and so $c_{v_{j-1}} = c_{v_i} b_j = c_{v_j} b_j = c_{v_j}$, which is a contradiction.

Suppose (e) holds, that is, $\lambda = c_{v_i} - \cdots - e_1 - c_s$, $1 \le i \le k$. Then $c_{v_k} = c_{v_i} e_1 = c_s e_1 = c_s$, which is a contradiction.

We have exhausted all possibilities and obtained that $\lambda$ must be equal to $\tau_1$ or $\tau_2$. The result follows. □

**Lemma 3.9.** *The path $\pi = a_1 - \cdots - a_k - b_1 - \cdots - b_k - e_1$ is a unique minimal $l$-path in $\mathcal{G}(S_0^k)$ with at least one endpoint that is not a constant.*

*Proof.* We have that $\pi$ is an $l$-path by Lemmas 3.3 and 3.6. Suppose that $\lambda = e - \cdots - f$ is a minimal $l$-path in $\mathcal{G}(S_0^k)$ such that $e$ or $f$ is not a constant.

We claim that $\lambda$ does not contain any constant $c_z$. By Lemma 3.7, there is no constant $c_z$ such that $\lambda = e - \cdots - c_z - \cdots - f$ (since otherwise $\lambda$ would not be minimal). We may assume that $f$ is not a constant. But then $e$ is not a constant either since otherwise we would have that $ef$ is a constant and $ff = f$ is not a constant. But this is impossible since $\lambda$ is an $l$-path, and so $ef = ff$. The claim has been proved.

Thus all elements in $\lambda$ are generators from list (3.3). We may assume that $e$ is to the left of $f$ (according to the ordering in (3.3)). Since $\lambda$ is an $l$-path, $e = ee = fe$. Hence, by Lemma 3.3, $e = a_p$ and $f = b_p$ (for some $p \in \{1, \ldots, k\}$) or $e = b_1$ and $f = e_1$ or $e = a_1$ and $f = e_1$.

Suppose that $e = a_p$ and $f = b_p$ for some $p$. Then, by Corollary 3.5, $\lambda = a_p - \cdots - a_k - b_1 - \cdots - b_p$. (Note that $\lambda = a_p - a_{p-1} - \cdots - a_1 - e_1 - b_k - \cdots - b_p$ is impossible since $a_1 e_1 \neq e_1 a_1$.) If $p > 1$ then, by Lemma 3.3, $c_{v_0} = a_p b_1 = b_p b_1 = c_{v_1}$, which is a contradiction. If $p = 1$, then $c_{y_{k-1}} = a_1 a_k = b_1 b_k = c_{y_k}$, which is again a contradiction.

Suppose that $e = b_1$ and $f = e_1$. Then $\lambda = b_1 - \cdots - b_k - e_1$, and so $c_{v_{k-1}} = b_1 b_k = e_1 b_k = c_{v_k}$, which is a contradiction.

Hence we must have $e = a_1$ and $f = e_1$. But then, by Corollary 3.5, $\lambda = a_1 - \cdots - a_k - b_1 - \cdots - b_k - e_1 = \pi$. The result follows. □

**Theorem 3.10.** *For every even integer $n \ge 2$, there is a band $S$ with knit degree $n$.*

*Proof.* Let $n = 2$. Consider the band $S = \{a, b, c, d\}$ defined by the following Cayley table:

| | $a$ | $b$ | $c$ | $d$ |
|---|---|---|---|---|
| $a$ | $a$ | $b$ | $c$ | $d$ |
| $b$ | $b$ | $b$ | $b$ | $b$ |
| $c$ | $a$ | $b$ | $c$ | $d$ |
| $d$ | $d$ | $d$ | $d$ | $d$ |

It is easy to see that the center of $S$ is empty and $a - b - c$ is a shortest $l$-path in $\mathcal{G}(S)$. Thus $\mathrm{kd}(S) = 2$.

Let $n = 2k$ where $k \ge 2$. Consider the semigroup $S_0^k$ defined by (3.2). Then, by Corollary 3.4, $S_0^k$ is a band. The paths $\tau_1$, $\tau_2$, and $\pi$ from Lemmas 3.8 and 3.9 are the only minimal $l$-paths in $\mathcal{G}(S_0^k)$. Since $\tau_1$ has length $2k+1 = n+1$, $\tau_2$ has length $2k+2 = n+2$, and $\pi$ has length $2k = n$, it follows that $\mathrm{kd}(S_0^k) = n$. □

## 3.2 The Odd Case

Suppose $n = 2k + 1 \ge 5$ is odd. We will obtain a band $S$ of knit degree $n$ by slightly modifying the construction of the band $S_0^k$ from Definition 3.2. Recall that $S_0^k$ has knit degree $2k$ (see the proof of Theorem 3.10). We will obtain a band of knit degree $n = 2k + 1$ by simply removing transformations $e_1$ and $c_s$ from $S_0^k$.

**Definition 3.11.** Let $k \geq 2$ be an integer. Consider the following subset of the semigroup $S_k^0$ from Definition 3.2:

$$S_k^1 = S_k^0 - \{e_1, c_s\} = \{a_1, \dots, a_k, b_1, \dots, b_k, c_{y_0}, c_{y_1}, \dots, c_{y_k}, c_{v_1}, \dots, c_{v_k}\}. \tag{3.4}$$

By Lemmas 3.3 and 3.6, $S_k^1$ is a subsemigroup of $S_k^0$.

**Remark 3.12.** Note that $r$ and $s$, which still occur in the domain (but not the image) of each element of $S_1^k$, are now superfluous. We can remove them from the domain of each element of $S_k^1$ and view $S_k^1$ as a semigroup of transformations on the set

$$X = \{y_0, y_1, \dots, y_k = v_0, v_1, \dots, v_k, x_1, \dots, x_k, u_1, \dots, u_k\}.$$

It is clear from the definition of $S_1^k$ that the multiplication table for $S_1^k$ is the multiplication table for $S_0^k$ (see Lemmas 3.3 and 3.6) with the rows and columns $e_1$ and $c_s$ removed. This new multiplication table is given by Lemmas 3.3 and 3.6 if we ignore the multiplications involving $e_1$ or $c_s$. Therefore, the following lemma follows immediately from Corollary 3.4 and Lemmas 3.8 and 3.9.

**Lemma 3.13.** *Let $S_1^k$ be the semigroups defined by (3.4). Then $S_1^k$ is a band and $\tau = c_{y_0} - a_1 - \cdots - a_k - b_1 - \cdots - b_k - c_{v_k}$ is the only minimal l-path in $\mathcal{G}(S_1^k)$.*

**Theorem 3.14.** *For every odd integer $n \geq 5$, there is a band $S$ of knit degree $n$.*

*Proof.* Let $n = 2k + 1$ where $k \geq 2$. Consider the semigroup $S_1^k$ defined by (3.4). Then, by Lemma 3.13, $S_1^k$ is a band and $\tau = c_{y_0} - a_1 - \cdots - a_k - b_1 - \cdots - b_k - c_{v_k}$ is the only minimal $l$-path in $\mathcal{G}(S_1^k)$. Since $\tau$ has length $2k + 1 = n$, it follows that $\mathrm{kd}(S_1^k) = n$. □

The case $n = 3$ remains unresolved.

**Open Question.** Is there a semigroup of knit degree 3?

# 4 Commuting Graphs with Arbitrary Diameters

In Section 2, we showed that, except for some special cases, the commuting graph of any ideal of the semigroup $T(X)$ has diameter 5. In this section, we use the constructions of Section 3 to show that there are semigroups whose commuting graphs have any prescribed diameter. We note that the situation is (might be) quite different in group theory: it has been conjectured that there is an upper bound for the diameters of the connected commuting graphs of finite non-abelian groups [9, Conjecture 2.2].

**Theorem 4.1.** *For every $n \geq 2$, there is a semigroup $S$ such that the diameter of $\mathcal{G}(S)$ is $n$.*

*Proof.* Let $n \in \{2, 3, 4\}$. The commuting graph of the band $S$ defined by the Cayley table in the proof of Theorem 3.10 is the cycle $a - b - c - d - a$. Thus the diameter of $\mathcal{G}(S)$ is 2. Consider the semigroup $S$ defined by the following table:

| | a | b | c | d |
|---|---|---|---|---|
| a | a | a | a | a |
| b | a | b | c | c |
| c | c | c | c | c |
| d | c | d | c | c |

Note that $Z(S) = \emptyset$ and $\mathcal{G}(S)$ is the chain $a - b - c - d$. Thus the diameter of $\mathcal{G}(S)$ is 3. The diameter of $\mathcal{G}(J_4)$ is 4 (where $J_4$ is an ideal of $T(X)$ with $|X| = 5$).

Let $n \geq 5$. Suppose $n$ is even. Then $n = 2k+2$ for some $k \geq 2$. Consider the band $S_0^k$ from Definition 3.2. Since $c_{y_0}$ and $a_1$ are the only elements of $S_0^k$ whose image contains $y_0$, they are the only elements of $S_0^k$ commuting with $c_{y_0}$ (see Lemma 3.1). Similarly, $e_1$ and $c_s$ are the only elements commuting with $c_s$. Therefore, it follows from Corollary 3.5 that $c_{y_0} - a_1 - \cdots - a_k - b_1 - \cdots - b_k - e_1 - c_s$ is a shortest path in $\mathcal{G}(S_0^k)$ from $c_{y_0}$ to $c_s$, that is, the distance between $c_{y_0}$ and $c_s$ is $2k+2 = n$. Since $a_1 - \cdots - a_k - b_1 - \cdots - b_k - e_1$ is a path in $\mathcal{G}(S_0^k)$, $c_{y_i}a_i = a_ic_{y_i}$ and $c_{v_i}b_i = b_ic_{v_i}$ $(1 \leq i \leq k)$, it follows that the distance between any two vertices of $\mathcal{G}(S_0^k)$ is at most $2k+2$. Hence the diameter of $\mathcal{G}(S_0^k)$ is $n$.

Suppose $n$ is odd. Then $n = 2k+1$ for some $k \geq 2$. Consider the band $S_1^k$ from Definition 3.11. Then $c_{y_0} - a_1 - \cdots - a_k - b_1 - \cdots - b_k - c_{v_k}$ is a shortest path in $\mathcal{G}(S_1^k)$ from $c_{y_0}$ to $c_{v_k}$, that is, the distance between $c_{y_0}$ and $c_{v_k}$ is $2k+1 = n$. As for $S_0^k$, we have $c_{y_i}a_i = a_ic_{y_i}$ and $c_{v_i}b_i = b_ic_{v_i}$ $(1 \leq i \leq k)$. Thus the distance between any two vertices of $S_1^k$ is at most $2k+1$, and so the diameter of $\mathcal{G}(S_1^k)$ is $n$. □

# 5 Schein's Conjecture

The results obtained in Section 3 enable us to settle a conjecture formulated by B.M. Schein in 1978 [14, p. 12]. Schein stated his conjecture in the context of the attempts to characterize the $r$-semisimple bands.

A right congruence $\tau$ on a semigroup S is said to be modular if there exists an element $e \in S$ such that $(ex)\tau x$ for all $x \in S$. The radical $R_r$ on a band $S$ is the intersection of all maximal modular right congruences on $S$ [11]. A band $S$ is called *$r$-semisimple* if its radical $R_r$ is the identity relation on $S$.

In 1969, B.D. Arendt announced a characterization of $r$-semisimple bands [3, Theorem 18]. In 1978, B.M Schein pointed out that Arendt's characterization is incorrect and proved [14, p. 2] that a band $S$ is $r$-semisimple if and only if it satisfies infinitely many quasi-identities: (1) and $(A_n)$ for all integers $n \geq 1$, where

$$\begin{aligned} &(1) \quad zx = zy \Rightarrow xy = yx, \\ &(A_n) \quad x_1x_2 = x_2x_1 \wedge x_2x_3 = x_3x_2 \wedge \ldots \wedge x_{n-1}x_n = x_nx_{n-1} \wedge \\ &\qquad \wedge\, x_1x_1 = x_nx_1 \wedge x_1x_2 = x_nx_2 \wedge \ldots \wedge x_1x_n = x_nx_n \Rightarrow x_1 = x_n. \end{aligned}$$

Schein observed that $(A_1)$ and $(A_2)$ are true in every band, that $(A_3)$ easily follows from (1), and that Arendt's characterization of $r$-semisimple bands is equivalent to (1). He used the last observation to show that Arendt's characterization is incorrect by providing an example of a band $T$ for which (1) holds but $(A_4)$ does not. We note that Schein's example is incorrect since the Cayley table in [14, p. 10], which is supposed to define $T$, does not define a semigroup because the operation is not associative: $(4*1)*1 = 10 \neq 8 = 4*(1*1)$. However, Schein was right that it is not true that condition (1) implies $(A_n)$ for all $n$. The semigroup $S_0^2$ (see Table 2) satisfies (1) but it does not satisfy $(A_5)$ since $a_1 - a_2 - b_1 - b_2 - e_1$ is an $l$-path (so the premise of $(A_5)$ holds) but $a_1 \neq e_1$.

At the end of the paper, Schein formulates his conjecture [14, p. 12]:

**Schein's Conjecture.** For every $n > 1$, $(A_n)$ does not imply $(A_{n+1})$.

The reason that Section 3 enables us to settle Schein's conjecture is the following lemma.

**Lemma 5.1.** *Let $n \geq 1$ and let $S$ be a band with no central elements. Then $S$ satisfies $(A_n)$ if and only if $\mathcal{G}(S)$ has no $l$-path of length $< n$.*

*Proof.* First note that $(A_n)$ can be expressed as: for all $x_1, \ldots, x_n \in S$,

$$x_1 - \cdots - x_n \text{ and } x_1x_i = x_nx_i \ (1 \leq i \leq n) \Rightarrow x_1 = x_n. \tag{5.1}$$

(Here, we allow $x - x$ and do not require that $x_1, \dots, x_n$ be distinct.)

Assume $S$ satisfies $(A_n)$. Suppose to the contrary that $\mathcal{G}(S)$ has an $l$-path $\lambda = x_1 - \cdots - x_k$ of length $< n$, that is, $k \leq n$. Then $x_1 - \cdots - x_k - x_{k+1} - \cdots - x_n$, where $x_i = x_k$ for every $i \in \{k+1, \dots, n\}$, and so $x_1 = x_n = x_k$ by (5.1). This is a contradiction since $\lambda$ is a path.

Conversely, suppose that $\mathcal{G}(S)$ has no $l$-path of length $< n$. Let $x_1 - \cdots - x_n$ and $x_1 x_i = x_n x_i$ $(1 \leq i \leq n)$. Suppose to the contrary that $x_1 \neq x_n$. If there are $i$ and $j$ such that $1 \leq i < j \leq n$ and $x_i = x_j$, we can replace $x_1 - \cdots - x_i - \cdots - x_j - \cdots - x_n$ with $x_1 - \cdots - x_i - x_{j+1} - \cdots - x_n$. Therefore, we can assume that $x_1, \dots, x_n$ are pairwise distinct. Recall that $S$ has no central elements, so all $x_i$ are vertices in $\mathcal{G}(S)$. Thus $x_1 - \cdots - x_n$ is an $l$-path in $\mathcal{G}(S)$ of length $n-1$, which is a contradiction. □

First, Schein's conjecture is false for $n = 3$.

**Proposition 5.2.** $(A_3) \Rightarrow (A_4)$.

*Proof.* Suppose a band $S$ satisfies $(A_3)$, that is,

$$x_1x_2 = x_2x_1 \wedge x_2x_3 = x_3x_2 \wedge x_1x_1 = x_3x_1 \wedge x_1x_2 = x_3x_2 \wedge x_1x_3 = x_3x_3 \Rightarrow x_1 = x_3. \quad (5.2)$$

To prove that $S$ satisfies $(A_4)$, suppose that

$$y_1y_2 = y_2y_1 \wedge y_2y_3 = y_3y_2 \wedge y_3y_4 = y_4y_3 \wedge y_1y_1 = y_4y_1 \wedge y_1y_2 = y_4y_2 \wedge y_1y_3 = y_4y_3 \wedge y_1y_4 = y_4y_4.$$

Take $x_1 = y_1$, $x_2 = y_2y_3$, and $x_3 = y_4$. Then $x_1, x_2, x_3$ satisfy the premise of (5.2):

$$\begin{aligned}
&x_1x_2 = y_1y_2y_3 = y_1y_3y_2 = y_4y_3y_2 = y_3y_4y_2 = y_3y_1y_2 = y_3y_2y_1 = y_2y_3y_1 = x_2x_1,\\
&x_2x_3 = y_2y_3y_4 = y_2y_4y_3 = y_2y_1y_3 = y_1y_2y_3 = y_4y_2y_3 = x_3x_2,\\
&x_1x_1 = y_1y_1 = y_4y_1 = x_3x_1,\ x_1x_2 = y_1y_2y_3 = y_4y_2y_3 = x_3x_2,\ x_1x_3 = y_1y_4 = y_4y_4 = x_3x_3.
\end{aligned}$$

Thus, by (5.2), $y_1 = x_1 = x_3 = y_4$, and so $(A_4)$ holds. □

Second, Schein's conjecture is true for $n \neq 3$.

**Proposition 5.3.** *If $n > 1$ and $n \neq 3$, then $(A_n)$ does not imply $(A_{n+1})$.*

*Proof.* Consider the band $S = \{e, f, 0\}$, where 0 is the zero, $ef = f$, and $fe = e$. Then $e - 0 - f$, $ee = fe$, $e0 = f0$, $ef = ff$, and $e \neq f$. Thus $S$ does not satisfy $(A_3)$. But $S$ satisfies $(A_2)$ since $(A_2)$ is true in every band. Hence $(A_2)$ does not imply $(A_3)$.

Let $n \geq 4$. Then, by Theorems 3.10 and 3.14 and their proofs, the band $S$ constructed in Definition 3.2 (if $n$ is even) or Definition 3.11 (if $n$ is odd) has knit degree $n$. By Lemmas 3.3 and 3.6, $S$ has no central elements. Since $\mathrm{kd}(S) = n$, there is an $l$-path in $\mathcal{G}(S)$ of length $n$ and there is no $l$-path in $\mathcal{G}(S)$ of length $< n$. Hence, by Lemma 5.1, $S$ satisfies $(A_n)$ and $S$ does not satisfy $(A_{n+1})$. Thus $(A_n)$ does not imply $(A_{n+1})$. □

# 6 Problems

We finish this paper with a list of some problems concerning commuting graphs of semigroups.

(1) Is there a semigroup with knit degree 3? Our guess is that such a semigroup does not exist.

(2) Classify the semigroups whose commuting graph is eulerian (proposed by M. Volkov). The same problem for hamiltonian and planar graphs.

(3) With the exception of the complete graph, is it true that for all finite connected graphs $\Gamma$, there is a semigroup $S$ such that $\mathcal{G}(S) \cong \Gamma$?

(4) Is it true that for all natural numbers $n \geq 3$, there is a semigroup $S$ such that the clique number (girth, chromatic number) of $\mathcal{G}(S)$ is $n$?

(5) Classify the semigroups $S$ such that the clique and chromatic numbers of $\mathcal{G}(S)$ coincide.

(6) Calculate the clique and chromatic numbers of the commuting graphs of $T(X)$ and $\mathrm{End}(V)$, where $X$ is a finite set and $V$ is a finite-dimensional vector space over a finite field.

(7) Let $\mathcal{G}(S)$ be the commuting graph of a finite non-commutative semigroup $S$. An *rl-path* is a path $a_1 - \cdots - a_m$ in $\mathcal{G}(S)$ such that $a_1 \neq a_m$ and $a_1 a_i a_1 = a_m a_i a_m$ for all $i = 1, \ldots, m$. For *rl*-paths, prove the results analogous to the results for *l*-paths contained in this paper.

(8) Find classes of finite non-commutative semigroups such that if $S$ and $T$ are two semigroups in that class and $\mathcal{G}(S) \cong \mathcal{G}(T)$, then $S \cong T$.

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