AN ELEGANT 3-BASIS FOR INVERSE SEMIGROUPS

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Abstract. It is well known that in every inverse semigroup the binary operation and the unary operation of inversion satisfy the following three identities:

$$x = (xx')x \qquad (xx')(y'y) = (y'y)(xx') \qquad (xy)z = x(yz'')\,.$$

The goal of this note is to prove the converse, that is, we prove that an algebra of type $\langle 2, 1\rangle$ satisfying these three identities is an inverse semigroup and the unary operation coincides with the usual inversion on such semigroups.

1. Introduction

A unary semigroup is a triple $(S, \cdot, ')$, where $(S, \cdot)$ is a semigroup and $'$ is a unary operation on S. We say that a set of n identities, in the language of a binary operation and a unary operation, is an n-basis for inverse semigroups, if those identities define the variety of unary inverse semigroups and the unary operation coincides with the natural inversion. This last requirement is needed in order to avoid having several copies of the same semigroup (as an algebra with one binary operation) in the variety defined by those identities.

Denoting by x' the inverse of an element x in an inverse semigroup, we then have $x = (xx')x$ (as inverse semigroups are regular semigroups) and $(xx')(y'y) = (y'y)(xx')$ (as both xx' and $y'y$ are idempotents, and idempotents commute in inverse semigroups). Thus we might be tempted to think that the following identities provide a 3-basis for unary inverse semigroups:

$$x = (xx')x, \qquad (xx')(y'y) = (y'y)(xx') \qquad \text{and} \qquad (xy)z = x(yz)\,. \tag{1.1}$$

However, for $S = \{0, 1\}$ with $xy = 0$, except for $11 = 1$, and defining $x' = 1$, we have the previous identities satisfied, but $0' \neq 0'00'$ and hence $'$ does not coincide with the natural inversion in $(S, \cdot)$.

B.M. Schein [3] repaired the *defect* of (1.1) by adjoining two additional identities: $x'' = x$ and $(xy)' = y'x'$. The resulting set of five identities provides indeed a 4-basis for inverse semigroups. (As observed in, for instance, [1], the identity $(xy)' = y'x'$ is dependent upon the others, and hence can be discarded.) Therefore the natural question to ask would be: *is it possible to find a 3-basis for inverse semigroups?* This question was first answered in the affirmative in [1], but the 3-basis given there requires an extremely complicated proof (it is still an open problem to provide a reasonable proof for that result).

The aim of this note is to repair (1.1) by providing the following easy, transparent and *elegant* 3-basis for inverse semigroups:

$$(\mathbf{E}_1) \quad x = (xx')x, \qquad (\mathbf{E}_2) \quad (xx')(y'y) = (y'y)(xx'), \qquad (\mathbf{E}_3) \quad (xy)z = x(yz'')\,.$$

2. Proof

In this section we prove that the identities $(\mathbf{E}_1)$–$(\mathbf{E}_3)$ imply Schein's 4-basis for inverse semigroups. As the converse is obvious, the equivalence of the two bases will follow.

Throughout this section let $(S, \cdot, ')$ be an algebra of type $\langle 2, 1\rangle$ satisfying $(\mathbf{E}_1)$–$(\mathbf{E}_3)$. We start by proving a few handy identities.

Lemma 2.1. *The following identities hold.*

$$x'x'' = x'x \tag{2.1}$$

$$(xy')y = x(y'y) \tag{2.2}$$

$$x = x(x'x) \tag{2.3}$$

$$x'' = (x''x')x = x''(x'x) \tag{2.4}$$

$$x'''x = x'''x'' = x'''x^{(4)} \tag{2.5}$$

Proof. Firstly, for (2.1), we have

$$x'x'' \overset{(\mathbf{E}_1)}{=} x'[(x''x''')x''] \overset{(\mathbf{E}_3)}{=} [x'(x''x''')]x \overset{(\mathbf{E}_3)}{=} [(x'x'')x']x \overset{(\mathbf{E}_1)}{=} x'x\,.$$

Next, for (2.2), we compute $(xy')y \overset{(\mathbf{E}_3)}{=} x(y'y'') \overset{(2.1)}{=} x(y'y)$.

Regarding (2.3), we have $x(x'x) \overset{(2.2)}{=} (xx')x \overset{(\mathbf{E}_1)}{=} x$.

Then for (2.4), we compute $x'' \overset{(2.3)}{=} x''(x'''x'') \overset{(\mathbf{E}_3)}{=} (x''x''')x \overset{(2.1)}{=} (x''x')x \overset{(2.2)}{=} x''(x'x)$.

Finally, for (2.5), we have

$$x'''x \overset{(2.3)}{=} [x'''(x''''x''')]x \overset{(\mathbf{E}_3)}{=} x'''[(x''''x''')x''] \overset{(2.4)}{=} x'''x'''' \overset{(2.1)}{=} x'''x''\,.$$

□

The next two lemmas are the key tools in the proof that the identities $(\mathbf{E}_1)$–$(\mathbf{E}_3)$ imply $x'' = x$.

Lemma 2.2. $(x'x)x''' = x'''$.

Proof. We start with two observations. Firstly, as

$$[x(y'''y)]y' \overset{(\mathbf{E}_3)}{=} x[(y'''y)y''] \overset{(2.5)}{=} x[(y'''y'''')y'''] \overset{(\mathbf{E}_1)}{=} xy'''\,,$$

we have

$$(x(y'''y))y' = xy'''\,. \tag{2.6}$$

Secondly,

$$(x'x)(x'''x) \overset{(2.5)}{=} (x'x)(x'''x'''') \overset{(\mathbf{E}_2)}{=} (x'''x'''')(x'x) \overset{(2.5)}{=} (x'''x'')(x'x) \overset{(2.2)}{=} [(x'''x'')x']x \overset{(2.4)}{=} x'''x\,,$$

so that

$$(x'x)(x'''x) = x'''x. \tag{2.7}$$

Now we have all we need to prove the lemma.

$$x''' \overset{(2.4)}{=} (x'''x'')x' \overset{(2.5)}{=} (x'''x)x' \overset{(2.7)}{=} [(x'x)(x'''x)]x' \overset{(2.6)}{=} (x'x)x'''.$$

□

Lemma 2.3. $(xy)z' = x(yz')$.

Proof. We start by proving that

$$x''' = x' \,. \tag{2.8}$$

In fact we have $xx' \overset{(2.3)}{=} [x(x'x)]x' \overset{(\mathbf{E}_3)}{=} x[(x'x)x'''] = xx'''$, using Lemma 2.2 in the last equality. Thus

$$xx''' = xx' \,. \tag{2.9}$$

Now, by Lemma 2.2,

$$x''' = (x'x)x''' \overset{(2.1)}{=} (x'x'')x''' \overset{(\mathbf{E}_3)}{=} x'(x''x^{(5)}) \overset{(2.9)}{=} x'(x''x''') \overset{(\mathbf{E}_3)}{=} (x'x'')x' \overset{(\mathbf{E}_1)}{=} x' \,.$$

Replacing z by z' in $(\mathbf{E}_3)$, we get

$$(xy)z' = x(yz''') = x(yz') \,,$$

where the last equality follows from (2.8). The lemma is proved. □

We have everything we need to prove our main result.

Theorem 2.4. *The identities* $(\mathbf{E}_1)$–$(\mathbf{E}_3)$ *imply* $x'' = x$ *and the associative law.*

Proof. First, we have

$$\begin{aligned} x''x' &\overset{(2.4)}{=} [(x''x')x]x' = (x''x')(xx') \overset{(\mathbf{E}_2)}{=} (xx')(x''x') \\ &= [(xx')x'']x' = [x(x'x'')]x' = x[(x'x'')x'] \overset{(\mathbf{E}_1)}{=} xx' \,, \end{aligned}$$

where we have used Lemma 2.3 in the unlabeled equalities. Thus

$$x''x' = xx' \,. \tag{2.10}$$

Now $x'' \overset{(2.4)}{=} (x''x')x \overset{(2.10)}{=} (xx')x \overset{(\mathbf{E}_1)}{=} x$, as claimed.

Associativity now follows easily: $(xy)z \overset{(\mathbf{E}_1)}{=} x(yz'') = x(yz)$. □

3. Other Sets of Axioms

It is natural to ask how sensitive the axioms $(\mathbf{E}_1)$–$(\mathbf{E}_3)$ are to certain modifications, such as shifting the parentheses in $(\mathbf{E}_1)$ or changing the placement of the double inverse in $(\mathbf{E}_3)$.

If, for instance, we leave $(\mathbf{E}_2)$ intact, replace $(\mathbf{E}_1)$ with $x(x'x) = x$ and replace $(\mathbf{E}_3)$ with $(x''y)z = x(yz)$, then we obtain a set of identities which are dual to $(\mathbf{E}_1)$–$(\mathbf{E}_3)$. By an argument dual to that in §2, this set of identities is another 3-basis for inverse semigroups.

Thus to dispense with these sorts of obvious dualities, we will assume that both $(\mathbf{E}_1)$ and $(\mathbf{E}_2)$ are left intact, and consider only alternative placement of the double inverse in $(\mathbf{E}_3)$. Using Prover9, we found that each of the following identities can substitute for $(\mathbf{E}_3)$ to give another 3-basis for inverse semigroups:

$$\begin{aligned} (xy)z &= x''(yz) \qquad & (xy)z &= x(y''z) \\ x(yz) &= (xy'')z \qquad & x(yz) &= (xy)z''. \end{aligned}$$

The remaining possibility, $x(yz) = (x''y)z$, does not work. Using Mace4, we found the counterexample given by the following tables. It satisfies $(\mathbf{E}_1)$, $(\mathbf{E}_2)$ and $x(yz) = (x''y)z$, but the binary operation is not associative ($(0 \cdot 0) \cdot 0 = 1 \cdot 0 = 7 \neq 6 = 0 \cdot 1 = 0 \cdot (0 \cdot 0)$), and the unary operation clearly fails to satisfy $x'' = x$.

$\cdot$	0	1	2	3	4	5	6	7	8	9	10	11
0	1	6	5	7	3	8	4	2	0	4	4	4
1	7	2	6	0	8	4	5	1	3	5	5	5
2	5	8	3	6	1	7	0	4	2	0	0	0
3	8	0	7	4	6	2	1	3	5	1	1	1
4	3	7	1	8	5	6	2	0	4	2	2	2
5	6	4	8	2	7	0	3	5	1	3	3	3
6	0	1	2	3	4	5	6	7	8	6	6	6
7	4	3	0	5	2	1	7	8	6	7	7	7
8	2	5	4	1	0	3	8	6	7	8	8	8
9	0	1	2	3	4	5	6	7	8	9	10	6
10	0	1	2	3	4	5	6	7	8	10	9	6
11	0	1	2	3	4	5	6	7	8	6	6	11

$'$	0	1	2	3	4	5	6	7	8	9	10	11
	1	2	3	4	5	0	6	8	7	9	10	11

4. Problem

As noted in the Introduction, the identity $(xy)' = y'x'$ is implied by Schein's 4-basis and hence, by our 3-basis. So we ask

Does there exist an independent n-basis for inverse semigroups containing the identity $(xy)' = y'x'$?

We guess that the answer is no.

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