QUANTUM MEASURES AND THE COEVENT INTERPRETATION

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Abstract

This paper first reviews quantum measure and integration theory. A new representation of the quantum integral is presented. This representation is illustrated by computing some quantum $(\text{Lebesgue})^2$ integrals. The rest of the paper only considers finite spaces. Anhomomorphic logics are discussed and the classical domain of a coevent is studied. Pure quantum measures and coevents are considered and it is shown that pure quantum measures are strictly contained in the extremal elements for the set of quantum measures bounded above by one. Moreover, we prove that any quantum measure on a finite event space $\mathcal{A}$ can be transferred to an ordinary measure on an anhomomorphic logic $\mathcal{A}^*$. In this way, the quantum dynamics on $\mathcal{A}$ can be described by a classical dynamics on the larger space $\mathcal{A}^*$.

Keywords: quantum measures, anhomomorphic logics, coevent interpretation.

1 Introduction

Quantum measures and the coevent interpretation were introduced by R. Sorkin in his studies of the histories approach to quantum mechanics and quantum gravity [11, 12, 14, 15, 16]. Since then a considerable amount of literature has appeared on these subjects [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 17, 18]. A quantum measure μ describes the dynamics of a quantum system in the sense that $\mu(A)$ gives the propensity that the event A occurs. Denoting the set of events by $\mathcal{A}$, a coevent is a potential reality for the system given by a truth function $\phi\colon \mathcal{A} \to \mathbb{Z}_2$ where $\mathbb{Z}_2$ is the two element Boolean algebra $\{0, 1\}$. Since coevents need not be Boolean homomorphisms, the set $\mathcal{A}^*$ of coevents is called an *anhomomorphic logic* [2, 3, 4, 8, 9, 10, 14, 17]. We refer to the study of anhomomorphic logics as the coevent interpretation of quantum mechanics. One of the goals of this field is to find the "actual reality" in $\mathcal{A}^*$ [2, 8, 10, 14, 15, 17].

In Section 2, we first review quantum measure and integration theory. A new representation of the quantum integral is presented. As an example, this representation is employed to compute quantum (Lebesgue)^2 integrals. Although general event spaces are treated in Section 2, only finite event spaces are considered in the rest of the paper. Section 3 discusses anhomomorphic logics. The center and classical domain of a coevent are studied. Section 4 first considers pure quantum measures and coevents. We show that the pure quantum measures are strictly contained in the extremal elements for the convex set of quantum measures bounded above by one. This result is then employed to prove an important connection between certain quantum measures and anhomomorphic logics. Moreover, we show that any quantum measure on a (finite) event space $\mathcal{A}$ can be transferred to an ordinary measure on $\mathcal{A}^*$. In this way the quantum dynamics on $\mathcal{A}$ can be described by a classical dynamics on the larger space $\mathcal{A}^*$. Variations of this transference result show that pure and quadratic anhomomorphic logics have better mathematical properties than additive or multiplicative ones.

2 Quantum Measures and Integrals

Let $(\Omega, \mathcal{A})$ be a measurable space, where Ω is a set of *outcomes* and $\mathcal{A}$ is a σ-algebra of subsets of Ω called *events* for a physical system. If $A, B \in \mathcal{A}$ are disjoint, we denote their union by $A \mathbin{\dot\cup} B$. A nonnegative set function $\mu\colon \mathcal{A} \to \mathbb{R}^+$ is *grade-2 additive* if

$$\mu\left(A \mathbin{\dot\cup} B \mathbin{\dot\cup} C\right) = \mu\left(A \mathbin{\dot\cup} B\right) + \mu\left(A \mathbin{\dot\cup} C\right) + \mu\left(B \mathbin{\dot\cup} C\right) - \mu(A) - \mu(B) - \mu(C) \tag{2.1}$$

for all mutually disjoint $A, B, C \in \mathcal{A}$. It follows by induction that if μ satisfies (2.1) then

$$\mu\left(\dot{\bigcup_{i=1}^{n}} A_i\right) = \sum_{i<j=1}^{n} \mu\left(A_i \,\dot{\cup}\, A_j\right) - (n-2)\sum_{i=1}^{n} \mu(A_i) \tag{2.2}$$

A *q-measure* is a grade-2 additive set function $\mu\colon \mathcal{A} \to \mathbb{R}^+$ that satisfies the following two continuity conditions.

(C1) If $A_1 \subseteq A_2 \subseteq \cdots$ is an increasing sequence in $\mathcal{A}$, then
$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcup_{i=1}^{\infty} A_i\right)$$

(C2) If $A_1 \supseteq A_2 \supseteq \cdots$ is a decreasing sequence in $\mathcal{A}$, then
$$\lim_{n\to\infty} \mu(A_n) = \mu\left(\bigcap_{i=1}^{\infty} A_i\right)$$

Due to quantum interference, a q-measure need not satisfy the usual additivity condition of an ordinary measure but satisfies the more general grade-2 additivity condition (2.1) instead [2, 14, 15]. For example, if $\nu\colon \mathcal{A} \to \mathbb{C}$ is a complex measure corresponding to a quantum amplitude, then $\mu(A) = |\nu(A)|^2$ becomes a q-measure. For another example, let $D\colon \mathcal{A} \times \mathcal{A} \to \mathbb{C}$ be a decoherence functional as studied in the histories approach to quantum mechanics [11, 12, 14, 15]. Then $\mu(A) = D(A, A)$ becomes a q-measure. If μ is a q-measure on $\mathcal{A}$, we call $(\Omega, \mathcal{A}, \mu)$ a *q-measure space*.

A signed measure $\lambda\colon \mathcal{A} \times \mathcal{A} \to \mathbb{R}$, where $\mathcal{A} \times \mathcal{A}$ is the product σ-algebra on $\Omega \times \Omega$, is *symmetric* if $\lambda(A \times B) = \lambda(B \times A)$ for all $A, B \in \mathcal{A}$ and is *diagonally positive* if $\lambda(\mathcal{A} \times \mathcal{A}) \geq 0$ for all $A \in \mathcal{A}$. It can be shown that if $\lambda\colon \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ is a symmetric diagonally positive signed measure, then $\mu(A) = \lambda(A \times A)$ is a q-measure on $\mathcal{A}$. Conversely, if μ is a q-measure on $\mathcal{A}$, then there exists a unique symmetric signed measure $\lambda\colon \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ such that $\mu(A) = \lambda(A \times A)$ for all $A \in \mathcal{A}$ [6].

Let $\Omega_n = \{\omega_1, \ldots, \omega_n\}$ be a finite outcome space. In this case we take $\mathcal{A}_n$ to be the power set $P(\Omega_n) = 2^{\Omega_n}$. Any grade-2 additive set function $\mu\colon \mathcal{A}_n \to \mathbb{R}^+$ is a q-measure because (C1) and (C2) hold automatically. We call $(\Omega_n, \mathcal{A}_n, \mu)$ a *finite q-measure space*. The study of finite q-measure spaces is simplified because by (2.2) the q-measure μ is determined by its values on singleton sets $\mu(\omega_i) = \mu(\{\omega_i\})$, $i = 1, \ldots, n$ and doubleton sets $\mu(\{\omega_i, \omega_j\})$, $i < j$, $i, j = 1, \ldots, n$. We call

$$I_{ij}^{\mu} = \mu(\{\omega_i, \omega_j\}) - \mu(\omega_i) - \mu(\omega_j) \tag{2.3}$$

the *ij-interference term*, $i < j$, $i, j = 1, \ldots, n$ [5]. The *Dirac measure* δ_i defined by

$$\delta_i(A) = \begin{cases} 1 & \text{if } \omega_i \in A \\ 0 & \text{if } \omega_i \notin A \end{cases}$$

is clearly a q-measure on $\mathcal{A}_n$. The product $\widehat{\delta}_{ij} = \delta_i \delta_j$, $i \neq j$, is not a measure because $\widehat{\delta}_{ij}(\omega_i) = \widehat{\delta}_{ij}(\omega_j) = 0$ and $\widehat{\delta}_{ij}(\{\omega_i, \omega_j\}) = 1$. However, we have the following .

Lemma 2.1. *The product $\widehat{\delta}_{ij}$, $i \neq j$, is a q-measure on $\mathcal{A}_n$.*

Proof. Notice that

$$\widehat{\delta}_{ij}(A) = \begin{cases} 1 & \text{if } \{\omega_i, \omega_j\} \subseteq A \\ 0 & \text{if } \{\omega_i, \omega_j\} \not\subseteq A \end{cases}$$

If $\delta_{ij}(A \uplus B \uplus C) = 0$, then $\{\omega_i, \omega_j\} \not\subseteq A \uplus B \uplus C$. Hence, $\{\omega_i, \omega_j\}$ is not a subset of A, B, C, $A \uplus B$, $A \uplus C$ or $B \uplus C$. Hence,

$$\widehat{\delta}_{ij}(A \uplus B) + \widehat{\delta}_{ij}(A \uplus C) + \widehat{\delta}_{ij}(B \uplus C) - \widehat{\delta}_{ij}(A) - \widehat{\delta}_{ij}(B) - \widehat{\delta}_{ij}(C) = 0$$

If $\widehat{\delta}_{ij}(A \uplus B \uplus C) = 1$, then $\{\omega_i, \omega_j\} \subseteq A \uplus B \uplus C$. If $\{\omega_i, \omega_j\} \subseteq A$, then

$$\begin{aligned} &\widehat{\delta}_{ij}(A \uplus B) + \widehat{\delta}_{ij}(A \uplus C) + \widehat{\delta}_{ij}(B \uplus C) - \widehat{\delta}_{ij}(A) - \widehat{\delta}_{ij}(B) - \widehat{\delta}_{ij}(C) \\ &\quad = 1 + 1 + 0 - 1 - 0 - 0 = 1 \end{aligned}$$

If $\omega_i \in A$, $\omega_j \in B$, then

$$\begin{aligned} &\widehat{\delta}_{ij}(A \uplus B) + \widehat{\delta}_{ij}(A \uplus C) + \widehat{\delta}_{ij}(B \uplus C) - \widehat{\delta}_{ij}(A) - \widehat{\delta}_{ij}(B) - \widehat{\delta}_{ij}(C) \\ &\quad = 1 + 0 + 0 - 0 - 0 - 0 = 1 \end{aligned}$$

The other cases hold by symmetry. □

A more general proof than that in Lemma 2.1 shows that the product of any two measures is a q-measure. A *signed q-measure* is a set function $\mu\colon \mathcal{A}_n \to \mathbb{R}^+$ that satisfies (2.1). Letting $\mathcal{S}(\mathcal{A}_n)$ be the set of signed q-measures, it is clear that $\mathcal{S}(\mathcal{A}_n)$ is a real linear space. Also, a signed q-measure is determined by its values on singleton and doubleton sets.

Theorem 2.2. *The q-measures $\delta_1,\ldots,\delta_n,\widehat{\delta}_{12},\widehat{\delta}_{13},\ldots,\widehat{\delta}_{n-1,n}$ form a basis for $\mathcal{S}(\mathcal{A}_n)$.*

Proof. To show linear independence, suppose that

$$\sum_{i=1}^n c_i\delta_i+\sum_{i<j=1}^n d_{ij}\widehat{\delta}_{ij}=0$$

Evaluating at $\{\omega_k\}$ gives $c_k=0$, $k=1,\ldots,n$. Hence,

$$\sum_{i<j=1}^n d_{ij}\widehat{\delta}_{ij}=0$$

and evaluating at $\{\omega_r,\omega_s\}$ gives $d_{rs}=0$, $r<s$. This proves linear independence. To show that these q-measures span $\mathcal{S}(\mathcal{A}_n)$, let $\mu\in\mathcal{S}(\mathcal{A}_n)$. Defining I_{ij}^μ as in (2.3), we have that

$$\mu=\sum_{i=1}^n\mu(\omega_i)\delta_i+\sum_{i<j=1}^n I_{ij}^\mu\widehat{\delta}_{ij} \tag{2.4}$$

because both sides agree on singleton and doubleton sets. □

We conclude from Theorem 2.2 that

$$\dim\mathcal{S}(\mathcal{A}_n)=n+\binom{n}{2}=n(n+1)/2$$

Also, if μ is a q-measure on $\mathcal{A}_n$, then μ has the unique representation given by (2.4). The next theorem proves a result that we already mentioned for the particular case of a finite Ω.

Theorem 2.3. *If μ is a q-measure on $\mathcal{A}_n$, then there exists a unique symmetric signed measure λ on $\mathcal{A}_n\times\mathcal{A}_n$ such $\mu(A)=\lambda(A\times A)$ for all $A\in\mathcal{A}_n$.*

Proof. Define the function $\alpha\colon\Omega_n\times\Omega_n\to\mathbb{R}$ by $\alpha(\omega_i,\omega_i)=\mu(\omega_i)$, $i=1,\ldots,n$, and

$$\alpha(\omega_i,\omega_j)=\alpha(\omega_j,\omega_i)=\tfrac{1}{2}I_{ij}^\mu$$

$i<j$, $i,j=1,\ldots,n$. Define the signed measure λ on $\mathcal{A}_n\times\mathcal{A}_n$ by

$$\lambda(\Delta)=\left\{\sum\alpha(\omega_i,\omega_j)\colon(\omega_i,\omega_j)\in\Delta\right\}$$

Now λ is symmetric because

$$\begin{aligned}\lambda(A\times B)&=\left\{\sum\alpha(\omega_i,\omega_j)\colon(\omega_i,\omega_j)\in A\times B\right\}\\&=\left\{\sum\alpha(\omega_i,\omega_j)\colon\omega_i\in A,\omega_j\in B\right\}\\&=\left\{\sum\alpha(\omega_j,\omega_i)\colon\omega_i\in A,\omega_j\in B\right\}\\&=\left\{\sum\alpha(\omega_j,\omega_i)\colon(\omega_j,\omega_i)\in B\times A\right\}=\lambda(B\times A)\end{aligned}$$

We now show that $\lambda(A\times A)=\mu(A)$ for all $A\in\mathcal{A}_n$. We can assume without loss of generality that $A=\{\omega_1,\dots,\omega_m\}$, $2\le m\le n$. It follows from (2.2) that

$$\mu(A)=\sum_{i<j=1}^{m}\mu\left(\{\omega_i,\omega_j\}\right)-(m-2)\sum_{i=1}^{m}\mu(\omega_i)$$

Moreover, we have that

$$\begin{aligned}\lambda(A\times A)&=\left\{\sum\alpha(\omega_i,\omega_j)\colon\omega_i,\omega_j\in A\right\}\\&=\sum_{i=1}^{m}\alpha(\omega_i,\omega_i)+2\sum_{i<j=1}^{m}\alpha(\omega_i,\omega_j)\\&=\sum_{i=1}^{m}\mu(\omega_i)+\sum_{i<j=1}^{m}I_{ij}^{\mu}\\&=\sum_{i=1}^{m}\mu(\omega_i)+\sum_{i<j=1}^{m}\mu\left(\{\omega_i,\omega_j\}\right)-\sum_{i<j=1}^{m}\left[\mu(\omega_i)-\mu(\omega_j)\right]\\&=\sum_{i=1}^{m}\mu(\omega_i)+\sum_{i<j=1}^{m}\mu\left(\{\omega_i,\omega_j\}\right)-(m-1)\sum_{i=1}^{m}\mu(\omega_i)\\&=\sum_{i<j=1}^{m}\mu\left(\{\omega_i,\omega_j\}\right)-(m-2)\sum_{i=1}^{m}\mu(\omega_i)=\mu(A)\end{aligned}$$

To prove uniqueness, suppose $\lambda'\colon\mathcal{A}_n\times\mathcal{A}_n\to\mathbb{R}$ is a symmetric signed measure satisfying $\lambda'(A\times A)=\mu(A)$ for every $A\in\mathcal{A}_n$. We then have that

$$\lambda'\left[(\omega_i,\omega_i)\right]=\lambda'\left[\{\omega_i\}\times\{\omega_i\}\right]=\mu(\omega_i)=\lambda\left[(\omega_i,\omega_i)\right]$$

for $i = 1, \dots, n$. Moreover, letting $A = \{(\omega_i, \omega_j)\}$ we have that

$$\begin{aligned}\mu(A) &= \lambda'(A \times A) = \lambda'\left[(\omega_i, \omega_i)\right] + \lambda'\left[(\omega_j, \omega_j)\right] + 2\lambda'\left[(\omega_i, \omega_j)\right] \\ &= \lambda\left[(\omega_i, \omega_i)\right] + \lambda\left[(\omega_j, \omega_j)\right] + 2\lambda\left[(\omega_i, \omega_j)\right]\end{aligned}$$

Hence, $\lambda'\left[(\omega_i, \omega_j)\right] = \lambda\left[(\omega_i, \omega_j)\right]$ and since signed measures are determined by their values on singleton sets, we have that $\lambda' = \lambda$. □

Let $(\Omega, \mathcal{A}, \mu)$ be a q-measure space and let $f\colon \Omega \to \mathbb{R}$ be a measurable function. The *q-integral* of f is defined by

$$\int f d\mu = \int_0^\infty \mu\left[f^{-1}(\lambda, \infty)\right] d\lambda - \int_0^\infty \mu\left[f^{-1}(-\infty, -\lambda)\right] d\lambda \tag{2.5}$$

where $d\lambda$ denotes Lebesgue measure on $\mathbb{R}$ [7]. Any measurable function $f\colon \Omega \to \mathbb{R}$ has a unique representation $f = f_1 - f_2$ where $f_1, f_2 \geq 0$ are measurable and $f_1 f_2 = 0$. It follows that

$$\int f d\mu = \int f_1 d\mu - \int f_2 d\mu \tag{2.6}$$

Because of (2.6) we only need to consider q-integrals of nonnegative functions. As usual in integration theory, if $A \in \mathcal{A}$ we define $\int_A f d\mu = \int f \chi_A d\mu$ where χ_A is the characteristic function for A.

Any nonnegative simple measurable function $f\colon \Omega \to \mathbb{R}^+$ has the canonical representation

$$f = \sum_{i=1}^n \alpha_i \chi_{A_i} \tag{2.7}$$

where $A_i \cap A_j = \emptyset$, $i \neq j$, $\cup A_i = \Omega$ and $0 \leq \alpha_1 < \alpha_2 \cdots < \alpha_n$. It follows from (2.5) that

$$\begin{aligned}\int f d\mu = \alpha_1 &\left[\mu\left(\mathop{\dot{\bigcup}}_{i=1}^n A_i\right) - \mu\left(\mathop{\dot{\bigcup}}_{i=2}^n A_i\right)\right] \\ &+ \cdots + \alpha_{n-1}\left[\mu\left(A_{n-1} \mathbin{\dot{\cup}} A_n\right) - \mu(A_n)\right] + \alpha_n \mu(A_n)\end{aligned} \tag{2.8}$$

Example 1. Let $f\colon \Omega_n \to \mathbb{R}^+$ be a nonnegative function on Ω_n and let $\mu\colon \mathcal{A}_n \to \mathbb{R}^+$ be a q-measure. Also, let $\lambda\colon \mathcal{A}_n \times \mathcal{A}_n \to \mathbb{R}$ be the unique symmetric signed measure such that $\mu(A) = \lambda(A \times A)$ for every $A \in \mathcal{A}_n$. We can assume without loss of generality that

$$0 \leq f(\omega_1) \leq f(\omega_2) \leq \cdots \leq f(\omega_n)$$

By (2.8) we have for $i < j$ that

$$\begin{aligned}\int f d\widehat{\delta}_{ij} &= f(\omega_1)\left[\widehat{\delta}_{ij}\left(\{\omega_1,\ldots,\omega_n\}\right) - \widehat{\delta}_{ij}\left(\{\omega_2,\ldots,\omega_n\}\right)\right] \\ &\quad + \cdots + f(\omega_{n-1})\left[\widehat{\delta}_{ij}\left(\{\omega_{n-1},\omega_n\}\right) - \widehat{\delta}_{ij}(\omega_n)\right] + f(\omega_n)\widehat{\delta}_{ij}(\omega_n) \\ &= f(\omega_i) = \min\left(f(\omega_i), f(\omega_j)\right)\end{aligned}$$

By (2.4) and the proof of Theorem 2.3, it follows that

$$\begin{aligned}\int f d\mu &= \int f d\left[\sum_{i=1}^{n}\mu(\omega_i)\delta_i + \sum_{i<j=1}^{n} I_{ij}^{\mu}\widehat{\delta}_{ij}\right] \\ &= \sum_{i=1}^{n}\mu(\omega_i)f(\omega_i) + \sum_{i<j=1}^{n} I_{ij}^{\mu}\min\left(f(\omega_i), f(\omega_j)\right) \\ &= \sum_{i,j=1}^{n}\min\left(f(\omega_i), f(\omega_j)\right)\lambda\left[(\omega_i,\omega_j)\right] \\ &= \int \min\left(f(\omega), f(\omega')\right) d\lambda(\omega,\omega')\end{aligned} \tag{2.9}$$

Example 2. Let $\nu\colon \mathcal{A}_n \to \mathbb{C}$ be a complex measure and define the q-measure $\mu\colon \mathcal{A}_n \to \mathbb{R}^+$ by $\mu(A) = |\nu(A)|^2$. Then for $i < j$ we have

$$I_{ij}^{\mu} = |\nu(\omega_i) + \nu(\omega_j)|^2 - |\nu(\omega_i)|^2 - |\nu(\omega_j)|^2 = 2\mathrm{Re}\,\nu(\omega_i)\overline{\nu(\omega_j)}$$

By (2.4) we conclude that

$$\begin{aligned}\mu &= \sum_{i=1}^{n}\mu(\omega_i)\delta_i + \sum_{i<j=1}^{m}\left[2\mathrm{Re}\,\nu(\omega_i)\overline{\nu(\omega_j)}\right]\widehat{\delta}_{ij} \\ &= \sum_{i=1}^{n}|\nu(\omega_i)|^2\delta_i + 2\mathrm{Re}\sum_{i<j=1}^{n}\nu(\omega_i)\overline{\nu(\omega_j)}\delta_i\delta_j \\ &= \left|\sum_{i=1}^{n}\nu(\omega_i)\delta_i\right|^2\end{aligned}$$

As in Example 1, we obtain

$$\int f d\mu = \sum_{i=1}^{n} \mu(\omega_i) f(\omega_i) + \sum_{i<j=1}^{n} \left[2\mathrm{Re}\, \nu(\omega_i)\overline{\nu(\omega_j)} \right] \min\left(f(\omega_i), f(\omega_j)\right)$$

Equation (2.9) suggests the following new characterization of the q-integral. If $\int |f|\, d\mu < \infty$, then f is *integrable.*

Theorem 2.4. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a q-measure space and let* $f\colon \Omega \to \mathbb{R}^+$ *be integrable. If* $\lambda\colon \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ *is the unique symmetric signed measure such that* $\mu(A) = \lambda(A \times A)$ *for all* $A \in \mathcal{A}$*, then*

$$\int f d\mu = \int \min\left(f(\omega), f(\omega')\right) d\lambda(\omega, \omega') \tag{2.10}$$

Proof. Let f be a nonnegative simple function on Ω with canonical representation (2.7). If $g(\omega, \omega') = \min\left(f(\omega), f(\omega')\right)$, then when $\omega \in A_i$, $\omega' \in A_j$ we have that

$$g(\omega, \omega') = \begin{cases} \alpha_i & \text{if } i \le j \\ \alpha_j & \text{if } j < i \end{cases}$$

Hence,

$$g = \sum_{i \le j=1}^{n} \alpha_i \chi_{A_i \times A_j} + \sum_{i>j=1}^{n} \alpha_j \chi_{A_i \times A_j}$$

It follows that

$$\begin{aligned}\int g(\omega, \omega') d\lambda(\omega, \omega') &= \sum_{i \le j=1}^{n} \alpha_i \lambda(A_i \times A_i) + \sum_{i>j=1}^{n} \alpha_j \lambda(A_i \times A_j) \\ &= \alpha_1 \left[\lambda(A_1 \times A_1) + 2\lambda(A_1 \times A_2) + \cdots + 2\lambda(A_1 \times A_n)\right] \\ &\quad + \alpha_2 \left[\lambda(A_2 \times A_2) + 2\lambda(A_2 \times A_3) + \cdots + 2\lambda(A_2 \times A_n)\right] \\ &\quad \vdots \\ &\quad + \alpha_{n-1} \left[\lambda(A_{n-1} \times_{n-1}) + 2\lambda(A_{n-1} \times A_n)\right] \\ &\quad + \alpha_n \lambda(A_n \times A_n)\end{aligned}$$

On the other hand, by (2.8) and grade-2 additivity we have

$$
\begin{aligned}
\int f d\mu = {} & \alpha_1 \left[\mu(A_1 \mathbin{\dot\cup} A_2) + \cdots + \mu(A_1 \mathbin{\dot\cup} A_n)\right. \\
& \left. -(n-1)\mu(A_1) - \mu(A_2) - \cdots - \mu(A_n)\right] \\
& + \alpha_2 \left[\mu(A_2 \mathbin{\dot\cup} A_3) + \cdots + \mu(A_2 \mathbin{\dot\cup} A_n)\right. \\
& \left. -(n-2)\mu(A_2) - \mu(A_3) - \cdots - \mu(A_n)\right] \\
& \vdots \\
& + \alpha_{n-1} \left[\mu(A_{n-1} \mathbin{\dot\cup} A_n) - \mu(A_n)\right] + \alpha_n \mu(A_n) \\
= {} & \alpha_1 \left[\lambda(A_1 \mathbin{\dot\cup} A_2 \times A_1 \mathbin{\dot\cup} A_2) + \cdots + \lambda(A_1 \mathbin{\dot\cup} A_n \times A_1 \mathbin{\dot\cup} A_n)\right. \\
& \left. -(n-1)\mu(A_1) - \mu(A_2) - \cdots - \mu(A_n)\right] \\
& + \alpha_2 \left[\lambda(A_2 \mathbin{\dot\cup} A_3 \times A_2 \mathbin{\dot\cup} A_3) + \cdots + \lambda(A_2 \mathbin{\dot\cup} A_n \times A_2 \mathbin{\dot\cup} A_n)\right. \\
& \left. -(n-2)\mu(A_2) - \mu(A_3) - \cdots - \mu(A_n)\right] \\
& \vdots \\
& + \alpha_{n-1} \left[\lambda(A_{n-1} \mathbin{\dot\cup} A_n \times A_{n-1} \mathbin{\dot\cup} A_n) - \mu(A_n)\right] + \alpha_n \mu(A_n) \\
= {} & \alpha_1 \left[\lambda(A_1 \times A_1) + 2\lambda(A_1 \times A_2) + \lambda(A_2 \times A_2) + \cdots + \lambda(A_1 \times A_1)\right. \\
& +2\lambda(A_1 \times A_n) + \lambda(A_n \times A_n) \\
& \left. -(n-1)\mu(A_1) - \mu(A_2) - \cdots - \mu(A_n)\right] \\
& + \alpha_2 \left[\lambda(A_2 \times A_2) + 2(A_2 \times A_3) + \lambda(A_3 \times A_3) + \cdots + \lambda(A_2 \times A_2)\right. \\
& +2\lambda(A_2 \times A_n) + \lambda(A_n \times A_n) \\
& \left. -(n-2)\mu(A_2) - \mu(A_3) - \cdots - \mu(A_n)\right] \\
& \vdots \\
& + \alpha_{n-1} \left[\lambda(A_{n-1} \times A_{n-1}) + 2\lambda(A_{n-1} \times A_n) + \lambda(A_n \times A_n) - \mu(A_n)\right] \\
& + \alpha_n \mu(A_n)
\end{aligned}
$$

which reduces to the expression given for $\int g(\omega,\omega')d\lambda(\omega,\omega')$. We conclude that (2.10) holds for nonnegative measurable simple functions. Since f is the limit of an increasing sequence of such functions, the result follows from the quantum dominated monotone convergence theorem [7]. □

We now apply Theorem 2.4 to compute some q-integrals for an interesting q-measure. Let $\Omega = [0,1] \subseteq \mathbb{R}$, let ν be Lebesgue measure on Ω and define the q-measure μ on the Borel σ-algebra $\mathcal{B}(\mathbb{R})$ by $\mu(A) = \nu(A)^2$. We call μ the *quantum (Lebesgue)^2 measure*. By Theorem 2.4, if $f\colon \Omega \to \mathbb{R}^+$ is integrable and $0 \le a < b \le 1$, we have

$$\int_a^b f d\mu = \int_{[a,b]} f d\mu = \int_a^b \int_a^b \min\left(f(x), f(y)\right) dx dy \tag{2.11}$$

Suppose f is increasing on Ω and let $g_y(x) = \min\left(f(x), f(y)\right)$. Then

$$g_u(x) = \begin{cases} f(x) & \text{for } x \le y \\ f(y) & \text{for } x \ge y \end{cases}$$

Hence, by (2.11) we have

$$\int_a^b f d\mu = \int_a^b \int_a^b g_y(x) dx dy = \int_a^b \left[\int_a^y f(x)dx + f(y)(b-y)\right] dy \tag{2.12}$$

Since

$$\int_a^b \int_a^y f(x) dx dy = \int_a^b \int_x^b f(x) dy dx = \int_a^b f(x)(b-x)dx$$

we conclude from (2.12) that

$$\int_a^b f d\mu = 2\int_a^b \int_a^y f(x) dx dy = 2\int_a^b f(x)(b-x)dx \tag{2.13}$$

Similarly, if f is decreasing, then

$$\int_a^b f d\mu = \int_a^b \left[f(y)(y-a) + \int_y^b f(x)dx\right] dy \tag{2.14}$$

Since

$$\int_a^b \int_y^b f(x) dx dy = \int_a^b \int_a^x f(x) dy dx = \int_a^b f(x)(x-a)dx$$

(2.14) becomes

$$\int_a^b f d\mu = 2\int_a^b \int_y^b f(x) dx dy = 2\int_a^b f(x)(x-a)dx \tag{2.15}$$

Example 3. For $n \geq 0$, since $f(x) = x^n$ is increasing, by (2.13) we have

$$\begin{aligned}\int_a^b x^n d\mu &= 2\int_a^b x^n(b-x)dx \\ &= \frac{2}{(n+1)(n+2)}\left[b^{n+2} - a^{n+1}\left((n+2)b-(n+1)a\right)\right]\end{aligned}$$

Example 4. Since $f(x) = e^x$ is increasing, by (2.13) we have

$$\int_a^b e^x dx = 2\int_a^b e^x(b-x)dx = 2e^b - 2e^a(b-a+1)$$

Example 5. For $n \geq 3$, since $f(x) = x^{-n}$ is decreasing, by (2.15) we have

$$\begin{aligned}\int_a^b x^{-n} d\mu &= 2\int_a^b x^{-n}(x-a)dx \\ &= \frac{2}{(n-1)(n-2)}\left[b^{-n+1}\left((n-2)a-(n-1)b\right) + a^{-n+2}\right]\end{aligned}$$

3 Anhomomorphic Logics

In this and the next section we shall restrict our attention to a finite outcome space $\Omega_n = \{\omega_1, \ldots, \omega_n\}$ and its corresponding set of events $\mathcal{A}_n = P(\Omega_n)$. Let $\mathbb{Z}_2$ be the two element Boolean algebra $\{0,1\}$ with the usual multiplication and with addition given by $0 \oplus 1 = 1 \oplus 0 = 1$ and $0 \oplus 0 = 1 \oplus 1 = 0$. A *coevent* on $\mathcal{A}_n$ is a truth function $\phi\colon \mathcal{A}_n \to \mathbb{Z}_2$ such that $\phi(\emptyset) = 0$ [2, 14, 15]. A coevent ϕ corresponds to a potential reality for a physical system in the sense that $\phi(A) = 1$ if A occurs and $\phi(A) = 0$ if A does not occur. For a classical system a coevent ϕ is taken to be a *homomorphism* by which we mean that

(H1) $\phi(\Omega_n) = 1$ (unital)

(H2) $\phi(A \mathbin{\dot\cup} B) = \phi(A) \oplus \phi(B)$ (additive)

(H3) $\phi(A \cap B) = \phi(A)\phi(B)$ (multiplicative)

Since there are various quantum systems for which the truth function is not a homomorphism [2, 14], at least one of these condition must fail. Denoting the set of coevents by $\mathcal{A}_n^*$, since the elements of $\mathcal{A}_n^*$ are not all homomorphisms we call $\mathcal{A}_n^*$ the *full anhomomorphic logic* [8]. Notice that the cardinality $|\mathcal{A}_n^*| = 2^{2^n-1}$ is very large.

Corresponding to $\omega_i \in \Omega_n$ we define the *evaluation map* $\omega_i^*\colon \mathcal{A}_n \to \mathbb{Z}_2$ by $\omega_i^*(A) = 1$ if and only if $\omega_i \in A$. It can be shown that a coevent ϕ is a homomorphism if and only if $\phi = \omega_i^*$ for some $i = 1, \ldots, n$ [2, 8]. Thus, there are only n possible realities for a classical system. For two coevents ϕ, ψ we define their sum and product in the usual way by $(\phi \oplus \psi)(A) = \phi(A) \oplus \psi(A)$ and $(\phi\psi)(A) = \phi(A)\psi(A)$ for all $A \in \mathcal{A}_n$. It can be shown that any coevent has a unique representation (up to order of the terms) as a polynomial in the evaluation maps.

A coevent that satisfies (H2) is called *additive* and ϕ is additive if and only if it has the form

$$\phi = c_1\omega_1^* \oplus \cdots \oplus c_n\omega_n^*$$

where $c_i = 0$ or 1, $i = 1, \ldots, n$ [2, 8, 14]. Denoting the set of additive coevents by $\mathcal{A}_{n,a}^*$, we see that $\left|\mathcal{A}_{n,a}^*\right| = 2^n$. A coevent that satisfies (H3) is called *multiplicative* and ϕ is multiplicative if and only if it has the form

$$\phi = \omega_{i_1}^*\omega_{i_2}^* \cdots \omega_{i_m}^*$$

where, by convention, $\phi = 0$ if there are no terms in the product [2, 8, 17]. Denoting the set of multiplicative coevents by $\mathcal{A}_{n,m}^*$ we again have that $\left|\mathcal{A}_{n,m}^*\right| = 2^n$. A coevent ϕ is *quadratic* or *grade*-2 if it satisfies

$$\phi(A \cup B \cup C) = \phi(A \cup B) \oplus \phi(A \cup C) \oplus \phi(B \cup C) \oplus \phi(A) \oplus \phi(B) \oplus \phi(C)$$

It can be shown that ϕ is quadratic if and only if it has the form

$$\phi = c_1\omega_1^* \oplus \cdots \oplus c_n\omega_n^* \oplus d_{12}\omega_1^*\omega_2^* \oplus \cdots \oplus d_{n-1,n}\omega_{n-1}^*\omega_n^*$$

where $c_i = 0$ or 1, $i = 1, \ldots, n$, and $d_{ij} = 0$ or 1, $i < j$, $i, j = 1, \ldots, n$ [2, 8]. Denoting the set of quadratic coevents by $\mathcal{A}_{n,q}^*$, we have that $\left|\mathcal{A}_{n,q}^*\right| = 2^{n(n+1)/2}$.

Let $\mathcal{B}$ be a Boolean algebra of subsets of a set. The set-theoretic operations on $\mathcal{B}$ are $\cup$, $\cap$ and $'$ where A' denotes the complement of $A \in \mathcal{B}$. A subset $\mathcal{B}_0 \subseteq \mathcal{B}$ is called a *subalgebra* (or *subring*) of $\mathcal{B}$ if $\emptyset \in \mathcal{B}_0$, $A \cup B$, $A \cap B$, $A \smallsetminus B \in \mathcal{B}_0$ whenever $A, B \in \mathcal{B}_0$ where $A \smallsetminus B = A \cap B'$. If $\mathcal{B}_0$ is finite, then $\mathcal{B}_0$ has a largest element C and $\mathcal{B}_0$ is itself a Boolean algebra under the operations $\cup$, $\cap$ and complement $A' = C \smallsetminus A$.

For $\phi \in \mathcal{A}_n^*$, it is of interest to find subalgebras of $\mathcal{A}_n$ on which ϕ acts classically. If $\mathcal{B}_0$ is a subalgebra of $\mathcal{A}_n$ and the restriction $\phi \mid \mathcal{B}_0$ is a homomorphism, then $\mathcal{B}_0$ is a *classical subdomain* for ϕ. A *classical domain* $\mathcal{B}$ for ϕ is a maximal classical subdomain for ϕ. That is, $\mathcal{B}$ is a classical subdomain for ϕ and if $\mathcal{C}$ is a classical subdomain for ϕ with $\mathcal{B} \subseteq \mathcal{C}$, then $\mathcal{B} = \mathcal{C}$. Any classical subdomain is contained in a classical domain $\mathcal{B}$ but $\mathcal{B}$ need not be unique. The rest of this section is devoted to the study of classical domains. For $\phi \in \mathcal{A}_n^*$, the *ϕ-center* Z_ϕ is the set of elements $A \in \mathcal{A}_n$ such that

$$\phi(B) = \phi(B \cap A) \oplus \phi(B \cap A') \tag{3.1}$$

for all $B \in \mathcal{A}_n$.

Theorem 3.1. *Z_ϕ is a subalgebra of $\mathcal{A}_n$ and $\phi \mid Z_\phi$ is additive.*

Proof. It is clear that $\emptyset, \Omega_n \in Z_\phi$ and that $A' \in Z_\phi$ whenever $A \in Z_\phi$. Suppose that $A, B \in Z_\phi$. We shall show that $A \cap B \in Z_\phi$. Since $B \in Z_\phi$, we have that

$$\phi(C \cap A) = \phi(C \cap A \cap B) \oplus \phi(C \cap A \cap B')$$

Hence,

$$\phi(C \cap A \cap B) = \phi(C \cap A) \oplus \phi(C \cap A \cap B')$$

for all $C \in \mathcal{A}_n$. It follows that

$$\begin{aligned}
\phi(C \cap A \cap B) \oplus \phi\left(C \cap (A \cap B)'\right) &= \phi(C \cap A \cap B) \oplus \phi\left(C \cap (A' \cup B')\right)\\
&= \phi(C \cap A) \oplus \phi(C \cap A \cap B') \oplus \phi\left[(C \cap A') \cup (C \cap B')\right]\\
&= \phi(C) \oplus \phi(C \cap A') \oplus \phi(C \cap A \cap B') \oplus \phi\left[(C \cap A') \cup (C \cap B')\right]\\
&= \phi(C) \oplus \phi(C \cap A') \oplus \phi(C \cap A \cap B') \oplus \phi(C \cap A \cap B')\\
&\quad \oplus \phi\left[(C \cap A') \cup (C \cap A' \cap B')\right]\\
&= \phi(C) \oplus \phi(C \cap A') \oplus \phi(C \cap A') = \phi(C)
\end{aligned}$$

Hence, $A \cap B \in Z_\phi$. Moreover, if $A, B \in Z_\phi$, then $A', B' \in Z_\phi$ so $A' \cap B' \in Z_\phi$. Hence, $A \cup B = (A' \cap B')' \in Z_\phi$. It follows that Z_ϕ is a subalgebra of $\mathcal{A}_n$. To show that $\phi \mid Z_\phi$ is additive, suppose that $A, B \in Z_\phi$ with $A \cap B = \emptyset$. We conclude that

$$\phi(A \mathbin{\dot\cup} B) = \phi\left[(A \mathbin{\dot\cup} B) \cap A\right] \oplus \phi\left[(A \mathbin{\dot\cup} B) \cap A'\right] = \phi(A) \oplus \phi(B) \qquad \square$$

A general $\phi \in \mathcal{A}_n^*$ may have many classical domains and these seem to be tedious to find. However, Z_ϕ is relatively easy to find and we now give a method for constructing classical subdomains within Z_ϕ. Since $\phi \mid Z_\phi$ is additive, we are partly there and only need to find a subalgebra Z_ϕ^1 of Z_ϕ on which ϕ is multiplicative and not 0. Indeed, the unital condition (H1) then holds because there exists an A such that $\phi(A) \neq 0$ and hence,

$$\phi(A) = \phi(A \cap C) = \phi(A)\phi(C)$$

which implies that $\phi(C) = 1$ where C is the largest element of Z_ϕ^1.

An *atom* in a Boolean algebra is a minimal nonzero element. Let $A_1, \ldots, A_m$ be the atoms of Z_ϕ. Then $A_1, \ldots, A_m$ are mutually disjoint, nonempty and $\mathbin{\dot\cup} A_i = \Omega_n$. Moreover, every nonempty set in Z_ϕ has the form $B = \mathbin{\dot\cup}_{j=1}^r A_{i_j}$. Define $A_i^* \in \mathcal{A}_n^*$ by $A_i^*(A) = 1$ if and only if $A_i \subseteq A$, $i = 1, \ldots, m$. Since ϕ is additive on Z_ϕ, it follows that ϕ has the form

$$\phi = A_{i_1}^* \oplus \cdots \oplus A_{i_r}^*$$

on Z_ϕ. We can assume without loss of generality that

$$\phi = A_1^* \oplus \cdots \oplus A_r^*$$

Let Z_ϕ^i be the subalgebra of Z_ϕ generated by $A_i, A_{r+1}, \ldots, A_m$, $i = 1, \ldots, r$. Then $\phi \mid Z_\phi^i = A_i^*$, $i = 1, \ldots, r$.

Corollary 3.2. *If $\phi \neq 0$, then Z_ϕ^i is a classical subdomain for ϕ, $i = 1, \ldots, r$.*

Proof. For simplicity, we work with Z_ϕ^1 and the other Z_ϕ^i, $i = 2, \ldots, r$ are similar. Now Z_ϕ^1 is a subalgebra of $\mathcal{A}_n$ with largest element

$$B_1 = A_1 \mathbin{\dot\cup} A_{r+1} \mathbin{\dot\cup} A_{r+2} \mathbin{\dot\cup} \cdots \mathbin{\dot\cup} A_m$$

Now $\phi \mid Z_\phi^1 = A_1^*$, A_1^* is additive and $A_1^*(B_1) = 1$. To show that A_1^* is multiplicative, we have for $A, B \in Z_\phi^1$ that $A_1^*(A \cap B) = 1$ if and only if $A_1 \subseteq A \cap B$. Since $A_1^*(A)A_1^*(B) = 1$ if and only if $A_1 \subseteq A$ and $A_1 \subseteq B$, we have that $A_1^*(A)A_1^*(B) = 1$ if and only if $A_1 \subseteq A \cap B$. Hence, $A_1^*(A \cap B) = A_1^*(A)A_1^*(B)$. We conclude that $\phi \mid Z_\phi^1 = A_1^*$ is a homomorphism on Z_ϕ^1. $\square$

Example 6. We consider some coevents in $\mathcal{A}_3^*$. For $\phi = \omega_1^*\omega_2^*$,

$$Z_\phi = \{\phi, \{\omega_3\}, \{\omega_1, \omega_2\}, \Omega_3\}$$

and Z_ϕ is the unique classical domain for ϕ. For $\psi = \omega_1^* \oplus \omega_2^*$, $Z_\psi = \mathcal{A}_3$ and the two classical domains for ψ are

$$\{\phi, \{\omega_1\}, \{\omega_3\}, \{\omega_1, \omega_3\}\}, \quad \{\phi, \{\omega_2\}, \{\omega_3\}, \{\omega_2, \omega_3\}\}$$

For $\gamma = \omega_1^* \oplus \omega_2^* \oplus \omega_3^*$, $Z_\gamma = \mathcal{A}_3$, $Z_\gamma^1 = \{\phi, \{\omega_1\}\}$, $Z_\gamma^2 = \{\phi, \{\omega_2\}\}$, $Z_\gamma^3 = \{\phi, \{\omega_3\}\}$. The classical domains for γ are

$$\{\phi, \{\omega_1\}, \{\omega_2, \omega_3\}, \Omega_3\}, \quad \{\phi, \{\omega_2\}, \{\omega_1, \omega_3\}, \Omega_3\}, \quad \{\phi, \{\omega_3\}, \{\omega_1, \omega_2\}, \Omega_3\}$$

For $\delta = \omega_1^* \omega_2^* \omega_3^*$, the unique classical domain is $Z_\delta = \{\phi, \Omega_3\}$.

4 Transferring Quantum Measures

This section provides a connection between the previous two sections. We study a method for transferring a q-measure μ on $\mathcal{A}_n$ to a measure on $\mathcal{A}_n^*$. An event $A \in \mathcal{A}_n$ is *μ-precluded* if $\mu(A) = 0$. Since a precluded event A occurs with zero propensity, if $\phi \in \mathcal{A}_n^*$ is a potential reality then A should not happen in this reality so that $\phi(A) = 0$. If $\phi(A) = 0$ whenever $\mu(A) = 0$ we say that ϕ is *μ-preclusive* [14, 15]. It is reasonable to assume that an actual reality for a system described by a q-measure μ is μ-preclusive.

If $\mathcal{A}_{n,0}^* \subseteq \mathcal{A}_n^*$, we can place a measure ν on $\mathcal{A}_{n,0}^*$ by specifying $\nu(\phi) = \nu(\{\phi\}) \geq 0$ on $\{\phi\}$ for every $\phi \in \mathcal{A}_{n,0}^*$ and extending ν to the power set $P(\mathcal{A}_{n,0}^*)$ of $\mathcal{A}_{n,0}^*$ by additivity. A q-measure μ on $\mathcal{A}_n$ *transfers* to $\mathcal{A}_{n,0}^*$ if there exists a measure ν on $P(\mathcal{A}_{n,0}^*)$ such that

$$\nu\left(\left\{\phi \in \mathcal{A}_{n,0}^* \colon \phi(A) = 1\right\}\right) = \mu(A) \tag{4.1}$$

for all $A \in \mathcal{A}_n$ [4]. The motivation for (4.1) is as follows. The set

$$\left\{\phi \in \mathcal{A}_{n,0}^* \colon \phi(A) = 1\right\}$$

is the "dual" of the event $A \in \mathcal{A}_n$ in $\mathcal{A}_{n,0}^*$ and this gives a method for transferring events in $\mathcal{A}_n$ to events in $P(\mathcal{A}_{n,0}^*)$. Then (4.1) gives a corresponding transfer of the q-measure μ on $\mathcal{A}_n$ to a measure ν on $P(\mathcal{A}_{n,0}^*)$. The transferred measure ν is not unique, in general. If μ transfers to ν, then the quantum dynamics given by μ can be described by a classical dynamics given by ν. Moreover, it follows from (4.1) that the set of nonpreclusive coevents in $\mathcal{A}_{n,0}^*$ has ν measure zero.

It is clear that if μ transfers to $\mathcal{A}_{n,0}^* \subseteq \mathcal{A}_n^*$ then μ transfers to $\mathcal{A}_{n,1}^*$ for any $\mathcal{A}_{n,0}^* \subseteq \mathcal{A}_{n,1}^* \subseteq \mathcal{A}_n^*$. For a q-measure μ on $\mathcal{A}_n$ we define the *μ-preclusive anhomomorphic logic* $\mathcal{A}_{n,\mu}^*$ by

$$\mathcal{A}_{n,\mu}^* = \{\phi \in \mathcal{A}_n^* \colon \phi \text{ is } \mu\text{-preclusive}\}$$

A coevent $\phi \in \mathcal{A}_n^*$ can be considered as a map $\phi\colon \mathcal{A}_n \to \{0,1\}$ where we view $\{0,1\} \subseteq \mathbb{R}$ with the usual addition and multiplication. For $A \in \mathcal{A}_n$ with $A \neq \emptyset$ we define $\phi_A \in \mathcal{A}_n^*$ by $\phi_A(B) = 1$ if and only if $B = A$.

Theorem 4.1. *Let μ be a q-measure on $\mathcal{A}_n$.* (a) *For $\mathcal{A}_{n,0}^* \subseteq \mathcal{A}_n^*$, μ transfers to ν on $P(\mathcal{A}_{n,0}^*)$ if and only if*

$$\mu = \sum \left\{\nu(\phi)\phi \colon \phi \in \mathcal{A}_{n,0}^*\right\} \tag{4.2}$$

(b) *μ transfers to $\mathcal{A}_{n,\mu}^*$.*

Proof. (a) If μ has the form (4.2) then for any $A \in \mathcal{A}_n$ we have

$$\begin{aligned}\mu(A) &= \sum \left\{\nu(\phi)\phi(A) \colon \phi \in \mathcal{A}_{n,0}^*\right\} = \sum \left\{\nu(\phi) \colon \phi \in \mathcal{A}_{n,0}^*, \phi(A) = 1\right\} \\ &= \nu\left(\left\{\phi \in \mathcal{A}_{n,0}^* \colon \phi(A) = 1\right\}\right)\end{aligned}$$

Conversely, if μ transfers to ν on $P(\mathcal{A}_{n,0}^*)$ then

$$\begin{aligned}\mu(A) &= \nu\left(\left\{\phi \in \mathcal{A}_{n,0}^* \colon \phi(A) = 1\right\}\right) = \sum \left\{\nu(\phi) \colon \phi \in \mathcal{A}_{n,0}^*, \phi(A) = 1\right\} \\ &= \sum \left\{\nu(\phi)\phi(A) \colon \phi \in \mathcal{A}_{n,0}^*\right\}\end{aligned}$$

We conclude that (4.2) holds.
(b) It is clear that

$$\mu = \sum \{\mu(A)\phi_A \colon A \in \mathcal{A}, \mu(A) \neq 0\} \tag{4.3}$$

Define the measure ν on $P(\mathcal{A}_{n,\mu}^*)$ by $\nu(\phi_A) = \mu(A)$ if $\mu(A) \neq 0$ and $\nu(\phi) = 0$ otherwise. Applying (4.3) we have

$$\mu = \sum \left\{\nu(\phi_A)\phi_A \colon \phi_A \in \mathcal{A}_{n,\mu}^*\right\} = \sum \left\{\nu(\phi)\phi \colon \phi \in \mathcal{A}_{n,\mu}^*\right\}$$

Hence, (4.2) holds and the result follows from (a). $\square$

We conclude from Theorem 4.1(b) that any q-measure μ on $\mathcal{A}_n$ transfers to $\mathcal{A}_n^*$. However, it is desirable to transfer μ to $\mathcal{A}_{n,0}^*$ where $\mathcal{A}_{n,0}^* \subseteq \mathcal{A}_n^*$ is as small as possible. This is because such an $\mathcal{A}_{n,0}^*$ will give the simplest classical dynamics. We can always reduce to preclusive coevents in the sense that if μ transfers to $\mathcal{A}_{n,0}^*$ then μ transfers to $\mathcal{A}_{n,0}^* \cap \mathcal{A}_{n,\mu}^*$. Let $\mathcal{A}_{n,b}^*$ be the anhomomorphic logic given by

$$\mathcal{A}_{n,b}^* = \{\phi_A \colon A \in \mathcal{A}_n, A \neq \emptyset\}$$

It follows from the proof of Theorem 4.1(b) that any q-measure μ on $\mathcal{A}_n$ transfers to $\mathcal{A}_{n,b}^*$. We conclude that μ transfers to $\mathcal{A}_{n,b}^* \cap \mathcal{A}_{n,\mu}^*$.

The next Corollary shows what happens if the q-measure μ turns out to be a measure.

Corollary 4.2. *Suppose $\mathcal{A}_{n,0}^*$ satisfies $\mathcal{A}_{n,a}^* \subseteq \mathcal{A}_{n,0}^* \subseteq \mathcal{A}_n^*$. If μ is a measure on $\mathcal{A}_n$, then μ transfers to a measure ν on $\mathcal{A}_{n,0}^*$ satisfying $\nu(\phi) = 0$ unless ϕ is additive. Conversely, if a measure ν on $\mathcal{A}_{n,0}^*$ satisfies $\mu(\phi) = 0$ unless ϕ is additive, then there is a measure on $\mathcal{A}_n$ that transfers to ν.*

Proof. If μ is a measure on $\mathcal{A}_n$ then μ has the form

$$\mu = \sum_{i=1}^{n} \lambda_i \delta_{\omega_i} \tag{4.4}$$

where $\lambda_i \geq 0$, $i = 1, \ldots, n$. Hence, μ transfers to

$$\nu = \sum_{i=1}^{n} \lambda_i \delta_{\omega_i^*} \tag{4.5}$$

on $\mathcal{A}_{n,0}^*$ where $\nu(\phi) = 0$ unless ϕ is additive. Conversely, if ν is a measure on $\mathcal{A}_{n,0}^*$ satisfying $\nu(\phi) = 0$ unless ϕ is additive, then ν has the form (4.5). Hence, the measure μ given by (4.4) transfers to ν. □

The next theorem shows that the multiplicative anhomomorphic logic $\mathcal{A}_{n,m}^*$ is not adequate for transferring q-measures. This result was proved in [2, 4]. However, our proof is simpler and more direct.

Theorem 4.3. *If a q-measure μ on $\mathcal{A}_n$ transfers to a measure ν on $\mathcal{A}_{n,m}^*$, then $\nu(\phi) = 0$ unless ϕ is quadratic.*

Proof. We have that

$$\mathcal{A}_{n,m}^* = \left\{0, \omega_1^*, \dots, \omega_n^*, \omega_1^*\omega_2^*, \dots, \omega_{n-1}^*\omega_n^*, \omega_1^*\omega_2^*\omega_3^* \dots, \omega_1^*\omega_2^* \cdots \omega_n^*\right\}$$

Since $\mu(A) = \nu\left(\left\{\phi \in \mathcal{A}_{n,m}^* \colon \phi(A) = 1\right\}\right)$ we have that $\mu(\omega_i) = \nu(\omega_i^*)$, $i = 1, \dots, n$ and

$$\mu\left(\{\omega_i, \omega_j\}\right) = \nu(\omega_i^*) + \nu(\omega_j^*) + \nu(\omega_i^*\omega_j^*)$$

$i, j = 1, \dots, n$. Now

$$\begin{aligned}\mu\left(\{\omega_1, \omega_2, \omega_3\}\right) &= \nu(\omega_1^*) + \nu(\omega_2^*) + \nu(\omega_3^*) + \nu(\omega_1^*\omega_2^*) + \nu(\omega_1^*\omega_3^*) \\ &\quad + \nu(\omega_2^*\omega_3^*) + \nu(\omega_1^*\omega_2^*\omega_3^*)\end{aligned} \tag{4.6}$$

Since μ is grade-2 additive we have that

$$\begin{aligned}\mu\left(\{\omega_1, \omega_2, \omega_3\}\right) &= \sum_{i<j=1}^{3} \mu\left(\{\omega_i, \omega_j\}\right) - \sum_{i=1}^{3} \mu(\omega_i) \\ &= \sum_{i<j=1}^{3} \nu(\omega_i^*\omega_j^*) + \sum_{i=1}^{n} \nu(\omega_i^*)\end{aligned} \tag{4.7}$$

Comparing (4.6) and (4.7) shows that $\nu(\omega_1^*\omega_2^*\omega_3^*) = 0$. In a similar way, we conclude that $\nu(\omega_i^*\omega_j^*\omega_k^*) = 0$, $i, j, k = 1, \dots, n$, $i < j < k$. Next,

$$\begin{aligned}\mu\left(\{\omega_1, \omega_2, \omega_3, \omega_4\}\right) &= \nu(\omega_1^*) + \cdots + \nu(\omega_4^*) + \nu(\omega_1^*\omega_2^*) \\ &\quad + \cdots + \nu(\omega_3^*\omega_4^*) + \nu(\omega_1^*\omega_2^*\omega_3^*\omega_4^*)\end{aligned} \tag{4.8}$$

Since μ is grade-2 additive we have that

$$\begin{aligned}\mu\left(\{\omega_1, \omega_2, \omega_3, \omega_4\}\right) &= \sum_{i<j=1}^{4} \mu\left(\{\omega_i, \omega_j\}\right) - 2\sum_{i=1}^{4} \mu(\omega_i) \\ &= \sum_{i<j=1}^{4} \nu(\omega_i^*\omega_j^*) + \sum_{i=1}^{4} \nu(\omega_i^*)\end{aligned} \tag{4.9}$$

Comparing (4.8) and (4.9) shows that $\nu(\omega_1^*\omega_2^*\omega_3^*\omega_4^*) = 0$. In a similar way, we conclude that $\nu(\omega_i^*\omega_j^*\omega_k^*\omega_l^*) = 0$, $i, j, k, l = 1, \dots, n$, $i < j < k < l$. Continuing by induction, we have that $\nu(\phi) = 0$ unless ϕ is quadratic. $\square$

The result (and proof) in Theorem 4.3 holds if $\mathcal{A}_{n,m}^*$ is replaced by any subset $\mathcal{A}_{n,0}^* \subseteq \mathcal{A}_{n,m}^*$. Theorem 4.3 also shows that for $\mathcal{A}_{n,0}^* \subseteq \mathcal{A}_{n,m}^*$, if a q-measure μ transfers from $\mathcal{A}_n$ to $\mathcal{A}_{n,0}^*$ then

$$\mu\left(\{\omega_i, \omega_j\}\right) \geq \mu(\omega_i) + \mu(\omega_j)$$

$i, j = 1, \ldots, n$, $i \neq j$. Hence, not all q-measures can be transferred to $\mathcal{A}_{n,m}^*$.

We now investigate a method for finding a "small" $\mathcal{A}_{n,0}^*$ to which a q-measure on $\mathcal{A}_n$ transfers. As we shall see this method only works for a specific but large class of q-measures. A q-measure μ on $\mathcal{A}_n$ is *pure* if the values of μ are contained in $\{0,1\}$. Let $\mathcal{M}(\mathcal{A}_n)$ be the set of q-measures μ on $\mathcal{A}_n$ such that

$$\max\left\{\mu(A)\colon A \in \mathcal{A}_n\right\} \leq 1$$

We have seen in Section 2 that the set of signed q-measures on $\mathcal{A}_n$ forms a finite dimensional real linear space $\mathcal{S}(\mathcal{A}_n)$ and it is clear that $\mathcal{M}(\mathcal{A}_n)$ is a convex subset of $\mathcal{S}(\mathcal{A}_n)$. Letting $\mathcal{P}(\mathcal{A}_n)$ be the set of pure q-measures, we now show that the elements of $\mathcal{P}(\mathcal{A}_n)$ are extremal in $\mathcal{M}(\mathcal{A}_n)$. Let $\mu \in \mathcal{P}(\mathcal{A}_n)$ and suppose that $\mu = \lambda\mu_1 + (1-\lambda)\mu_2$ for $0 < \lambda < 1$ and $\mu_1, \mu_2 \in \mathcal{M}(\mathcal{A}_n)$. If $\mu(A) = 0$, then

$$\lambda\mu_1(A) + (1-\lambda)\mu_2(A) = 0$$

so that $\mu_1(A) = \mu_2(A) = 0$. If $\mu(A) = 1$, then

$$\lambda\mu_1(A) + (1-\lambda)\mu(A) = 1$$

so that $\mu_1(A) = \mu_2(A) = 1$. Hence, $\mu_1 = \mu_2 = \mu$ which shows that μ is extremal.

Denoting the set of extremal elements of $\mathcal{M}(\mathcal{A}_n)$ by $\operatorname{Ext}\mathcal{M}(\mathcal{A}_n)$ we have shown that $\mathcal{P}(\mathcal{A}_n) \subseteq \operatorname{Ext}\mathcal{M}(\mathcal{A}_n)$. Our main question now is whether $\mathcal{P}(\mathcal{A}_n) = \operatorname{Ext}\mathcal{M}(\mathcal{A}_n)$. We first illustrate that $\mathcal{P}(\mathcal{A}_2) = \operatorname{Ext}\mathcal{M}(\mathcal{A}_2)$. Let $\mu \in \operatorname{Ext}\mathcal{M}(\mathcal{A}_2)$. If $\mu \notin \mathcal{P}(\mathcal{A}_2)$ then at least one of the numbers $\mu(\omega_1)$, $\mu(\omega_2)$, $\mu(\Omega_2)$ is not 0 or 1. Suppose, for example that $0 < \mu(\omega_1) < 1$. Then there exists an $\varepsilon > 0$ such that $\varepsilon < \mu(\omega_1) < 1 - \varepsilon$. Define $\mu_1\colon \mathcal{A}_2 \to \mathbb{R}^+$ by $\mu_1(A) = \mu(A)$ if $A \neq \{\omega_1\}$ and $\mu_1(\omega_1) = \mu(\omega_1) + \varepsilon$. Also, define $\mu_2\colon \mathcal{A}_2 \to \mathbb{R}^+$ by $\mu_2(A) = \mu(A)$ if $A \neq \{\omega_1\}$ and $\mu_2(\omega_1) = \mu(\omega_1) - \varepsilon$. Then $\mu_1, \mu_2 \in \mathcal{M}(\mathcal{A}_2)$, $\mu_1 \neq \mu_2$ and $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$. This contradicts the fact that μ is extremal. Hence, $\mu \in \mathcal{P}(\mathcal{A}_2)$. The other cases are similar so $\mathcal{P}(\mathcal{A}_2) = \operatorname{Ext}\mathcal{M}(\mathcal{A}_2)$.

Theorem 4.4. $\mathcal{P}(\mathcal{A}_3) = \operatorname{Ext}\mathcal{M}(\mathcal{A}_3)$.

Proof. To show that $\operatorname{Ext}\mathcal{M}(\mathcal{A}_3) \subseteq \mathcal{P}(\mathcal{A}_3)$ suppose that $\mu \in \operatorname{Ext}\mathcal{M}(\mathcal{A}_3)$ and $\mu \notin \mathcal{P}(\mathcal{A}_3)$. Then one of the numbers $\mu(A)$, $A \in \mathcal{A}\ _3 \smallsetminus \{\emptyset\}$ is not 0 or 1. Since

$$\mu(\Omega_3) = \mu\left(\{\omega_1,\omega_2\}\right) + \mu\left(\{\omega_1,\omega_3\}\right) + \mu\left(\{\omega_2,\omega_3\}\right) - \mu(\omega_1) - \mu(\omega_2) - \mu(\omega_3) \tag{4.10}$$

a second of these numbers is not 0 or 1. There are various possibilities, all of which are similar, and we shall consider two of them. Suppose, for example, that $0 < \mu\left(\{\omega_1,\omega_2\}\right) < 1$ and $0 < \mu(\omega_3) < 1$. Then there exists an $\varepsilon > 0$ such that $\varepsilon < \mu\left(\{\omega_2,\omega_2\}\right) < 1 - \varepsilon$ and $\varepsilon < \mu(\omega_3) < 1 - \varepsilon$. Define $\mu_1 \colon \mathcal{A}_3 \to \mathbb{R}^+$ by $\mu_1(A) = \mu(A)$ for $A \neq \{\omega_1,\omega_2\}$ or $\{\omega_3\}$ and $\mu_1(\omega_3) = \mu(\omega_3) + \varepsilon$, $\mu_1\left(\{\omega_1,\omega_2\}\right) = \mu\left(\{\omega_1,\omega_2\}\right) + \varepsilon$. Thus, μ_1 satisfies (4.10) so $\mu_1 \in \mathcal{M}(\mathcal{A}_3)$. Define $\mu_2 \colon \mathcal{A}_3 \to \mathbb{R}^+$ by $\mu_2(A) = \mu(A)$ for $A \neq \{\omega_1,\omega_2\}$ or $\{\omega_3\}$ and $\mu_2(\omega_3) = \mu(\omega_3) - \varepsilon$, $\mu_2\left(\{\omega_1,\omega_2\}\right) = \mu\left(\{\omega_1,\omega_2\}\right) - \varepsilon$. Again μ_2 satisfies (4.10) so $\mu_2 \in \mathcal{M}(\mathcal{A}_3)$. Also, $\mu_1 \neq \mu_2$ and $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ which contradicts the fact that $\mu \in \operatorname{Ext}\mathcal{M}(\mathcal{A}_3)$. As another case, suppose that $0 < \mu(\omega_1) < 1$ and $0 < \mu(\Omega_3) < 1$. Then there exists an $\varepsilon < \mu(\omega) < 1 - \varepsilon$ and $\varepsilon < \mu(\Omega_3) < 1 - \varepsilon$. Define $\mu_1, \mu_2 \colon \mathcal{A}_3 \to \mathbb{R}^+$ by $\mu_1(A) = \mu_2(A) = \mu(A)$ for all $A \neq \{\omega_1\}$ or Ω_3 and $\mu_1(\omega_1) = \mu(\omega_1) + \varepsilon$, $\mu_1(\Omega_3) = \mu(\Omega_3) - \varepsilon$, $\mu_2(\omega_1) = \mu(\omega_1) - \varepsilon$, $\mu_2(\Omega_3) + \varepsilon$. Then μ_1, μ_2 satisfy (4.10) so $\mu_1, \mu_2 \in \mathcal{M}(\mathcal{A}_3)$. Moreover, $\mu_1 \neq \mu_2$ and $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$ which contradicts the fact that $\mu \in \operatorname{Ext}\mathcal{M}(\mathcal{A}_3)$. Since this method applies to all the cases, we conclude that $\operatorname{Ext}\mathcal{M}(\mathcal{A}_3) \subseteq \mathcal{P}(\mathcal{A}_3)$ and the result follows. □

Even though $\mathcal{P}(\mathcal{A}_n) = \operatorname{Ext}\mathcal{M}(\mathcal{A}_n)$ for $n = 2, 3$ and the counterpart of this result for measures holds for all n, the result does not hold for all n. The next example shows that $\mathcal{P}(\mathcal{A}_{16}) \neq \operatorname{Ext}\mathcal{M}(\mathcal{A}_{16})$ It would be interesting to find the smallest n such that $\mathcal{P}(\mathcal{A}_n) \neq \operatorname{Ext}\mathcal{M}(\mathcal{A}_n)$ and describe the form of elements in $\operatorname{Ext}\mathcal{M}(\mathcal{A}_n) \smallsetminus \mathcal{P}(\mathcal{A}_n)$.

Example 7. It follows from an example in Section 6.1.4 [2] that there is a $\mu \in \mathcal{M}(\mathcal{A}_{16})$ with $\mu(\Omega_{16}) \neq 0$ and there exist no unital, μ-preclusive $\phi \in \mathcal{A}_{16,q}^*$. Now suppose that $\mathcal{P}(\mathcal{A}_{16}) = \operatorname{Ext}\mathcal{M}(\mathcal{A}_{16})$. Then it follows from the Krein-Milman theorem that μ has the form $\mu = \sum \lambda_i\mu_i$, $\lambda_i > 0$, $\mu_i \in \mathcal{P}(\mathcal{A}_{16})$, $i = 1, \ldots, n$. Now μ_i can be considered as a coevent and our next result shows that μ_i is quadratic, $i = 1, \ldots, n$. Moreover, it is clear from the form of μ that μ_i is μ-preclusive, $i = 1, \ldots, n$. It follows that μ_i is not unital so $\mu_i(\Omega_{16}) = 0$, $i = 1, \ldots, n$. But this contradicts the fact that $\mu(\Omega_{16}) \neq 0$. Hence, $\mathcal{P}(\mathcal{A}_{16}) \neq \operatorname{Ext}\mathcal{M}(\mathcal{A}_{16})$.

If $\phi \in \mathcal{A}_n^*$ satisfies the condition $\phi \in \mathcal{M}(\mathcal{A}_n)$ or equivalently $\phi \in \mathcal{P}(\mathcal{A}_n)$, then we say that ϕ is a *pure coevent*. Conversely, if μ is a pure q-measure, then we call the map $\widehat{\mu}\colon \mathcal{A}_n \to \mathbb{Z}_2$ with the same values as μ the *corresponding* pure coevent. The set of all pure coevents in $\mathcal{A}_n^*$ is denoted by $\mathcal{A}_{n,p}^*$ and is called the *pure anhomomorphic logic*. It is clear that every coevent in $\mathcal{A}_2^*$ is pure so that $\mathcal{A}_{2,p}^* = \mathcal{A}_2^*$. However, there are only 34 pure coevents in $\mathcal{A}_3^*$ out of a total $2^{2^3-1} = 128$ coevents [8].

Example 8. Examples of pure coevents in $\mathcal{A}_3^*$ are ω_1^*, $\omega_1^* \oplus \omega_2^*$, $\omega_1^* \oplus \omega_1^*\omega_2^*$ $\omega_1^*\omega_2^*$, $\omega_1^* \oplus \omega_2^* \oplus \omega_1^*\omega_2^*$, $\omega_1^* \oplus \omega_1^*\omega_2^* \oplus \omega_2^*\omega_3^*$, $\omega_1^* \oplus \omega_2^* \oplus \omega_1^*\omega_2^* \oplus \omega_1^*\omega_3^*$, $\omega_1^* \oplus \omega_1^*\omega_2^* \oplus \omega_1^*\omega_3^* \oplus \omega_2^*\omega_3^*$, $\omega_1^* \oplus \omega_2^* \oplus \omega_3^* \oplus \omega_1^*\omega_2^* \oplus \omega_1^*\omega_3^* \oplus \omega_2^*\omega_3^*$ and the rest are obtained by symmetry. An example of a $\phi \in \mathcal{A}_3^*$ that is not pure is $\phi = \omega_1^* \oplus \omega_2^* \oplus \omega_3^*$. Indeed, $\phi(\Omega_3) = 1$ and

$$\phi\left(\{\omega_1, \omega_2\}\right) + \phi\left(\{\omega_1, \omega_3\}\right) + \phi\left(\{\omega_2, \omega_3\}\right) - \phi(\omega_1) - \phi(\omega_2) - \phi(\omega_3) = -3$$

Another example of a nonpure element of $\mathcal{A}_3^*$ is $\psi = \omega_1^* \oplus \omega_2^*\omega_3^*$. Indeed $\psi(\Omega_3) = 0$ and

$$\psi\left(\{\omega_1, \omega_2\}\right) + \psi\left(\{\omega_1, \omega_3\}\right) + \psi\left(\{\omega_2, \omega_3\}\right) - \psi(\omega_1) - \psi(\omega_2) - \psi(\omega_3) = 2$$

Lemma 4.5. *If $\phi \in \mathcal{A}_{n,p}^*$, then ϕ is quadratic.*

Proof. We must show that if ϕ satisfies

$$\phi(A \cup B \cup C) = \phi(A \cup B) + \phi(A \cup C) + \phi(B \cup C) - \phi(A) - \phi(B) - \phi(C) \quad (4.11)$$

then ϕ satisfies

$$\phi(A \cup B \cup C) = \phi(A \cup B) \oplus \phi(A \cup C) \oplus \phi(B \cup C) \oplus \phi(A) \oplus \phi(B) \oplus \phi(C) \quad (4.12)$$

Suppose the left hand side of (4.12) is 1. Then there are an odd number of 1s on the right hand side of (4.11). Hence, the right hand side of (4.12) is 1. Suppose the left hand side of (4.12) is 0. Then there are an even number of 1s on the right hand side of (4.11). Hence, the right hand side of (4.12) is 0. We conclude that (4.12) holds so ϕ is quadratic. □

Let $\mathcal{C}(\mathcal{A}_n)$ be the positive cone generated by the set $\mathcal{P}(\mathcal{A}_n)$. Thus $\mu \in \mathcal{C}(\mathcal{A}_n)$ if and only if μ has the form $\mu = \sum \lambda_i \mu_i$, $\lambda_i > 0$, $\mu_i \in \mathcal{P}(\mathcal{A}_n)$. The next result follows from Theorem 4.1(a).

Corollary 4.6. *A q-measure μ transfers to $\mathcal{A}_{n,p}^*$ if and only if $\mu \in \mathcal{C}(\mathcal{A}_n)$.*

Motivated by Corollary 4.6 one might conjecture that if every q-measure in $\mathcal{C}(\mathcal{A}_n)$ transfers to $\mathcal{A}_{n,0}^*$, then $\mathcal{A}_{n,p}^* \subseteq \mathcal{A}_{n,0}^*$; that is, $\mathcal{A}_{n,p}^*$ is the smallest subset of $\mathcal{A}_n^*$ to which every q-measure in $\mathcal{C}(\mathcal{A}_n)$ transfers. The next example shows that this conjecture does not hold.

Example 9. We have that

$$\mathcal{A}_{2,p}^* = \mathcal{A}_2^* = \{0, \omega_1^*, \omega_2^*, \omega_1^* \oplus \omega_2^*, \omega_1^*\omega_2^*, \omega_1^* \oplus \omega_1^*\omega_2^*, \omega_2^* \oplus \omega_1^*\omega_2^*, 1\}$$

Let $\phi_1 = \omega_1^*$, $\phi_2 = \omega_2^*$, $\phi_3 = \omega_1^* \oplus \omega_2^*$, $\phi_4 = \omega_1^*\omega_2^*$, $\phi_5 = \omega_1^* \oplus \omega_1\omega_2^*$ and $\phi_6 = \omega_2^* \oplus \omega_1^*\omega_2^*$. If μ is an arbitrary q-measure on $\mathcal{A}_2$, then μ transfers to a measure ν on $\mathcal{A}_2^*$ and we have that

$$\begin{aligned}
\mu(\omega_1) &= \nu\left(\{\phi \in \mathcal{A}_2^* \colon \phi(\omega_1) = 1\}\right) = \nu(\phi_1) + \nu(\phi_3) + \nu(\phi_5) + \nu(1)\\
\mu(\omega_2) &= \nu\left(\{\phi \in \mathcal{A}_2^* \colon \phi(\omega_2) = 1\}\right) = \nu(\phi_2) + \nu(\phi_3) + \nu(\phi_6) + \nu(1)\\
\mu(\Omega_2) &= \nu\left(\{\phi \in \mathcal{A}_2^* \colon \phi(\Omega_2) = 1\}\right) = \nu(\phi_1) + \nu(\phi_2) + \nu(\phi_4) + \nu(1)
\end{aligned}$$

Letting $\mathcal{A}_{2,0}^* = \{\phi_4, \phi_5, \phi_6\}$, we have that every q-measure on $\mathcal{A}_2$ transfers to $\mathcal{A}_{2,0}^*$. In fact, if μ is a q-measure on $\mathcal{A}_2$, then μ transfers to the measure ν on $\mathcal{A}_{2,0}^*$ given by $\nu(\phi_4) = \mu(\Omega_2)$, $\nu(\phi_5) = \mu(\omega_1)$, $\nu(\phi_6) = \mu(\omega_2)$. This example also shows that the measure that μ transfers to need not be unique. For instance, let μ be the q-measure on $\mathcal{A}_2$ given by $\mu(\omega_1) = 1$, $\mu(\omega_2) = 1$, $\mu(\Omega_2) = 0$. Then μ transfers to ν_1 on $\mathcal{A}_2^*$ given by $\nu_1(\phi_5) = \nu_1(\phi_6) = 1$ and $\nu(\phi) = 0$, $\phi \neq \phi_5, \phi_6$. Also, μ transfers to ν_2 on $\mathcal{A}_2^*$ given by $\nu_2(\phi_3) = 1$ and $\nu(\phi) = 0$ for $\phi \neq \phi_3$.

Example 10. We use the same notation as in Example 9. We first show that a q-measure μ on $\mathcal{A}_2$ transfers to

$$\mathcal{A}_{2,m}^* = \{0, \phi_1, \phi_2, \phi_4\}$$

if and only if $\mu(\Omega_2) \geq \mu(\omega_1) + \mu(\omega_2)$. If μ transfers to ν, then $\mu(\omega_1) = \nu(\phi_1)$, $\mu(\omega_2) = \nu(\phi_2)$ and

$$\mu(\Omega_2) = \nu(\phi_1) + \nu(\phi_2) + \nu(\phi_4)$$

Hence, $\mu(\Omega_2) \geq \mu(\omega_1 + \mu(\omega_2)$. Conversely, if $\mu(\Omega_2) \geq \mu(\omega_1) + \mu(\omega_2)$ then letting $\nu(\phi_1) = \mu(\omega_1)$, $\nu(\phi_2) = \mu(\omega_2)$, $\nu(0) = 0$ and

$$\nu(\phi_4) = \mu(\Omega_2) - \mu(\omega_1) - \mu(\omega_2)$$

we see that μ transfers to ν on $\mathcal{A}_{2,m}^*$. We next show that a q-measure μ on $\mathcal{A}_2$ transfers to

$$\mathcal{A}_{2,a}^* = \{0, \phi_1, \phi_2, \phi_3\}$$

if and only if

$$|\mu(\omega_1) - \mu(\omega_2)| \leq \mu(\Omega_2) \leq \mu(\omega_1) + \mu(\omega_2)$$

If μ transfers to ν, then $\mu(\omega_1) = \nu(\phi_1) + \nu(\phi_3)$, $\mu(\omega_2) = \nu(\phi_2) + \nu(\phi_3)$ and $\mu(\Omega_2) = \nu(\phi_1) + \nu(\phi_2)$. Hence, $\mu(\Omega_2) \leq \mu(\omega_1) + \mu(\omega_2)$ and

$$\mu(\omega_1) - \mu(\omega_2) = \nu(\phi_1) + \nu(\phi_2) \leq \mu(\Omega_2)$$
$$\mu(\omega_2) - \mu(\omega_1) = \nu(\phi_2) - \nu(\phi_1) \leq \mu(\Omega_2)$$

so the given inequalities hold. Conversely, if the inequalities hold, then letting $\nu(0) = 0$ and

$$\nu(\phi_1) = \tfrac{1}{2}\left[\mu(\Omega_2) + \mu(\omega_1) - \mu(\omega_2)\right]$$
$$\nu(\phi_2) = \tfrac{1}{2}\left[\mu(\Omega_2) - \mu(\omega_1) + \mu(\omega_2)\right]$$
$$\nu(\phi_3) = \tfrac{1}{2}\left[\mu(\omega_1) + \mu(\omega_2) - \mu(\Omega_2)\right]$$

we see that μ transfers to ν on $\mathcal{A}_{2,a}^*$.

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