ISOMORPHISMS OF NONCOMMUTATIVE DOMAIN ALGEBRAS II

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Abstract. This paper extends the results of the previous work of the authors on the classification on noncommutative domain algebras up to completely isometric isomorphism. Using Sunada's classification of Reinhardt domains in $\mathbb{C}^n$, we show that aspherical noncommutative domain algebras are isomorphic if and only if their defining symbols are equivalent, in the sense that one can be obtained from the other via permutation and scaling of the free variables. Our result also shows that the automorphism groups of aspherical noncommutative domain algebras consists of a subgroup of some finite dimensional unitary group. We conclude by illustrating how our methods can be used to extend to noncommutative domain algebras some results from analysis in $\mathbb{C}^n$ with the example of Cartan's lemma.

1. Introduction

Noncommutative domain algebras, introduced by Popescu [5], provide a generalization of noncommutative disk algebras and serve as universal operator algebras for a large class of noncommutative domains, i.e. noncommutative analogues of domains in $\mathbb{C}^n$. This paper extends the results of the previous work of the authors [1] on the classification on noncommutative domain algebras up to completely isometric isomorphism. Our present work uses the classification of Reinhardt domains by Sunada in [8] to provide a complete classification a large class of noncommutative domain algebras in terms of their defining symbol. This class consists of the aspherical noncommutative domain algebras whose symbol is polynomial, as we shall define in the first section of this paper.

Let us introduce the objects of interest in this paper, as well as the notations we will use throughout our exposition. Popescu noncommutative domains are defined by means of a symbol, which is a special type of formal power series. Let $\mathbb{F}_n^+$ be the free semigroup on n generators $g_1,\ldots,g_n$ and identity g_0. If $X_1,\ldots,X_n$ belong to any ring, and if $\alpha\in\mathbb{F}_n^+$ is written as $\alpha = g_{i_1}\cdots g_{i_n}$, then we write $X_\alpha = X_{i_1}\cdots X_{i_n}$. With these notations, a free formal power series $f=\sum_{\alpha\in\mathbb{F}_n^+} a_\alpha X_\alpha$

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with real coefficients a_α ($\alpha \in \mathbb{F}_n^+$) is *regular positive* if:

$$\begin{cases} a_{g_0} = 0, \\ a_{g_i} > 0 \text{ if } i = 1, \ldots, n, \\ a_\alpha \geq 0 \text{ for all } \alpha \in \mathbb{F}_n^+ \\ \sup_{n \in \mathbb{N}^*} \left(\left| \sum_{|\alpha|=n} a_\alpha^2 \right|^{\frac{1}{n}} \right) < \infty. \end{cases} \tag{1.1}$$

Let us be given a regular positive free formal power series $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha X_\alpha$ in n indeterminates, and a Hilbert space $\mathcal{H}$. We define the noncommutative domain:

$$\mathcal{D}_f(\mathcal{H}) = \{(T_1, \ldots, T_n) \in \mathcal{B}(\mathcal{H}) : \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha T_\alpha T_\alpha^* \leq 1\}$$

where $\mathcal{B}(\mathcal{H})$ is the Von Neumann algebra of all bounded linear operators on $\mathcal{H}$ and 1 is its identity. The interior of $\mathcal{D}_f(\mathbb{C}^k)$ will be denoted by $\mathbb{D}_f^k$. Popescu proved in [5] that certain weighted shifts on the full Fock space $\ell^2(\mathbb{F}_n^+)$ of $\mathbb{C}^n$ provide a model for all n-tuples in all domains $\mathcal{D}_f(\mathcal{H})$. Specifically, as shown in [5], one can find positive real numbers $(b_\alpha)_{\alpha \in \mathbb{F}_n^+}$ and define linear operators W_i^f on $\ell^2(\mathbb{F}_n^+)$ such that:

$$W_i^f \delta_\alpha = \sqrt[2]{\frac{b_\alpha}{b_{g_i \alpha}}} \delta_{g_i \alpha},$$

where $\{\delta_\alpha : \alpha \in \mathbb{F}_n^+\}$ is the canonical basis of $\ell^2(\mathbb{F}_n^+)$. Then $(W_1^f, \ldots, W_n^f) \in \mathcal{D}_f(\ell^2(\mathbb{F}_n^+))$ and the coefficients $(a_\alpha)_{\alpha \in \mathbb{F}_n^+}$ can be recovered from the coefficients $(b_\alpha)_{\alpha \in \mathbb{F}_n^+}$. We then define:

$$\mathcal{A}(\mathcal{D}_f) = \overline{\operatorname{span}\{W_\alpha : \alpha \in \mathbb{F}_n^+\}}$$

i.e. $\mathcal{A}(\mathcal{D}_f)$ is the norm closure in $\mathcal{B}(\mathcal{H})$ of the algebra generated by $W_1^f, \ldots, W_n^f$. The fundamental property of this algebra is that:

Proposition 1.1 (Popescu)**.** *Let $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha X_\alpha$ be a regular positive free power series in n indeterminates. Let $\mathcal{H}$ be a Hilbert space. Let $(T_1, \ldots, T_n) \in \mathcal{D}_f(\mathcal{H})$. Then there exists a (necessarily unique) completely contractive unital algebra morphism $\Phi : \mathcal{A}(\mathcal{D}_F) \longrightarrow \mathcal{B}(\mathcal{H})$ such that $\Phi(W_j^f) = T_j$ for $j = 1, \ldots, n$.*

The algebra $\mathcal{A}(\mathcal{D}_f)$ is the *noncommutative domain algebra of symbol f*. When $f = X_1 + \cdots + X_n$, the algebra $\mathcal{A}(\mathcal{D}_f)$ is the disk algebra in n -generators. In [1], we used techniques of complex analysis on domains in $\mathbb{C}^n$ to study the isomorphism problem for noncommutative domain algebras. In our context, the category NCD of noncommutative domain algebras consists of the algebras $\mathcal{A}(\mathcal{D}_f)$ for all positive, regular n-free formal power series f for objects, and *completely isometric* unital algebra isomorphisms for arrows. We then constructed for each $k \in \mathbb{N}$ a contravariant functor $\mathbb{D}^k$ from NCD to the category $\mathrm{HD}_{\mathrm{kn}^2}$ of connected open subsets of $\mathbb{C}^{kn^2}$ with holomorphic maps. Given a regular positive formal n-free formal power series f, the functor $\mathbb{D}^k$ associates to $\mathcal{A}(\mathcal{D}_f)$ the domain $\mathbb{D}_f^k$. The key observation of [1] is that any isomorphism $\Phi : \mathcal{A}(\mathcal{D}_f) \to \mathcal{A}(\mathcal{D}_g)$ in NCD gives rise to a biholomophic map $\widehat{\Phi_k} = \mathbb{D}^k(\Phi)$ from $\mathbb{D}_g^k$ onto $\mathbb{D}_f^k$, and this construction is functorial (contravariant). Thus, $\mathbb{D}^k$ give fundamental invariants of noncommutative domain algebras. We showed in [1, Theorem 3.18] that if $\Phi : \mathcal{A}(\mathcal{D}_f) \to \mathcal{A}(\mathcal{D}_g)$ is an isomorphism in

NCD such that $\widehat{\Phi_1}(0) = 0$ then there is an invertible matrix $M \in M_{n\times n}(\mathbb{C})$ such that $\begin{bmatrix} W_1^g \\ \vdots \\ W_n^g \end{bmatrix} = (M \otimes 1_{\ell(\mathbb{F}_n^+)}) \begin{bmatrix} W_1^f \\ \vdots \\ W_n^f \end{bmatrix}$. We could then use this result to characterize the disk algebra among all noncommutative domain algebras [1, sec 4.3]: $\mathcal{A}(\mathcal{D}_f)$ is isomorphic to the n-disk algebra in NCD if and only if $f = X_1 + \ldots + X_n$ after possible rescaling (i.e. replacing each indeterminate with a multiple of itself). We could also distinguish between the instructive examples $f = X_1 + X_2 + X_1X_2$ and $g = X_1 + X_2 + \frac{1}{2}X_1X_2 + \frac{1}{2}X_2X_1$ in NCD [1, sec. 4.2].

In this paper, we extend our results to characterize a large class of noncommutative domain algebras in terms of their symbol. For instance, we strengthen [1, sec. 4.2] by showing that, given $f = X_1 + X_2 + X_1X_2$, the algebra $\mathcal{A}(\mathcal{D}_f)$ is isomorphic in NCD to $\mathcal{A}(\mathcal{D}_g)$ if and only if, after permuting and rescaling the indeterminates in g, we have $f = g$.

In [7, Theorem 4.5], Popescu proved that two noncommutative domain algebras are isomorphic in NCD if and only if their corresponding noncommutative domains are biholomorphic, in the generalized sense of Popescu. Thus, our result extends to the biholomorphic classification of noncommutative domains in terms of their defining symbols. We note that Popescu's result applies to a larger class of domain algebras and noncommutative domains associated with the Berezin transform. We refer to [7, Theorem 4.5] for details.

Thus, this paper continues the program initiated in [1] with three new results. First, we show that Sunada's classification of Reinhardt domains [8] allows one to prove that for a large class of noncommutative domains, the only possible automorphisms are linear. Second, we show that for that class of domains, the isomorphism problem is fully solved, as isomorphisms correspond to a simple form of equivalence on regular positive n-free formal power series . As a third matter, we show that the same techniques allow us to obtain a generalization of the Cartan's lemma to noncommutative domain by different means than [6, 7]. We expect that many results of complex analysis in several variables can be generalized to noncommutative domain algebras in a similar manner.

2. ASPHERICAL NONCOMMUTATIVE DOMAIN ALGEBRAS

This first section applies Sunada's classification of Reinhardt domains to the classification of noncommutative domain algebras. In [1], we showed how to classify all noncommutative domain algebras isomorphic to the disk algebras. In this paper, we will provide a complete classification for the class of polynomial aspherical domains as defined below.

Definition 2.1. Let f, g be two regular positive free power series in n indeterminates. We say that f and g are *permutation-rescaling equivalent* if there exists a permutation σ of $\{1, \ldots, n\}$ and a nonzero positive numbers $\lambda_1, \ldots, \lambda_n$ such that $g(\lambda_1 X_{\sigma(1)}, \ldots, \lambda_n X_{\sigma(n)}) = f(X_1, \ldots, X_n)$ where $X_1, \ldots, X_n$ are the indeterminates.

One checks easily that permutation-rescaling equivalence is an equivalence relation of regular positive free power series. It was shown in [1, Lemma 4.4] that if f and g are regular positive free formal power series in n indeterminates, and f and g are permutation-rescaling equivalent, then $\mathcal{A}(\mathcal{D}_f)$ and $\mathcal{A}(\mathcal{D}_g)$ are isomorphic.

Definition 2.2. Let f be a positive regular n-free formal power series. Then f is spherical if it is permutation-rescaling to some g such that:

$$\mathbb{D}_g^1 = \{(z_1, \ldots, z_n) \in \mathbb{C}^n : |z_1|^2 + \cdots + |z_n|^2 < 1\}.$$

Definition 2.3. Let f be a positive regular n-free formal power series. Then f is aspherical if f is not spherical.

Definition 2.4. A free power series f is a free polynomial if, writing $f = \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha X_\alpha$, the set $\{\alpha \in \mathbb{F}_n^+ : a_\alpha \neq 0\}$ is finite. We shall write n-free polynomial for a free polynomial in n-indeterminates.

Definition 2.5. The noncommutative domain $\mathcal{A}(\mathcal{D}_f)$ is aspherical when f is aspherical, and polynomial when f is a free polynomial.

It should be noted that there are domains which are not aspherical but not isomorphic to the disk algebras. Let us give a few examples to illustrate our definition.

Example 2.6. Let $f = \frac{1}{2}X_1 + \frac{1}{2}X_2 + \frac{1}{2}X_1^2 + \frac{1}{2}X_2^2 + X_1X_2$. Then $\mathbb{D}_f^1$ is the open unit ball of $\mathbb{C}^2$. Yet, by [1, Theorem 4.7], we know that $\mathcal{A}(\mathcal{D}_f)$ is not a disk algebra.

Example 2.7. Let $f = \frac{1}{2}X_1 + \frac{1}{2}X_2 + \frac{1}{2}(X_1 + X_2)^2$. Using [1, Theorem 4.7], we see again that $\mathcal{A}(\mathcal{D}_f)$ is not isomorphic to a disk algebra in NCD. However, there is a completely bounded isomorphism from the disk algebra onto $\mathcal{A}(\mathcal{D}_f)$. This illustrates the importance of our choice of isomorphisms as *completely isometric* unital algebra isomorphisms.

Example 2.8. Let $f = X_1 + X_2 + 3X_1X_2$, $g = 2X_1 + X_2 + 6X_2X_1$ and $h = X_1 + 2X_2 + X_1^2$. All three symbols are polynomial and aspherical. We will show in Theorem (2.15) that $\mathcal{A}(\mathcal{D}_f)$ and $\mathcal{A}(\mathcal{D}_g)$ are isomorphic, but $\mathcal{A}(\mathcal{D}_f)$ and $\mathcal{A}(\mathcal{D}_g)$ are not.

It is of course quite easy to produce examples of polynomial aspherical symbols, so our work will apply to a large class of examples.

It is immediate, by definition, that:

Observation 2.9. *Let f be a positive regular free formal power series in n indeterminates. Then $\mathbb{D}_f^1$ is a bounded Reinhardt domain in $\mathbb{C}^{n^2}$.*

Now, Sunada in [8, Theorem A] shows that up to rescaling-permutation of the canonical basis vectors of $\mathbb{C}^n$, all bounded Reinhardt domains can be written in a normalized form. We cite this theorem with the necessary notations added in the statement.

Theorem 2.10 (Sunada, Theorem A)**.** *Let D be a bounded Reinhardt domain in $\mathbb{C}^n$. Up to applying to D a map of the form $(z_1, \ldots, z_n) \in \mathbb{C}^n \mapsto (\lambda_1 z_{\sigma(1)}, \ldots, \lambda_n z_{\sigma(n)})$ with σ some permutation of $\{1, \ldots, n\}$ and $\lambda_1, \ldots, \lambda_n$ some nonzero complex numbers, D can be described as follows. There exists integers $0 = n_0 < n_1 < \ldots <$*

$n_s = n$, integers $p, r \in \{1, \ldots, n\}$ with $n_r = p$ and a bounded Reinhard domain D_1 in $\mathbb{C}^{n-p}$ such that, if for any $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ we set $\mathbf{z}_j = (z_{n_{j-1}+1}, \ldots, z_{n_j})$ and $r = n - p$, then:

- $D \cap (\mathbb{C}^p \times \{0\}) = \{(\mathbf{z}_1, \ldots, \mathbf{z}_s) \in \mathbb{C}^n : |\mathbf{z}_1| < 1 \wedge \cdots \wedge |\mathbf{z}_r| < 1 \wedge (\mathbf{z}_{r+1}, \ldots, \mathbf{z}_s) = (0, \ldots, 0)\}$
- $\{0\} \times D_1 = D \cap (\{0\} \times \mathbb{C}^{n-p})$,
- *We have:*

$$D = \{(\mathbf{z}_1, \ldots, \mathbf{z}_s) : |\mathbf{z}_1| < 1 \wedge \ldots \wedge |\mathbf{z}_r| < 1 \wedge \left(\frac{\mathbf{z}_{r+1}}{\prod_{j=1}^{r} (1 - |\mathbf{z}_j|^2)^{q_{r+1,j}}}, \ldots, \frac{\mathbf{z}_s}{\prod_{j=1}^{r} (1 - |\mathbf{z}_j|^2)^{q_{s,j}}} \right) \in D_1 \right\}$$

where $q_{r+1,1}, \ldots, q_{s,r}$ are positive nonzero real numbers distinct from 1.

We also observe that if f is a regular positive n-free power series, then $\mathbb{D}_f^1 \cap (\mathbb{C}^p \times \{0\}) = \mathbb{D}_g^1$ for the power series g obtained by evaluating $X_{p+1}, \ldots, X_n$ to 0. One readily checks that g is a regular positive p-free power series. This simple observation will prove useful.

Our strategy to prove our main theorem of this section, Theorem (2.14), is as follows. Let f be a regular positive aspherical n-free polynomial.

(1) Since f is aspherical, $n_1 < n$ and thus $r < s$ in Theorem (2.10) for $\mathbb{D}_f^1$.
(2) We first show that $\mathbb{D}_f^1$ can not be a product of bounded Reinhardt domains. This implies that $r \in \{0, 1\}$ in Theorem (2.10) for $\mathbb{D}_f^1$.
(3) We then show that if $n = 2$, i.e. f has only two indeterminates, then $\mathbb{D}_f^1$ can not be a Thullen domain [9]. This implies that $r = 0$ in Theorem (2.10) for $\mathbb{D}_f^1$ for an arbitrary n.
(4) We then conclude, using ([8, Corollary 2, sec. 6. p. 126]), that the automorphism group of $\mathbb{D}_f^1$ fixes the origin.

To implement this approach, we start with a few lemmas.

Lemma 2.11. *Let f be a positive regular n-free formal polynomial. Then $\mathcal{D}_f(\mathbb{C})$ is not a Cartesian product of bounded domains.*

Proof. Up to rescaling, we assume that $a_\alpha = 1$ for all words α of length 1. Assume that there are two domains D and D', respectively in $\mathbb{C}^p$ and $\mathbb{C}^q$ with $p + q = n$, $p, q > 0$ and such that:

$$\mathbb{D}_f^1 = D \times D'.$$

Let $(z_1, \ldots, z_p) \in D$ be a boundary point. Then $(z_1, \ldots, z_p, 0, \ldots, 0) \in \mathbb{C}^n$ is also a boundary point of $\mathbb{D}_f^1$. Hence:

$$\sum_{\alpha \in \mathbb{F}_p^+} a_\alpha |z_\alpha|^2 = 1.$$

Let $(w_1, \ldots, w_q) \in D' \setminus \{0\}$. Then $(z_1, \ldots, z_p, w_1, \ldots, w_q) \in \mathbb{D}_f^1$, so (if we let $z_{p+1} = w_1, \ldots, z_n = w_q$):

$$\begin{aligned} 1 &\geq \sum_{\alpha \in \mathbb{F}_n^+} a_\alpha |z_\alpha|^2 \geq \sum_{\alpha \in \mathbb{F}_p^+} a_\alpha |z_\alpha|^2 + |w_1|^2 + \cdots + |w_q|^2 \\ &\geq 1 + |w_1|^2 + \cdots + |w_q|^2 > 1 \end{aligned}$$

which is a contradiction. Hence $\mathbb{D}_f^1$ is not a product of domains in proper subspaces of $\mathbb{C}^n$. □

Thullen [9] proved that all bounded Reinhardt domains containing the origin in $\mathbb{C}^2$ are biholomorphic to either the unit ball, the polydisk, a domain whose automorphism group fixes 0, or to a Thullen domain:

$$\{(z,w)\in\mathbb{C}^2 : |z|^2+|w|^{2q}\leq 1\}$$

for some $q\in(0,1)\cup(1,\infty)$. This result, of course, is generalized in Theorem (2.10). We will start by showing that in two dimensions, Thullen domains are not in the range of the object map of the functor s $\mathbb{D}^1$.

Lemma 2.12. *Let f be an aspherical positive regular 2-free formal polynomial. Then $\mathcal{D}_f(\mathbb{C})$ is not a Thullen domain* [9].

Proof. Assume $\mathcal{D}_f(\mathbb{C})$ is a Thullen domain [9]. Up to applying a rescaling and permutation of the free variables of f ([1, Lemma 4.4]), there exists $q\in(0,1)\cup(1,\infty)$ such that, writing $\tau:(z_1,z_2)\mapsto |z_1|^2+|z_2|^{2q}-1$, we have:

$$\mathbb{D}_f^1=\left\{(z,w)\in\mathbb{C}^2 : \tau(z,w)<0\right\}$$

and the boundary $\partial\mathbb{D}_f^1$ of $\mathbb{D}_f^1$ is then

$$\left\{(z_1,z_2)\in\mathbb{C}^2 : \tau(z_1,z_2)=0\right\}.$$

We shall adopt the standard notation that given $(z_1,z_2)\in\mathbb{C}^2$, we have $z_j=x_j+iy_j$ for $x_j,y_j\in\mathbb{R}$. We identity $\mathbb{C}^2$ with $\mathbb{R}^4$. Then:

$$\tau:(x_1,y_1,x_2,y_2)\mapsto x_1^2+y_1^2+(x_2^2+y_2^2)^q-1.$$

Let ∇ denote the gradient operator on $\mathbb{R}^4$ (i.e., with usual abuse of notations, $\nabla=\begin{bmatrix}\frac{\partial}{\partial x_1}\\ \frac{\partial}{\partial y_1}\\ \frac{\partial}{\partial x_2}\\ \frac{\partial}{\partial y_2}\end{bmatrix}$). Then for all $(z_1,z_2)\in\partial\mathbb{D}_f^1$):

$$\nabla\tau(z_1,z_2)=\begin{bmatrix}2x_1\\ 2y_1\\ 2qx_2(x_2^2+y_2^2)^{q-1}\\ 2qy_2(x_2^2+y_2^2)^{q-1}\end{bmatrix}.$$

On the other hand, by definition, the boundary of $\mathbb{D}_f^1$ is:

$$\left\{(z_1,z_2)\in\mathbb{C}^2 : \sum a_\alpha|z_\alpha|^2-1=0\right\}.$$

To simplify notations, we shall introduce the coefficients $(c_{n,m})_{n,m\in\mathbb{N}}$ as follows:

$$c_{n,m}=\sum\left\{a_\alpha : \alpha\in\mathbb{F}_2^+,\quad |\alpha|_1=n\wedge|\alpha|_2=m\right\}$$

where $|\alpha|_i$ is the number of times the generator g_i appears in the word α ($i=1,2$). Thus we can write for all $(z_1,z_2)\in\mathbb{C}^2$:

$$\sum_{\alpha\in\mathbb{F}_2^+}a_\alpha|z_\alpha|^2=\sum_{n,m\in\mathbb{N}}c_{n,m}|z_1|^{2n}|z_2|^{2m}.$$

Now, let:

$$\rho : (z_1, z_2) \in \mathbb{C}^2 \mapsto \sum_{n,m\in\mathbb{N}} c_{n,m} |z_1|^{2n} |z_2|^{2m} - 1.$$

We thus have:

$$\rho(x_1, y_1, x_2, y_2) = \sum_{n,m\in\mathbb{N}} c_{n,m}(x_1^2 + y_1^2)^n (x_2^2 + y_2^2)^m - 1.$$

Then for all $(z_1, z_2) \in \partial\mathbb{D}_f^1$:

$$\begin{aligned}
\nabla\rho(z_1, z_2) &= \begin{bmatrix} \sum_{n,m\in\mathbb{N},(n,m)\neq(0,0)} c_{n,m} 2nx_1(x_1^2+y_1^2)^{n-1}(x_2^2+y_2^2)^m \\ \sum_{n,m\in\mathbb{N},(n,m)\neq(0,0)} c_{n,m} 2ny_1(x_1^2+y_1^2)^{n-1}(x_2^2+y_2^2)^m \\ \sum_{n,m\in\mathbb{N},(n,m)\neq(0,0)} c_{n,m} 2mx_2(x_1^2+y_1^2)^{n}(x_2^2+y_2^2)^{m-1} \\ \sum_{n,m\in\mathbb{N},(n,m)\neq(0,0)} c_{n,m} 2my_2(x_1^2+y_1^2)^{n}(x_2^2+y_2^2)^{m-1} \end{bmatrix} \\
&= \begin{bmatrix} 2x_1\left(c_{1,0} + \sum_{n,m\in\mathbb{N},n>1} nc_{n,m}(x_1^2+y_1^2)^{n-1}(x_2^2+y_2^2)^m\right) \\ 2y_1\left(c_{1,0} + \sum_{n,m\in\mathbb{N},n>1} nc_{n,m}(x_1^2+y_1^2)^{n-1}(x_2^2+y_2^2)^m\right) \\ 2x_2\left(c_{0,1} + \sum_{n,m\in\mathbb{N},m>1} mc_{n,m}(x_1^2+y_1^2)^{n}(x_2^2+y_2^2)^{m-1}\right) \\ 2y_2\left(c_{0,1} + \sum_{n,m\in\mathbb{N},m>1} mc_{n,m}(x_1^2+y_1^2)^{n}(x_2^2+y_2^2)^{m-1}\right) \end{bmatrix}
\end{aligned}$$

The tangent plane in $\mathbb{R}^4$ to a boundary point (z_1, z_2) (where $z_2 \neq 0$ so that we work at a regular point for τ) of $\mathbb{D}_f^1$ is the orthogonal space to any normal vector to the boundary of $\mathbb{D}_f^1$ at that point, namely it is the orthogonal of $\nabla\rho(z_1, z_2)$, as well as the orthogonal space to $\nabla\tau(z_1, z_2)$ (see, for instance, [4, chapter 3]). Thus, these two vectors must be co-linear. In particular, let us focus on the first and third coordinates. It is therefore necessary that if $x_1 \neq 0, x_2 \neq 0$:

$$\begin{aligned}
&\left(c_{1,0} + \sum_{n,m\in\mathbb{N},n>1} nc_{n,m}(x_1^2+y_1^2)^{n-1}(x_2^2+y_2^2)^m\right) q(x_2^2+y_2^2)^{q-1} \\
= \;& c_{0,1} + \sum_{n,m\in\mathbb{N},m>1} mc_{n,m}(x_1^2+y_1^2)^n(x_2^2+y_2^2)^{m-1}.
\end{aligned}$$

which is equivalent to:

$$\begin{aligned}
&\left(c_{1,0} + \sum_{n,m\in\mathbb{N},n>1} nc_{n,m}|z_1|^{2n-2}|z_2|^{2m}\right) q|z_2|^{2q-2} \\
= \;& c_{0,1} + \sum_{n,m\in\mathbb{N},m>1} mc_{n,m}|z_1|^{2n}|z_2|^{2m-2}.
\end{aligned}$$

Now, since (z_1, z_2) is on the boundary of the Thullen domain $\mathbb{D}_f^1$, we have $\tau(z_1, z_2) = 0$ i.e. $|z_1|^2 = 1 - |z_2|^{2q}$. Hence:

$$\begin{aligned}
&\left(c_{1,0} + \sum_{n,m\in\mathbb{N},n>1} nc_{n,m}\left(1 - |z_2|^{2q}\right)^{n-1} |z_2|^{2m}\right) q\,|z_2|^{2q-2} \\
= \;& c_{0,1} + \sum_{n,m\in\mathbb{N},m>1} mc_{n,m}\left(1 - |z_2|^{2q}\right)^{n} |z_2|^{2m-2}.
\end{aligned}$$

This identity must be valid for all (z_1, z_2) on the boundary of $\mathbb{D}_f^1$ except when $z_2 = 0$. Thus it is true on a neighborhood of 0, except at 0. Hence both sides of this identity must be continuous at 0 since the right hand side is as a polynomial in $\mathbb{R}^2$. This precludes that $q < 1$. Hence $q > 1$ and we get at the limit when $z_2 \to 0$ that $0 = c_{0,1}$, which contradicts the definition of f regular positive. So $\mathcal{D}_f(\mathbb{C})$ is not a Thullen domain. □

Remark 2.13. An important observation is that Thullen domains are a special case of the domains described in Theorem (2.10). Indeed, assume, in the notations of 2.10, that $n = 2$, $s = 2$, $r = 1$. Then D_1, which is a bounded Reinhard domain in $\mathbb{C}$ which we can put in standard form, is just the unit disk in $\mathbb{C}$, so:

$$D = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 1 \wedge |z_2| < (1 - |z_1|^2)^q\} \tag{2.1}$$

$$= \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2/q} < 1\} \tag{2.2}$$

which is to say that D is a Thullen domain [9]. In particular, if n is now arbitrary, $s > r > 1$, then the intersection of D with the plane spanned by the first and $(p+1)^{th}$ canonical basis vectors of $\mathbb{C}^n$ is a Thullen domain.

Theorem 2.14. *Let f be a aspherical regular positive n-free polynomial. Then the automorphism group of $\mathcal{D}_f(\mathbb{C})$ fixes 0.*

Proof. By [1, Proposition 3.11], $\mathbb{D}_f^1$ is a bounded Reinhardt domain in $\mathbb{C}^n$.

Up to replacing f by a permutation-scaling equivalent symbol, we can assume that $\mathbb{D}_f^1$ is in standard form, i.e. of the form given in Theorem (2.10). We recall from [1, Lemma 4.4] that if two symbols are permutation-scaling equivalent, then their associated noncommutative domain algebras are isomorphic.

We shall now use the notations of Theorem (2.10) applied to $\mathbb{D}_f^1$.

By definition of aspherical, $\mathbb{D}_f^1$ is not the open unit ball of $\mathbb{C}^n$, so $n_1 < n$, $r < s$ and $p < n$.

Assume now that $r \geq 1$. Then $\mathbb{D}_f^1 \cap (\mathbb{C}^p \times \{0\})$ is a product of open unit balls. Yet, if g is obtained from f by mapping $X_{p+1}, \ldots, X_n$ by 0, then g is a regular positive p-free polynomial such that $\mathbb{D}_g^1 \times 0 = \mathbb{D}_f^1 \cap (\mathbb{C}^p \times \{0\})$ by construction. By Lemma (2.11), $\mathbb{D}_g^1$ is not a proper product, so it must be at most one unit ball. Hence, $r = 1$.

Therefore, in general, $r \in \{0, 1\}$. Assume that $r = 1$, so $1 \leq p < n$ (since $r < s$ as well). Let h be the regular formal 2-free polynomial obtained from f by mapping $X_2, \ldots, X_p, X_{p+2}, \ldots, X_n$ to 0 (once again it is immediate to check that indeed h is a regular positive polynomial in indeterminates X_1, X_{p+1}). In particular, observe that $\mathbb{D}_h^1$ is the intersection of $\mathbb{D}_f^1$ by the plane spanned by the first and $(p+1)^{th}$ canonical basis vector in $\mathbb{C}^n$. By Remark (2.13), $\mathbb{D}_h^1$ is a Thullen domain. By Lemma (2.12), this is impossible. So we have reached a contradiction.

Hence $r = 0$. By [8, Corollary 2 of Theorem B, sec 6. p.126], all automorphisms of $\mathbb{D}_f^1$ fix 0. Our theorem is proven.

□

Applying [1, Theorem 3.18], we get the following important result:

Theorem 2.15. *Let f and g be regular positive n-free polynomials, with f aspherical. Then if $\Phi : \mathcal{A}(\mathcal{D}_f) \rightarrow \mathcal{A}(\mathcal{D}_g)$ is an isomorphism, then $\widehat{\Phi}(0) = 0$ and therefore, there exists $M \in M_{n\times n}(\mathbb{C})$ such that:*

$$\begin{bmatrix} \Phi(W_1^f) \\ \vdots \\ \Phi(W_n^f) \end{bmatrix} = (M \otimes 1_{\ell^2(\mathbb{F}_n)}) \begin{bmatrix} W_1^g \\ \vdots \\ W_n^g \end{bmatrix}.$$

Proof. Let $\lambda = \widehat{\Phi}(0)$. Assume $\lambda \neq 0$. Let $j \in \{1, \ldots, n\}$ such that $\lambda_j \neq 0$ where $\lambda = (\lambda_1, \ldots, \lambda_n)$. Let ρ be the linear transformation of $\mathbb{C}^n$ given by the matrix $\begin{bmatrix} 1_{j-1} & & \\ & -1 & \\ & & 1_{n-j} \end{bmatrix}$ in the canonical basis of $\mathbb{C}^n$ (where 1_k is the identity of order k for any $k \in \mathbb{N}$). Then $\rho(\lambda) \neq \lambda$. By [1, Lemma 4.4], there exists a unique automorphism $\widetilde{\rho}$ of $\mathcal{A}(\mathcal{D}_g)$ such that $\mathbb{D}^1(\widetilde{\rho}) = \rho$ since ρ only rescale the coordinates. Let $\tau = \Phi^{-1} \circ \widetilde{\rho} \circ \Phi$: by construction, τ is an automorphism of $\mathcal{A}(\mathcal{D}_f)$, and moreover $\widehat{\tau}(0) = \widehat{\Phi}^{-1}(\rho(\lambda)) \neq 0$ since $\widehat{\Phi}^{-1}$ is injective. Thus $\widehat{\tau}$ is an automorphism of $\mathcal{D}_f(\mathbb{C})$ which does not fix 0. By Theorem (2.14), since f is aspherical, this is a contradiction. Hence $\widehat{\Phi}(0) = 0$. The theorem follows from [1, Theorem 3.18]. □

Corollary 2.16. *Let f, g be aspherical regular positive n-free polynomials. Then there exist an isomorphism Φ from $\mathcal{A}(\mathcal{D}_f)$ onto $\mathcal{A}(\mathcal{D}_g)$ if and only if there exists an invertible matrix $M \in M_{n\times n}(\mathbb{C})$ such that:*

$$\begin{bmatrix} W_1^f \\ \vdots \\ W_n^f \end{bmatrix} = (M \otimes 1_{\ell^2(\mathbb{F}_n)}) \begin{bmatrix} W_1^g \\ \vdots \\ W_n^g \end{bmatrix}.$$

In particular, the automorphism group of aspherical polynomial noncommutative domain algebras consists only of invertible linear transformation on the generators, in contrast with the disk algebras which have the full automorphism group of the unit ball as a normal subgroup of automorphism [3]. We also contrast this result to the computation of other nontrivial automorphism groups associated to the disk algebras in [2].

Corollary 2.17. *Let f be an aspherical regular positive n-free polynomial. The automorphism group of $\mathcal{A}(\mathcal{D}_f)$ is a subgroup of the unitary group $U(n)$.*

We can now see immediately that Theorem (2.15) solves Example (2.8).

3. Classification of polynomial aspherical noncommutative domains algebras

This section establishes an explicit equivalence on aspherical regular positive n-free polynomials which corresponds to isomorphism of the associated noncommutative domain algebra. This result generalizes [1, sec 4]

We defined in [1] dual maps associated to an isomorphism in NCD as follows. Fix f and g two regular, positive n-free formal power series, and let $\Phi : \mathcal{A}(\mathcal{D}_f) \rightarrow \mathcal{A}(\mathcal{D}_g)$ be an isomorphism in NCD. Let $k \in \mathbb{N}$, $k > 0$. For any $\lambda = (\lambda_1, \cdots, \lambda_n) \in$

$\mathbb{D}_f^k \subseteq (M_{k\times k}(\mathbb{C}))^n$, there exists a unique completely contractive representation $\pi_\lambda : \mathcal{A}(\mathcal{D}_g) \to M_{n\times n}(\mathbb{C})$ such that $\pi(W_j^g) = \lambda_j$ for $j = 1, \ldots, n$. Now, $\rho = \pi_\lambda \circ \Phi$ is a completely contractive representation of $\mathcal{A}(\mathcal{D}_f)$ on $M_{n\times n}(\mathbb{C})$, so there exists $(\mu_1, \ldots, \mu_n) \in \mathbb{D}_f^k$ such that $\rho(W_j^f) = \mu_j$ for $j = 1, \ldots, n$. We set $\widehat{\Phi_k}(\lambda_1, \ldots, \lambda_n) = (\mu_1, \ldots, \mu_n)$. We showed in [1] that these maps are biholomorphic and that this construction is functorial and contravariant.

We start with a combinatorial result.

Proposition 3.1. *Let* $f = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha^f X_\alpha$ *and* $g = \sum_{\alpha\in\mathbb{F}_n^+} a_\alpha^g X_\alpha$ *be regular positive* n*-free polynomials. Up to rescaling, we assume* $a_\alpha^f = a_\alpha^g = 1$ *for all words* α *of length 1. Let* $\Phi : \mathcal{A}(\mathcal{D}_f) \longrightarrow \mathcal{A}(\mathcal{D}_g)$ *be an isomorphism such that* $\widehat{\Phi}(0) = 0$*. Let* $U = [u_{ij}]_{1\leq i,j\leq n} \in M_{n\times n}(\mathbb{C})$ *be the unitary matrix such that for all* $i \in \{1, \ldots, n\}$*:*

$$\Phi\left(W_i^f\right) = \sum_{j=1}^n u_{ij} W_j^g. \tag{3.1}$$

We define the support function s_Φ *of* Φ *by setting, for any subset* A *of* $\{1, \ldots, n\}$*:*

$$s_\Phi(A) = \{j \in \{1, \ldots, n\} : \exists i \in A \quad u_{ij} \neq 0\}.$$

Then there exists partitions $\{\sigma_1, \ldots, \sigma_p\}$ *and* $\{\psi_1, \ldots, \psi_p\}$ *of* $\{1, \ldots, n\}$ *such that for all* $i \in \{1, \ldots, p\}$ *we have* $s_\Phi(\sigma_i) = \psi_i$*,* $s_{\Phi^{-1}}(\psi_i) = \sigma_i$ *and* $|\sigma_i| = |\psi_i|$*. Moreover, if* $A \subseteq \{1, \ldots, n\}$ *satisfies* $s_{\Phi^{-1}} \circ s_\Phi(A) = A$ *, then:*

$$A = \bigcup \{\sigma_i : i \in \{1, \ldots, p\} \wedge \sigma_i \subseteq A\}.$$

Proof. Since Φ is an isomorphism such that $\widehat{\Phi}(0) = 0$, we conclude by [1, Theorem 3.18] that there exists a unitary $U = [u_{ij}]_{1\leq i,j\leq n} \in M_{n\times n}(\mathbb{C})$ such that Equality (3.1) holds for $i \in \{1, \ldots, n\}$. Note that by definition $s_\Phi(A) = \bigcup_{i\in A} s_\Phi(\{i\})$. Moreover, for the same reason, there exists $V = [v_{ij}]_{1\leq i,j\leq n} \in M_{n\times n}(\mathbb{C})$ such that $\Phi^{-1}(W_i^g) = \sum_{j=1}^n v_{ij} W_j^f$ and it is immediate that $V = U^*$ i.e.:

$$\Phi^{-1}(W_i^g) = \sum \overline{u_{ji}} W_j^f.$$

This implies that for $A \subseteq \{1, \ldots, n\}$ we have:

$$\begin{aligned} s_{\Phi^{-1}}(A) &= \{j \in \{1, \ldots, n\} : \exists i \in A \quad v_{ij} \neq 0\} \text{ by definition,} \\ &= \{j \in \{1, \ldots, n\} : \exists i \in A \quad u_{ji} \neq 0\} \text{ since } V = U^*. \end{aligned}$$

In particular:

$$k \in s_\Phi(\{i\}) \iff i \in s_{\Phi^{-1}}(\{k\}).$$

Thus $i \in s_{\Phi^{-1}} \circ s_\Phi(\{i\})$.

We now observe that given any partition $\{\sigma_1, \ldots, \sigma_n\}$, if we set $\psi_i = s_\Phi(\sigma_i)$ for all $i \in \{1, \ldots, n\}$, then $|\psi_i| \geq |\sigma_i|$ for all $i \in \{1, \ldots, n\}$. Indeed, fix $i \in \{1, \ldots, n\}$. Let $j \in \sigma_i$. By definition:

$$\Phi(W_j^f) = \sum_{k\in\psi_i} u_{jk} W_k^g.$$

Since Φ is injective and $\left\{W_1^f,\ldots,W_n^f\right\}$ and $\{W_1^g,\ldots,W_n^g\}$ are linearly independent sets, we have:

$$\begin{aligned} |\sigma_i| &= \dim\operatorname{span}\{W_j : j\in\sigma_i\} = \dim\operatorname{span}\left\{\Phi(W_j^f) : j\in\sigma_i\right\} \\ &\leq \dim\operatorname{span}\{W_k^g : k\in\psi_i\} = |\psi_i|\,. \end{aligned}$$

If, moreover, $\sigma_i = s_{\Phi^{-1}}(\psi_i)$ for all $i\in\{1,\ldots,n\}$, then one gets $|\sigma_i| = |\psi_i|$ for $i\in\{1,\ldots,n\}$.

Now, we turn to the construction of a partition $\{\sigma_1,\ldots,\sigma_n\}$ of $\{1,\ldots,n\}$ such that $s_{\Phi^{-1}}\circ s_\Phi(\sigma_i)=\sigma_i$ for all $i\in\{1,\ldots,n\}$. Let $i\in\{1,\ldots,n\}$. Set $A_0=\{i\}$ and $A_{m+1} = s_{\Phi^{-1}}\circ s_\Phi(A_m)$ for all $m\in\mathbb{N}$. Let $k\in A_m$ for some $m\in\mathbb{N}$. Then there exists $j\in s_\Phi(A_m)$ such that $u_{kj}\neq 0$. Since $j\in s_\Phi(\{k\})$ we have $k\in s_{\Phi^{-1}}(\{j\})\subseteq s_{\Phi^{-1}}\circ s_\Phi(A_m)$. Hence $k\in A_{m+1}$. So $(A_m)_{m\in\mathbb{N}}$ is a sequence of subsets of $\{1,\ldots,n\}$ increasing for the inclusion. Since $\{1,\ldots,n\}$ is finite, there exists $N\in\mathbb{N}$ such that $A_N = A_{N+1}$. We define $\operatorname{cl}(i) = A_N$.

We thus construct subsets $\operatorname{cl}(1),\ldots,\operatorname{cl}(n)$ of $\{1,\ldots,n\}$ such that $i\in\operatorname{cl}(i)$ for all $i\in\{1,\ldots,n\}$, so $\bigcup_{i\in\{1,\ldots,n\}}\operatorname{cl}(i)=\{1,\ldots,n\}$. To show $\{\operatorname{cl}(1),\ldots,\operatorname{cl}(n)\}$ is a partition, it is thus sufficient to show that if, for some $i,j\in\{1,\ldots,n\}$ $\operatorname{cl}(i)\cap\operatorname{cl}(j)\neq\emptyset$ then $\operatorname{cl}(i)=\operatorname{cl}(j)$.

To do so, let us assume that $k\in\operatorname{cl}(i)$. Since $s_{\Phi^{-1}}\circ s_\Phi(\operatorname{cl}(i))=\operatorname{cl}(i)$ by definition, we conclude that $\operatorname{cl}(k)\subseteq\operatorname{cl}(i)$. On the other hand, by construction, there exists $j_1,\ldots,j_q$ (for $q\leq n$) such that $j_1=i$, $j_q=k$ and $j_{m+1}\in s_{\Phi^{-1}}\circ s_\Phi(\{j_m\})$. Fix $m\in\{1,\ldots,q\}$. Since $j_{m+1}\in s_{\Phi^{-1}}\circ s_\Phi(\{j_m\})$ there exists $r_m\in s_\Phi(\{j_m\})$ such that $j_{m+1}\in s_{\Phi^{-1}}(\{r_m\})$ so $r_m\in s_\Phi(\{j_{m+1}\})$. Since $r_m\in s_\Phi(\{j_m\})$ we have $j_m\in s_{\Phi^{-1}}(\{r_m\})$ and therefore $j_m\in s_{\Phi^{-1}}\circ s_\Phi(j_{m+1})$. Hence, $i\in\operatorname{cl}(k)$. Therefore, $\operatorname{cl}(k)=\operatorname{cl}(i)$.

Assume now that $\operatorname{cl}(i)\cap\operatorname{cl}(j)\neq\emptyset$ for some $i,j\in\{1,\ldots,n\}$. Let $k\in\operatorname{cl}(i)\cap\operatorname{cl}(j)$. We have shown that $\operatorname{cl}(i)=\operatorname{cl}(k)=\operatorname{cl}(j)$. Therefore, $\{\operatorname{cl}(1),\ldots,\operatorname{cl}(n)\}$ is a partition of $\{1,\ldots,n\}$ such that $s_{\Phi^{-1}}\circ s_\Phi(\operatorname{cl}(i))=\operatorname{cl}(i)$ for $i=1,\ldots,n$. We rewrite this partition as $\{\sigma_1,\ldots,\sigma_p\}$ (the order is unimportant). Setting $\psi_i = s_\Phi(\sigma_i)$ for $i=1,\ldots,p$, we have found a partition satisfying our theorem.

Note, at last, that if $A\subseteq\{1,\ldots,n\}$ such that $s_{\Phi^{-1}}\circ s_\Phi(A)=A$ then for $i\in A$ then

$$s_{\Phi^{-1}}\circ s_\Phi(i)\subseteq s_{\Phi^{-1}}\circ s_\Phi(A)=A$$

and thus $\operatorname{cl}(i)\subseteq A$, as desired. □

We now show that the permutation-rescaling equivalence on symbols corresponds precisely to isomorphism of polynomial aspherical noncommutative domain algebras. To this end, we shall recall the following observations. Let $f=\sum_\alpha a_\alpha X_\alpha$ be a regular positive n-free formal power series.

(1) There exists positive real numbers b_α for all $\alpha\in\mathbb{F}_n^+$ such that W_j maps the canonical basis vector δ_α at $\alpha\in\mathbf{F}_n^+$ to $\sqrt{\frac{b_\alpha}{b_{g_j\alpha}}}\delta_{g_j\alpha}$ for $j=1,\ldots,n$ [5].
(2) The correspondence between the coefficients $(a_\alpha)_{\alpha\in\mathbb{F}_n^+}$ and $(b_\alpha)_{\alpha\in\mathbb{F}_n^+}$ is bijective [5].
(3) For any $\alpha\in\mathbb{F}_n^+$ we have $\|W_\alpha^f\| = \frac{1}{b_\alpha^f}$ [5].

(4) The Pythagorean identity holds for the weighted shifts W_α^f in the following sense:

$$\|\sum_{j=1}^{k} c_j W_{\alpha_j}^f\|^2 = \sum_{j-1}^{n} |c_j|^2 \|W_{\alpha_j}^f\|^2$$

where $c_1, \ldots, c_k$ are arbitrary complex numbers and $\alpha_1, \ldots, \alpha_n \in \mathbb{F}_n^+$ of are all words *of the same length* [1].

We now can show:

Theorem 3.2. *Let f, g be two regular, positive n-free polynomials. There exists an isomorphism $\Phi : \mathcal{A}(\mathcal{D}_f) \longrightarrow \mathcal{A}(\mathcal{D}_g)$ such that $\widehat{\Phi}(0) = 0$ if and only if f and g are permutation-rescaling equivalent.*

Proof. Write $f = \sum a_\alpha^f X_\alpha$ and $g = \sum a_\alpha^g X_\alpha$. Up to rescaling (see [1, Lemma 4.4]), we assume $a_\alpha^f = a_\alpha^g = 1$ for all words α of length 1. We first assume that there exists an isomorphism $\Phi : \mathcal{A}(\mathcal{D}_f) \longrightarrow \mathcal{A}(\mathcal{D}_g)$ such that $\widehat{\Phi}(0) = 0$. By Proposition (3.1), we can find a partition $\{\sigma_1, \ldots, \sigma_p\}$ of $\{1, \ldots, n\}$ such that $s_{\Phi^{-1}} \circ s_\Phi(\sigma_i) = \sigma_i$ for all $i \in \{1, \ldots, p\}$, where s_Φ (resp. $s_{\Phi^{-1}}$) is the support function for Φ (resp. Φ^{-1}). Up to permutation of the free variables in g we may assume that $s_\Phi(\sigma_i) = \sigma_i$ for $i \in \{1, \ldots, p\}$.

Let $l \in \mathbb{N}$ with $l > 1$ be given. The finite set $\left\{b_\alpha^f, b_\alpha^g : \alpha \in \mathbb{F}_n^+ \wedge |\alpha| = l\right\}$ of real numbers has a smallest number, which we note b_ω^f (if the minimum is reached for a coefficient for g instead, we flip the notations for f and g for this case), with $\omega \in \mathbb{F}_n^+$ and $|\omega| = l$. We write $\omega = g_{i_1} \cdots g_{i_l}$ where $g_1, \ldots, g_n$ are the canonical generators of $\mathbb{F}_n^+$. Now, since Φ is an isometry and is implemented on the generators $W_1^f, \ldots, W_n^f$ by a scalar unitary $[u_{ij}]_{1 \leq i,j \leq n}$ acting on $(W_1^g, \ldots, W_n^g)$ (see Proposition (3.1) for notations), we have:

$$\begin{aligned}
\frac{1}{b_\omega^f} &= \left\|W_\omega^f\right\|^2 = \left\|\Phi\left(W_\omega^f\right)\right\|^2 \\
&= \left\|\prod_{k=1}^{l} \Phi\left(W_{i_k}^f\right)\right\|^2 \\
&= \left\|\prod_{k=1}^{l} \sum_{r_k \in s_\Phi(\{i_k\})} u_{i_k r_k} W_{r_k}^g\right\|^2 \\
&= \left\|\sum_{r_1 \in s_\Phi(\{i_1\})} \cdots \sum_{r_l \in s_\Phi(\{i_l\})} u_{i_1 r_1} u_{i_2 r_2} \cdots u_{i_l r_l} W_{r_1}^g \cdots W_{r_l}^g\right\|^2 \\
&= \sum_{r_1 \in s_\Phi(\{i_1\})} \cdots \sum_{r_l \in s_\Phi(\{i_l\})} \left|u_{i_1 r_1}\right|^2 \left|u_{i_2 r_2}\right|^2 \cdots \left|u_{i_l r_l}\right|^2 \frac{1}{b_{g_{r_1} \cdots g_{r_l}}^g}.
\end{aligned}$$

Thus, since U is unitary, we conclude that $\frac{1}{b_\omega^f}$ is a convex combination of elements in $\left\{\frac{1}{b_\alpha^f}, \frac{1}{b_\alpha^g} : \alpha \in \mathbb{F}_n^+ \wedge |\alpha| = l\right\}$, though it is its maximum. This is only possible if:

$$\forall r_1 \in s_\Phi(\{i_1\}) \quad \cdots \quad \forall r_l \in s_\Phi(\{i_1\}) \quad \frac{1}{b_{g_{r_1} \cdots g_{r_l}}^g} = \frac{1}{b_\omega^f}.$$

Let us adopt the following notation. A word α will be of type $\sigma_{i_1}\cdots\sigma_{i_l}$ if $\alpha = g_{r_1}\cdots g_{r_n}$ for $r_j \in \sigma_j$ with $j = 1,\ldots,l$. Let $\sigma_{v_1}\cdots\sigma_{v_l}$ be the type of the ω as above. By repeating the above argument, we can thus show that $b^f_\alpha = b^g_\alpha = b^f_\omega$ for all α of type ω.

We can now repeat the proof above by picking the minimum of

$$\left\{\frac{1}{b^f_\alpha}, \frac{1}{b^g_\alpha} : \alpha \in \mathbb{F}_n^+ \wedge |\alpha| = l \wedge \alpha \text{ is not of the same type as } \omega\right\}$$

and so forth until we exhaust the set $\left\{\frac{1}{b^f_\alpha}, \frac{1}{b^g_\alpha} : \alpha \in \mathbb{F}_n^+ \wedge |\alpha| = l\right\}$ to show that $b^f_\alpha = b^g_\alpha = b^f_\omega$ for all $\alpha \in \mathbb{F}_n^+$ with $|\alpha| = l$. This completes our proof. □

Corollary 3.3. *Let Let f, g be two regular positive n-free polynomials, and assume f is aspherical. Then $\mathcal{A}(\mathcal{D}_f)$ and $\mathcal{A}(\mathcal{D}_g)$ are isomorphic if and only if f and g are scale-permutation equivalent.*

Proof. This results follows from Theorem (2.14) and Theorem (3.2). □

We can summarize the current understanding of classification for noncommutative domain algebra:

Theorem 3.4. *Let f be a regular, positive n-free polynomial. Then:*

- *If $f = \sum c_i X_i$ for some $c_1,\ldots,c_n \in (0,\infty)$ then $\mathcal{A}(\mathcal{D}_f)$ is isomorphic to the disk algebra $\mathcal{A}_n$,*
- *If f is aspherical, then $\mathcal{A}(\mathcal{D}_g)$ is isomorphic to $\mathcal{A}(\mathcal{D}_f)$ if and only if $f = g$ after rescaling/permutation of the free variables of g.*

This leaves the matter of classifying $\mathcal{A}(\mathcal{D}_f)$ when f is spherical, i.e. $\mathcal{D}_f(\mathbb{C})$ is the closed unit ball of $\mathbb{C}^n$, yet f is not of degree 1.

4. Cartan's Lemma

In [6, Theorem 1.4] and [7, Theorem 4.5], Popescu establishes a generalization of Cartan's Lemma, first in the context of the unit ball of $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$, then for a large class of noncommutative domains that includes the domains considered in this paper. We propose to illustrate in this section that our methods, as implemented in this paper and in [1], can be used to obtain a simpler proof of these results. We refer to [5, 6, 7] for definitions and the general theory of holomorphic functions in the context of noncommutative domains. We shall only use the following special case of holomorphic functions:

Definition 4.1. Let f be a positive, regular n-free formal power series. A holomorphic map F with domain and codomain $\mathcal{D}_f$ is the given of a family of complex coefficients $(c_\alpha)_{\alpha\in\mathbb{F}_n^+}$ such that, for any Hilbert space $\mathcal{H}$, the function:

$$F_{\mathcal{H}} : (T_1,\ldots,T_n) \in \text{interior}\,\mathcal{D}_f(\mathcal{H}) \longmapsto \left(\sum_{\alpha\in\mathbb{F}_n^+} c_{1,\alpha}T_\alpha, \ldots, \sum_{\alpha\in\mathbb{F}_n^+} c_{1,\alpha}T_\alpha\right)$$

is well-defined and its range is a subset of the interior of $\mathcal{D}_f(\mathcal{H})$.

Remark 4.2. The domain and codomain of a holomorphic function, in this context, is the noncommutative domain $\mathcal{D}_f$ which, itself, is not a set, but a map from separable Hilbert spaces to subsets of the algebra of linear bounded operators on the given Hilbert space. Note moreover that the domain and codomain of the maps induced on various noncommutative domains by holomorphic maps are the *interior* of the noncommutative domains.

Definition 4.3. Let f be a positive, regular, n-free formal power series. A biholomorphic map F on $\mathcal{D}_f$ is a holomorphic map from $\mathcal{D}_f$ to $\mathcal{D}_f$ such that there exists a holomorphic map G from $\mathcal{D}_f$ to $\mathcal{D}_f$ such that $F \circ G$ and $G \circ F$ both induce the identity on $\mathcal{D}_F(\mathcal{H})$ for all Hilbert spaces $\mathcal{H}$.

Remark 4.4. By definition, the map induced by a holomorphic map on a specific noncommutative domain is a holomorphic map on the interior of this domain.

Theorem 4.5 (Cartan's Lemma)**.** *Let f be a regular positive n-free formal power series. Let F be a biholomorphic map of D_f such that $F(0) = 0$. Then F is linear.*

Proof. By definition, there exists complex coefficients $(c_{j,\alpha})_{j\in\{1,\ldots n\},\alpha\in\mathbb{F}_n^+}$ such that for all Hilbert space $\mathcal{H}$ and all $(T_1, \ldots, T_n) \in \operatorname{interior} \mathcal{D}_f(\mathcal{H})$ we have:

$$F_{\mathcal{H}}(T_1, \ldots, T_n) = \left(\sum_{\alpha\in\mathbb{F}_n^+} c_{1,\alpha} T_\alpha, \ldots, \sum_{\alpha\in\mathbb{F}_n^+} c_{1,\alpha} T_\alpha \right) \in \operatorname{interior} \mathcal{D}_f(\mathcal{H}).$$

When $\mathcal{H} = \mathbb{C}^k$, we will write F_k for the biholomorphic map induced by F on $\mathbb{D}_f^k$ for $k \in \mathbb{N}$, $k > 0$.

Since this proof is essentially the same as [1, Theorem 3.18], we shall only deal with the case where $\alpha \in \mathbb{F}_n^+, |\alpha| \le 2$ and $n = 2$.

By definition, F induces a biholomorphic map F_1 on $\mathbb{D}_f^1$. This map fixes 0 and $\mathbb{D}_f^1$ is a circular domain (even Reinhardt) in $\mathbb{C}^n$, Cartan's lemma [4] implies that F_1 is linear.

Now, the j^{th} coordinate of $F_1(z_1, z_2)$ for an arbitrary $(z_1, z_2) \in \mathbb{D}_f^1$ is:

$$c_{j,g_1} z_1 + c_{j,g_2} z_2 + c_{j,g_1 g_1} z_1^2 + c_{j,g_2 g_2} z_2^2 + (c_{j,g_1 g_2} + c_{j,g_2 g_1}) z_1 z_2 + \cdots$$

so the linearity of F_1 implies that:

$$c_{j,g_1 g_1} z_1^2 + c_{j,g_2 g_2} z_2^2 + (c_{j,g_1 g_2} + c_{j,g_2 g_1}) z_1 z_2 + \cdots = 0$$

which in turns shows that $c_{j,g_1 g_1} = c_{j,g_2 g_2} = c_{j,g_1 g_2} + c_{j,g_2 g_1} = 0$. To show that $c_{j,g_1 g_2} = c_{j,g_2 g_1} = 0$ we go to higher dimensions. Again by definition, F induces a biholomorphic map on $\mathbb{D}_f^2$. The later domain is circular (not Reinhardt in general), and thus again by Cartan's lemma [4], since $F_2(0) = 0$ we conclude that F_2 is linear.

Now, a quick computation shows that the $(2, 1)$ component of the j^{th} coordinate of $F_2(M, N)$ where $M = \begin{bmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{bmatrix}$ and $N = \begin{bmatrix} \lambda_5 & \lambda_6 \\ \lambda_7 & \lambda_8 \end{bmatrix}$ are 2×2 complex matrices in $\mathbb{D}_f^2$ is given by:

$$\begin{aligned} c_{j,g_1}\lambda_2 + c_{j,g_2}\lambda_6 \quad &+ \quad c_{j,g_1 g_2}(\lambda_1\lambda_6 + \lambda_2\lambda_8) + c_{j,g_2 g_1}(\lambda_2\lambda_5 + \lambda_4\lambda_6) \\ &+ \quad \text{terms of higher degrees in } \lambda_1, \ldots, \lambda_8 \end{aligned}$$

so linearity of F_2 implies that $c_{j,g_1 g_2} = c_{j,g_2 g_1} = 0$.

The proof for higher terms is similar and undertaken in [1, Theorem 3.18].

□

We expect that the same method of reduction to finite dimension can yield other generalizations of results from the study of domains in complex analysis to the framework of Popescu's noncommutative domains.

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