Classical Mechanics in Hilbert Space, Part 1

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Abstract

We consider the Hamilton formulation as well as the Hamiltonian flows on a symplectic (phase) space. These symplectic spaces are derivable from the Lie group of symmetries of the physical system considered. In Part 2 of this work, we then obtain the Hamiltonian formalism in the Hilbert spaces of square integrable functions on the symplectic spaces so obtained.

1 Introduction

There has been a long history of classical mechanics, culminating in a differentio-geometric formulation in what is called a phase space in physics or a symplectic space in mathematics. In the "historic formulation" of classical mechanics, one takes the position(s), q, and momentum(a), p, for a massive particle(s) to make up the phase space, and the time is added on as a parameter. These ps and qs are augmented by the equations of time propagation via a Lagrangian or a Hamiltonian as the propagator of the time. These equations are Lagrange's or Hamilton's equations, and are usually presented by variational techniques. Meanwhile, (non-relativistic, spinless) quantum mechanics was formulated as a Hilbert space of functions on configuration space (or on momentum space). In 1931 [1], there was an effort to put classical mechanics also on a Hilbert space. This was the work of B. O. Koopman who succeeded in putting the (non-relativistic, spinless) phase space formulation in a Hilbert space of functions of the ps and qs. Then the world waited until V. Guillemin and S. Sternberg in 1984 published a book "Symplectic Techniques in Physics". [2] In this book, they proved that every symplectic space, Γ, with a symplectic symmetry group, G, which is a Lie group, arises as (the orbit of) a quotient of G with certain subgroups H. Thus we have $\Gamma = \cup_{g\in G} G/(g \circ H \circ g^{-1})$ or $\Gamma = G/H$. The second was used by one of the present authors [3] to make progress in that every quantum realization was an irreducible representation of G obtained from the Hilbert space of Koopman, suitably generalized, via the orbit in G of a coherent state, η, on the quantum Hilbert space. This means that, for any $\psi \in \mathcal{H}_{\text{quantum}}$ with $< \cdot, \cdot >$ as the inner product, $< U(\sigma(x))\eta, \psi >$ is in $\mathcal{H}_{Koopman}$, where $\sigma : G/H \to G$ is a (Borel) section, and U is a representation of G on H_{quantum}. (There are more requirements on η. See [3].) The part that $\mathcal{H}_{Koopman}$ played in this was merely its existence. There was no effort in [3] to put in the details of the classical mechanics into the Koopman formalism in general, and then compare with the resultant quantum mechanical representation(s). We shall do the first of these here and in Part 2 when G is the Heisenberg, the rotation, the Galilei, the Lorentz, and the Poincaré groups. For other groups central to the classical mechanics of interacting systems, see the forthcoming paper(s) by J. J. Sławianowski and F. E. Schroeck [4].

We have two facts of which many in the physics community will have to beware. In this formalism on phase space, we will have to work on the Hilbert space in the following fashion:

First: Many physicists are used to thinking that all actions of G on a manifold M are from the left ($W(g)x = gx$), but here this will not "work" except for the simplest of cases. This "simplest case" is the Heisenberg group, most other cases having only the action "on the right" to embed classical mechanics as we will show. We will illustrate that this holds in the case of $G =$ the Heisenberg group but that this does not hold for the rotation group or other groups in general. The action on the right is one of two possibilities: For every $x \in G/H$, there is at least one $g_x \in G$ such that $x = g_x \circ H$. We will define either (a) $W(g)x = (g_x \circ g^{-1}) \circ H$, or (b) $W(g)x = g_x \circ H \circ g^{-1}$. We will show that (a) is not well defined in general, but that (b) is, if we write it as $W(g)x = g_x \circ g^{-1} \circ (g \circ H \circ g^{-1})$. Thus, we will have to work in the orbit of the symplectic space G/H in the classical case. The action of g on the right of $g_x \circ H$ is necessitated by the requirement that G/H is a phase space and we will search for canonical variables, which we will explain. Hence, by the action on the right we obtain classical mechanics (and on the left we obtain quantum mechanics as in [3]). This is similar to action on the left versus right having the following physical interpretation for internal degrees of freedom of which many other physicists may be cognizant: We paraphrase, "The action on the left corresponds to the physical space and has Euler variables; the action on the right corresponds to the 'material space' and has Lagrange variables." [6] We also see that by the trick of having the action on the right for canonical variables, we frequently avoid the problem of making up "canonical variables" from the Lie algebra relations which simply does not work out. See for example [7].

Second: We will show that there is an intimate connection between the phase space formulation and the representation on $\mathcal{H}_{Koopman} = L^2(\textit{phase space})$ and its irreducible components. This hosts some surprising results, in that there are many "results" of quantum mechanics that say you can not do somethings; we shall do them all. It furthermore does not matter whether we do them classically or quantum mechanically.

In the next section we will consider the historical Hamiltonian formalism.

In Section 3 we shall define a general symplectic (phase) space. In Section 4 we make a connection between a symplectic space and the orbit of G/H, where G is the group of symmetries and H is a certain subgroup of G. Then in Part 2, we will illustrate all of this with several examples and then continue with the connection with Hilbert spaces.

2 Historical Hamiltonian Formalism

We let $x = (p, q)$ denote a general element of *any* phase space, Γ, where p, q are vectors of dimension equal to half the dimension of the phase space (symplectic space). The ps and qs are to be canonical coordinates of the phase space. We let $H(p, q)$ denote the Hamiltonian of the system. Then we have Hamilton's equations of motion:

$$\frac{\partial H}{\partial p_j} = \frac{dq_j}{dt}, \ \frac{\partial H}{\partial q_j} = -\frac{dp_j}{dt}, \tag{1}$$

where t is the (proper) time. These express the fact that these coordinates are in fact canonical coordinates. For any functions $f, h \in C^\infty(\Gamma)$, we define the Poisson bracket by

$$\{f, h\} = \sum_j \left(\frac{\partial f}{\partial q_j}\frac{\partial h}{\partial p_j} - \frac{\partial f}{\partial p_j}\frac{\partial h}{\partial q_j} \right). \tag{2}$$

We note that in Γ there are many sets of canonical coordinates, but that we have the same results for any canonical coordinates. [5, pp 254-255] Also we have for all f, h, and $k \in C^\infty(\Gamma)$,

$$\begin{aligned}
\{f, h\} &= -\{h, f\}, \\
\{f, f\} &= 0 \text{ (a redundant equation)}, \\
\{f, c\} &= 0 \text{ for any } c \text{ that does not depend on } p \text{ or } q, \\
\{f, h + k\} &= \{f, h\} + \{f, k\}, \\
\{f, hk\} &= \{f, h\}k + h\{f, k\}, \qquad (3)
\end{aligned}$$

and also

$$\{f, \{h, k\}\} + \{h, \{k, f\}\} + \{k, \{f, h\}\} = 0, \tag{4}$$

the last of which is called Jacobi's identity. We have, for any function of the ps and qs, $u \in C^\infty(\Gamma)$, and time, t, as well,

$$\frac{du}{dt} = \{H, u\} + \frac{\partial u}{\partial t}.$$

Thus if $\frac{\partial u}{\partial t} = 0$, then u is a constant of the motion iff $\{H, u\} = 0$. From Jacobi's identity,

$$\text{if } \{H, f\} = 0 = \{H, h\}, \text{ then } \{H, \{f, h\}\} = 0;$$

so, $\{f, h\}$ is also a constant of the motion.

There are important relations between the canonical ps and qs, namely

$$\begin{aligned}\{q_j, p_k\} &= \delta_{jk}, \\ \{q_j, q_k\} &= 0 = \{p_j, p_k\}\end{aligned} \tag{5}$$

for any components of the ps and qs. Futhermore, for any $K \in C^\infty(\Gamma)$,

$$\{q_j, K\} = \frac{\partial K}{\partial p_j}, \text{ and } \{p_j, K\} = -\frac{\partial K}{\partial q_j} \tag{6}$$

from (2).

To treat Γ as a phase (or symplectic) space, we first define $T_x\Gamma$ as the tangent space at a point $x \in \Gamma$, and $T\Gamma = \cup_{x\in\Gamma} T_x\Gamma$. Then we define the tangent vectors by the following: For $f \in C^\infty(\Gamma)$, define $X_f \in T\Gamma$ by

$$X_f = \sum_{k=1}^{(\dim\Gamma)/2} \left(\frac{\partial f}{\partial q_k}\frac{\partial}{\partial p_k} - \frac{\partial f}{\partial p_k}\frac{\partial}{\partial q_k} \right). \tag{7}$$

Let $[A, B] \equiv AB - BA$ for A, $B \in T\Gamma$. Then

$$[X_f, X_h] = X_{\{f,h\}},$$

and

$$[X_f, [X_h, X_k]] + [X_h, [X_k, X_f]] + [X_k, [X_f, X_h]] = 0$$

for all $f, h, k \in C^\infty(\Gamma)$. That is, the set of the X_fs satisfy the Jacobi identities also.

Furthermore, we have that, for all $u \in C^\infty(\Gamma)$,

$$X_f u = \{f, u\}. \tag{8}$$

The canonical coordinates have the following representations:

$$\begin{aligned} X_{p_j} &= \sum_{k=1}^{(\dim\Gamma)/2} \left(\frac{\partial p_j}{\partial q_k}\frac{\partial}{\partial p_k} - \frac{\partial p_j}{\partial p_k}\frac{\partial}{\partial q_k} \right) = -\frac{\partial}{\partial q_j}, \\ X_{q_j} &= \sum_{k=1}^{(\dim\Gamma)/2} \left(\frac{\partial q_j}{\partial q_k}\frac{\partial}{\partial p_k} - \frac{\partial q_j}{\partial p_k}\frac{\partial}{\partial q_k} \right) = \frac{\partial}{\partial p_j}. \end{aligned} \tag{9}$$

Consequently, these operators commute. Alternatively, by the Jacobi identity, for any $u \in C^\infty(\Gamma)$,

$$\{q_j, \{p_k, u\}\} + \{p_k, \{u, q_j\}\} + \{u, \{q_j, p_k\}\} = 0.$$

But the term $\{u, \{q_j, p_k\}\} = 0$ for all q_j, p_k; so,

$$\{q_j, \{p_k, u\}\} = \{p_k, \{q_j, u\}\},$$

from which we obtain the same commutations.

The fact that the operators X_{p_j} and X_{q_k} commute allows us to define the "flows" of any $u \in C^\infty(\Gamma)$ with respect to X_f; i.e., with respect to f. Let $u = u(0)$. Then,

$$[\exp(-tX_f)u]\,(0) = \left(1 + (-t)X_f + \left[(-t)^2/2\right] {X_f}^2 + \cdots\right) u, \tag{10}$$

if $u(t)$ is an analytic function of t. In particular,

$$\begin{aligned} [\exp(-tX_H)u]\,(0) &= \left(1 + (-t)X_H + \left[(-t)^2/2\right] {X_H}^2 + \cdots\right) u \\ &= u + t\frac{du}{dt} + t^2/2\frac{d^2u}{d^2t} + \cdots \\ &= u(t). \end{aligned} \tag{11}$$

If one takes the commutation relations for the canonical coordinates to be always zero, then there is a difficulty with making a connection with any group for which the Lie algebra has commutations which do not vanish. We will have more to say on this after we obtain a group representation on Γ. Assuming this as a special case, we see that

$$\exp(-tX_H)q_j = q_j(t), \text{ and } \exp(-tX_H)p_j = p_j(t). \tag{12}$$

Hence X_H is the generator of the map from the point (p, q) at time zero to the point $(p(t), q(t))$ at time t. With the aid of the last equation in (3), we also have for any analytic function of the ps, qs, and t,

$$[u(p, q)](t) = u(p(t), q(t)). \tag{13}$$

Consequently, X_H is also the generator of the map from u at time zero to $u(t)$ given by (13) at time t.

Similarly, on u an analytic function on Γ and denoting $\sum_j a_j X_{q_j}$ by $a \cdot X_q$, etc., we have

$$\begin{aligned} [\exp(a \cdot X_q)u](p, q) &= u(p + a, q), \\ [\exp(b \cdot X_p)u](p, q) &= u(p, q - b) \end{aligned} \tag{14}$$

for all $a, b \in \mathbb{R}^n$, $n = \dim(\Gamma)/2$. Thus $-X_{q_j}$ is the generator for translations in p_j and X_{p_j} is the generator for translations in q_j.

Since the operators X_{p_j} and X_{q_k} commute, we obtain

$$\exp(a \cdot X_q) \exp(b \cdot X_p) = \exp(a \cdot X_q + b \cdot X_p).$$

Therefore, we have a representation of $\mathbb{R}^{2n}$ on the analytic vectors of Γ. We may extend to all of $C^\infty(\Gamma)$.

Now, rewriting X_f as

$$\begin{aligned} X_f &= \sum_{k=1}^{(\dim\Gamma)/2} \left(\frac{\partial f}{\partial q_k} X_{q_k} + \frac{\partial f}{\partial p_k} X_{p_k}\right) \\ &= \nabla_q f \cdot X_q + \nabla_p f \cdot X_p, \end{aligned}$$

we may write

$$\begin{aligned} & \exp(-tX_f)\exp(a\cdot X_q)\exp(b\cdot X_p) \\ = \; & \exp([a-t\nabla_q f]\cdot X_q+[b-t\nabla_p f]\cdot X_p). \end{aligned} \tag{15}$$

For example, if we have a Hamiltonian for our system, H_0+V, for $H_0=\sum_j(2m_j)^{-1}p_j^2$ and for $V=$ a function of q only, then choosing $f=H_0+V$, we obtain

$$\begin{aligned} & \exp(-tX_{H_0+V})\exp(a\cdot X_q)\exp(b\cdot X_p) \\ = \; & \exp([a-t\nabla_q V]\cdot X_q+[b-t\nabla_p H_0]\cdot X_p). \end{aligned} \tag{16}$$

Now, for $m_jv_j=p_j$, $b_j-t\frac{\partial H_0}{\partial p_j}=b_j-t\frac{p_j}{m_j}=b_j-tv_j$. Thus,

$$\begin{aligned} & [\exp(-tX_{H_0+V})\exp(a\cdot X_q)\exp(b\cdot X_p)u](p,q) \\ = \; & u(p+a-t\nabla_q V, q-b+tv), \end{aligned} \tag{17}$$

which is physically interpretable, and is a key element of the Hamiltonian formalism.

3 General Symplectic Spaces

We wish to generalize from the phase space $\mathbb{R}^{2n}$ to a general phase space, Γ. Hence, let M be a locally compact manifold and let its tangent space be TM. We presume that TM has a finite dimensional basis $\{X_j\}$. Let the set of linear functionals on the linear space TM be denoted T^*M. Let $\{X_j^*\}$ (or $\{dX_j\}$) be the dual basis of T^*M; i.e.,

$$X_j^*(X_k)=\delta_{j,k}=dX_j(X_k).$$

We shall use the notation $\{X_j^*\}$ for the dual basis henceforth. Define $\wedge^0(T^*M)=C^\infty(M)$, $\wedge^1(T^*M)=T^*M$, and $\wedge^m(T^*M)$ as the linear space of m-fold products of elements of T^*M that are antisymmetric in any two places. Elements of $\wedge^m(T^*M)$ are called m-forms. The coboundary operator, δ, is defined as the graded operator

$$\delta:\wedge^m(T^*M)\to\wedge^{m+1}(T^*M)$$

for every $m\in\mathbb{N}\cup\{0\}$, given by

$$\begin{aligned} \delta f(x) &= \sum_{j=1}^n\frac{\partial f}{\partial x_j}(x)X_j^*|_x \text{ for } f\in C^\infty(M), \text{ and } x\in M; \\ \delta\left(\sum_{j=1}^n f_j(x)X_j^*|_x\right) &= \sum_{j,k=1}^n\frac{\partial f_j}{\partial x_k}(x)X_k^*|_x\wedge X_j^*|_x \text{ for } f_j\in C^\infty(M); \\ \text{and } \delta(X\wedge Y) &= (\delta X)\wedge Y+(-1)^qX\wedge\delta Y \end{aligned} \tag{18}$$

for X a q-form and Y an r-form. Since, for any $f \in C^\infty(M)$ we have $\frac{\partial^2 f}{\partial x_j \partial x_k} = \frac{\partial^2 f}{\partial x_k \partial x_j}$, then $\delta^2 = 0$. Consequently

$$\mathrm{Im}(\delta|_{(m-1)-\mathrm{forms}}) \subseteq \ker(\delta|_{m-\mathrm{forms}}).$$

We next define

$$\begin{aligned} \text{the set of } m\text{-cocycles} &= \text{ the set of closed } m\text{-forms} \\ &= Z^m(T^*M) \\ &= \ker(\delta|_{m-\mathrm{forms}}) \\ &= \{m - \text{forms } \omega \mid \delta\omega = 0\}, \end{aligned} \tag{19}$$

and

$$\begin{aligned} \text{the set of } m\text{-coboundaries} &= \text{ the set of exact } (m-1)\text{-forms} \\ &= B^m(T^*M) \\ &= \mathrm{Im}(\delta|_{(m-1)-\mathrm{forms}}), \end{aligned} \tag{20}$$

and furthermore, by the inclusion in the previous sentence,

$$\begin{aligned} \text{the } n\text{th de Rham cohomology (group)} &= H^m(T^*M) \\ &= Z^m(T^*M)/B^m(T^*M). \end{aligned} \tag{21}$$

A 2-form ω is said to be non-degenerate if $\omega(X, Y) = 0$ for all $Y \in TM$ implies $X = 0$. Finally, a symplectic vector space is defined as a finite dimensional vector space with a closed, non-degenerate 2-form (called the symplectic form). A symplectic manifold, M, is a manifold such that for each $x \in M$, T_xM is a symplectic vector space. We will write $M = \Gamma$ in this case.

By performing the following induction [2, pp 152-153] one may show that every symplectic manifold has even dimension $= 2n$, $n \in \mathbb{N}$, and that any 2-form ω is associated with a basis $\{X_j^*\}$ such that

$$\omega = X_1^* \wedge X_2^* + \cdots + X_{2n-1}^* \wedge X_{2n.}^* \tag{22}$$

Proof: Let V be an n-dimensional vector space and let ω be a 2-form on V expressed in terms of a basis $\{Y_1^*, \cdots, Y_n^*\}$:

$$\omega = \sum_{1 \le i < j \le n} a_{ij} Y_i^* \wedge Y_{j.}^*$$

If $\omega \neq 0$, then by rearranging the basis if necessary, $a_{12} \neq 0$. Define

$$\omega = (Y_1^* - \frac{a_{23}}{a_{12}} Y_3^* - \cdots - \frac{a_{2n}}{a_{12}} Y_n^*) \wedge (a_{12} Y_2^* + a_{13} Y_3^* + \cdots + a_{1n} Y_n^*) + \omega'$$

where ω' does not contain any expression involving Y_1^* or Y_2^*. Define

$$\begin{aligned} X_1^* &= Y_1^* - \frac{a_{23}}{a_{12}} Y_3^* - \cdots - \frac{a_{2n}}{a_{12}} Y_n^*, \\ X_2^* &= a_{12} Y_2^* + a_{13} Y_3^* + \cdots + a_{1n} Y_n^* \end{aligned}$$

and obtain

$$\omega = X_1^* \wedge X_2^* + \omega'.$$

Iterate. $\blacksquare$

Next, given a connected Lie group G, we obtain elements of the Lie algebra $\mathfrak{g}$ in the following fashion: Let $\varphi : [-\varepsilon, \varepsilon] \to G$ be a C^∞ curve in G such that $\varphi(0) = \mathbf{1}$. Then on $u \in C^\infty(G)$, $X \in \mathfrak{g}$ is defined by $[Xu](g) = \frac{d}{dt}u(g \circ \varphi(t))|_{t=0}$, as $t \mapsto g \circ \varphi(t) \equiv l_g\varphi(t)$ is another curve such that $g \circ \varphi(0) = g$. We define l_g as the operation from the left:

$$l_g : G \to G,\ l_g(g_1) = g \circ g_1 \ \forall g, g_1 \in G.$$

Then for $u \in C^\infty(G)$, we denote by the same symbol, $[l_g u](g') = u(l_g g') = u(g \circ g')$. We obtain that the vector fields $X \in TG$ as defined above are left invariant, meaning $l_g X = X l_g$ as operators on $C^\infty(G)$, and have the "same" commutation relations from point g to point g'. From the definition, we compute the left action of $g \in G$ on $X \in \mathfrak{g}$ written as $(l_g)_* : TG \to TG$. We define

$$(l_g)^*\omega(X, Y, \cdots) = \omega((l_g)_*X, (l_g)_*Y, \cdots),$$

for any m-form ω. We similarly define the right action of G by

$$\begin{aligned} r_g \quad &: \quad G \to G,\ r_g(g_1) = g_1 g^{-1}; \\ (r_g)_* \quad &: \quad TG \to TG; \\ (r_g)^*\omega(X, Y, \cdots) \quad &= \quad \omega((r_g)_*X, (r_g)_*Y, \cdots). \end{aligned}$$

Then $l_g r_g$ corresponds to conjugation in the group, and we have a special name for the analog of $(l_g)_*$ and $(l_g)^*$ or $(r_g)_*$ and $(r_g)^*$, namely g_* and g^*.

We define a symmetry group, or a symplectic group, as a group G, $G : \Gamma \to \Gamma$, with 2-form ω satisfying left invariance:

$$(l_g^*)\omega = \omega \text{ for all } g \in G. \tag{23}$$

We note that for the left invariant vector fields, this is automatic.

For $X \in TM$, the contraction ι_X is defined by

$$\begin{aligned} \iota_X \quad &: \quad \wedge^{m+1}(T^*M) \to \wedge^m(T^*M), \\ [\iota_X\omega](Y_1, \cdots, Y_m) \quad &= \quad \omega(X, Y_1, \cdots, Y_m), \end{aligned} \tag{24}$$

for all $m \in \mathbb{N}$.

Definition 1 *A vector field X is Hamiltonian iff there is a function $f \in C^\infty(M)$ such that*

$$\iota_X\omega = \delta f, \tag{25}$$

where ω is the symplectic 2-form. Note that X_f is automatically Hamiltonian for $f \in C^\infty(M)$.

It may be shown that this is a generalization of the Hamiltonian structure in the last section. We will call such Hamiltonian vector fields as the/a Hamiltonian on the phase space $(M,\omega) = (\Gamma,\omega)$.

Furthermore, this symplectic space is exactly the phase space we have in Section 2 for classical mechanics with

$$\begin{aligned} X_{2j} &= X_{p_j},\ X_{2j-1} = X_{q_j}, \\ \omega(X,Y) &= \sum_{j=1}^{n} \left(X^*_{q_j}(X) X^*_{p_j}(Y) - X^*_{q_j}(Y) X^*_{p_j}(X) \right), \end{aligned} \tag{26}$$

$X, Y \in T\Gamma$. With X_f satisfying (7), we then have

$$\omega(X_f, X_h) = \{f, h\},\ f, h \in C^\infty(\Gamma).$$

In this fashion, we have generalized from the phase space being $\mathbb{R}^{2n}$ to being a general symplectic manifold.

4 Symplectic Space and G/H

We present a series of results that assert that we may consider a symplectic space (Γ,Ω) where Ω is the symplectic 2-form on Γ as a manifold $(G/H,\omega)$ where ω is a symplectic 2-form on G/H. Then we form the orbit of these $(G/H,\omega)$, and we obtain the leaves of (Γ,Ω). That is, we present the general theory of classical mechanics from the standpoint of a general locally compact Lie group, G, of symmetries of a physical system. We follow [2].

We start with a differentiable manifold M on which G acts on the left. Thus for all $g \in G$, we have that

$$\varphi_g : M \to M,\ \varphi_g(m) = gm \tag{27}$$

is a diffeomorphism. Similarly, for all $m \in M$, we have the diffeomorphism

$$\psi_m : G \to M,\ \psi_m(g) = gm. \tag{28}$$

Since the operation satisfies $g_1(gm) = (g_1 \circ g)m$ for all $g_1, g \in G$, we obtain

$$(\varphi_g \psi_m)(g_1) = g(g_1 m) = (\psi_m l_g)(g_1)$$

or

$$\varphi_g \psi_m = \psi_m l_g. \tag{29}$$

We also have $\psi_{gm}(g_1) = g_1(gm) = (g_1 \circ g)m = \psi_m r_{g^{-1}}(g_1)$, or

$$\psi_{gm} = \psi_m r_{g^{-1}}. \tag{30}$$

We want M to be a phase space, Γ, and that implies that it has a G-left invariant 2-form, Ω. Now, the G-left invariance of the 2-form is equivalent to the G-left invariant Poisson bracket: Let $(\varphi_g)_*$ denote the left action of g on $X \in T\Gamma$, and the action of $g \in G$ on any 2-form by φ_g^*:

$$[\varphi_g^*\Omega][X_f, X_h] = \Omega[(\varphi_g)_* X_f, (\varphi_g)_* X_h]. \tag{31}$$

Then, if we have left invariance of the Poisson bracket, we have left invariance of the 2-form Ω:

$$\Omega[(\varphi_g)_* X_f, (\varphi_g)_* X_h] = X_{\{\varphi_g f, \varphi_g h\}} = X_{\{f,h\}} = \Omega[X_f, X_h]. \tag{32}$$

The converse is proven similarly.

With this action, we have just proved

$$\varphi_g^*\Omega = \Omega. \tag{33}$$

Now, for each point $x \in \Gamma$, the 2-form on G, $\psi_x^*\Omega$, satisfies

$$l_g^*\psi_x^*\Omega = \psi_x^*\varphi_g^*\Omega = \psi_x^*\Omega; \tag{34}$$

i.e., $\psi_x^*\Omega$ is also left invariant.

Since the group action on the left commutes with the action on the right $[(l_g r_h)(k) = (r_h l_g)(k)\ \forall g, h, k \in G]$, we obtain

$$r_h^* l_g^* = l_g^* r_h^*.$$

Then for any left-invariant form ω on G, we obtain

$$l_g^* r_h^* \omega = r_h^* l_g^* \omega = r_h^* \omega;$$

so, $r_h^*\omega$ is another left-invariant form on G. Also, $(l_g r_g)(k) = g \circ k \circ g^{-1}$ is the action of the group commutator, from which we obtain the action of $g \in G$ on the curves in G, from which we obtain the action of $g \in G$ on elements in TG. Hence

$$g \mapsto (r_{g^{-1}})^* \tag{35}$$

defines a representation of G on the set of left-invariant 2-forms.

Finally, by

$$\psi_{gx} = \psi_x r_{g^{-1}},$$

we obtain

$$\psi_{gx}^*\Omega = r_{g^{-1}}^*\psi_x^*\Omega.$$

Thus we have proven

Theorem 2 *([2, p 173]) A left-invariant 2-form Ω on Γ defines a map*

$$\Psi : \Gamma \to \{\textit{2-forms on } G\}$$

given by

$$\Psi(x) = \psi_x^*\Omega,$$

and furthermore, this map has a representation of G:

$$\Psi(gx) = r_{g^{-1}}^*\Psi(x). \tag{36}$$

We also have

$$\delta\Psi(x) = \delta(\psi_x^*\Omega) = \psi_x^*\delta\Omega = 0;$$

i.e., $\Psi(x)$ is also a closed 2-form. Hence, $\Psi : \Gamma \to Z^2(\mathfrak{g}^*)$.

Consequently, we have

Theorem 3 *([2, p 173]) Given a symplectic space (Γ, Ω) with a (Lie) group of symplectic symmetries G, we define a G morphism, $\Psi : \Gamma \to Z^2(\mathfrak{g}^*)$. Since the map Ψ is a G morphism, $\Psi(\Gamma)$ is a union of G orbits in $Z^2(\mathfrak{g}^*)$. In particular, if the action of G on Γ is transitive, then the image of Ψ consists of a single G orbit in $Z^2(\mathfrak{g}^*)$.*

In view of the comment we made after equation (11), we may have trouble when we try to connect elements of $\mathfrak{g}$ with the canonical coordinates in Γ. We note, however, that we will be making a connection by operating on the right consistent with equation (30)!

By the next result, we may perform the inverse mapping as well: Now let G be a locally compact Lie group with its Lie algebra $\mathfrak{g}$ which is taken to be finite dimensional. Let $\omega \in Z^2(\mathfrak{g}^*)$. (If ω is not in the form of equation (22), it may be put in that form using the induction in the proof.) Define

$$\mathfrak{h}_\omega = \{X \in \mathfrak{g} \mid \iota_X\omega = 0\}.$$

Hence, if $Y \in \mathfrak{g}$ is such that $\iota_Y\omega = 0$, then

$$D_Y\omega = \iota_Y\delta\omega + \delta(\iota_Y\omega) = 0$$

since $\delta\omega = 0$, D_Y denoting the derivative in the direction of Y. Now if both X and Y are in $\mathfrak{h}_\omega$, we have

$$0 = D_X(\iota_Y\omega) = \iota_{D_X(Y)}\omega + \iota_Y D_X(\omega) = \iota_{D_X(Y)}\omega.$$

But one can show that $D_X(Y) = [X, Y]$; so, $\iota_{[X,Y]}\omega = 0$; that is, $\mathfrak{h}_\omega$ is a Lie algebra and a Lie subalgebra of $\mathfrak{g}$.

Next, form H_ω, a subgroup of G, by exponentiating $\mathfrak{h}_\omega$. *We assume that H_ω is a closed subgroup of G.* Hence G/H_ω is a manifold. Then, we may prove in one of two ways that G/H_ω is a symplectic space:

Method 1: There is a map (a submersion) $\rho : G \to G/H_\omega$ such that the leaves of the foliation are all of the form $\rho^{-1}(x)$ for $x \in G/H_\omega$. ρ is defined by using local coordinates $(x_1, \cdots, x_{\dim G})$ such that the leaves of the foliation are given by $x_1 = \text{constant}, \cdots, x_k = \text{constant}$, and so that the tangent space to G/H_ω is spanned by $\frac{\partial}{\partial x_{k+1}} = X_{k+1}, \cdots, \frac{\partial}{\partial x_{\dim G}} = X_{\dim G}$. Now write $\omega = \sum^{\dim G} a_{j,l} X_j^* \wedge X_l^*$ where the $a_{j,l}$ are real-valued functions. Then the condition

$$\iota\left(\frac{\partial}{\partial x_{k+1}}\right)\omega = 0, \cdots, \iota\left(\frac{\partial}{\partial x_{\dim G}}\right)\omega = 0$$

implies that $a_{j,l} = 0$ for j or l greater than k. The condition $\delta\omega = 0$ implies that $a_{j,l}$ cannot depend on the coordinates $x_{k+1}, \cdots, x_{\dim G}$. Thus we may use the coordinates $x_1, \cdots, x_k$ as local coordinates on G/H_ω. Therefore, we have $\rho(x_1, \cdots, x_{\dim G}) = (x_1, \cdots, x_k)$, and locally $\omega = \rho^* \overline{\omega}$ where

$$\overline{\omega} = \sum^{\dim G/H_\omega} a_{j,l} X_j^* \wedge X_l^*.$$

$\overline{\omega}$ is non-degenerate and closed; i.e., symplectic.

Method 2: From the Campbell-Baker-Hausdorff Theorem, we see that, for all $X \in \mathfrak{g}$ and $Y \in \mathfrak{h}_\omega$, $e^X \circ e^Y = e^{X+Z}$ for some $Z \in \mathfrak{h}_\omega$. But $e^X \circ e^Y$ is any element in G/H_ω, for G connected. Now $\omega(Te^X, Te^W) = \omega(X, W)$ for X, $W \in \mathfrak{g}$. Consequently, for $Y, U \in \mathfrak{h}_\omega$, there exist $V, Z \in \mathfrak{h}_\omega$ such that $e^X \circ e^Y = e^{X+V}$, and $e^W \circ e^U = e^{W+Z}$. Hence,

$$\begin{aligned} \omega(T(e^X \circ e^Y), T(e^W \circ e^U)) &= \omega(T(e^{X+V}), T(e^{W+Z})) \\ &= \omega(X+V, W+Z) \\ &= \omega(X, W). \end{aligned}$$

Thus we have an element of $Z^2(T^*(G/H_\omega))$.

We will henceforth use $(G/H_\omega, \overline{\omega})$ or its orbit as the phase space (Γ, Ω).

We should prove the remark that a general action of G on G/H_ω from the left, denoted here by $\cdot$, or from the right, denoted here by $*$, is a representation of G:

$$\begin{aligned} (l \circ k) \cdot (gH_\omega) &= [(l \circ k) \circ g]H_\omega \\ &= [l \circ (k \circ g)]H_\omega \\ &= l \cdot [(k \circ g)H_\omega] \\ &= l \cdot k \cdot [gH_\omega], \end{aligned} \tag{37}$$

and

$$\begin{aligned} (l \circ k) * (gH_\omega) &= (gH_\omega) \circ (l \circ k)^{-1} \\ &= (gH_\omega) \circ (k^{-1} \circ l^{-1}) \\ &= [(gH_\omega) \circ k^{-1}] \circ l^{-1} \\ &= l * [(gH_\omega) \circ k^{-1}] \\ &= l * [k * (gH_\omega)], \end{aligned} \tag{38}$$

for all $l, k, g \in G$. We will address the fact that this right action as it presently stands, is not an action on G/H_ω at all. It, however, will be seen to be an action on the orbit of G/H_ω.

In the special case of a Lie group in which the subgroup, H_ω, is normal, (38) may be rewritten:

$$\begin{aligned} (l \circ k) * (gH_\omega) &= [g \circ (l \circ k)^{-1}]H_\omega \\ &= [g \circ k^{-1} \circ l^{-1}]H_\omega \\ &= l * [(g \circ k^{-1})H_\omega] \\ &= l * k * (gH_\omega). \end{aligned}$$

But we will have a difficulty in pushing the last through for all Lie groups, as the element "$k * (gH_\omega)$" is not well-defined. If you have $g \circ H_\omega = g' \circ H_\omega$ then you have $g' = g \circ h$ for some $h \in H_\omega$ and then $(g \circ k^{-1}) \circ H_\omega = k * (g \circ H_\omega) = k * (g' \circ H_\omega) = k * (g \circ h \circ H_\omega) = (g \circ h \circ k^{-1}) \circ H_\omega$, but $g \circ h \circ k^{-1}$ is not equal to $g \circ k^{-1}$ even up to a multiplication on the left by some $h' \in H_\omega$. (Unless H_ω is normal in G.)

Now we may address the fact that the right action is not an action on G/H_ω at all. We will take for an action on the right,

$$k * (g \circ H_\omega) = (g \circ H_\omega) \circ k^{-1} = (g \circ k^{-1}) \circ (k \circ H_\omega \circ k^{-1}) \tag{39}$$

where we may use $k \circ H_\omega \circ k^{-1}$ as a new "$H_{\omega'}$". If H_ω is a closed subgroup of G, then so is $k \circ H_\omega \circ k^{-1}$. But this change from G/H_ω to $G/(k \circ H_\omega \circ k^{-1})$ is in accordance with the fact that we have a symplectic *manifold* M, T_xM, etc., and which is not simply a symplectic vector space. Consequently we have $k \circ H_\omega \circ k^{-1} = H_{k^*\omega} = H_{r_k^* l_k^* \omega} = H_{r_k^*\omega}$, consistant with the form of the symplectic manifold structure. This, in turn, says that we do not have a single symplectic vector space, G/H_ω, on which we work, but rather the following for the action $*$ on the right:

$$\begin{aligned} * \ &: \ (G, \cup'_{k \in G} G/H_{r_k^*\omega}) \to \cup'_{k \in G} G/H_{r_k^*\omega}, \\ * \ &: \ (g_0, g \circ H_{r_k^*\omega}) \mapsto (g \circ g_0^{-1}) \circ H_{r_{g_0}^* r_k^* \omega}, \end{aligned}$$

where $\cup'$ denotes the disjoint union and $g, g_0 \in G$.

We will augment the above paragraphs with the following: Instead of $l_g^*\omega = \omega$, suppose that you have $g^*\omega = \omega$. Then, for $X \in \mathfrak{h}_\omega$ and for all $Y \in \mathfrak{g}$, $0 = \omega(X, Y) = g^*\omega(X, Y) = \omega(g_*X, g_*Y)$. But the map $Y \mapsto g_*Y$ is one-to-one; so, this makes $\omega(g_*X, Y) = 0$ for all $Y \in \mathfrak{g}$ That is, $g_*X \in \mathfrak{h}_\omega$. In this way we obtain that $g^*\mathfrak{h}_\omega = \mathfrak{h}_\omega$ for all $g \in G$, and then $g \circ H_\omega \circ g^{-1} = H_\omega$; i.e., H_ω is normal in G. Hence we will get just one symplectic manifold $(G/H_\omega, \omega)$ which is just a symplectic vector space! The converse is also valid. As an application of this result, the Heisenberg group is commutative; so, every subgroup of the Heisenberg group is normal and we obtain just one symplectic manifold that is a symplectic vector space given any non-degenerate, closed 2-form. This, however, does not generalize to other Lie groups.

Also, we should remark that for the case of the left regular representation (quantum mechanics), it is enough to have a single $\omega \in Z^2(\mathfrak{g})$ and hence a single G/H_ω as $l_g G/H_\omega = G/H_\omega$.

4.0.1 Comment on the Right Action

For x with a group action on both the left or the right, we have from the left

$$x \mapsto \frac{d}{dt} e^{tX} x \mid_{t=0} \equiv Xx$$

and from the right

$$x \mapsto \frac{d}{dt} x e^{-tX} \mid_{t=0} \equiv xX.$$

In particular, if $x = gH_\omega$ and $X \in \mathfrak{h}_\omega$, then for the right action, we have

$$-xX = -gH_\omega X = -\frac{d}{dt} gH_\omega e^{-tX} \mid_{t=0} = -\frac{d}{dt} gH_\omega \mid_{t=0} = 0.$$

Thus, on G/H_ω, every element of $\mathfrak{h}_\omega$ has a right action equal to 0. However, $\mathfrak{h}_\omega$ has a different character when operating from the left: $Xx = XgH_\omega$. You cannot do anything to simplify the action in general, because of the intervening "g" which is variable. Similarly, if X and Y have $[X, Y] = Z \in \mathfrak{h}_\omega$, then when operating from the right on any $x \in G/H_\omega$, we have $x[X, Y] = xZ = 0$; that is,

$$xXY = xYX.$$

Thus the X and Y may be represented by partial derivatives when operating on the right on G/H_ω. We would like to generalize this result.

Now for classical mechanics, we wish to have canonical variables on $\cup' G/H_{r_g^*\omega}$; i.e., we wish to have elements of $T(\cup' G/H_{r_g^*\omega})$ that have commutation relations (Lie brackets) that are zero on $\cup' G/H_{r_g^*\omega}$. In view of the disjoint union, it suffices to make that a zero on G/H_ω and then perform the map $X \mapsto gXg^{-1}$ on the functions on $T(\cup' G/H_{r_g^*\omega})$. We have just shown that some of the elements of $T(G/H_\omega)$ are of the form of elements that are in $\mathfrak{g}$ and are not in $\mathfrak{h}_\omega$; i.e., we would like to have $X, Y \in \mathfrak{g}/\mathfrak{h}_\omega$ such that $[X, Y] \in \mathfrak{h}_\omega$ for all X, Y in a basis for $T(G/H_\omega)$. But $[\mathfrak{g}/\mathfrak{h}_\omega, \mathfrak{g}/\mathfrak{h}_\omega]$ is not in $\mathfrak{h}_\omega$ for a general Lie algebra. For example, in the rotation group in three dimensions (see Part 2), one basis of generators is $\{J_j \mid j \in \{1, 2, 3\}\}$ with $[J_1, J_2] = J_3$ and cyclically. You may choose ω such that $\mathfrak{h}_\omega = \{aJ_3 \mid a \in \mathbb{R}\}$. Then $[J_1, J_2] \in \mathfrak{h}_\omega$, but $[J_1 + J_3, J_2] \notin \mathfrak{h}_\omega$; i.e., canonical coordinates are not $\{J_1 + J_3, J_2\}$ but are $\{J_1, J_2\}$.

Definition 4 *(Tentative) Elements X_j of a basis of $\mathfrak{g}/\mathfrak{h}_\omega$ such that $[X_j, X_k] \in \mathfrak{h}_\omega$ will be denoted by $X_j \in \mathfrak{g} - \mathfrak{h}_\omega$.*

Then we would have

$$[\mathfrak{g} - \mathfrak{h}_\omega, \mathfrak{g} - \mathfrak{h}_\omega] \subseteq \mathfrak{h}_\omega \tag{40}$$

implies "the commutator equals zero" on G/H_ω.

This condition does not cover all cases in which X and Y apparently commute when acting on G/H_ω. Suppose we have a representation π on the right of G on G/H_ω. Denote the representation of $\mathfrak{g}$ on G/H_ω by π also. Then we have $[\pi(X), \pi(Y)] = \pi(Z)$, and $\pi(\mathfrak{h}_\omega) = 0$. Next, suppose in addition we have a $Z \in \mathfrak{g}$ that has $\pi(Z)$ having an eigenvalue, λ, on every vector in $\pi(G/H_\omega)$; i.e., $\pi(Z) = \lambda 1$. (Equivalently, we are working in a central extension of G with $\lambda 1$ denoting the point (λ, e) in the extension. In this case, automatically we have $\lambda 1 \in \mathfrak{h}_\omega$.) We now have $\lambda \mathfrak{h}_\omega = \mathfrak{h}_\omega$; so, for every such representation π of G on G/H_ω, we have $\mathfrak{h}_\omega[\pi(X), \pi(Y)] = \mathfrak{h}_\omega \pi(Z) = \lambda \mathfrak{h}_\omega = \mathfrak{h}_\omega$. Consequently, we have $(G/H_\omega)e^Z = G/H_\omega$. Now, if Z has a denumerable set of eigenvalues, we could use the spectral theorem and decompose $\pi(G/H_\omega)$ accordingly. We obtain the following:

Definition 5 *Let G be a Lie group with a finite dimensional Lie algebra $\mathfrak{g}$; let $\omega \in Z^2(\mathfrak{g})$ and let $\mathfrak{h}_\omega$ be its kernel. Assume that H_ω is a closed subgroup of G generated by the elements of $\mathfrak{h}_\omega$. The set of all elements X and Y of $\mathfrak{g}$ that, in a representation π of $\mathfrak{g}$ (or G) on G/H_ω, have*

$$[\pi(X), \pi(Y)] \subseteq \pi(\mathfrak{h}_\omega) + \mathbb{R}1 = \mathbb{R}1 \tag{40'}$$

form the set $\pi(\mathfrak{g} - \mathfrak{h}_\omega)$. In case $\pi(\mathfrak{g} - \mathfrak{h}_\omega)$ is not all of $T(G/H_\omega)$ because some of the pairs of elements have commutators that are operators that have non-singleton spectrum, then we (direct) sum over the spectrum. The elements of $\pi(\mathfrak{g} - \mathfrak{h}_\omega)$ are recognized potentially as a basis for $T(G/H_\omega)$. They will be referred to as "being a set of canonical variables in $\pi(G/H_\omega)$."

We emphasize that if $[\pi(X), \pi(Y)] = \pi(Z) \neq 0$, then we have no representation π in which $\pi(X) = \lambda 1$ or $\pi(Y) = \lambda 1$ for $\lambda \neq 0$. Consequently, the problem becomes one of choosing which of the canonical variables we may choose to be diagonalizable in terms of an eigenspectrum.

Furthermore, we may write (40') as {for all $Z \in \mathfrak{g}$, $[[\pi(X), \pi(Y)], \pi(Z)] = 0$} and interpret this as a condition that "$[X, Y]$ acts as a zero" on $\pi(G/H_\omega)$. This result is reminiscent of G. W. Mackey's work on the Galilei group to obtain interactions in quantum mechanics. [8]

In case (40') fails to hold for all elements $\pi(X)$ in a basis for $\pi(\mathfrak{g}/\mathfrak{h}_\omega)$, we may still proceed chosing a partial basis for $\pi(G/H_\omega)$ consisting of the $\pi(X)$s for which (40') hold; then make the Hamiltonian in terms of them, the rest of the basis for $\pi(G/H_\omega)$ being just orthogonal coordinates to complete the partial basis to a basis. This will generate the canonical variables in general. We will not have to resort to this case for most of the examples we will treat here.

For $g(G/H_\omega)g^{-1} = G/H_{r_g^*\omega}$, we have $[X_j, X_k] \in \mathfrak{h}_\omega + \mathbb{R}1$ (maybe doing a central extension of G) implies $[(r_g)_* X_j, (r_g)_* X_k] \in (r_g)_*(\mathfrak{h}_\omega + \mathbb{R}1)$. Consequently, supposing the set $\{X_j\}$ is a basis of canonical variables for G/H_ω, then the set $\{(r_g)_* X_j\}$ will be a basis of canonical variables on $g(G/H_\omega)g^{-1} = G/H_{r_g^*\omega}$. By taking the disjoint direct sum over all $g \in G$ we will obtain the canonical variables for the symplectic manifold. Similarly for $\pi(g(G/H_\omega)g^{-1})$, etc.

For completeness, we note that we only ask that (40') holds and not that the commutators of things in $P_n(\mathfrak{g})$, the polynomials in $\mathfrak{g}$ of order n, are in $\pi(\mathfrak{h}_\omega) + \mathbb{R}1$, even for $n = 2$. In the case of the Heisenberg group, generalizing (40), and not (40'), to $P_n(\mathfrak{g})$ is what Santille [9] calls the "problematic aspects of the quantization of Hamilton's equations into Heisenberg's equations," or in the language of Groenwald [10], von Hove [11], and others, a "full quantization" does not exist. In this "quantization" a map from the classical variables to the self-adjoint operators on some Hilbert space is assumed. (The recognition that the operators may operate from the right or the left doesn't seem to have been made.) In Groenwald's Theorem, it is shown that an inconsistency occurs when polynomials of order 3 are considered. Here, we are interested only in satisfying (40'). "Quantization" of arbitrary functions of the ps and qs are treated in [3] as well as in Section 3 of Part 2, but that presupposes that we have a phase space already.

5 Summary

In Part 1, we have shown that any locally compact Lie group has, in theory, classical interactions, H, on the classical variables defined on the appropriate G/H_ω by $g_0 H_\omega \mapsto g_0 H_\omega e^{-itH} = g_0 e^{-itH} H_\omega$. Then, on $g(G/H_\omega)g^{-1} = G/H_{r_g^*\omega}$ we have the corresponding form of the Hamiltonian given by $g_0 H_{r_g^*\omega} \mapsto g_0 H_{r_g^*\omega} e^{-it(r_g)_* H} = g_0 e^{-it(r_g)_* H} H_{r_g^*\omega}$. Note the dependence on $g \in G$.

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