Cayley's and Holland's Theorems for Idempotent Semirings and Their Applications to Residuated Lattices

Nikolaos Galatos
Department of Mathematics
University of Denver
ngalatos@du.edu

Rostislav Horčík
Institute of Computer Sciences
Academy of Sciences of the Czech Republic
horcik@cs.cas.cz

Abstract

We extend Cayley's and Holland's representation theorems to idempotent semirings and residuated lattices, and provide both functional and relational versions. Our analysis allows for extensions of the results to situations where conditions are imposed on the order relation of the representing structures. Moreover, we give a new proof of the finite embeddability property for the variety of integral residuated lattices and many of its subvarieties.

1 Introduction

Cayley's theorem states that every group can be embedded in the (symmetric) group of permutations on a set. Likewise, every monoid can be embedded into the (transformation) monoid of self-maps on a set. C. Holland [10] showed that every lattice-ordered group can be embedded into the lattice-ordered group of order-preserving permutations on a totally-ordered set. Recall that a *lattice-ordered group* (*ℓ-group*) is a structure $\boldsymbol{G} = \langle G, \vee, \wedge, \cdot, ^{-1}, 1\rangle$, where $\langle G, \cdot, ^{-1}, 1\rangle$ is group and $\langle G, \vee, \wedge\rangle$ is a lattice, such that multiplication preserves the order (equivalently, it distributes over joins and/or meets). An analogous representation was proved also for distributive lattice-ordered monoids in [2, 11]. We will prove similar theorems for residuated lattices and idempotent semirings in Sections 2 and 3. Section 4 focuses on the finite embeddability property (FEP) for various classes of idempotent semirings and residuated lattices. In particular, we present a new proof of the FEP for integral residuated lattices, which extends to classes of residuated lattices which were not known to have the FEP. Recall that if a finitely axiomatizable class of algebras has the FEP then its universal theory is decidable (see [4]).

An algebraic structure $\boldsymbol{R} = \langle R, +, \cdot, 1\rangle$ is called a *(unital) semiring* if $\langle R, +\rangle$ is a commutative semigroup, $\langle R, \cdot, 1\rangle$ is a monoid and multiplication distributes over addition, i.e., we have $a(b+c) = ab+ac$ and $(b+c)a = ba+ca$, for all $a, b, c \in R$. The additive reduct $\langle R, +\rangle$ of $\boldsymbol{R}$ is denoted by $\boldsymbol{R}^+$. A semiring $\boldsymbol{R}$ is called *idempotent* if it satisfies $a + a = a$, for all $a \in R$.

In that case $\boldsymbol{R}^+$ forms a semilattice. Thus one can introduce a partial order making R into a join-semilattice by letting $a \leq b$ iff $a + b = b$ (i.e., $+$ becomes the join). In this paper, we will be focusing only on idempotent semirings. Hence we will feel free to omit the modifier 'idempotent' from semirings (and semimodules over them).

Example 1.1 Let $\boldsymbol{L} = \langle L, \vee \rangle$ be a join-semilattice. Then the set $\mathrm{End}(\boldsymbol{L})$ of all join-semilattice endomorphisms on $\boldsymbol{L}$ forms an idempotent semiring $\mathbf{End}(\boldsymbol{L}) = \langle \mathrm{End}(\boldsymbol{L}), \vee, \circ, id \rangle$, where $\vee$ is computed pointwise, $\circ$ is the functional composition and id is the identity map on L.

A *residuated lattice* is an algebra $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1 \rangle$, where $\langle A, \wedge, \vee \rangle$ is a lattice, $\langle A, \cdot, 1 \rangle$ is a monoid and the following condition holds:

$$x \cdot y \leq z \quad \text{iff} \quad y \leq x \backslash z \quad \text{iff} \quad x \leq z / y \,.$$

It follows from the last equivalence that multiplication distributes over all existing (hence in particular over all finite) joins; for this and other facts about residuated lattices, see for example [7]. Thus, $\langle A, \vee, \cdot, 1 \rangle$ is an idempotent semiring.

Recall that one can define groups as structures $\langle G, \cdot, \backslash, /, 1 \rangle$, by the term equivalence: $x \backslash y = x^{-1} y$, $y / x = y x^{-1}$, and $x^{-1} = 1/x$. Therefore, ℓ-groups are term equivalent to special residuated lattices. In particular, if $\boldsymbol{G}$ is an ℓ-group, then $\langle G, \vee, \cdot, 1 \rangle$ is an idempotent semiring.

Let $\boldsymbol{P} = \langle P, \leq \rangle$ be a partially ordered set (poset). Its dual poset is denoted $\boldsymbol{P}^\partial = \langle P, \leq^\partial \rangle$ (i.e., $\leq^\partial = \geq$). A subset $U \subseteq P$ is called an *upset* if it is upward closed. Dually $D \subseteq P$ is a *downset* if it is downward closed. Given a subset $S \subseteq P$, the upset (resp. downset) generated by S is denoted $\uparrow S$ (resp. $\downarrow S$). An upset U is said to be finitely generated if $U = \uparrow\{u_1, \ldots, u_k\}$ for some $u_1, \ldots, u_k \in U$ (analogously for downsets). We adopt the convention of writing $\downarrow x$, $\uparrow x$ instead of $\downarrow\{x\}$, $\uparrow\{x\}$.

Let $\boldsymbol{P}$, $\boldsymbol{Q}$ be posets. A map $f \colon P \to Q$ is said to be *residuated* if there is a map $f^\dagger \colon Q \to P$ such that for all $x \in P$ and $y \in Q$ we have

$$f(x) \leq y \quad \text{iff} \quad x \leq f^\dagger(y) \,.$$

The map $f^\dagger$ is called a *residual* of f. The set of all residuated maps from $\boldsymbol{P}$ to $\boldsymbol{Q}$ is denoted $\mathrm{Res}(\boldsymbol{P}, \boldsymbol{Q})$. If $\boldsymbol{P} = \boldsymbol{Q}$ then we call f a residuated map on $\boldsymbol{P}$. We denote by $\mathrm{Res}(\boldsymbol{P})$ the set of all residuated maps on $\boldsymbol{P}$. Note that $\{f^\dagger \mid f \in \mathrm{Res}(\boldsymbol{P})\} = \mathrm{Res}(\boldsymbol{P}^\partial)$ are the residuated maps on the dual poset $\boldsymbol{P}^\partial$.

Residuated maps are closed under composition and they preserve arbitrary existing joins. On the other hand their residuals preserve arbitrary existing meets. Maps on complete lattices are residuated iff they preserve arbitrary joins.

Example 1.2 Let $\boldsymbol{L} = \langle L, \vee \rangle$ be a join-semilattice. The set $\mathrm{Res}(\boldsymbol{L})$ of all residuated maps on $\boldsymbol{L}$ forms a subsemiring $\mathbf{Res}(\boldsymbol{L})$ of $\mathbf{End}(\boldsymbol{L})$ since residuated maps are closed under composition and pointwise join.

Recall that an idempotent semiring $\boldsymbol{R}$ such that $\boldsymbol{R}^+$ is a complete lattice forms a residuated lattice iff its multiplication distributes over arbitrary joins from both sides (see e.g. [7]), i.e., for all $a \in R$ and $S \subseteq R$ we have:

$$a \cdot (\bigvee S) = \bigvee \{a \cdot s \mid s \in S\} \,, \qquad (\bigvee S) \cdot a = \bigvee \{s \cdot a \mid s \in S\} \,.$$

Example 1.3 Let $\boldsymbol{L}$ be a complete lattice. Then $\mathbf{Res}(\boldsymbol{L})$ is a complete idempotent semiring. Moreover it is a residuated lattice because for any $\{f\} \cup \{g_i \mid i \in K\} \subseteq \mathrm{Res}(\boldsymbol{L})$ we have

$$(f \circ \bigvee_{i\in K} g_i)(x) = f(\bigvee_{i\in K} g_i(x)) = \bigvee_{i\in K} f(g_i(x)),$$

$$((\bigvee_{i\in K} g_i) \circ f)(x) = (\bigvee_{i\in K} g_i)(f(x)) = \bigvee_{i\in K} g_i(f(x)).$$

2 Cayley-type representation theorems

The proof of Cayley's theorem relies on viewing an action of a group on a set as a group homomorphism into the symmetric group. More generally, assume that $\boldsymbol{M} = \langle M, \cdot, 1\rangle$ is a monoid, X a set, and $\star : M \times X \to X$ a left action of $\boldsymbol{M}$ on X, namely $1 \star x = x$ and $(m \cdot n) \star x = m \star (n \star x)$, for all $x \in X$ and $m, n \in M$. We refer to the structure $\langle X, \star\rangle$ as an *$\boldsymbol{M}$-set*. Then, the map $f_\star : M \to End(X)$ is a monoid homomorphism, where $End(X)$ is the set of all self-maps on X and $(f_\star(m))(x) = m\star x$. Conversely, given a monoid homomorphism $f : M \to End(X)$, then $\langle X, \star_f\rangle$ becomes an $\boldsymbol{M}$-set, where $m \star_f x = (f(m))(x)$. In this section we establish similar results for idempotent semirings and residuated lattices. We also provide residuated and relational versions of these theorems.

2.1 Cayley's endomorphism representation for idempotent semirings

We easily observe in this subsection that the situation generalizes in case we assume an additional (idempotent) semigroup structure.

Let $\boldsymbol{R} = \langle R, +, \cdot, 1\rangle$ be a (not necessarily idempotent) semiring. A *(left) $\boldsymbol{R}$-semimodule* $\boldsymbol{M}$ is a commutative semigroup $\langle M, +\rangle$ together with a map $\star\colon R \times M \to M$ such that for all $r, r' \in R$ and $m, m' \in M$ the following identities hold:

- $r \star (m + m') = r \star m + r \star m'$,
- $(r + r') \star m = r \star m + r' \star m$,
- $r \star (r' \star m) = (r \cdot r') \star m$,
- $1 \star m = m$.

Note that the map $\star$ is a left action of $\boldsymbol{R}$ on M. *Right $\boldsymbol{R}$-semimodules* are defined analogously replacing the left action $\star$ by a right action. Given a semiring $\boldsymbol{R}$, the *opposite semiring* $\boldsymbol{R}^{op} = \langle R, +, \odot, 1\rangle$ differs from $\boldsymbol{R}$ only in its multiplication, which is defined by $x \odot y = y \cdot x$. Clearly, every right $\boldsymbol{R}$-semimodule $\boldsymbol{M}$ can be viewed as a left $\boldsymbol{R}^{op}$-semimodule. Let $\boldsymbol{M}$ be a left or right $\boldsymbol{R}$-semimodule. We denote by $\boldsymbol{M}^+ = \langle M, +\rangle$ the scalar-free reduct of $\boldsymbol{M}$. Given $\boldsymbol{R}$-semimodules $\boldsymbol{M}$ and $\boldsymbol{N}$, a map $f\colon M \to N$ is an $\boldsymbol{R}$-semimodule homomorphism if $f(m + m') = f(m) + f(m')$ and $f(r \star m) = r \star f(m)$.

An $\boldsymbol{R}$-semimodule $\boldsymbol{M}$ is said to be *idempotent* if $m + m = m$ for all $m \in M$. Then $\boldsymbol{M}^+$ forms a semilattice where $m \leq n$ iff $m+n = n$ (i.e., $+$ is the corresponding join). If $\boldsymbol{M}^+$ forms a complete lattice, then we call $\boldsymbol{M}$ a *complete* $\boldsymbol{R}$-semimodule. Since we focus exclusively on idempotent semimodules over idempotent semirings, every semimodule in the paper will be assumed to be idempotent.

Example 2.1 Every semiring $\boldsymbol{R} = \langle R, \vee, \cdot, 1\rangle$ gives rise to an $\boldsymbol{R}$-semimodule $\langle R, \vee\rangle$, where multiplication serves as the action. On the other hand, every join-semilattice $\boldsymbol{L} = \langle L, \vee\rangle$ can be viewed as an $\mathbf{End}(\boldsymbol{L})$-semimodule, where the action $\star\colon \mathrm{End}(\boldsymbol{L}) \times L \to L$ is defined by $f \star m = f(m)$.

A subset E of an $\boldsymbol{R}$-semimodule $\boldsymbol{M}$ is called a *separating set* if for all $r, s \in R$ we have the following implication:

$$(\forall e \in E)(r \star e = s \star e) \quad \Rightarrow \quad r = s\,.$$

In that case the join-semilattice reduct of $\boldsymbol{R}$ embeds into the direct product of $|E|$-many copies of $\boldsymbol{M}^+$ via $r \mapsto \langle r \star e\rangle_{e\in E}$. In particular, if E is a singleton $\{e\}$, then the join-semilattice reduct of $\boldsymbol{R}$ embeds into $\boldsymbol{M}^+$ via $r \mapsto r \star e$. Note that separating sets are preserved by embeddings of $\boldsymbol{R}$-semimodules, i.e., if $f\colon \boldsymbol{M} \to \boldsymbol{N}$ is an embedding of $\boldsymbol{R}$-semimodules and $E \subseteq M$ a separating set in $\boldsymbol{M}$, then $f[E]$ is a separating set in $\boldsymbol{N}$.

Lemma 2.2 *Assume that $\boldsymbol{M}$ is an idempotent semigroup (i.e., a join semilattice) and that $\boldsymbol{R}$ is a semiring. Then $\boldsymbol{M}$ is an $\boldsymbol{R}$-semimodule with action $\star$ iff $\phi\colon \boldsymbol{R} \to \mathbf{End}(\boldsymbol{M}^+)$ is a semiring homomorphism, where $\star$ and ϕ are interdefinable by $(\phi(r))(m) = r \star m$. Moreover, $\boldsymbol{M}$ has a separating set iff ϕ is an embedding.*

Proof: It is easy to see that ϕ is a semiring homomorphism. Suppose that $E \subseteq M$ is a separating set and $r, s \in R$. If $r \neq s$ then there is $e \in E$ such that $r \star e \neq s \star e$. Thus ϕ is an embedding because $(\phi(r))(e) = r \star e \neq s \star e = (\phi(s))(e)$. The converse is equally easy. □

Since $\{1\}$ is a separating set in the semiring $\boldsymbol{R}$, viewed as an $\boldsymbol{R}$-semimodule, Lemma 2.2 shows that there is a semiring embedding $\phi\colon \boldsymbol{R} \to \mathbf{End}(\boldsymbol{R}^+)$, defined by $(\phi(r))(m) = rm$, for $r, m \in R$. Thus we obtain Cayley's theorem for idempotent semirings.

Theorem 2.3 (Cayley's theorem for idempotent semirings) *Every idempotent semiring $\boldsymbol{R}$ embeds into $\mathbf{End}(\boldsymbol{R}^+)$.*

2.2 Cayley's residuated representation for idempotent semirings

Let $\boldsymbol{R} = \langle R, \vee, \cdot, 1\rangle$ be an idempotent semiring. An $\boldsymbol{R}$-semimodule $\boldsymbol{M}$ is called *residuated* if there is a map $\backslash\colon R \times M \to M$ such that $r \star m \le n$ iff $m \le r\backslash n$. Note that $\backslash$ has to be a right action of $\boldsymbol{R}$ on M because one easily proves that $1\backslash m = m$ and $s\backslash(r\backslash m) = (rs)\backslash m$. Although $\backslash$ is a right action, we write is on the left hand side, because of its connection to residuated lattices. Thus $\boldsymbol{M}$ is residuated iff for every $r \in R$ the endomorphism $f_r\colon \boldsymbol{M}^+ \to \boldsymbol{M}^+$ defined by $f_r(m) = r \star m$ is residuated. Consequently, if $\boldsymbol{M}$ is complete, then $\boldsymbol{M}$ is residuated iff $r \star \bigvee S = \bigvee_{m\in S}(r \star m)$ for any $r \in R$ and $S \subseteq M$.

Let $\boldsymbol{M}$ be a residuated $\boldsymbol{R}$-semimodule such that $\boldsymbol{M}^+$ forms a lattice (e.g., if $\boldsymbol{M}^+$ is complete). Then $r\backslash(m \wedge n) = r\backslash m \wedge r\backslash n$ and $(r \vee s)\backslash m = r\backslash m \wedge s\backslash m$. Thus $(\boldsymbol{M}^+)^\partial$ together with the right action $\backslash$ forms an idempotent right $\boldsymbol{R}$-semimodule.

Example 2.4 Every join-semilattice $\boldsymbol{L} = \langle L, \vee\rangle$ can be turned into a residuated $\mathbf{Res}(\boldsymbol{L})$-semimodule whose right action is given by $f\backslash m = f^\dagger(m)$. It has also a separating set, namely L. Moreover, if $\boldsymbol{L}$ is additionally a lattice then $\boldsymbol{L}^\partial$ together with the right action $\backslash$ forms a right $\mathbf{Res}(\boldsymbol{L})$-semimodule.

Lemma 2.5 *In the notation of Lemma 2.2, $\boldsymbol{M}$ is residuated iff the image of ϕ is inside $\mathbf{Res}(\boldsymbol{M}^+)$.*

A subset I of a join-semilattice $\boldsymbol{L} = \langle L, \vee\rangle$ is said to be an *ideal* if for all $x, y \in L$:

- $y \in I$ and $x \leq y$ implies $x \in I$ (i.e., I is a downset),
- $x, y \in I$ implies $x \vee y \in I$.

The set $\mathcal{I}(\boldsymbol{L})$ of all ideals forms an algebraic closure system on $\boldsymbol{L}$, i.e., ideals are closed under arbitrary intersections and directed unions (see e.g. [6]). Thus the lattice $\mathcal{I}(\boldsymbol{L}) = \langle \mathcal{I}(\boldsymbol{L}), \cap, \vee\rangle$ is algebraic, where $I \vee J$ is the ideal generated by $I \cup J$. Note that $\emptyset$ is also considered to be an ideal.

Given a subset I of an idempotent $\boldsymbol{R}$-semimodule $\boldsymbol{M}$ and $r \in R$, we define a set $r \star I$ as follows:

$$r \star I = \{r \star m \mid m \in I\}\,.$$

Then $\mathcal{I}(\boldsymbol{M})$ denotes the join-semilattice $\langle \mathcal{I}(\boldsymbol{M}^+), \vee\rangle$ together with the map $*\colon R \times \mathcal{I}(\boldsymbol{M}^+) \to \mathcal{I}(\boldsymbol{M}^+)$ defined by $r * I = {\downarrow}(r \star I)$. It is easy to see that $r * I \in \mathcal{I}(\boldsymbol{M}^+)$ because if $a \leq r \star m$ and $b \leq r \star n$ for some $m, n \in I$ then $a \vee b \leq r \star (m \vee n) \in r \star I$. The following theorem shows that every $\boldsymbol{R}$-semimodule can be embedded into a complete, residuated one.

Theorem 2.6 *Let $\boldsymbol{M}$ be an $\boldsymbol{R}$-semimodule whose left action is $\star$. Then $\mathcal{I}(\boldsymbol{M})$ is a complete residuated $\boldsymbol{R}$-semimodule whose left action is $*$ and the residual right action is given by $r\backslash I = \{m \in M \mid r \star m \in I\}$.*

Moreover, $\boldsymbol{M}$ embeds into $\mathcal{I}(\boldsymbol{M})$ as an $\boldsymbol{R}$-semimodule via the map $m \mapsto {\downarrow}m$. In addition, if E is a separating set in $\boldsymbol{M}$ then $\{{\downarrow}e \mid e \in E\}$ is a separating set in $\mathcal{I}(\boldsymbol{M})$.

Proof: First, we check that $r\backslash I \in \mathcal{I}(\boldsymbol{M}^+)$ for all $r \in R$ and $I \in \mathcal{I}(\boldsymbol{M}^+)$. The set $r\backslash I$ is a downset because $n \leq m$ implies $r \star n \leq r \star m$. Let $a, b \in r\backslash I$, i.e., $r \star a, r \star b \in I$. Then $r \star (a \vee b) = r \star a \vee r \star b \in I$. Thus $a \vee b \in r\backslash I$.

Now we check that the axioms of $\boldsymbol{R}$-semimodules hold for $\mathcal{I}(\boldsymbol{M})$. Obviously, $1 * I = {\downarrow}(1 \star I) = {\downarrow}I = I$ for all $I \in \mathcal{I}(\boldsymbol{M}^+)$. It is straightforward to check that $(rs) * I = {\downarrow}((rs) \star I) = {\downarrow}(r \star {\downarrow}(s \star I)) = r * (s * I)$. Thus $*$ is a left action of $\boldsymbol{R}$ on $\mathcal{I}(\boldsymbol{M}^+)$. We will show that it is residuated, i.e., $r * I \subseteq J$ iff $I \subseteq r\backslash J$. Suppose that $r * I \subseteq J$ holds and let $a \in I$. Then $r \star a \in r * I \subseteq J$. Thus $a \in r\backslash J$. Conversely, suppose that $I \subseteq r\backslash J$. Let $a \in r * I$, i.e., $a \leq r \star m$ for some $m \in I$. Then $m \in r\backslash J$, i.e., $r \star m \in J$. Consequently, $a \in J$ because J is an ideal. Since $*$ is residuated, it follows that for every $r \in R$ the endomorphism $f_r\colon \mathcal{I}(\boldsymbol{M}^+) \to \mathcal{I}(\boldsymbol{M}^+)$ given by $f_r(I) = r * I$ is residuated. Thus $r * (I \vee J) = r * I \vee r * J$ because residuated maps preserve joins. To see that $(r \vee s) * I = r * I \vee s * I$, note that $r \leq r'$ implies $r * I \subseteq r' * I$. Thus $(r \vee s) * I \supseteq r * I \vee s * I$. Conversely, assume that $r * I, s * I \subseteq J$. Then $I \subseteq r\backslash J \cap s\backslash J = (r \vee s)\backslash J$. Thus $(r \vee s) * I \subseteq J$. Summing up, $\mathcal{I}(\boldsymbol{M})$ is a complete residuated $\boldsymbol{R}$-semimodule.

The map $f\colon M \to \mathcal{I}(\boldsymbol{M}^+)$ defined by $f(m) = {\downarrow}m$ is an $\boldsymbol{R}$-semimodule homomorphism. Indeed, the map f preserves finite joins because ${\downarrow}(m \vee n) = {\downarrow}m \vee {\downarrow}n$. Further, $f(r * m) = {\downarrow}(r \star m) = {\downarrow}(r \star {\downarrow}m) = r * f(m)$. The additional part follows from the fact that embeddings of $\boldsymbol{R}$-semimodules preserves separating sets. □

Corollary 2.7 *Let $\boldsymbol{M}$ be an $\boldsymbol{R}$-semimodule. Then there is a semiring homomorphism $\phi\colon \boldsymbol{R} \to \mathbf{Res}(\mathcal{I}(\boldsymbol{M}^+))$ given by $(\phi(r))(I) = r * I = {\downarrow}(r \star I)$. In addition, if $\boldsymbol{M}$ has a separating set then ϕ is an embedding.*

Proof: $\mathcal{I}(\boldsymbol{M})$ is a residuated $\boldsymbol{R}$-semimodule by Theorem 2.6, so there is a semiring homomorphism ϕ from $\boldsymbol{R}$ to $\mathbf{Res}(\mathcal{I}(\boldsymbol{M}^+))$ by Lemmas 2.2 and 2.5. The claim about separating sets follows from the corresponding part of Theorem 2.6. $\square$

Theorem 2.8 (Residuated Cayley's theorem for idempotent semirings) *Any idempotent semiring $\boldsymbol{R}$ is embeddable into $\mathbf{Res}(\mathcal{I}(\boldsymbol{R}^+))$.*

2.3 Cayley's relational representation for idempotent semirings

Note that one can identify a binary relation $R \subseteq A \times B$ with a function from A to $\mathcal{P}(B)$ mapping $a \in A$ to $R(a) = \{b \in B \mid \langle a, b\rangle \in R\}$. Furthermore, such a function lifts to a function from $\mathcal{P}(A)$ to $\mathcal{P}(B)$, defined by $R[X] = \bigcup_{x\in X} R(x)$; note that we abuse notation by overloading the symbol R. Actually, all such lifted functions are exactly the residuated maps from $\mathcal{P}(A)$ to $\mathcal{P}(B)$. So we identify relations from A to B with residuated maps from $\mathcal{P}(A)$ to $\mathcal{P}(B)$. Due to this, we compose relations like functions, i.e., for $R \subseteq A\times B$ and $S \subseteq B\times C$ the composition of R and S is $S \circ R = \{\langle a, c\rangle \in A \times C \mid (\exists b \in B)(\langle a, b\rangle \in R \text{ and } \langle b, c\rangle \in S)\}$.

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are join-semilattices the above maps restrict to maps from $\mathcal{I}(\boldsymbol{A})$ to $\mathcal{P}(B)$. We will focus on the case where this restrictions are actually residuated maps from $\mathcal{I}(\boldsymbol{A})$ to $\mathcal{I}(\boldsymbol{B})$; we denote the associated set by $\mathrm{Res}(\mathcal{I}(\boldsymbol{A}), \mathcal{I}(\boldsymbol{B}))$. Note that the this set forms a join semilattice under pointwise order. We will characterize the relations that give rise to residuated maps from $\mathcal{I}(\boldsymbol{A})$ to $\mathcal{I}(\boldsymbol{B})$.

A relation $R \subseteq A \times B$ is called *compatible* if for all $x \in A, y \in B$:

- $R(x) \in \mathcal{I}(\boldsymbol{B})$,
- $R(x \vee y) = R(x) \vee R(y)$, where the second join is computed in $\mathcal{I}(\boldsymbol{B})$.

In other words, they can be identified with join-semilattice homomorphisms from $\boldsymbol{A}$ to $\mathcal{I}(\boldsymbol{B})$, and as such they also form a complete (since $\mathcal{I}(\boldsymbol{B})$ is complete) join semilattice that we denote by $\mathrm{REnd}(\boldsymbol{A}, \boldsymbol{B})$. If $\boldsymbol{A} = \boldsymbol{B}$, we refer to R as a compatible relation on $\boldsymbol{A}$ and write $\mathrm{REnd}(\boldsymbol{A})$ for the above set.

For every compatible relation R we define the map $f_R \colon \mathcal{I}(\boldsymbol{A}) \to \mathcal{I}(\boldsymbol{B})$ by $f_R(I) = R[I]$.

Lemma 2.9 *Given join semilattices $\boldsymbol{A}$ and $\boldsymbol{B}$, then $\phi\colon \mathrm{REnd}(\boldsymbol{A}, \boldsymbol{B}) \to \mathrm{Res}(\mathcal{I}(\boldsymbol{A}), \mathcal{I}(\boldsymbol{B}))$, where $\phi(R) = f_R$, is a join-semilattice isomorphism.*

Proof: First we show that if R is a compatible relation, then $f_R \in \mathrm{Res}(\mathcal{I}(\boldsymbol{A}), \mathcal{I}(\boldsymbol{B}))$. Note that if I is an ideal of $\boldsymbol{A}$, then $R[I] = R[\bigcup_{x\in I}\{x\}] = \bigcup_{x\in I} R(x)$ is an ideal because the union is directed; we used that from the definition of a compatible relation it follows that $x \le y$ implies $R(x) \subseteq R(y)$. We now show that f_R preserves arbitrary joins, and thus is residuated. Indeed, let $\{I_i \in \mathcal{I}(\boldsymbol{A}) \mid i \in K\}$ be an indexed family of ideals. We have to show that $R[\bigvee_{i\in K} I_i] = \bigvee_{i\in K} R[I_i]$. The inclusion $R[\bigvee_{i\in K} I_i] \supseteq \bigvee_{i\in K} R[I_i]$ is obvious because $\bigvee_{i\in K} I_i \supseteq I_i$ for all $i \in K$. Conversely, let $a \in R[\bigvee_{i\in K} I_i]$. Then there is $x \in \bigvee_{i\in K} I_i$ (i.e., $x \le u_1 \vee\cdots\vee u_n$ for some $i_1, \dots, i_n \in K$ and $u_1 \in I_{i_1}, \dots, u_n \in I_{i_n}$) such that $a \in R(x)$. Since R is compatible, we have

$$a \in R(x) \subseteq R(u_1 \vee \cdots \vee u_n) = R(u_1) \vee \cdots \vee R(u_n) \subseteq R[I_{i_1}] \vee \cdots \vee R[I_{i_n}] \subseteq \bigvee_{i\in K} R[I_i]\,.$$

To show that ϕ is onto, let $f \in \mathrm{Res}(\mathcal{I}(\boldsymbol{A}), \mathcal{I}(\boldsymbol{B}))$. Note that the relation $R \subseteq A \times B$, where $R(x) = f(\downarrow x)$, is compatible since $f(\downarrow x)$ is an ideal and $f(\downarrow(x \vee y)) = f(\downarrow x \vee \downarrow y) = f(\downarrow x) \vee f(\downarrow y)$. We will show that $\phi(R) = f$. We have

$$(\phi(R))(I) = R[I] = R[\bigcup_{x \in I}\{x\}] = \bigcup_{x \in I} R(x) = \bigcup_{x \in I} f(\downarrow x)\,.$$

Observe that the last union is directed. Consequently,

$$\bigcup_{x \in I} f(\downarrow x) = \bigvee_{x \in I} f(\downarrow x) = f(\bigvee_{x \in I} \downarrow x) = f(I)\,.$$

To show that ϕ is 1-1, we prove that for all compatible relations R, $R(x) = R[\downarrow x]$. The forward inclusion is obvious. For the converse, if $y \in R[\downarrow x]$ then $y \in R(z)$ for some $z \leq x$. Since then $R(z) \subseteq R(x)$, we obtain $y \in R(x)$.

Now, we show that ϕ is a homomorphism, namely that $(R \vee S)[I] = R[I] \vee S[I]$ for any ideal $I \in \mathcal{I}(\boldsymbol{A})$. We have

$$(R \vee S)[I] = (R \vee S)[\bigcup_{x \in I}\{x\}] = \bigcup_{x \in I}(R \vee S)(x) = \bigcup_{x \in I} R(x) \vee S(x)\,.$$

Thus it suffices to prove that $\bigcup_{x \in I} R(x) \vee S(x) = R[I] \vee S[I]$. First, suppose that $R[I] = \emptyset$ or $S[I] = \emptyset$. Without any loss of generality we can assume that $R[I] = \emptyset$. Then $R(x) = \emptyset$ for every $x \in I$. Thus $R[I] \vee S[I] = S[I] = \bigcup_{i \in I} S(x) = \bigcup_{i \in I} R(x) \vee S(x)$. Second, assume that $R[I], S[I] \neq \emptyset$. Since $R(x) \subseteq R[I]$ and $S(x) \subseteq S[I]$ for every $x \in I$, we have $\bigcup_{x \in I} R(x) \vee S(x) \subseteq R[I] \vee S[I]$. Conversely, let $a \in R[I] \vee S[I]$. Then $a \leq u \vee v$ for some $u \in R[I]$ and $v \in S[I]$. Thus there are $y, y' \in I$ such that $u \in R(y)$ and $v \in S(y')$. Then $y \vee y' \in I$ as well and by the properties of compatible relations we obtain $u \in R(y \vee y')$ and $v \in S(y \vee y')$. Since $a \leq u \vee v$, we have

$$a \in R(y \vee y') \vee S(y \vee y') \subseteq \bigcup_{x \in I} R(x) \vee S(x)\,.$$

Thus the second inclusion holds as well. □

Lemma 2.10 *Given a join semilattice* $\boldsymbol{L} = \langle L, \vee \rangle$, $\mathbf{REnd}(\boldsymbol{L}) = \langle \mathrm{REnd}(\boldsymbol{L}), \vee, \circ, Id \rangle$ *is a semiring isomorphic to* $\mathbf{Res}(\mathcal{I}(\boldsymbol{L}))$*, where* $\circ$ *is the relational composition and* $Id(x) = \downarrow x$ *(i.e.,* $\langle x, y \rangle \in Id$ *iff* $x \geq y$*).*

Proof: To prove this, we first show that $\mathbf{REnd}(\boldsymbol{L})$ is a well-defined algebra, i.e., $\vee$, $\circ$ and Id are well-defined operations. Then we show that it is isomorphic to the semiring $\mathbf{Res}(\mathcal{I}(\boldsymbol{L}))$ (see Example 1.2).

By Lemma 2.9 we know that $\phi\colon \mathrm{REnd}(\boldsymbol{L}) \to \mathrm{Res}(\mathcal{I}(\boldsymbol{L}))$ defined by $\phi(R) = f_R$ (where $f_R(I) = R[I]$) is a join-semilattice isomorphism. Thus $\mathrm{REnd}(\boldsymbol{L})$ forms a join semilattice. Further, $Id \in \mathrm{REnd}(\boldsymbol{L})$ since $Id(x) = \downarrow x \in \mathcal{I}(\boldsymbol{L})$ and $\downarrow(x \vee y) = \downarrow x \vee \downarrow y$. Finally we check that compatible relations are closed under composition. Let $R, S \in \mathrm{REnd}(\boldsymbol{L})$. Clearly, $(S \circ R)(x) = S[R(x)] = f_S(R(x))$ is an ideal because $f_S \in \mathrm{Res}(\mathcal{I}(\boldsymbol{L}))$ maps ideals to ideals. To see the second condition from the definition of a compatible relation, note that $(S \circ R)(x \vee y) = S[R(x \vee y)] = S[R(x) \vee R(y)] = S[R(x)] \vee S[R(y)]$ because $f_S \in \mathrm{Res}(\mathcal{I}(\boldsymbol{L}))$ preserves joins.

Now it suffices to prove that ϕ preserves also $\circ$ and Id. We have $f_{S\circ R}(I) = S[R[I]] = f_S(f_R(I))$ and

$$f_{Id}(I) = Id[\bigcup_{x\in I}\{x\}] = \bigcup_{x\in I} Id(x) = \bigcup_{x\in I} {\downarrow}x = I\,.$$

□

We now restate Corollary 2.7 and Theorem 2.8 in view of the new presentation of $\mathbf{Res}(\mathcal{I}(\boldsymbol{L}^+))$.

Corollary 2.11 *Let $\boldsymbol{L}$ be an $\boldsymbol{R}$-semimodule. Then there is a semiring homomorphism $\phi\colon \boldsymbol{R} \to \mathbf{REnd}(\boldsymbol{L}^+)$ given by $(\phi(r))(x) = {\downarrow}(r \star x)$ for $x \in L$. In addition, if $\boldsymbol{L}$ has a separating set then ϕ is an embedding.*

Theorem 2.12 (Relational Cayley's theorem for idempotent semirings) *Any idempotent semiring $\boldsymbol{R}$ is embeddable into the semiring of relations $\mathbf{REnd}(\boldsymbol{R}^+)$.*

Note that similar kind of relational representations for quantales (i.e., complete idempotent semirings where multiplication distributes over arbitrary joins from both sides) were obtained in [5, 12].

2.4 Cayley's representation for residuated lattices

Recall that an *interior operator* on a poset $\boldsymbol{P}$ is a map $\sigma\colon P \to P$ which is contracting ($\sigma(x) \leq x$), idempotent ($\sigma(\sigma(x)) = \sigma(x)$) and monotone ($x \leq y$ implies $\sigma(x) \leq \sigma(y)$). The image of σ is denoted P_σ. Let $\boldsymbol{L}$ be a complete lattice and $S \subseteq L$. Then S induces an interior operator $\sigma\colon L \to L$ as follows:

$$\sigma_S(x) = \bigvee\{z \in S \mid z \leq x\} = \bigvee(S \cap {\downarrow}x)\,. \tag{1}$$

Let $\boldsymbol{A}$ be a residuated lattice. An interior operator σ on $\boldsymbol{A}$ is called a *conucleus* if $\sigma(1) = 1$ and $\sigma(x)\sigma(y) \leq \sigma(xy)$. Given a conucleus σ on a residuated lattice $\boldsymbol{A} = \langle A, \wedge, \vee, \cdot, \backslash, /, 1\rangle$, one can define a new residuated lattice $\boldsymbol{A}_\sigma = \langle A_\sigma, \wedge_\sigma, \vee, \cdot, \backslash_\sigma, /_\sigma, 1\rangle$, where $x \wedge_\sigma y = \sigma(x \wedge y)$, $x\backslash_\sigma y = \sigma(x\backslash y)$ and $x/_\sigma y = \sigma(x/y)$ (see [7]). The residuated lattice $\boldsymbol{A}_\sigma$ is called a *conuclear contraction*[1] of $\boldsymbol{A}$. Note that $\boldsymbol{A}_\sigma$ is a subsemiring of $\boldsymbol{A}$.

Lemma 2.13 *Let $\boldsymbol{A}$ be a complete residuated lattice and S a submonoid of $\boldsymbol{A}$. Then the interior operator σ_S on $\boldsymbol{A}$ given by (1) is a conucleus.*

Proof: Since $1 \in S$, we clearly have $\sigma_S(1) = 1$. It remains to check that $\sigma(x)\sigma(y) \leq \sigma(xy)$. Since multiplication distributes over arbitrary joins, we obtain

$$\sigma(x)\sigma(y) = \Big(\bigvee_{\substack{u\in S\\ u\leq x}} u\Big)\Big(\bigvee_{\substack{v\in S\\ v\leq y}} v\Big) = \bigvee_{\substack{u,v\in S\\ u\leq x\\ v\leq y}} uv \;\leq \bigvee_{\substack{z\in S\\ z\leq xy}} z = \sigma(xy)\,.$$

□

[1] Complete residuated lattices where multiplication coincides with meet give rise to *locales*, examples of which are formed by considering the open sets of a topological space. If $\boldsymbol{A}$ is the powerset of a topological space X, viewed as a residuated lattice, then the topological interior operator σ on X is a conucleus and $\boldsymbol{A}_\sigma$ is the locale of open sets of X. Note that if $S \subseteq A$ is a *subbasis* for X, then σ_S gives the associated interior operator for the topology.

Now we are going to prove a key lemma which allows us to transfer our results on semirings to residuated lattices. Before we do that, we recall the definition of a partial subalgebra. Let $\boldsymbol{A} = \langle A, \langle f_i^{\boldsymbol{A}} \rangle_{i\in K}\rangle$ be an algebra and $B \subseteq A$. Then $\boldsymbol{B} = \langle B, \langle f_i^{\boldsymbol{B}} \rangle_{i\in K}\rangle$ is a partial subalgebra of $\boldsymbol{A}$ where for every n-ary operation f_i, $i \in K$, we define

$$f_i^{\boldsymbol{B}}(a_1, \dots, a_n) = \begin{cases} f_i^{\boldsymbol{A}}(a_1, \dots, a_n) & \text{if } f_i^{\boldsymbol{A}}(a_1, \dots, a_n) \in B, \\ \text{undefined} & \text{otherwise.} \end{cases}$$

Given an algebra $\boldsymbol{C}$ of the same type as $\boldsymbol{A}$ and a one-to-one map $g\colon B \to C$, we say that g is an embedding of $\boldsymbol{B}$ into $\boldsymbol{C}$ if for every n-ary operation f_i, $i \in K$, and $a_1, \dots, a_n \in B$ we have

$$g(f_i^{\boldsymbol{B}}(a_1, \dots, a_n)) = f_i^{\boldsymbol{C}}(g(a_1), \dots, g(a_n)),$$

whenever $f_i^{\boldsymbol{B}}(a_1, \dots, a_n)$ is defined.

Lemma 2.14 *Let $\boldsymbol{A}, \boldsymbol{B}$ be residuated lattices such that $\boldsymbol{B}$ is complete and $\boldsymbol{C}$ a partial subalgebra of $\boldsymbol{A}$. Further, let $\boldsymbol{D}$ be the idempotent subsemiring of $\boldsymbol{A}$ generated by C. Suppose that there is a semiring homomorphism $f\colon \boldsymbol{D} \to \boldsymbol{B}$ such that $f(d) \leq f(c)$ implies $d \leq c$ for all $d \in D$ and $c \in C$. Then $f\colon \boldsymbol{C} \to \boldsymbol{B}_{\sigma_{f[D]}}$ is an embedding of residuated lattices.*

Proof: Since $f[D]$ is a submonoid of $\boldsymbol{B}$, $\sigma_{f[D]}$ is a conucleus by Lemma 2.13. We check that the map f preserves finite meets, i.e., $f(a \wedge b) = f(a) \wedge_{\sigma_{f[D]}} f(b) = \sigma_{f[D]}(f(a) \wedge f(b))$ for $a, b, a \wedge b \in C$. To prove it, we will show that $f(a \wedge b) = \max f[D] \cap {\downarrow}(f(a) \wedge f(b))$, i.e., $f(a \wedge b) = \sigma_{f[D]}(f(a) \wedge f(b))$. First, note that $f(a \wedge b) \in f[D] \cap {\downarrow}(f(a) \wedge f(b))$ because f preserves order. Let $f(d) \in f[D] \cap {\downarrow}(f(a) \wedge f(b))$. Then $f(d) \leq f(a) \wedge f(b)$, i.e., $f(d) \leq f(a)$ and $f(d) \leq f(b)$. Since $d \in D$ and $a, b \in C$, we have $d \leq a$ and $d \leq b$ by our assumption, i.e., $d \leq a \wedge b$. Thus $f(d) \leq f(a \wedge b)$.

Next we show that the map f preserves the left division, i.e., $f(a\backslash b) = f(a)\backslash_{\sigma_{f[D]}} f(b) = \sigma_{f[D]}(f(a)\backslash f(b))$ for $a, b, a\backslash b \in C$. Similarly as before we show that $f(a\backslash b) = \max f[D] \cap {\downarrow}(f(a)\backslash f(b))$. First, $a(a\backslash b) \leq b$. Thus $f(a)f(a\backslash b) \leq f(b)$, i.e., $f(a\backslash b) \leq f(a)\backslash f(b)$. Consequently, $f(a\backslash b) \in f[D] \cap {\downarrow}(f(a)\backslash f(b))$. Let $f(d) \in f[D] \cap {\downarrow}(f(a)\backslash f(b))$. Then $f(ad) = f(a)f(d) \leq f(b)$. Since $ad \in D$ and $b \in C$, we have $ad \leq b$ by our assumption, i.e., $d \leq a\backslash b$. Thus $f(d) \leq f(a\backslash b)$. The proof for the right division is analogous.

Finally, our assumption on f ensures that f is one-to-one when restricted to C. □

Theorem 2.15 *Let $\boldsymbol{A}, \boldsymbol{B}$ be residuated lattices such that $\boldsymbol{B}$ is complete. If $\boldsymbol{A}$ embeds into $\boldsymbol{B}$ via f as an idempotent semiring, then $\boldsymbol{A}$ embeds into $\boldsymbol{B}_{\sigma_{f[A]}}$ as a residuated lattice.*

Proof: Let $f\colon A \to B$ be the semiring embedding. Note that $f(a) \leq f(b)$ implies $a \leq b$ for all $a, b \in A$. Indeed, if $f(a) \leq f(b)$ then $f(a \vee b) = f(a) \vee f(b) = f(b)$. Consequently, $a \vee b = b$ since f is one-to-one. Thus by Lemma 2.14 the map $f\colon \boldsymbol{A} \to \boldsymbol{B}_{\sigma_{f[A]}}$ is an embedding of residuated lattices. □

Corollary 2.16 (Cayley's theorem for residuated lattices) *Let $\boldsymbol{A}$ be a residuated lattice and $\boldsymbol{A}^+$ its join-semilattice reduct. Then $\boldsymbol{A}$ embeds into a conuclear contraction of $\mathbf{Res}(\mathcal{I}(\boldsymbol{A}^+)) \cong \mathbf{REnd}(\boldsymbol{A}^+)$.*

In addition, if $\boldsymbol{A}$ is complete then $\boldsymbol{A}$ embeds into a conuclear contraction of $\mathbf{Res}(\boldsymbol{A}^+)$.

Proof: By Theorem 2.8 $\boldsymbol{A}$ embeds into $\mathbf{Res}(\mathcal{I}(\boldsymbol{A}^+)) \cong \mathbf{REnd}(\boldsymbol{A}^+)$ as a semiring. Thus the corollary follows by Theorem 2.15. To see the additional part, note that $\boldsymbol{A}$ embeds as a semiring into $\mathbf{Res}(\boldsymbol{A}^+)$ by Lemma 2.2 because $\boldsymbol{A}$ is a residuated lattice. Consequently, $\boldsymbol{A}$ embeds into a conuclear contraction of $\mathbf{Res}(\boldsymbol{A}^+)$ by Theorem 2.15. □

3 Holland-type representation theorems

Throughout this section $\boldsymbol{R}$ always denotes an idempotent semiring. A congruence on an $\boldsymbol{R}$-semimodule $\boldsymbol{M}$ is an equivalence $\sim$ on M such that $m \sim m'$ implies $r \star m \sim r \star m'$ and $m \vee n \sim m' \vee n$ for all $r \in R$ and $n \in M$. Every ideal $I \in \mathcal{I}(\boldsymbol{M}^+)$ induces a congruence $\sim_I$ on $\boldsymbol{M}$ defined as follows:

$$m \sim_I m' \qquad \text{iff} \qquad (\forall r \in R)(r \star m \in I \Leftrightarrow r \star m' \in I)\,.$$

Indeed, the relation $\sim_I$ is clearly an equivalence. Further, let $m, m', n \in M$ and $m \sim_I m'$. Then $r \star (m \vee n) = r \star m \vee r \star n \in I$ iff $r \star m, r \star n \in I$. Since $m \sim_I m'$, we have $r \star m \in I$ iff $r \star m' \in I$. Thus $r \star m \vee r \star n \in I$ iff $r \star m' \vee r \star n \in I$ because I is closed under finite joins. Finally, for any $r \in R$ we have $r \star (r' \star m) = (rr') \star m \in I$ iff $r \star (r' \star m') = (rr') \star m' \in I$. Thus $\sim_I$ is a congruence. The resulting quotient is denoted $\boldsymbol{M}/I$.

An ideal $I \in \mathcal{I}(\boldsymbol{M}^+)$ is called *linear* if $r \star m \in I$ and $s \star n \in I$ implies $r \star n \in I$ or $s \star m \in I$. The reason for this name is explained by the following lemma.

Lemma 3.1 *Let $\boldsymbol{M}$ be an idempotent $\boldsymbol{R}$-semimodule. Then $I \in \mathcal{I}(\boldsymbol{M}^+)$ is linear iff $\boldsymbol{M}/I$ is linearly ordered.*

Proof: $\boldsymbol{M}/I$ is linearly ordered iff for all $m, n \in M$ we have either $m \vee n \sim_I n$ or $m \vee n \sim_I m$. Observe that $r \star (m \vee n) = r \star m \vee r \star n \in I$ implies $r \star m, r \star n \in I$ for all $r \in R$. Thus $\boldsymbol{M}/I$ is linearly ordered iff either $r \star m \in I$ implies $r \star m \vee r \star n \in I$ for all $r \in R$ or $r \star n \in I$ implies $r \star m \vee r \star n \in I$ for all $r \in R$.

Assume that I is linear and there is $s \in R$ such that $s \star n \in I$ but $s \star m \vee s \star n \notin I$. It remains to show that $r \star m \in I$ implies $r \star m \vee r \star n \in I$ for all $r \in R$. Assume that $r \star m \in I$. By linearity of I we have $r \star n \in I$ or $s \star m \in I$. Obviously, the latter cannot be true otherwise $s \star m \vee s \star n \in I$. Thus $r \star n \in I$. Consequently, $r \star m \vee r \star n \in I$.

Conversely, assume that $\boldsymbol{M}/I$ is linearly ordered. Let $r \star m \in I, s \star n \in I$. By the assumption, either $r \star m \vee r \star n \in I$ or $s \star m \vee s \star n \in I$. If $r \star n \notin I$, then $r \star m \vee r \star n \notin I$, so $s \star m \vee s \star n \in I$, hence $s \star m \in I$. Consequently, I is linear. $\square$

Consider the following quasi-identity in the language of semimodules:

$$u \le h \vee c \star a \quad \text{and} \quad u \le h \vee d \star b \qquad \text{implies} \qquad u \le h \vee c \star b \vee d \star a\,. \tag{sl}$$

The same quasi-identity can be considered also in the language of semirings:

$$u \le h \vee ca \quad \text{and} \quad u \le h \vee db \qquad \text{implies} \qquad u \le h \vee cb \vee da\,. \tag{sl'}$$

We refer to semimodules and semirings satisfying these equations as *semilinear*. Clearly, every semilinear semiring $\boldsymbol{R}$ is semilinear also when viewed as an $\boldsymbol{R}$-semimodule. Furthermore, if an $\boldsymbol{R}$-semimodule $\boldsymbol{M}$ is semilinear then $\boldsymbol{R}$ is semilinear as well provided that $\boldsymbol{M}$ has a separating set.

Lemma 3.2 *Let $\boldsymbol{M}$ be a semilinear $\boldsymbol{R}$-semimodule with a separating set E. Then $\boldsymbol{R}$ is semilinear as well.*

Proof: Assume that $u \leq h \vee ca$ and $u \leq h \vee db$ for some $a, b, c, d, h, u \in R$. Then for every $e \in E$ we have $u \star e \leq (h \vee ca) \star e = h \star e \vee c \star (a \star e)$. Similarly we have $u \star e \leq h \star e \vee d \star (b \star e)$. Since $\boldsymbol{M}$ satisfies (sl), we obtain for all $e \in E$

$$u \star e \leq h \star e \vee c \star (b \star e) \vee d \star (a \star e) = (h \vee cb \vee da) \star e\,.$$

Since E is a separating set, we have $u \leq h \vee cb \vee da$. Thus $\boldsymbol{R}$ satisfies (sl'). □

Now we are going to show that semilinear $\boldsymbol{R}$-semimodules are exactly those $\boldsymbol{R}$-semimodules which are representable as subdirect products of linearly ordered $\boldsymbol{R}$-semimodules.

Lemma 3.3 *Every linearly ordered $\boldsymbol{R}$-semimodule $\boldsymbol{M}$ is semilinear.*

Proof: Assume that $u \leq h \vee c \star a$ and $u \leq h \vee d \star b$ for some $u, h, a, b \in M$ and $c, d \in R$. If $a \leq b$ then $u \leq h \vee c \star a \leq h \vee c \star b \leq h \vee c \star b \vee d \star a$. If $a > b$ then $u \leq h \vee d \star b \leq h \vee d \star a \leq h \vee c \star b \vee d \star a$. □

Lemma 3.4 *Let $\boldsymbol{M}$ be a semilinear $\boldsymbol{R}$-semimodule. Then every ideal of $\mathcal{I}(\boldsymbol{M}^+)$ maximal with respect to not containing a given element $u \in M$ is linear.*

Proof: Let $I \in \mathcal{I}(\boldsymbol{M}^+)$ maximal w.r.t. not containing u. We will show that $z \star x, w \star y \in I$ implies $z \star y \in I$ or $w \star x \in I$, for $x, y \in M$, and $z, w \in R$. We will show the contrapositive. If $z \star y \notin I$ and $w \star x \notin I$, then there are $h_1, h_2 \in I$ such that $u \leq h_1 \vee z \star y$ and $u \leq h_2 \vee w \star x$ because I is maximal w.r.t. not containing u. So, $h := h_1 \vee h_2 \in I$, and $u \leq h \vee z \star y$ and $u \leq h \vee w \star x$. By (sl) $u \leq h \vee z \star x \vee w \star y$, namely $z \star x \vee w \star y \notin I$, hence $z \star x \notin I$ or $w \star y \notin I$. □

Lemma 3.5 *Let $\boldsymbol{M}$ be a semilinear $\boldsymbol{R}$-semimodule. Then the following are equivalent:*

1. *There exists a nontrivial $\boldsymbol{R}$-semimodule homomorphism $f\colon \boldsymbol{M} \to \boldsymbol{N}$ for some linearly ordered $\boldsymbol{R}$-semimodule $\boldsymbol{N}$ (nontrivial means that $\ker(f) \neq M^2$).*

2. *$\boldsymbol{M}$ has a proper ideal (which can be chosen maximal w.r.t. not containing an element).*

Proof: (1⇒2): Since f is nontrivial, there are $m, n \in M$ such that $f(m) \neq f(n)$. Without any loss of generality we may assume that $f(m) \not\leq f(n)$. Consequently, $m \not\leq n$ as well since f is order-preserving. Consider the principal ideal $\downarrow n$. It does not contain m. Thus $\downarrow n$ is proper.

(2⇒1): Let $\emptyset \neq I \subsetneq M$ be a proper ideal. Then I does not contain an element $u \in M$. By Zorn's lemma we can extend I to a maximal ideal not containing u. Thus we will assume that I is maximal w.r.t. not containing u. By Lemma 3.4 I is linear. Hence we have the natural homomorphism from $\boldsymbol{M}$ to $\boldsymbol{M}/I$. It is nontrivial since $u \not\sim_I v$ for any $v \in I$. Indeed, we have $1 \star u = u \notin I$ and $1 \star v = v \in I$. □

Theorem 3.6 *Let $\boldsymbol{M}$ be an idempotent $\boldsymbol{R}$-semimodule. Then the following are equivalent:*

1. *$\boldsymbol{M}$ is embeddable into $\prod_{i \in K} \boldsymbol{N}_i$ for some family $\{\boldsymbol{N}_i \mid i \in K\}$ of linearly ordered idempotent $\boldsymbol{R}$-semimodules.*

2. *$\boldsymbol{M}$ is semilinear.*

Proof: (1⇒2) follows immediately from Lemma 3.3. To see (2⇒1), consider the collection $\{I_i \in \mathcal{I}(\boldsymbol{M}^+) \mid i \in K\}$ of all maximal ideals w.r.t. not containing an element in M. Then by Lemma 3.5 there are linearly ordered idempotent $\boldsymbol{R}$-semimodules $\boldsymbol{M}/I_i$ and homomorphisms $f_i\colon \boldsymbol{M} \to \boldsymbol{M}/I_i$. Thus we have a natural homomorphism $f\colon \boldsymbol{M} \to \prod_{i\in K} \boldsymbol{M}/I_i$. We will show that f is an embedding. Let $m, n \in M$ such that $m \neq n$. Without any loss of generality we may assume that $m \not\leq n$. Consider the principal ideal $\downarrow n$. It does not contain m. Moreover, we can extend $\downarrow n$ by Zorn's lemma to a maximal ideal I_i not containing m. Consequently, $m \not\sim_{I_i} n$ because $1 \star m = m \notin I_i$ and $1 \star n = n \in I_i$. Thus $f_i(m) \neq f_i(n)$ and so $f(m) \neq f(n)$. □

Corollary 3.7 *A quasi-variety* Q *of idempotent* $\boldsymbol{R}$*-semimodules is generated by chains iff (sl) holds in* Q*.*

Before we prove Holland's theorem for idempotent semirings, we have to introduce the ordinal sum construction of idempotent $\boldsymbol{R}$-semimodules. Let $\langle K, \leq\rangle$ be a linearly ordered set and $\{\boldsymbol{M}_i \mid i \in K\}$ be a family of idempotent $\boldsymbol{R}$-semimodules whose left actions are denoted $\star_i$. Then $\bigoplus_{i\in K} \boldsymbol{M}_i$ is an idempotent $\boldsymbol{R}$-semimodule, whose underlying join-semilattice is the ordinal sum of $\{\boldsymbol{M}_i^+ \mid i \in K\}$ and its left action is given by $r\star m = r\star_i m$ if $m \in M_i$. The only thing, one has to check, is $r\star(m\vee m') = r\star m\vee r\star m'$. If $m, m' \in M_i$ then it holds since $\boldsymbol{M}_i$ is an $\boldsymbol{R}$-semimodule. If $m \in M_i$ and $m' \in M_j$ with $i < j$. Then $r\star(m\vee m') = r\star m' = r\star m\vee r\star m'$ since $r \star m \in M_i$ and $r \star m' \in M_j$.

Theorem 3.8 *Let* $\boldsymbol{M}$ *be a semilinear* $\boldsymbol{R}$*-semimodule. Then there is a linearly ordered* $\boldsymbol{R}$*-semimodule* $\boldsymbol{N}$*. In addition, if* $\boldsymbol{M}$ *has a one-element separating set* $\{e\}$ *then* $\boldsymbol{N}$ *has a separating set* E *which is dually well ordered. Moreover,* $\boldsymbol{N}$ *can be chosen complete.*

Proof: By Theorem 3.6 we have an embedding of $\boldsymbol{R}$-semimodules $f\colon \boldsymbol{M} \to \prod_{i\in K} \boldsymbol{M}_i$ for some family $\{\boldsymbol{M}_i \mid i \in K\}$ of linearly ordered $\boldsymbol{R}$-semimodules. Let $f_i = \pi_i \circ f$ where π_i is the projection onto ith-component of $\prod_{i\in K} \boldsymbol{M}_i$. Recall that every set can be dually well ordered. Thus we may assume that K is dually well ordered (in particular K is linearly ordered). Then the ordinal sum $\boldsymbol{N} = \bigoplus_{i\in K} \boldsymbol{M}_i$ is a linearly ordered $\boldsymbol{R}$-semimodule with a left action $\star$.

To see the additional part, assume that $\{e\}$ is a separating set in $\boldsymbol{M}$. Then $E = \{f_i(e) \mid i \in K\}$ is a separating set in $\boldsymbol{N}$. Indeed, if $r \neq s$ then $r \star e \neq s \star e$ and there is $i \in K$ such that $f_i(r \star e) \neq f_i(s \star e)$ because f is an embedding. Consequently,

$$r \star f_i(e) = f_i(r \star e) \neq f_i(s \star e) = s \star f_i(e)\,.$$

Moreover, since K is dually well ordered, E is dually well ordered as well.

To see the last part, $\boldsymbol{N}$ can be embedded into a complete $\boldsymbol{R}$-semimodule by Theorem 2.6. Moreover, this embedding preserves linear order and the separating set E. □

Theorem 3.9 (Holland's theorem for idempotent semirings) *Let* $\boldsymbol{R}$ *be an idempotent semiring. Then the following are equivalent:*

1. $\boldsymbol{R}$ *is semilinear.*
2. $\boldsymbol{R}$ *is embeddable into* $\mathbf{End}(\boldsymbol{\Omega})$ *for some chain* $\boldsymbol{\Omega}$*.*
3. $\boldsymbol{R}$ *is embeddable into* $\mathbf{REnd}(\boldsymbol{\Omega}) \cong \mathbf{Res}(\mathcal{I}(\boldsymbol{\Omega}))$ *for some chain* $\boldsymbol{\Omega}$*.*

Proof: (1⇒2): Since $\boldsymbol{R}$ can be viewed as an $\boldsymbol{R}$-semimodule with a separating set $\{1\}$, there is a linear ordered $\boldsymbol{R}$-semimodule $\boldsymbol{N}$ by Theorem 3.8 with a linearly and dually well ordered separating set E. Let $\boldsymbol{\Omega} = \boldsymbol{N}^+$. Then $\boldsymbol{R}$ embeds as a semiring into $\mathbf{End}(\boldsymbol{\Omega})$ by Lemma 2.2.

(2⇒3): Clearly, $\boldsymbol{\Omega}$ can be viewed as an $\mathbf{End}(\boldsymbol{\Omega})$-semimodule. Thus by Corollary 2.7 there is a semiring homomorphism $\phi\colon \mathbf{End}(\boldsymbol{\Omega}) \to \mathbf{Res}(\mathcal{I}(\boldsymbol{\Omega})) \cong \mathbf{REnd}(\boldsymbol{\Omega})$ given by $(\phi(f))(I) = {\downarrow} f[I]$. Then $(\phi(f))({\downarrow} x) = {\downarrow} f(x)$ for all $x \in \Omega$. Assume that $\phi(f) = \phi(g)$. Consequently, ${\downarrow} f(x) = {\downarrow} g(x)$ for every $x \in \Omega$, i.e., $f = g$. Thus ϕ is an embedding.

(3⇒1): Since $\mathbf{REnd}(\boldsymbol{\Omega}) \cong \mathbf{Res}(\mathcal{I}(\boldsymbol{\Omega}))$, it is sufficient to show that $\mathbf{Res}(\mathcal{I}(\boldsymbol{\Omega}))$ satisfies (sl'). The chain $\mathcal{I}(\boldsymbol{\Omega})$ can be viewed as an $\mathbf{Res}(\mathcal{I}(\boldsymbol{\Omega}))$-semimodule having a separating set $\mathcal{I}(\boldsymbol{\Omega})$ (see Example 2.4). Then $\mathcal{I}(\boldsymbol{\Omega})$ is semilinear by Lemma 3.3. Consequently, $\mathbf{Res}(\mathcal{I}(\boldsymbol{\Omega}))$ is semilinear by Lemma 3.2. □

As a corollary of the above theorem, we can easily derive the original Holland's theorem for ℓ-groups from [10]. Given a chain $\boldsymbol{\Omega}$, the ℓ-group of all order-preserving bijections on $\boldsymbol{\Omega}$ is denoted $\mathbf{Aut}(\boldsymbol{\Omega})$.

Corollary 3.10 (Holland's theorem for ℓ-groups) *Every ℓ-group $\boldsymbol{G}$ is embeddable into $\mathbf{Aut}(\boldsymbol{\Omega})$ for some chain $\boldsymbol{\Omega}$.*

Proof: Since every ℓ-group satisfy (sl'), $\boldsymbol{G}$ embeds as an idempotent semiring into $\mathbf{End}(\boldsymbol{\Omega})$ via an embedding ϕ by Theorem 3.9. Since meet is definable in ℓ-groups as $x \wedge y = (x^{-1} \vee y^{-1})^{-1}$, ϕ is in fact an ℓ-group embedding. Finally, we have $\phi[G] \subseteq \mathrm{Aut}(\boldsymbol{\Omega})$ because ϕ gives rise to a group action of $\boldsymbol{G}$ on Ω.

□

Theorem 3.11 (Holland's theorem for residuated lattices) *Let $\boldsymbol{A}$ be a residuated lattice. The following are equivalent:*

1. *$\boldsymbol{A}$ satisfies $(h \vee ca) \wedge (h \vee db) \leq h \vee cb \vee da$.*
2. *$\boldsymbol{A}$ embeds into a conuclear contraction of $\mathbf{REnd}(\boldsymbol{\Omega})$ for a chain $\boldsymbol{\Omega}$.*
3. *$\boldsymbol{A}$ embeds into a conuclear contraction of $\mathbf{Res}(\boldsymbol{\Omega}')$ for a complete chain $\boldsymbol{\Omega}'$.*

Proof: (1⇒2) First, note that the quasi-identity (sl') is equivalent to an identity $(h \vee ca) \wedge (h \vee db) \leq h \vee cb \vee da$ in the presence of meet. Thus $\boldsymbol{A}$ embeds as an idempotent semiring into $\mathbf{REnd}(\boldsymbol{\Omega})$ for a chain $\boldsymbol{\Omega}$ by Theorem 3.9. Since $\mathbf{REnd}(\boldsymbol{\Omega}) \cong \mathbf{Res}(\mathcal{I}(\boldsymbol{\Omega}))$ is a complete residuated lattice, $\boldsymbol{A}$ embeds into a conuclear contraction of $\mathbf{REnd}(\boldsymbol{\Omega})$ by Theorem 2.15.

(2⇒3) It is obvious since $\mathcal{I}(\boldsymbol{\Omega})$ is a complete chain.

(3⇒1) Since $\mathbf{Res}(\boldsymbol{\Omega}')$ is a subsemiring of $\mathbf{End}(\boldsymbol{\Omega}')$, $\mathbf{Res}(\boldsymbol{\Omega}')$ satisfies (sl') by Theorem 3.9. Since $\mathbf{Res}(\boldsymbol{\Omega}')_\sigma$ is a subsemiring of $\mathbf{Res}(\boldsymbol{\Omega}')$ for any conucleus σ, $\boldsymbol{A}$ has to satisfy (sl') as well. □

We finish this section by several comments on applicability of the above theorem. A residuated lattice is called *prelinear* if it satisfies $1 = (x\backslash y \wedge 1) \vee (y\backslash x \wedge 1)$, *cancellative* is it satisfies $xy/y = x$ and $y\backslash yx = x$, and *semilinear* if it is a subdirect product of chains.

Corollary 3.12 *The following varieties of residuated lattices contain algebras that embed into a conuclear contraction of $\mathbf{Res}(\boldsymbol{\Omega})$ for a complete chain $\boldsymbol{\Omega}$.*

1. *Prelinear residuated lattices.*
2. *ℓ-groups.*
3. *Semilinear residuated lattices.*
4. *Commutative cancellative residuated lattices.*
5. *Distributive residuated lattices where multiplication distributes over meet.*

Proof: We verify that all varieties satisfy Theorem 3.11(1). Recall also that the identity $(h \vee ca) \wedge (h \vee db) \leq h \vee cb \vee da$ is equivalent to (sl') in the presence of meet.

For (1), if $u \leq h \vee ca$ and $u \leq h \vee db$ then $u(a\backslash b \wedge 1) \leq h(a\backslash b \wedge 1) \vee cb$ and $u(b\backslash a \wedge 1) \leq h(b\backslash a \wedge 1) \vee da$. Thus

$$u = u((b\backslash a \wedge 1) \vee (a\backslash b \wedge 1)) = u(b\backslash a \wedge 1) \vee u(a\backslash b \wedge 1) \leq h((b\backslash a \wedge 1) \vee (a\backslash b \wedge 1)) \vee cb \vee da = h \vee cb \vee da\,.$$

For (2) and (3), we note that ℓ-groups and semilinear residuated lattice are prelinear. For (4), if $u \leq h \vee ca$ and $u \leq h \vee db$ then $ub \leq hb \vee cab$ and $ua \leq ha \vee dba$. Thus

$$u(a \vee b) = ub \vee ua \leq hb \vee c(a \vee b)b \vee ha \vee d(a \vee b)a = (h \vee cb \vee da)(a \vee b)\,.$$

Consequently,

$$u = u(a \vee b)/(a \vee b) \leq (h \vee cb \vee da)(a \vee b)/(a \vee b) = h \vee cb \vee da\,.$$

Finally for (5), it suffices to verify $ca \wedge db \leq cb \vee da$ thanks to distributivity. We have

$$\begin{aligned}(ca \wedge db) \vee (cb \vee da) &= (ca \vee cb \vee da) \wedge (db \vee cb \vee da) = (c(a \vee b) \vee da) \wedge (cb \vee d(a \vee b)) = \\ &= (c(a \vee b) \wedge cb) \vee (c(a \vee b) \wedge d(a \vee b)) \vee (da \wedge cb) \vee (da \vee d(a \vee b)) = \\ &= cb \vee (c \wedge d)(a \vee b) \vee (da \wedge cb) \vee da = cb \vee (c \wedge d)(a \vee b) \vee da = \\ &= cb \vee (c \wedge d)a \vee (c \wedge d)b \vee da = cb \vee da\,.\end{aligned}$$

□

On the other hand, we note that there are residuated lattices where Holland's theorem is not applicable. One of the simplest examples can be constructed as follows. Let $\boldsymbol{Z}_2 = \langle \{0,1\}, +, 0 \rangle$ be the two-element group (ordered discretely). Then one can extend it by a top and bottom element $\top, \bot$ letting $\bot + x = \bot = x + \bot$ and $\top + x = \top = x + \top$ for $x \neq \top$. Then the resulting algebra is a residuated lattice whose lattice reduct is the distributive lattice $\mathbf{2}^2$ where $\mathbf{2}$ is the two-element chain. Moreover, $(h \vee ca) \wedge (h \vee db) \leq h \vee cb \vee da$ does not hold in this extension. Indeed, we have

$$\top = (1 \vee (0+0)) \wedge (1 \vee (1+1)) \not\leq 1 \vee (0+1) \vee (0+1) = 1\,.$$

4 Finite embeddability property

In this section we will show an application of the results from the previous sections. Recall that a semiring $\boldsymbol{R}$ is said to be *integral* if 1 is a top element with respect to the join-semilattice order on $\boldsymbol{R}$. We denote the variety of all integral idempotent semirings by ISR and Q its sub-quasivariety axiomatized by (sl'). We will show the finite embeddability property (FEP) for K and $\mathsf{K} \cap \mathsf{Q}$ where K is an arbitrary subvariety of IRS. Then we will use this result in order to prove the FEP for many varieties of integral residuated lattices. Recall that a class K of algebras in the same language has the FEP if every finite partial subalgebra is embeddable into a finite member of K (see [4]).

A poset $\boldsymbol{P} = \langle P, \leq \rangle$ is said to satisfy the *ascending chain condition* (ACC) if every ascending sequence $a_0 \leq a_1 \leq \cdots$ of elements from P is terminating, i.e., there is $n \in \mathbb{N}$ such that $a_n = a_{n+1} = \cdots$. Dually, $\boldsymbol{P}$ is said to have the *descending chain condition* (DCC) if every descending sequence is terminating.

Definition 4.1 ([1]) Let $\boldsymbol{P}$ be a poset. Then the following are equivalent conditions for $\boldsymbol{P}$ to be a *well partial order* (wpo):

1. $\boldsymbol{P}$ contains neither infinite strictly decreasing chains nor infinite antichains.
2. For every infinite sequence $a_1, a_2, \ldots$ of elements from P there are $n < k$ such that $a_n \leq a_k$.
3. Every infinite sequence of elements from P contains an infinite ascending subsequence.
4. Every upset $U \subseteq P$ is finitely generated.
5. The set of all upsets in $\boldsymbol{P}$ ordered by inclusion has the ACC.

Recall some well-known results on wpos. First, let $\boldsymbol{P}$, $\boldsymbol{Q}$ be wpos. Then $\boldsymbol{P} \times \boldsymbol{Q}$ (ordered component-wise) is a wpo as well. Second, let $\boldsymbol{P}$ be a wpo, $\boldsymbol{Q}$ a poset and $f \colon P \to Q$ an order-preserving surjection. Then $\boldsymbol{Q}$ has to be a wpo as well. Finally, let $\boldsymbol{P}$ be a wpo. Then Higman's lemma [9] states that the set P^* of all finite sequences of elements from P forms a wpo ordered as follows: $a_1, \ldots, a_n \sqsubseteq b_1, \ldots, b_m$ iff there are $1 \leq i_1 < \cdots < i_n \leq m$ such that $a_1 \leq b_{i_1}, \ldots, a_n \leq b_{i_n}$.

Recall that a poset $\boldsymbol{P}$ is called a *dual well partial order* (dwpo) if its dual $\boldsymbol{P}^\partial$ is a wpo. A *partially ordered monoid* (pomonoid) $\boldsymbol{M} = \langle M, \cdot, 1 \leq \rangle$ is a monoid whose multiplication is compatible with the order relation, i.e., $x \leq y$ implies $xz \leq yz$ and $zx \leq zy$. Moreover, a pomonoid is said to be *integral* if 1 is the top element of $\langle M, \leq \rangle$. Note that given a dwpo $\boldsymbol{P}$, it is easy to check that the free monoid $\boldsymbol{P}^*$ generated by P ordered by $\sqsubseteq^\partial$ makes $\boldsymbol{P}^*$ into an integral pomonoid. Thus Higman's lemma can be used in order to prove the following lemma on integral pomonoids generated by a dwpo.

Lemma 4.2 *Every integral pomonoid $\boldsymbol{M}$ generated by a dwpo $\boldsymbol{G}$ (in particular by a finite set) forms a dwpo.*

Proof: By Higman's lemma the integral pomonoid $\boldsymbol{G}^*$ generated by G forms a dwpo. Then it is easy to check that the natural homomorphism from $\boldsymbol{G}^*$ to $\boldsymbol{M}$ is order-preserving. Thus $\boldsymbol{M}$ has to be a dwpo as well. □

Lemma 4.3 *Let $\boldsymbol{L}$ be a bounded join-semilattice (i.e., it has a bottom element $\bot$) generated by a dwpo $\boldsymbol{W}$ such that $\bot \in W$. Then $\boldsymbol{L}$ is a complete lattice having the ACC. Moreover, every join is finite.*

Proof: Let $S \subseteq L$. If $S = \emptyset$ then $\bigvee S = \bot$. Assume that $S \neq \emptyset$. Without any loss of generality we may assume that S is a downset. Then $\downarrow(S \cap W)$ is finitely generated by some elements $m_1, \ldots, m_k \in S \cap W$ (see Definition 4.1). We claim that $\bigvee S = m_1 \vee \cdots \vee m_k$. Clearly, $m_1 \vee \cdots \vee m_k$ is less than or equal to any upper bound of S. Let $s \in S$. Then $s = n_1 \vee \cdots \vee n_l$ for some $n_1, \ldots, n_l \in W$. Since S is a downset, $n_1, \ldots, n_l \in S \cap W \subseteq \downarrow(S \cap W)$, i.e., $s \leq m_1 \vee \cdots \vee m_k$. Summing up, $\boldsymbol{L}$ is a complete lattice and every join is finite.

It is known that the complete lattice $\mathcal{D}(\boldsymbol{W})$ of all downsets on $\boldsymbol{W}$ has the ACC (see Definition 4.1). The map f sending D to $\bigvee D$ is onto and preserves joins. Indeed, we have

$$f(D_1 \cup D_2) = \bigvee(D_1 \cup D_2) = (\bigvee D_1) \vee (\bigvee D_2) = f(D_1) \vee f(D_2)\,.$$

Moreover, if $a = w_1 \vee \cdots \vee w_k \in L$ for some $w_1, \ldots, w_k \in W$ then $f(\downarrow\{w_1, \ldots, w_k\}) = a$. Assume that $f(D_1) \leq f(D_2) \leq f(D_3) \cdots$ is an ascending sequence in L. Then $D_1 \subseteq D_1 \cup D_2 \subseteq D_1 \cup D_2 \cup D_3 \subseteq \cdots$ is an ascending sequence in $\mathcal{D}(\boldsymbol{W})$, i.e., it has to terminate. Then $f(D_1) \leq f(D_2) \leq f(D_3) \cdots$ is terminating as well because $f(D_k) = \bigvee_{i=1}^k f(D_i) = f(\bigcup_{i=1}^k D_i)$. Thus L satisfies the ACC. $\square$

We continue by proving several auxiliary results.

Lemma 4.4 *Let $\boldsymbol{R}$ be a finitely generated integral semiring and $\boldsymbol{M}$ a left or right $\boldsymbol{R}$-semimodule generated by a dwpo $\boldsymbol{E}$. Then the join-semilattice $\boldsymbol{M}^+$ is generated by a dwpo $\boldsymbol{G}$ such that $E \subseteq G$.*

Proof: We will assume that $\boldsymbol{M}$ is a left $\boldsymbol{R}$-semimodule. The proof for a right $\boldsymbol{R}$-semimodule is analogous. Let C be the finite generating set for $\boldsymbol{R}$ and $\boldsymbol{S}$ the submonoid of $\boldsymbol{R}$ generated by C. Note that $\boldsymbol{M}^+$ is generated by $G = \{s \star e \mid s \in S, e \in E\}$. Indeed, every element in $m \in M$ is of the form $m = \bigvee_{i=1}^k r_i \star e_i$ for some $r_i \in R$ and $e_i \in E$. Since multiplication in $\boldsymbol{R}$ distributes over finite joins, every $r \in R$ is of the form $r = s_1 \vee \cdots \vee s_k$ for some $s_1, \ldots, s_k \in S$. Thus m is a join of elements from G. Further, S is a dwpo by Lemma 4.2. Thus $S \times E$ is a dwpo. It is easy to see that $\langle s, e\rangle \mapsto s \star e$ is an order-preserving map onto G. Thus G forms a dwpo as well. Moreover, $E \subseteq G$ because $e = 1 \star e \in G$ for every $e \in E$. $\square$

Theorem 4.5 *Let $\boldsymbol{R}$ be a finitely generated integral semiring and $\boldsymbol{M}$ a left or right $\boldsymbol{R}$-semimodule whose join-semilattice reduct $\boldsymbol{M}^+$ has a bottom element $\bot$. Moreover, assume that $\boldsymbol{M}$ is generated by a dwpo $\boldsymbol{E}$ containing $\bot$. Then $\boldsymbol{M}$ is residuated, $\boldsymbol{M}^+$ forms a complete lattice and has the ACC. In addition, if $\boldsymbol{M}^+$ has the DCC then M is finite.*

Proof: By Lemma 4.4 $\boldsymbol{M}^+$ is generated by a dwpo $\boldsymbol{G}$ such that $\bot \in E \subseteq G$. Hence $\boldsymbol{M}^+$ forms a complete lattice having the ACC by Lemma 4.3 where every join is finite. Consequently, $\boldsymbol{M}$ is residuated because the left/right action in $\boldsymbol{M}$ distributes over finite joins.

To see the additional part, it suffices to show that the generating set G for $\boldsymbol{M}^+$ is finite. Suppose that G is infinite. Since $\boldsymbol{G}$ is a dwpo, there is a strictly decreasing sequence $m_1 > m_2 > \cdots$ of elements from G. However, this is a contradiction with our assumption that $\boldsymbol{M}$ has DCC. $\square$

Now we are ready to prove the main theorem of this section.

Theorem 4.6 *Let* K *be a subvariety of* ISR *and* $\boldsymbol{R} \in \mathsf{K}$ *generated by a finite set* C*. Then there is a finite* $\boldsymbol{S} \in \mathsf{K}$ *and a surjective homomorphism* $\phi\colon \boldsymbol{R} \to \boldsymbol{S}$ *such that* $\phi(r) \leq \phi(c)$ *implies* $r \leq c$ *for all* $r \in R$ *and* $c \in C$*. In addition, if* $\boldsymbol{R} \in \mathsf{K} \cap \mathsf{Q}$ *then* $\boldsymbol{S} \in \mathsf{K} \cap \mathsf{Q}$ *as well.*

Proof: Without any loss of generality we may assume that $1 \in C$. Since $\boldsymbol{R}$ can be viewed as an $\boldsymbol{R}$-semimodule having a separating set $\{1\}$, there is a complete $\boldsymbol{R}$-semimodule $\boldsymbol{M}$ with a one-element separating set $E = \{e\}$ (see Theorem 2.6). Since $\boldsymbol{M}^+$ is a complete lattice, there is a bottom element $\bot \in M$ and a top element $\top \in M$. Since any superset of E is a separating set as well, we may enlarge E by $\bot$ and $\top$, i.e., $E = \{e, \bot, \top\}$.

Now consider the $\boldsymbol{R}$-subsemimodule $\boldsymbol{N}$ of $\boldsymbol{M}$ generated by E. Thanks to Theorem 4.5 $\boldsymbol{N}$ is complete, residuated having the ACC. Thus the left action $\star$ in $\boldsymbol{N}$ is residuated, i.e., there is a right action $\backslash$ such that $r \star m \leq n$ iff $m \leq r\backslash n$. Consequently, $(\boldsymbol{N}^+)^\partial = \langle N, \leq^\partial \rangle$ together with $\backslash$ forms a right $\boldsymbol{R}$-semimodule $\boldsymbol{N}^\partial$ having the DCC. Consider the right $\boldsymbol{R}$-subsemimodule $\boldsymbol{K}$ of $\boldsymbol{N}^\partial$ generated by the set $C\star E = \{c\star e \mid c \in C, e \in E\}$. Since C and E are finite, $C\star E$ is finite. Moreover, the bottom element of $(\boldsymbol{N}^+)^\partial$ is $\top$ and belongs to $C \star E$ because $\top = 1 \star \top$. Thus $\boldsymbol{K}$ is a finite complete residuated right $\boldsymbol{R}$-semimodule by Theorem 4.5. Consequently, there is a semiring homomorphism $\phi\colon \boldsymbol{R} \to \mathbf{Res}(\boldsymbol{K}^+)^{op}$ by Lemma 2.5 mapping $r \in R$ to a residuated map f_r on $\boldsymbol{K}^+$ defined by $f_r(m) = r\backslash m$ (note that $\mathbf{Res}(\boldsymbol{K}^+)^{op}$ is ordered pointwise by $\leq^\partial$). Since K is finite, $\mathbf{Res}(\boldsymbol{K}^+)^{op}$ has to be finite as well. Thus the image $\phi[R]$ forms a finite semiring $\boldsymbol{S}$ belonging to K because varieties are closed under homomorphic images.

It remains to show that $\phi(r) \leq^\partial \phi(c)$ implies $r \leq c$ for all $r \in R$ and $c \in C$. Assume that $r \not\leq c$. Then there is $e \in E$ such that $r \star e \not\leq c \star e$ because E is a separating set in $\boldsymbol{M}$. Then $e \not\leq r\backslash(c \star e)$. On the other hand, $c\backslash(c \star e) \geq e$. Thus $r\backslash(c \star e) \not\geq c\backslash(c \star e)$. Consequently, we have $\phi(r)(c \star e) = f_r(c \star e) = r\backslash(c \star e) \not\geq e$ and $\phi(c)(c \star e) = f_c(c \star e) = c\backslash(c \star e) \geq e$ for some $e \in E$. Thus $\phi(r) \not\geq \phi(c)$, i.e., $\phi(r) \not\leq^\partial \phi(c)$.

To see the additional part, if $\boldsymbol{R} \in \mathsf{Q}$ then we can replace the complete $\boldsymbol{R}$-semimodule $\boldsymbol{M}$ with a one-element separating set $E = \{e\}$ by a linearly ordered complete $\boldsymbol{R}$-semimodule $\boldsymbol{M}$ with a dually well ordered separating set E by Theorem 3.8. Then we can construct the finite right $\boldsymbol{R}$-semimodule $\boldsymbol{K}$ in the same way as above. Moreover, since $\boldsymbol{M}$ is linearly ordered, $\boldsymbol{K}$ is linearly ordered as well because it was constructed only by taking subsemimodules and duals. Hence $\mathbf{Res}(\boldsymbol{K}^+)^{op}$ belongs to Q by Theorem 3.9 and the fact that a semiring $\boldsymbol{A}$ satisfies (sl') iff $\boldsymbol{A}^{op}$ satisfies it. □

Corollary 4.7 *Let* K *be a subvariety of* ISR*. Then* K *and* $\mathsf{K} \cap \mathsf{Q}$ *have the finite embeddability property.*

Proof: Let $\boldsymbol{C}$ be a finite partial subalgebra of $\boldsymbol{R} \in \mathsf{K}$ (resp. $\boldsymbol{R} \in \mathsf{K} \cap \mathsf{Q}$). Without any loss of generality we may assume that $\boldsymbol{R}$ is finitely generated by C. By Theorem 4.6 there is a finite $\boldsymbol{S} \in \mathsf{K}$ (resp. $\boldsymbol{S} \in \mathsf{K} \cap \mathsf{Q}$) and a surjective homomorphism $\phi\colon \boldsymbol{R} \to \boldsymbol{S}$ such that $\phi(r) \leq \phi(c)$ implies $r \leq c$ for all $r \in R$ and $c \in C$. Thus ϕ is one-to-one when restricted to C because $\phi(c) = \phi(c')$ implies $c = c'$ for any $c, c' \in C$. □

Now we are ready to use the results on semirings in order to prove the FEP for various classes of residuated lattices. A residuated lattice is said to be *integral* if its semiring reduct is integral (i.e., 1 is a top element). We denote the variety of all integral residuated lattices by IRL.

Lemma 4.8 *Every finite integral semiring $\boldsymbol{R}$ forms a complete integral residuated lattice.*

Proof: Since $\boldsymbol{R}$ is a finite join semilattice, it is a complete lattice iff it has a bottom element. Let $R = \{a_1, \dots, a_k\}$. Then $a_1 \cdots a_k \leq a_i$ for all $1 \leq i \leq k$ because $\boldsymbol{R}$ is integral. Thus $\bot = a_1 \cdots a_k$ is the bottom element.

We claim that multiplication distributes over arbitrary joins from both sides. Let $S \subseteq R$. If $S \neq \emptyset$ then $\bigvee S$ is a finite join. Thus the claim follows from the axioms of semirings. If $S = \emptyset$ then by integrality $a \cdot \bigvee \emptyset = a \cdot \bot = \bot = \bigvee \emptyset = \bigvee \{a \cdot s \mid s \in \emptyset\}$ and similarly $(\bigvee \emptyset) \cdot a = \bigvee \{s \cdot a \mid s \in \emptyset\}$. □

Theorem 4.9 *Let V_1 be a subvariety of IRL axiomatized by the set $\mathcal{E}$ of identities using only $\vee, \cdot, 1$. Further, let V_2 be the subvariety of V_1 relatively axiomatized by $(h \vee ca) \wedge (h \vee db) \leq h \vee cb \vee da$. Then V_1 and V_2 have the finite embeddability property.*

Proof: Let K be the subvariety of ISR axiomatized by $\mathcal{E}$ and Q the subquasivariety of ISR axiomatized by (sl'). Suppose that $\boldsymbol{A} \in \mathsf{V}_1$ (resp. $\boldsymbol{A} \in \mathsf{V}_2$). Then the semiring reduct of $\boldsymbol{A}$ belongs to K (resp. $\mathsf{K} \cap \mathsf{Q}$). Let $\boldsymbol{C}$ be a finite partial subalgebra of $\boldsymbol{A}$. Consider the subsemiring $\boldsymbol{R}$ generated by C. Then $\boldsymbol{R} \in \mathsf{K}$ (resp. $\mathsf{K} \cap \mathsf{Q}$). By Theorem 4.6 there is a finite semiring $\boldsymbol{S} \in \mathsf{K}$ (resp. $\boldsymbol{S} \in \mathsf{K} \cap \mathsf{Q}$) and a surjective semiring homomorphism $\phi \colon \boldsymbol{R} \to \boldsymbol{S}$ such that $\phi(r) \leq \phi(c)$ implies $r \leq c$ for every $r \in R$ and $c \in C$. Since $\boldsymbol{S}$ is finite and integral, it has to be a complete integral residuated lattice by Lemma 4.8. Consequently, it follows from Lemma 2.14 that $\boldsymbol{C}$ embeds as a residuated lattice into a conuclear contraction $\boldsymbol{S}_\sigma$ which has to be finite as well. Since $\boldsymbol{S}_\sigma$ is a subsemiring of $\boldsymbol{S}$, its semiring reduct belongs to K (resp. $\mathsf{K} \cap \mathsf{Q}$). Consequently, $\boldsymbol{S}_\sigma \in \mathsf{V}_1$ (resp. $\boldsymbol{S}_\sigma \in \mathsf{V}_2$). □

In the notation of the previous theorem, we point out that it was already known that V_1 has the FEP (see [4, 8]). However, we provide here a new proof of this fact. On the other hand, the fact that V_2 has the FEP is new.

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