CATEGORICAL SKEW LATTICES

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Abstract. Categorical skew lattices are a variety of skew lattices on which the natural partial order is especially well behaved. While most skew lattices of interest are categorical, not all are. They are characterized by a countable family of forbidden subalgebras. We also consider the subclass of strictly categorical skew lattices.

1. Introduction and Background

A ***skew lattice*** is an algebra $\mathbf{S} = (S; \vee, \wedge)$ where $\vee$ and $\wedge$ are associative, idempotent binary operations satisfying the absorption identities

$$x \wedge (x \vee y) = x = (y \vee x) \wedge x \qquad \text{and} \qquad x \vee (x \wedge y) = x = (y \wedge x) \vee x\,. \tag{1.1}$$

Given that $\vee$ and $\wedge$ are associative and idempotent, (1.1) is equivalent to the dualities:

$$x \wedge y = x \quad \text{iff} \quad x \vee y = y \qquad \text{and} \qquad x \wedge y = y \quad \text{iff} \quad x \vee y = x\,. \tag{1.2}$$

Every skew lattice has a ***natural preorder*** defined by

$$x \succeq y \quad \Leftrightarrow \quad x \vee y \vee x = x \quad \text{or equivalently} \quad y \wedge x \wedge y = y\,. \tag{1.3}$$

This preorder is refined by the ***natural partial order*** defined by

$$x \geq y \quad \Leftrightarrow \quad x \vee y = x = y \vee x \quad \text{or equivalently} \quad x \wedge y = y = y \wedge x\,. \tag{1.4}$$

In what follows, any mentioned preordering or partial ordering of a skew lattice is assumed to be natural. Of course $x > y$ means $x \geq y$ but $x \neq y$; likewise, $x \succ y$ means $x \succeq y$ but not $y \succeq x$.

Every skew lattice is ***regular*** in that the identity $x \circ y \circ x \circ z \circ x = x \circ y \circ z \circ x$ holds for both $\circ = \vee$ and $\circ = \wedge$ (see [8, Theorem 1.15] or [12, Theorem 1.11]). As a consequence, one quickly gets:

$$x \vee y \vee x' \vee z \vee x'' = x \vee y \vee z \vee x'' \qquad \text{if} \quad x' \preceq x, x'' \tag{1.5a}$$

and

$$x \wedge y \wedge x' \wedge z \wedge x'' = x \wedge y \wedge z \wedge x'' \qquad \text{if} \quad x' \succeq x, x''\,. \tag{1.5b}$$

In any lattice, $\geq$ and $\succeq$ are identical, with $\vee$ and $\wedge$ determined by $s \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$. For skew lattices, the situation is more complicated. To see what happens, we must first recall several fundamental aspects of skew lattices. The preorder $\succeq$ induces a natural equivalence $\mathcal{D}$ defined by $x\,\mathcal{D}\,y$ if $x \succeq y \succeq x$. This is one of three ***Green's relations*** defined by:

$$x\,\mathcal{R}\,y \;\Leftrightarrow\; (x \wedge y = y \;\&\; y \wedge x = x) \;\Leftrightarrow\; (x \vee y = x \;\&\; y \vee x = y)\,. \tag{$\mathcal{R}$}$$

$$x\,\mathcal{L}\,y \;\Leftrightarrow\; (x \wedge y = x \;\&\; y \wedge x = y) \;\Leftrightarrow\; (x \vee y = y \;\&\; y \vee x = x)\,. \tag{$\mathcal{L}$}$$

$$x\,\mathcal{D}\,y \;\Leftrightarrow (x \wedge y \wedge x = x \;\&\; y \wedge x \wedge y = y) \;\Leftrightarrow (x \vee y \vee x = x \;\&\; y \vee x \vee y = y)\,. \tag{$\mathcal{D}$}$$

$\mathcal{R}$, $\mathcal{L}$ and $\mathcal{D}$ are congruences on any skew lattice, with $\mathcal{L} \vee \mathcal{R} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L} = \mathcal{D}$ and $\mathcal{L} \cap \mathcal{R} = \Delta$, the identity equivalence. Their congruence classes (called $\mathcal{R}$-classes, $\mathcal{L}$-classes or $\mathcal{D}$-classes) are all rectangular subalgebras. (A skew lattice is ***rectangular*** if $x \wedge y \wedge x = x = x \wedge y \wedge x$, or equivalently,

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$x \wedge y = y \vee x$ holds. These are precisely the anti-commutative skew lattices in that $x \wedge y = y \wedge x$ or $x \vee y = y \vee x$ imply $x = y$. See [8, §1] or recently, [7, §1].) The Green's congruence classes of a an element x are denoted, respectively, by $\mathcal{R}_x$, $\mathcal{L}_x$ or $\mathcal{D}_x$.

The First Decomposition Theorem for Skew Lattices [8, Theorem 1.7] states: *Given a skew lattice* $\mathbf{S}$, *each* $\mathcal{D}$*-class is a maximal rectangular subalgebra of* $\mathbf{S}$ *and* $\mathbf{S}/\mathcal{D}$ *is the maximal lattice image of* $\mathbf{S}$. In brief, *every skew lattice is a lattice of rectangular [anticommutative] subalgebras* in that it looks roughly like a lattice whose points are rectangular skew lattices. Clearly $x \succeq y$ in $\mathbf{S}$ if and only if $D_x \geq D_y$ in

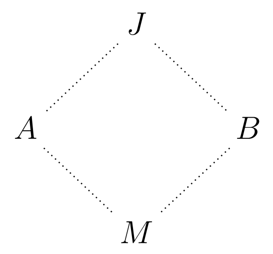


FIGURE 1. A, B, J, & M are maximal rectangular subalgebras

the lattice $\mathbf{S}/\mathcal{D}$ where D_x and D_y are the $\mathcal{D}$-classes of x and y, respectively. Given $a \in A$ and $b \in B$ for $\mathcal{D}$-classes A and B, $a \vee b$ just lie in their join $\mathcal{D}$-class J; similarly $a \wedge b$ must lie in their meet $\mathcal{D}$-class M.

Our interest in this paper is in ***skew chains*** that consist of totally ordered families of $\mathcal{D}$-classes: $A > B > \cdots > X$. As a (sub-)skew lattice, a skew chain T is ***totally preordered***: given $x, y \in T$, either $x \preceq y$ or $y \preceq x$. Of special interest are skew chains of length 1 ($A > B$) called ***primitive skew lattices***, and skew chains of length 2 ($A > B > C$) that occur in skew lattices.

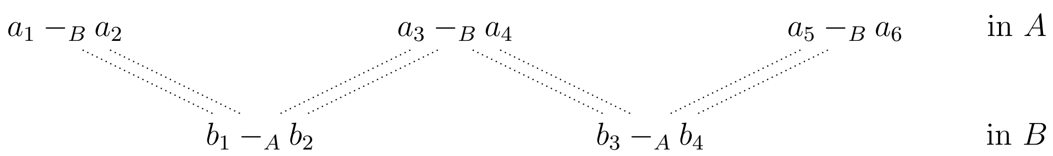
Given a primitive skew lattice with $\mathcal{D}$-class structure $A > B$, an $\mathbf{A}$***-coset in*** $\mathbf{B}$ is any subset of B of the form

$$A \wedge b \wedge A = \{a \wedge b \wedge a' \mid a, a' \in A\} = \{a \wedge b \wedge a \mid a \in A\}$$

for some $b \in B$. (The second equality follows from (1.5b).) Any two A-cosets in B are either identical or else disjoint. Since b must lie in $A \wedge b \wedge A$ for all $b \in B$, *the* A*-cosets in* B *form a partition of* B. Dually a B***-coset in*** A is a subset of A of the form

$$B \vee a \vee B = \{b \vee a \vee b \mid b, b' \in B\} = \{b \vee a \vee b \mid b \in B\}$$

for some $a \in A$. Again, the B-cosets in A partition A. Given a B-coset X in A and an A-coset Y in B, the natural partial ordering induces a ***coset bijection*** $\varphi : X \to Y$ given by $\varphi(a) = b$ for $a \in X$ and $b \in Y$ if and only if $a > b$, in which case $b = \varphi(a) = a \wedge y \wedge a$ for *all* $y \in Y$ and $a = \varphi^{-1}(b) = b \vee x \vee b$ for *all* $x \in X$. Cosets are rectangular subalgebras of their $\mathcal{D}$-classes; moreover, all coset bijections are isomorphisms between these subalgebras. *All* A*-cosets in* B *and all* B*-cosets in* A *thus share a common size and structure.* If $a, a' \in A$ lie in a common B-coset, we denote this by $a -_B a'$; likewise $b -_A b'$ in B if b and b' lie in a common A-coset. This is illustrated in the partial configuration below where $\ddots$ and $\iddots$ indicate $>$ between a's and b's. (The coset bijections from $\{a_1, a_2\}$ to $\{b_3, b_4\}$ and from $\{a_5, a_6\}$ to $\{b_1, b_2\}$ are not shown.)



Cosets and their bijections determine $\vee$ *and* $\wedge$ *in this situation.* Given $a \in A$ and $b \in B$:

$$a \vee b = a \vee a' \text{ and } b \vee a = a' \vee a \text{ in } A \text{ where } a' -_B a \text{ is such that } a' \geq b\,. \tag{1.6a}$$

$$a \wedge b = b' \wedge b \text{ and } b \wedge a = b \wedge b \text{ in } B \text{ where } b' -_A b \text{ is such that } a \geq b\,. \tag{1.6b}$$

(See [11, Lemma 1.3].) This explains how $\geq$ determines $\vee$ and $\wedge$ in the primitive case. How this is extended to the general case where A and B are incomparable $\mathcal{D}$-classes is explained in [11, §3]; see also [12].

This paper focuses on skew chains of $\mathcal{D}$-classes $A > B > C$ in a skew lattice and their three primitive subalgebras: $A > B$, $B > C$ and $A > C$. Viewing coset bijections as partial bijections between the relevant $\mathcal{D}$-classes one may ask: *is the composite* $\psi\varphi$ *of coset bijections* $\varphi : A \to B$ *and* $\psi : B \to C$, *if nonempty, a coset bijection from* A *to* C? If the answer is always yes, the skew chain is called ***categorical***. (Since including identity maps on $\mathcal{D}$-classes and empty partial bijections if needed creates a category with $\mathcal{D}$-classes for objects, coset bijections for morphisms and composition being that of partial bijections.) If this occurs for all skew chains in a skew lattice $\mathbf{S}$, then $\mathbf{S}$ is categorical. If such compositions are also always nonempty, the skew chain [skew lattice] is ***strictly categorical***.

Both categorical and strictly categorical skew lattices form varieties. (See [11, Theorem 3.16] and Corollary 4.3 below.) We will see that distributive skew lattices are categorical, and in particular skew lattices in rings are categorical. All skew Boolean algebras [10] are strictly categorical. Categorical skew lattices were introduced in [11]. Here we take an alternatively approach.

In all this, individual ordered pairs $a > b$ are bundled to form coset bijections. We first look at how this "bundling" process (parallelism) extends from the $A - B$ and $B - C$ settings to the $A - C$ settings in the next section.

2. Parallel ordered pairs

Suppose $A > B$ is a (primitive) skew chain and $\varphi : X \to Y$ is a fixed coset bijection where X is a B-coset in A and Y is an A-coset in B. Viewing the function φ as a binary relation, let us momentarily identify it with the set of strictly ordered pairs $a > b$ where $a \in X$, $b \in Y$ are such that $\varphi(a) = b$. Suppose $a > b$ and $a' > b'$ are two such pairs. Since $b' = \varphi(a') = a' \wedge y \wedge a'$ for all $y \in Y$, we certainly have $b' = a' \wedge b \wedge a$ and similarly $b = a \wedge b' \wedge a$. Since $a' = \varphi^{-1}(b') = b' \vee x \vee b'$ for all $x \in X$, we have $a' = b' \vee a \vee b'$ and similarly $a = b \vee a' \vee b$. These observations motivate the following definition.

Strictly ordered pairs $a > b$ and $a' > b'$ in a skew lattice $\mathbf{S}$ are said to be ***parallel***, denoted $a > b \;//\; a' > b'$, if $a \,\mathcal{D}\, a'$, $b \,\mathcal{D}\, b'$, $a' = b' \vee a \vee b'$ and $b' = a' \wedge b \wedge a'$. In this case, (1.5a) and (1.5b) imply that $a = b \vee a' \vee b$ and $b = a \wedge b' \wedge a$ also, so that the concept is symmetric with respect to both inequalities. In fact, the two pairs are parallel precisely when both lie in a common coset bijection φ, when considered to be a binary relation. Indeed, $a > b \;//\; a' > b'$ implies that both a and a' share a common $\mathcal{D}_b$-coset in $\mathcal{D}_a$, and b and b' share a common $\mathcal{D}_a$-coset in $\mathcal{D}_b$, making both pairs belong to a common φ. Conversely, if $a > b$ and $a' > b'$ lie in a common coset bijections so that a, a' share a $\mathcal{D}_b$-coset in $\mathcal{D}_a$ and b, b' share a $\mathcal{D}_a$-coset in $\mathcal{D}_b$, then $a' = b' \vee a \vee b'$ and $b' = a' \wedge b \wedge a'$ must follow so that $a > b \;//\; a' > b'$. Thus:

Proposition 2.1. *Parallelism is an equivalence relation on the set of all partially ordered pairs* $a > b$ *in a skew lattice* $\mathbf{S}$, *the equivalence classes of which form coset bijections when the latter are viewed as binary relations. Moreover:*

i) *If* $a > b \;//\; a' > b'$, *then* $a = a'$ *if and only if* $b = b'$;
ii) *If* $a > b \;//\; a' > b'$ *and* $b > c \;//\; b' > c'$, *then* $a > c \;//\; a' > c'$;
iii) *Given just* $a \succ b$, *then* $a > a \wedge b \wedge a \;//\; b \vee a \vee b > b$.

Proof. The first claim is routine, and (i)-(iii) follow from basic properties of coset bijections: their being bijections indeed, their composition and their connections to their particular cosets of relevance. □

Now we return to the point of view that for a skew chain $A > B$, a coset bijection $\varphi : X \to Y$, $X \subseteq A$, $Y \subseteq B$, is a partial bijection $\varphi : A \to B$ of the $\mathcal{D}$-classes. Let $A > B > C$ be a 3-term skew chain and suppose $\varphi : A \to B$ and $\psi : B \to C$ are coset (partial) bijections. Suppose that the composite partial bijection $\psi \circ \varphi : A \to C$ is nonempty, say $a > b > c$ with $b = \varphi(a)$ and $c = \psi(b)$. Then there is a uniquely determined partial bijection $\chi : A \to C$ defined on its coset domain by $\chi(u) = u \wedge c \wedge u$ such that $\psi \circ \varphi \subseteq \chi$. Later we shall see instances where the inclusion is proper. We are interested in characterizing equality.

In terms of parallelism and the fixed triple $a > b > c$, the situation we have described so far is that *if* $a > b \mathbin{//} a' > b'$ and $b > c \mathbin{//} b' > c'$, *then* $a > c \mathbin{//} a' > c'$. We see that $\chi = \psi \circ \varphi$ precisely when the converse holds, that is, *if* $a > c \mathbin{//} a' > c'$, *then* there exists a (necessarily) unique $b' \in B$ such that $a > b \mathbin{//} a' > b'$ and $b > c \mathbin{//} b' > c'$. In particular, b' must equal both $a' \wedge b \wedge a'$ and $c' \vee b \vee c'$. This gives the following Hasse configuration of parallel pairs.

$$
\begin{array}{ccl}
a & — & a' = c' \vee a \vee c' = b' \vee a \vee b' \\
\vdots & & \qquad\qquad \vdots \\
b & — & b' = c' \vee b \vee c' = a' \wedge b \wedge a' \\
\vdots & & \qquad\qquad \vdots \\
c & — & c' = a' \wedge c \wedge a' = b' \wedge c \wedge b'
\end{array}
\tag{2.1}
$$

Now considering this for all possible coset bijections in a skew lattice, we obtain the following characterization.

Proposition 2.2. *A skew lattice* $\mathbf{S}$ *is categorical if and only if, given* $a > b > c$ *with* $a > c \mathbin{//} a' > c'$, *there exists a unique* $b' \in S$ *such that* $a > b \mathbin{//} a' > b'$ *and* $b > c \mathbin{//} b' > c'$.

Theorem 2.3. *For a skew lattice* $\mathbf{S}$, *the following are equivalent.*

i) $\mathbf{S}$ *is categorical;*
ii) *For all* $x, y, z \in S$,

$$x \geq y \succeq z \quad \Rightarrow \quad x \wedge (z \vee y \vee z) \wedge x = (x \wedge z \wedge x) \vee y \vee (x \wedge z \wedge x)\,; \tag{2.2}$$

iii) *For all* $x, y, z \in S$,

$$x \succeq y \geq z \quad \Rightarrow \quad z \vee (x \wedge y \wedge x) \vee z = (z \vee x \vee z) \wedge y \wedge (z \vee x \vee z)\,. \tag{2.3}$$

Proof. Assume (i) holds and let $a \geq b \succeq c$ be given. If $a = b$ or if $b \,\mathcal{D}\, c$, then their insertion into (2.2) produces a trivial identity. Thus we may assume the comparisons to be strict: $a > b \succ c$. Proposition 2.1(iii) gives $a > a \wedge c \wedge a \mathbin{//} c \vee a \vee c > c$. Since $c \vee a \vee c > c \vee b \vee c > c$, (2.1) gives

$$a \wedge (c \vee b \vee c \wedge a = (a \wedge c \wedge a) \vee c \vee b \vee c \vee (a \wedge c \wedge a)\,.$$

From $c \,\mathcal{D}\, a \wedge c \wedge a$, (1.5a) reduces the right side to $(a \wedge c \wedge a) \vee b \vee (a \wedge c \wedge a)$ and so (2.2) holds. We have established (i)⇒(ii).

Conversely assume that (ii) holds, and let both $a > c \mathbin{//} a' > c'$ and $a > b > c$. Since $b > c \,\mathcal{D}\, c'$, $b \succ c'$. Thus $a > b \succ c'$, and so by (2.2),

$$a \wedge (c' \vee b \vee c') \wedge a = (a \wedge c' \wedge a) \vee v \vee (a \wedge c' \wedge a) = c \vee b \vee c = b\,,$$

since $a > b$ and $a \wedge c' \wedge a = c$. Taking two-sided meets with a' gives

$$\begin{aligned}
a' \wedge b \wedge a' &= a' \wedge a \wedge (c' \vee b \vee c') \wedge a \wedge a' \\
&= a' \wedge a \wedge a' \wedge (c' \vee b \vee c') \wedge a' \wedge a \wedge a' && \text{(by regularity)} \\
&= a' \wedge (c' \vee b \vee c') \wedge a' && \text{(since } a \,\mathcal{D}\, a'\text{)} \\
&= (c' \vee a \vee c') \wedge (c' \vee b \vee c') \wedge (c' \vee a \vee c') && \text{(since } a' > c'\text{)} \\
&= (c' \vee a \vee b \vee c') \wedge (c' \vee b \vee c') \wedge (c' \vee b \vee a \vee c') && \text{(by (1.5a))} \\
&= (c' \vee a \vee c' \vee b \vee c') \wedge (c' \vee b \vee c') \wedge (c' \vee b \vee c' \vee a \vee c') && \text{(by (1.5a))} \\
&= c' \vee b \vee c && \text{(by (1.1))}\,.
\end{aligned}$$

Thus (2.1) holds and **S** is categorical.

We have established (i)⇔(ii). The proof of (i)⇔(iii) is dual to this, exchanging $\wedge$ and $\vee$ as needed. □

Next we will show that categorical skew lattices form a variety by giving characterizing identities. This was already done in [11, Theorem 3.16], but the identity given there is rather long. Here we give two new ones, the first being the shortest we know and the second exhibiting a certain amount of symmetry in the variables. First we recall more basic notions.

A skew lattice is ***right-handed*** [respectively, ***left-handed***] if it satisfies the identities

$$x \wedge y \wedge x = y \wedge x \quad \text{and} \quad x \vee y \vee x = x \vee y\,. \tag{2.4a}$$

$$[x \wedge y \wedge x = x \wedge y \quad \text{and} \quad x \vee y \vee x = y \vee x]\,. \tag{2.4b}$$

Equivalently, $x \wedge y = y$ and $x \vee y = x$ [$x \wedge y = x$ and $x \vee y = y$] hold in each $\mathcal{D}$-class, thus reducing $\mathcal{D}$ to $\mathcal{R}$ [or $\mathcal{L}$]. Useful right- and left-handed variants of (2.4a) and (2.4b) are

$$x \succeq x' \quad \Rightarrow \quad x \wedge y \wedge x' = y \wedge x' \quad \text{and} \quad x \vee y \vee x' = x \vee y\,; \tag{2.5a}$$

$$x \succeq x' \quad \Rightarrow \quad x' \wedge y \wedge x = x' \wedge y \quad \text{and} \quad x' \vee y \vee x = y \vee x'\,; \tag{2.5b}$$

The *Second Decomposition Theorem* [8, Theorem 1.15] states that *given any skew lattice* **S**, $\mathbf{S}/\mathcal{R}$ *and* $\mathbf{S}/\mathcal{L}$ *are its respective maximal left- and right-handed images, and* **S** *is isomorphic to their fibred product (pullback)* $\mathbf{S}/\mathcal{R} \times_{\mathbf{S}/\mathcal{D}} \mathbf{S}/\mathcal{L}$ *over their maximal lattice image under the map* $x \mapsto (\mathcal{R}_x, \mathcal{L}_x)$. Thus a skew lattice **S** *belongs to a variety* $\mathcal{V}$ *of skew lattices if and only if both* $\mathbf{S}/\mathcal{R}$ *and* $\mathbf{S}/\mathcal{L}$ *do.* (See also [5, 12].)

Theorem 2.4. *Let* **S** *be a skew lattice. The following are equivalent.*

i) **S** *is categorical.*

ii) *For all* $x, y, z \in S$,

$$x \wedge [(x \wedge y \wedge z \wedge y \wedge x) \vee y \vee (x \wedge y \wedge z \wedge y \wedge x)] \wedge x = x \wedge y \wedge x\,. \tag{2.6}$$

iii) *For all* $x, y, z \in S$,

$$x \wedge [(x \wedge z \wedge x) \vee y \vee (x \wedge z \wedge x)] \wedge x = x \wedge [(z \wedge x \wedge z) \vee y \vee (z \wedge x \wedge z)] \wedge x\,. \tag{2.7}$$

Proof. Assume first that **S** is a left-handed categorical skew lattice. Suppose (i) holds. By Theorem 2.3, **S** satisfies the left-handed version of (**??**):

$$x \geq y \succeq z \quad \Rightarrow \quad x \wedge (y \vee z) = y \vee (x \wedge z)\,. \tag{2.8}$$

Note that $x \vee y \geq y \succeq (y \vee x) \wedge y \wedge z$. We may thus apply (2.8). The right side becomes

$$y \vee [(x \vee y) \wedge (y \vee x) \wedge y \wedge z] = y \vee [(x \vee y) \wedge y \wedge z] = y \vee [y \wedge z] = y\,,$$

using left-handedness and absorption. Therefore the identity

$$(x \vee y) \wedge [y \vee ((y \vee x) \wedge y \wedge z)] = y \tag{2.9}$$

holds. Taking the meet of both sides on the left with x, we get

$$x \wedge [y \vee ((y \vee x) \wedge y \wedge z)] = x \wedge y\,. \tag{2.10}$$

Now replace y with $y \wedge x$. The left side of (2.10) becomes

$$x \wedge [(y \wedge x) \vee (((y \wedge x) \vee x) \wedge y \wedge z)] = x \wedge [(y \wedge x) \vee (x \wedge y \wedge z)]\,,$$

and the right side becomes $x \wedge y \wedge x = x \wedge y$. Thus we have the identity

$$x \wedge [(y \wedge x) \vee (x \wedge y \wedge z)] = x \wedge y\,. \tag{2.11}$$

Now meet both sides of (2.11) on the left with $x \wedge (y \vee (x \wedge y \wedge z))$. On the right side, we get

$$x \wedge (y \vee (x \wedge y \wedge z)) \wedge x \wedge y = x \wedge (y \vee (x \wedge y \wedge z)) \wedge y = x \wedge (y \vee (x \wedge y \wedge z))\,,$$

since $y = y \vee y \succeq y \vee (x \wedge y \wedge z)$. The left side becomes

$$\begin{aligned} &x \wedge [y \vee (x \wedge y \wedge z)] \wedge x \wedge [(y \wedge x) \vee (x \wedge y \wedge z)] \\ &= x \wedge [y \vee (y \wedge x) \vee (x \wedge y \wedge z)] \wedge [(y \wedge x) \vee (x \wedge y \wedge z)] \\ &= x \wedge [(y \wedge x) \vee (x \wedge y \wedge z)] \\ &= x \wedge y\,, \end{aligned}$$

where the last step is an application of (2.11). Thus we have established

$$x \wedge (y \vee (x \wedge y \wedge z)) = x \wedge y\,, \tag{2.12}$$

which is the left-handed version of (2.6). This proves (i)⇒(ii) for all left-handed skew lattices.

Continuing to assume $\mathbf{S}$ is left-handed, suppose (ii) holds. Replace y with $y \vee z$ in (2.12). On the left side, we obtain

$$x \wedge (y \vee z \vee (x \wedge (y \vee z) \wedge z)) = x \wedge (y \vee z \vee (x \wedge z))\,.$$

On the right side, we get $x \wedge (y \vee z)$, and so we have

$$x \wedge (y \vee z \vee (x \wedge z)) = x \wedge (y \vee z)\,. \tag{2.13}$$

Now in (2.13), replace z with $z \wedge x$. On the left side, we get

$$x \wedge (y \vee (z \wedge x) \vee (x \wedge z \wedge x)) = x \wedge (y \vee (z \wedge x \wedge z) \vee (x \wedge z)) = x \wedge (y \vee (x \wedge z))\,.$$

On the right side, we get $x \wedge (y \vee (z \wedge x))$, and thus we obtain the identity

$$x \wedge (y \vee (x \wedge z)) = x \wedge (y \vee (z \wedge x))\,, \tag{2.14}$$

which is the left-handed version of (2.7). This proves (ii)⇒(iii) in left-handed skew lattices.

Still assuming $\mathbf{S}$ is left-handed, suppose (iii) holds. Fix $a, b, c \in S$ satisfying $a \geq b \succeq c$. Then

$$\begin{aligned} a \wedge (b \vee c) &= a \wedge (b \vee (c \wedge a)) && \text{(since } a \succeq c\text{)} \\ &= a \wedge (b \vee (a \wedge c)) && \text{(by (2.14))} \\ &= (a \vee (a \wedge c)) \wedge (b \vee (a \wedge c)) \\ &= (a \vee b \vee (a \wedge c)) \wedge (b \vee (a \wedge c)) && \text{(since } a \geq b\text{)} \\ &= b \vee (a \wedge c)\,. \end{aligned}$$

Thus (2.8) holds and so by Theorem 2.3, $\mathbf{S}$ is categorical. This proves (iii)⇒(i) for left-handed skew lattices.

In general, if $\mathbf{S}$ is a skew lattice, then conditions (i), (ii) and (iii) are equivalent for the maximal left-handed image $\mathbf{S}/\mathcal{R}$. The left-right (horizontal) dual of the whole argument implies that the same is true for $\mathbf{S}/\mathcal{L}$. It follows that (i), (ii) and (iii) are equivalent for $\mathbf{S}$ itself. □

Corollary 2.5. *Categorical skew lattices form a variety.*

Of course, categorical skew lattices are also characterized by the $\vee - \wedge$ duals of (2.6) and (2.7).

Recall that a skew lattice is ***distributive*** if the following dual pair of identities holds:

$$x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)\,, \tag{2.15}$$

$$x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x)\,. \tag{2.16}$$

Many important classes of skew lattices are distributive, in particular, skew lattices in rings and skew Boolean algebras [1, 2, 3, 4, 8, 10, 13, 15]. Since (2.15) implies (2.2), we have:

Corollary 2.6. *Distributive skew lattices are categorical.*

3. Forbidden subalgebras

Clearly what occurs in the middle class of a 3-term skew chain $A > B > C$ is significant. Two elements $b, b' \in B$ are **AC-*connected*** if a finite sequence $b = b_0, b_1, \ldots, b_n = b'$ in B exists such that $b_i -_A b_{i+1}$ or $b_i -_C b_{i+1}$ for all $i \leq n-1$. A maximally AC-connected subset of B is an **AC-*component*** of B (or just ***component*** if the context is clear). Given a component B' in the middle class B, a sub-skew chain is given by $A > B' > C$. Indeed, if A_1 and C_1 are B-cosets in A and C respectively, then $A_1 > B' > C_1$ is an even smaller sub-skew chain.

Furthermore, let X denote an A-coset in B (thus $X = A \wedge b \wedge A$ for any $b \in X$) and let Y denote a C-coset in B (thus $Y = C \vee b \vee C$ for any $b \in Y$). If $X \cap Y \neq \emptyset$, it is called an **AC-*coset*** in B. *When* $\mathbf{S}$ *is categorical,* $(X \cap Y) \vee a \vee (X \cap Y)$ *is a C-coset in A and dually,* $(X \cap Y) \wedge c \wedge (X \cap Y)$ *is an A-coset in C for all* $a \in A$, $c \in C$. *Conversely, when* $\mathbf{S}$ *is categorical, given a C-coset U in A, for all* $b \in B$, $U \wedge b \wedge U$ *is an AC-coset in B; likewise given any A-coset V in C,* $V \vee b \vee V$ *is an AC-coset in B for all* $b \in B$. *In both cases we get the unique AC-coset in B containing b.* An extended discussion of these matters occurs in [14, §2].

We start our characterization of categorical skew lattices in terms of forbidden subalgebras with a relevant lemma.

Lemma 3.1. *Let* $A > B > C$ *be a left-handed skew chain with* $a > c \;//\; a' > c'$ *where* $a \neq a' \in A$ *and* $c \neq c' \in C$. *Set* $A^* = \{a, a'\}$, $B^* = \{x \in B \mid a > x > c \text{ or } a' > x > c'\}$ *and* $C^* = \{c, c'\}$. *Then* $A^* > B^* > C^*$ *is a sub-skew chain. In particular,*

i) $a' > x > c'$ *for* $x \in B^*$ *implies:* $a >$ *both* $a \wedge x$ *and* $x \vee c > c$ *with* $a \wedge x -_{A^*} x -_{C^*} x \vee c$.

ii) $a > x > c$ *for* $x \in B^*$ *implies:* $a' >$ *both* $a' \wedge x$ *and* $x \vee c' > c'$ *with* $a' \wedge x -_{A^*} x -_{C^*} x \vee c'$.

All A^**-cosets and all* C^**-cosets in* B^* *are of order* 2. *An* A^*C^**-component in* B^* *is either a subset* $\{b, b'\}$ *that is simultaneously an* A^**-coset and* C^**-coset in* B^* *or else it is a larger subset with all* A^*C^**-cosets having size* 1 *and having the alternating coset form*

$$\cdots -_{A^*} \bullet -_{C^*} \bullet -_{A^*} \bullet -_{C^*} \bullet -_{A^*} \bullet -_{C^*} \cdots$$

Only the former case can occur if the skew chain is categorical.

Proof. Being left-handed, we need only check the mixed outcomes, say $a \wedge x$, $x \wedge a$, $c \vee x$ and $x \vee c$ where $a' > x > c'$ for case (i). Trivially $x \wedge a = x = c \vee x$. As for $a \wedge x$, $a \wedge (a \wedge x) = a \wedge x = (a \wedge x) \wedge a$, due to left-handedness, so that $a > a \wedge x$; likewise $c \wedge (a \wedge x) = c$, while

$$(a \wedge x) \wedge c = a \wedge x \wedge a \wedge c' = a \wedge x \wedge c' = a \wedge c' = c$$

by left-handedness and parallelism. Hence $a \wedge x > c$ also, so that $a \wedge x$ is in B^*. The dual argument gives $a > x \vee c > c$, so that $x \vee c \in B^*$ also. Similarly (ii) holds and we have a sub-skew chain.

Clearly the A^*-cosets in B^* either all have order 1 or all have order 2. If they have order 1, then $a, a' >$ all elements in B^*, and by transitivity, $a, a' >$ both c, c', so that $a > c$ is not parallel to $a' > c'$. Thus all A^*-cosets in B^* have order 2 and likewise all C^*-cosets in B^* have order 2. In an A^*C*-component in B^*, if the first case does not occur, a situation $x -_{C^*} y -_{A^*} z$ with x, y, z distinct develops. Since A^*-cosets and C^*-cosets have size 2, it extends in an alternating coset pattern in both directions, either doing so indefinitely or eventually connecting to form a cycle of even length. $\square$

A complete set of examples with B^* being a single A^*C^*-component is as follows.

Example 3.2. Consider the class of skew chains $A > B_n > C$ for $1 \leq n \leq \omega$, where

$$A = \{a_1, a_2\}, C = \{c_1, c_2\} \text{ and}$$
$$B_n = \{b_1, b_2, \dots, b_{2n}\} \text{ or } \{\dots, b_{-2}, b_{-1}, b_0, b_1, b_2, \dots\} \text{ if } n = \omega\,.$$

The partial order is given by parity: $a_1 > b_{\text{odd}} > c_1$ and $a_2 > b_{\text{even}} > c_2$. Both A and C are full B-cosets as well as full cosets of each other. A-cosets and C-cosets in B are given respectively by:

$$\{b_1, b_2 \mid b_3, b_4 \mid \cdots \mid b_{2n-1}, b_{2n}\} \quad \text{and} \quad \{b_{2n}, b_1 \mid b_2, b_3 \mid \cdots \mid b_{2n-2}, b_{2n-1}\} \quad \text{for } n < \omega\,.$$

For $n > 1$, B_n has the following alternating coset structure (modulo n when n is finite):

$$\cdots -_A b_{2k-2} -_C b_{2k-1} -_A b_{2k} -_C b_{2k+1} -_A b_{2k+2} -_C \cdots\,.$$

Clearly B_n is a single component. We denote the left-handed skew chain thus determined by $\mathbf{X}_n$ and its right-handed dual by $\mathbf{Y}_n$ for $n \leq \omega$. Their Hasse diagrams for $n = 1, 2$ are given in Figure 2.

$$\begin{array}{ccc} a_1 & - & a_2 \\ \vdots & & \vdots \\ b_1 & - & b_2 \\ \vdots & & \vdots \\ c_1 & - & c_2 \end{array} \qquad\qquad \begin{array}{ccccccccccc} & & a_1 & & - & & a_2 & & & & \\ & \cdot^{\cdot^{\cdot}} & & \ddots & & \cdot^{\cdot^{\cdot}} & & \ddots & & & \\ b_1 & -_A & b_2 & & -_C & & b_3 & -_A & b_4 & -_C & (b_1) \\ & \ddots & & \cdot^{\cdot^{\cdot}} & & \ddots & & \cdot^{\cdot^{\cdot}} & & & \\ & & c_1 & & - & & c_2 & & & & \end{array}$$

FIGURE 2. Hasse diagrams for $\mathbf{X}_n/\mathbf{Y}_n$, $n = 1, 2$

Applying (1.6a) and (1.6b) above, instances of left-handed operations on $\mathbf{X}_2$ are given by

$$a_1 \vee c_2 = a_2 = a_1 \vee a_2, \quad a_1 \wedge b_4 = b_3 \wedge b_4 = b_3, \quad \text{and} \quad b_1 \vee c_2 = b_1 \vee b_4 = b_4\,.$$

Except for $\mathbf{X}_1$ and $\mathbf{Y}_1$, none of these skew lattices is categorical. In $\mathbf{X}_n$ for $n \geq 2$, $a_1 > b_1 > c_1$, $a_2 \wedge c_1 = c_2$, $a_1 \vee c_2 = a_2$, but $a_2 \wedge b_1 = b_2$, while $b_1 \vee c_2$ is either b_{2n} or b_0. Note that while all A-cosets and all C-cosets in B_n have order 2, the AC-cosets have order 1.

Theorem 3.3. *A left-handed skew lattice is categorical if and only if it contains no copy of $\mathbf{X}_n$ for $2 \leq n \leq \omega$. Dually, a right-handed skew lattice is categorical if and only if it contains no copy of $\mathbf{Y}_n$ for $2 \leq n \leq \omega$. In general, a skew lattice is categorical if and only if it contains no copy of any of these algebras. Finally, none of these algebras is a subalgebra of another one.*

Proof. We begin with a skew chain $A > B > C$ in a left-handed skew lattice $\mathbf{S}$. Given $a > b > c$ in $\mathbf{S}$, where $a \in A$, $b \in B$ and $c \in C$, let $a > c \mathbin{//} a' > c'$ with $a \neq a'$. In the skew chain of Lemma 3.1, $A^* > B^* > C^*$ where $A^* = \{a, a'\}$ and $C^* = \{c, c'\}$, we obtain the following configuration.

$$\begin{array}{ccccc} a' & - & a & - & a' \\ \vdots & & \vdots & & \vdots \\ b \vee c' & -_C & b & -_A & a' \wedge b \\ \vdots & & \vdots & & \vdots \\ c' & - & c & - & c' \end{array}$$

When $a' \wedge c = b \vee c'$, the situation is compatible with **S** being categorical. Otherwise, in the A^*C^*-component of b in B^*, the middle row in the above configuration extends to an alternating coset pattern of the type in Lemma 3.1, giving us a copy of $\mathbf{X}_n$ where $2 \leq n \leq \omega$. If **S** is not categorical, such a situation must occur. Conversely, any left-handed skew lattice containing a copy of $\mathbf{X}_n$ for $n \geq 2$ is not categorical. The first assertion now follows. The nature of the middle row implies that no $\mathbf{X}_m$ can be embedded in any $\mathbf{X}_n$ for $n > m$.

The right-handed case is similar. Clearly, a categorical skew lattice contains no $\mathbf{X}_n$ or $\mathbf{Y}_n$ copy for $n \geq 2$. Conversely, if a skew lattice **S** contains copies of none of them, then neither does $\mathbf{S}/\mathcal{R}$ or $\mathbf{S}/\mathcal{L}$ since every skew chain with three $\mathcal{D}$-classes in either $\mathbf{S}/\mathcal{R}$ or $\mathbf{S}/\mathcal{L}$ can be lifted to an isomorphic subalgebra of **S**. (Indeed, given any skew chain T: $A > B > C$, one easily finds $a > b > c$ with $a \in A$, $b \in B$ and $c \in C$. Then, *e.g.*, the sub-skew chain $\mathcal{R}_a > \mathcal{R}_b > \mathcal{R}_c$ of $\mathcal{R}$-classes in T is isomorphic to $T/\mathcal{L}$. See [5].) Thus $\mathbf{S}/\mathcal{R}$ and $\mathbf{S}/\mathcal{L}$ are categorical, and hence so is **S**. □

A skew chain $A > B > A'$ is ***reflective*** if (1) A and A' are full cosets of each other in themselves, making $A \equiv A'$ with both being full B-cosets in themselves, and (2) B consists of a single AA'-component. All $\mathbf{X}_n$ and $\mathbf{Y}_n$ are reflective. If B is both an A-coset and an A'-coset for every reflective skew chain in a skew lattice **S** (making the skew chain a direct product of a chain $a > b > a'$ and a rectangular subalgebra), then **S** is categorical. Indeed, copies of $\mathbf{X}_n$ or $\mathbf{Y}_n$ for $n \geq 2$ are eliminated as subalgebras, while $\mathbf{X}_1$ and $\mathbf{Y}_1$ clearly factor as stated.

The converse is also true. Consider a reflective skew chain $A > B > A'$ in a categorical skew lattice. Let $\varphi : A \to B$ be a coset bijection of A onto an A-coset in B and let $\psi : B \to A'$ be a coset bijection of B onto A' such that the composition $\psi \circ \varphi$ is the unique coset bijection of A onto A'. As partial bijections, the only way for $\psi \circ \varphi$ to be both one-to-one and onto is for φ and ψ to be full bijections between A and B, and between B and A', respectively, thus making B both a full A-coset and a full A'-coset within itself. We thus have:

Proposition 3.4. *A skew lattice* **S** *is categorical if and only if every reflective skew chain* $A > B > A'$ *in* **S** *factors as a direct product of a chain,* $a > b > a'$*, and a rectangular skew lattice.*

4. Strictly categorical skew lattices

Recall that a categorical skew lattice **S** is ***strictly categorical*** if for every skew chain of $\mathcal{D}$-classes $A > B > C$ in **S**, each A-coset in B has nonempty intersection with each C-coset in B, making both B an entire AC-component and empty coset bijections unnecessary. Examples are:

a) *Normal* skew lattices characterized by the conditions: $x \wedge y \wedge z \wedge w = x \wedge z \wedge y \wedge w$; equivalently, every subset $[e]\downarrow = \{x \in S \mid e \geq x\} = \{e \wedge x \wedge e \mid x \in S\}$ is a sublattice;
b) *Conormal* skew lattices satisfying the dual condition $x \vee y \vee z \vee w = x \vee z \vee y \vee w$; equivalently, every subset $[e]\uparrow = \{x \in S \mid e \leq x\} = \{e \vee x \vee e \mid x \in S\}$ is a sublattice;
c) *Primitive* skew lattices consisting of two $\mathcal{D}$-classes: $A > B$ and rectangular skew lattices.
d) *Skew diamonds* in cancellative skew lattices, and in particular, skew diamonds in rings. (A skew diamond is a skew lattice $\{J > A, B > M\}$ consisting of two incomparable $\mathcal{D}$-classes A and B along with their join $\mathcal{D}$-class J and their meet $\mathcal{D}$-class M.) See [7].

See [7] for general results on normal skew lattices. Their importance is due in part to skew Boolean algebras being normal as skew lattices [1, 2, 12, 13, 15]. Some nice counting theorems for categorical and strictly categorical skew lattices are given in [14].

Theorem 4.1. *Let* $A > B > C$ *be a strictly categorical skew chain. Then:*

i) *For any* $a \in A$*, all images of* a *in* B *lie in a unique* C*-coset in* B*;*
ii) *For any* $c \in C$*, all images of* c *in* B *lie in a unique* A*-coset in* B*;*

iii) *Given* $a > c$ *with* $a \in A$ *and* $c \in C$, *a unique* $b \in B$ *exists such that* $a > b > c$. *This* b *lies jointly in the* C*-coset in* B *containing all images of* a *in* B *and in the* A*-coset in* B *containing all images of* c *in* B.

Proof. To verify (i) we assume without loss of generality that C is a full B-coset within itself. If $a \wedge C \wedge a = \{c \in C \mid a > c\}$ is the image set of a in C parameterizing the A-cosets in C and $b \in B$ is such that $a > b$, then $\{c \vee b \vee c \mid c \in a \wedge C \wedge a\}$, the set of all images of a in the C-coset $C \vee b \vee C$ in B, parameterizes the AC-cosets in B lying in $C \vee b \vee C$ (since AC-cosets in $C \vee b \vee C$ are inverse images of the A-cosets in C under the coset bijection of $C \vee b \vee C$ onto C). By assumption, all A-cosets X in B are in bijective correspondence with all these AC-cosets under the map $X \mapsto X \cap C \vee b \vee C$. Thus each element x in $\{c \vee b \vee c \mid c \in a \wedge C' \wedge a\}$ is the (necessarily) unique image of a in the A-coset in B which x belongs, and as we traverse through these x's, every such A-coset occurs as $A \wedge x \wedge A$. Thus all images of a in B lie within the C-coset $C \vee b \vee C$ in B. In similar fashion one verifies (ii). Finally, given $a > c$ with $a \in A$ and $c \in C$, a unique AC-coset U exists that is the intersection of the A-coset containing all images of c in B and the C-coset containing all images of a in B. In particular, U contains unique elements u, v such that $a > u$ and $v > c$. Consider $b = a \wedge v \wedge a$ in B. Clearly $a > b > c$ so that b is a simultaneous image of a and c in B (since $b -_A v$) and thus is in U; moreover, by uniqueness of u and v in U, we have $u = b = v$. □

This leads to the following multiple characterization of strictly categorical skew lattices.

Theorem 4.2. *The following seven conditions on a skew lattice* $\mathbf{S}$ *are equivalent.*

i) $\mathbf{S}$ *is strictly categorical;*

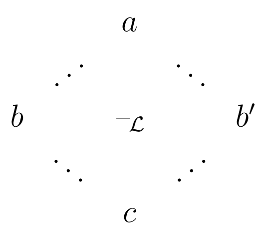
ii) $\mathbf{S}$ *satisfies*

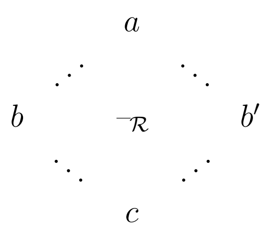
$$x > y > z \quad \& \quad x > y' > z \quad \& \quad y\,\mathcal{D}\,y' \quad \Rightarrow \quad y = y'\,;$$

iii) $\mathbf{S}$ *satisfies*

$$x \geq y \geq z \quad \& \quad x \geq y' \geq z \quad \& \quad y\,\mathcal{D}\,y' \quad \Rightarrow \quad y = y'\,;$$

iv) $\mathbf{S}$ *has no subalgebra isomorphic to either of the following* 4*-element skew chains.*





v) *If* $a > b$ *in* $\mathbf{S}$, *the interval subalgebra* $[a, b] = \{x \in S \mid a \geq x \geq b\}$ *is a sublattice.*

vi) *Given* $a \in S$, $[a]\!\uparrow = \{x \in S \mid x \geq a\}$ *is a normal subalgebra of* $\mathbf{S}$ *and* $[a]\!\downarrow = \{x \in S \mid a \geq x\}$ *is a conormal subalgebra of* S.

vii) $\mathbf{S}$ *is categorical and given any skew chain* $A > B > C$ *of* $\mathcal{D}$*-classes in* S, *for each coset bijection* $\varphi : A \to C$, *there exist unique coset bijections* $\psi : A \to B$ *and* $\chi : B \to C$ *such that* $\varphi = \chi \circ \psi$.

viii) *Every reflective skew chain* $A > B > C$ *is an isochain.*

Proof. Theorem 4.1(iii) gives us (i)⇒(ii). Conversely, if $\mathbf{S}$ satisfies (ii) then no subalgebra of $\mathbf{S}$ can be one of the forbidden subalgebras of the last section, making $\mathbf{S}$ categorical. We next show that given $x, y \in B$, there exist $u, v \in B$ such that $x -_A u -_C y$ and $x -_C v -_A y$. This guarantees that in B, every A-coset meets every C-coset. Indeed, pick $a \in A$ and $c \in C$ so that $a > x > c$. Note that $a > a \wedge (c \vee y \vee c) \wedge a$, $c \vee (a \wedge y \wedge a) \vee c > c$. But by assumption x is the unique element in B between a and c under $>$. Thus

$a \wedge (c \vee y \vee c) \wedge a = x = c \vee (a \wedge y \wedge a) \vee c$ so that both $x -_A c \vee y \vee c -_C y$ and $x -_C a \wedge y \wedge a -_A y$ in B, which gives (ii)⇒(i).

Next let $\mathbf{S}$ be categorical with $A > B > C$ as stated in (vii). The unique factorization in (vii) occurs precisely when (ii) holds, making (ii) and (vii) equivalent, with (viii) being a variant of (vii). Finally, (iii)-(vi) are easily seen to be equivalent variants of (ii). □

Corollary 4.3. *Strictly categorical skew lattices form a variety of skew lattices.*

Proof. We will show that strictly categorical skew lattices are characterized by the following identity (or its dual):

$$x \vee (y \wedge z \wedge u \wedge y) \vee x = x \vee (y \wedge u \wedge z \wedge y) \vee x\,. \tag{4.1}$$

Let e denote the left side and f denote the right side. Observe that $e\mathcal{D}f$ since $z \wedge u\,\mathcal{D}\,u \wedge z$. Note that $x \vee y \vee x \geq e, f \geq x$ by (1.1). Hence if a skew lattice $\mathbf{S}$ is strictly categorical, then (4.1) holds by Theorem 4.2(iii). Conversely, let (4.1) hold in $\mathbf{S}$ and suppose that $a \geq$ both $b, b' \geq c$ in S with $b\,\mathcal{D}\,b'$. Assigning $x \mapsto c$, $y \mapsto a$, $z \mapsto b \wedge b'$ and $u \mapsto b' \wedge b$ reduced (4.1) to $b = b \wedge b' \wedge b = b' \wedge b \wedge b'$ so that $\mathbf{S}$ is strictly categorical by Theorem 4.2(iii). □

While distributive skew lattices are categorical, they need not be strictly categorical, but *a strictly categorical skew lattice* $\mathbf{S}$ *is distributive iff* $\mathbf{S}/\mathcal{D}$ *is distributive.* (See [7, Theorem 5.4].)

It is natural to ask: *What is the variety generated jointly from the varieties of normal and conormal skew lattices?* To refine this question, we first proceed as follows.

A primitive skew lattice $A > B$ is ***order-closed*** if for $a, a' \in A$ and $b, b' \in B$, both $a, a' > b$ and $a > b, b'$ imply $a' > b'$. A primitive skew lattice $A > B$ is ***simply order-closed*** if $a > b$ for all $a \in A$

$$\begin{array}{cccccc} A & a & - & - & - & a' \\ & \vdots & \ddots & & \iddots & \\ & \vdots & \iddots & & \ddots & \\ B & b & - & - & - & b' \end{array}$$

and all $b \in B$. In this case the cosets of A and B in each other are singleton subsets. It is easy to verity that *a primitive skew lattice* $\mathbf{S}$ *is order-closed if and only if it factors into a product* $D \times T$ *where* D *is rectangular and* T *is simply order-closed and primitive.*

A skew lattice is ***order-closed*** if all its primitive subalgebras are thus. Examples include:

a) Normal skew lattices and conormal skew lattices;
b) The sequences of examples $\mathbf{X}_n$ and $\mathbf{Y}_n$ of section 3.

On the other hand, primitive skew lattices that are not order-closed are easily found. (See [11, §§1,2].)

Theorem 4.4. *Order-closed skew lattices form a variety of skew lattices.*

Proof. The following generic situation holds between comparable $\mathcal{D}$-classes in a skew lattice: where as

$$\begin{array}{cccccc} x \wedge y & - & - & - & (x \wedge y \wedge u \wedge v \wedge x \wedge y) \vee (y \wedge x) \vee (x \wedge y \wedge u \wedge v \wedge x \wedge y) \\ \vdots & \ddots & & \iddots & \\ \vdots & \iddots & & \ddots & \\ x \wedge y \wedge u \wedge v \wedge x \wedge y & - & - & - & x \wedge y \wedge v \wedge u \wedge x \wedge y \end{array}$$

usual, the dotted lines denote $\geq$ relationships. Being order-closed requires both expressions on the right side of the diagram to commute under $\vee$ (or $\wedge$). Commutativity under $\vee$ together with (1.1) gives

$$(x \wedge y \wedge \underbrace{v \wedge u} \wedge x \wedge y) \vee (y \wedge x) \vee (x \wedge y \wedge \underbrace{u \wedge v} \wedge x \wedge y) = (x \wedge y \wedge \underbrace{u \wedge v} \wedge x \wedge y) \vee (y \wedge x) \vee (x \wedge y \wedge \underbrace{v \wedge u} \wedge x \wedge y) \tag{4.2}$$

(or its dual) as a characterizing identity for order-closed skew lattices. □

Refining the above question, we ask:

Problem 4.5. *Do order-closed, strictly categorical skew lattices form the join variety of the varieties of normal skew lattices and their conormal duals?*

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