

# AN EINSTEIN EQUATION FOR DISCRETE QUANTUM GRAVITY

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## Abstract

The basic framework for this article is the causal set approach to discrete quantum gravity (DQG). Let  $Q_n$  be the collection of causal sets with cardinality not greater than  $n$  and let  $K_n$  be the standard Hilbert space of complex-valued functions on  $Q_n$ . The formalism of DQG presents us with a decoherence matrix  $D_n(x, y)$ ,  $x, y \in Q_n$ . There is a growth order in  $Q_n$  and a path in  $Q_n$  is a maximal chain relative to this order. We denote the set of paths in  $Q_n$  by  $\Omega_n$ . For  $\omega, \omega' \in \Omega_n$  we define a bidifference operator  $\nabla_{\omega, \omega'}^n$  on  $K_n \otimes K_n$  that is covariant in the sense that  $\nabla_{\omega, \omega'}^n$  leaves  $D_n$  stationary. We then define the curvature operator  $\mathcal{R}_{\omega, \omega'}^n = \nabla_{\omega, \omega'}^n - \nabla_{\omega', \omega}^n$ . It turns out that  $\mathcal{R}_{\omega, \omega'}^n$  naturally decomposes into two parts  $\mathcal{R}_{\omega, \omega'}^n = \mathcal{D}_{\omega, \omega'}^n + \mathcal{T}_{\omega, \omega'}^n$  where  $\mathcal{D}_{\omega, \omega'}^n$  is closely associated with  $D_n$  and is called the metric operator while  $\mathcal{T}_{\omega, \omega'}^n$  is called the mass-energy operator. This decomposition is a discrete analogue of Einstein's equation of general relativity. Our analogue may be useful in determining whether general relativity theory is a close approximation to DQG.

## 1 Causet Approach to DQG

A *causal set* (*causet*) is a finite partially ordered set  $x$ . Thus,  $x$  is endowed with an irreflexive, transitive relation  $<$  [1, 8, 10]. That is,  $a \not< a$  for all  $a \in x$

and  $a < b$ ,  $b < c$  imply that  $a < c$  for  $a, b, c \in x$ . The relation  $a < b$  indicates that  $b$  is in the causal future of  $a$ . Let  $\mathcal{P}_n$  be the collection of all causets with cardinality  $n$ ,  $n = 1, 2, \dots$ , and let  $\mathcal{P} = \cup \mathcal{P}_n$ . For  $x \in \mathcal{P}$ , an element  $a \in x$  is *maximal* if there is no  $b \in x$  with  $a < b$ . If  $x \in \mathcal{P}_n$ ,  $y \in \mathcal{P}_{n+1}$ , then  $x$  *produces*  $y$  if  $y$  is obtained from  $x$  by adjoining a single new element  $a$  to  $x$  that is maximal in  $y$ . In this way, there is no element of  $y$  in the causal future of  $a$ . If  $x$  produces  $y$ , we say that  $y$  is an *offspring* of  $x$  and write  $x \rightarrow y$ .

A *path* in  $\mathcal{P}$  is a string (sequence)  $x_1 x_2 \dots$  where  $x_i \in \mathcal{P}_i$  and  $x_i \rightarrow x_{i+1}$ ,  $i = 1, 2, \dots$ . An *n-path* in  $\mathcal{P}$  is a finite string  $x_1 x_2 \dots x_n$  where again  $x_i \in \mathcal{P}_i$  and  $x_i \rightarrow x_{i+1}$ . We denote the set of paths by  $\Omega$  and the set of  $n$ -paths by  $\Omega_n$ . We think of  $\omega \in \Omega$  as a possible universe (or universe history). The set of paths whose initial  $n$ -path is  $\omega_0 \in \Omega_n$  is called an *elementary cylinder set* and is denoted by  $\text{cyl}(\omega_0)$ . Thus, if  $\omega_0 = x_1 x_2 \dots x_n$ , then

$$\text{cyl}(\omega_0) = \{\omega \in \Omega: \omega = x_1 x_2 \dots x_n y_{n+1} y_{n+2} \dots\}$$

The *cylinder set generated* by  $A \subseteq \Omega_n$  is defined by

$$\text{cyl}(A) = \bigcup_{\omega \in A} \text{cyl}(\omega)$$

The collection  $\mathcal{A}_n = \{\text{cyl}(A): A \subseteq \Omega_n\}$  forms an increasing sequence of algebras  $\mathcal{A}_1 \subseteq \mathcal{A}_2 \subseteq \dots$  on  $\Omega$  and hence  $\mathcal{C}(\Omega) = \cup \mathcal{A}_n$  is an algebra of subsets of  $\Omega$ . We denote the  $\sigma$ -algebra generated by  $\mathcal{C}(\Omega)$  as  $\mathcal{A}$ .

It is shown [6, 11] that a classical sequential growth process (CSGP) on  $\mathcal{P}$  that satisfies natural causality and covariance conditions is determined by a sequence of nonnegative numbers  $c = (c_0, c_1, \dots)$  called *coupling constants*. The coupling constants specify a unique probability measure  $\nu_c$  on  $\mathcal{A}$  making  $(\Omega, \mathcal{A}, \nu_c)$  a probability space. The *path Hilbert space* is given by  $H = L_2(\Omega, \mathcal{A}, \nu_c)$ . If  $\nu_c^n = \nu_c \upharpoonright \mathcal{A}_n$  is the restriction of  $\nu_c$  to  $\mathcal{A}_n$ , then  $H_n = L_2(\Omega, \mathcal{A}_n, \nu_c^n)$  is an increasing sequence of closed subspaces of  $H$ .

A bounded operator  $T$  on  $H_n$  will also be considered as a bounded operator on  $H$  by defining  $Tf = 0$  for all  $f \in H_n^\perp$ . We denote the characteristic function of a set  $A \in \mathcal{A}$  by  $\chi_A$  and use the notation  $\chi_\Omega = 1$ . A *q-probability operator* is a bounded positive operator  $\rho_n$  on  $H_n$  that satisfies  $\langle \rho_n 1, 1 \rangle = 1$ . Denote the set of  $q$ -probability operators on  $H_n$  by  $\mathcal{Q}(H_n)$ . For  $\rho_n \in \mathcal{Q}(H_n)$  we define the *n-decoherence functional* [3, 5, 8]  $D_n: \mathcal{A}_n \times \mathcal{A}_n \rightarrow \mathbb{C}$  by

$$D_n(A, B) = \langle \rho_n \chi_B, \chi_A \rangle$$

The functional  $D_n(A, B)$  gives a measure of the interference between the events  $A$  and  $B$  when the system is described by  $\rho_n$ . It is clear that  $D_n(\Omega_n, \Omega_n) = 1$ ,  $D_n(A, B) = \overline{D_n(B, A)}$  and  $A \mapsto D_n(A, B)$  is a complex measure for all  $B \in \mathcal{A}_n$ . It is also well known that if  $A_1, \dots, A_n \in \mathcal{A}_n$ , then the matrix with entries  $D_n(A_j, A_k)$  is positive semidefinite. In particular the positive semidefinite matrix with entries

$$D_n(\omega_i, \omega_j) = D_n(\text{cyl}(\omega_i), \text{cyl}(\omega_j)), \quad \omega_i, \omega_j \in \Omega_n$$

is called the *n-decoherence matrix*.

We define the map  $\mu_n: \mathcal{A}_n \rightarrow \mathbb{R}^+$  given by

$$\mu_n(A) = D_n(A, A) = \langle \rho_n \chi_A, \chi_A \rangle$$

Notice that  $\mu_n(\Omega) = 1$ . Although  $\mu_n$  is not additive in general, it satisfies the *grade-2 additivity condition* [2, 3, 5, 7, 9]: if  $A, B, C \in \mathcal{A}_n$  are mutually disjoint then

$$\mu_n(A \cup B \cup C) = \mu_n(A \cup B) + \mu_n(A \cup C) + \mu_n(B \cup C) - \mu_n(A) - \mu_n(B) - \mu_n(C)$$

We call  $\mu_n$  the *q-measure* corresponding to  $\rho_n$  and interpret  $\mu_n(A)$  as the quantum propensity for the occurrence of the event  $A \in \mathcal{A}_n$ . A simple example is to let  $\rho_n = I$ ,  $n = 1, 2, \dots$ . Then

$$D_n(A, B) = \langle \chi_B, \chi_A \rangle = \nu_c^n(A \cap B)$$

and  $\mu_n(A) = \nu_c^n(A)$ , the classical measure. A more interesting example is to let  $\rho_n = |1\rangle\langle 1|$ ,  $n = 1, 2, \dots$ . Then

$$D_n(A, B) = \langle |1\rangle\langle 1| \chi_B, \chi_A \rangle = \nu_c^n(A) \nu_c^n(B)$$

and  $\mu_n(A) = [\nu_c^n(A)]^2$ , the classical measure square.

We call a sequence  $\rho_n \in \mathcal{Q}(H_n)$ ,  $n = 1, 2, \dots$ , *consistent* if

$$D_{n+1}(A, B) = D_n(A, B)$$

for all  $A, B \in \mathcal{A}_n$ . A *quantum sequential growth process* (QSGP) is a consistent sequence  $\rho_n \in \mathcal{Q}(H_n)$  [3, 4]. We consider a QSGP as a model for discrete quantum gravity. It is hoped that additional theoretical principles or experimental data will help determine the coupling constants and hence  $\nu_c$ , which is the classical part of the process, and also the  $\rho_n \in \mathcal{Q}(H_n)$ , which is

the quantum part. Moreover, it is believed that general relativity will eventually be shown to be a close approximation to this discrete model. Until now it has not been clear how this can be accomplished. However, in Section 3 we shall derive a discrete Einstein equation which might be useful in performing these tasks.

## 2 Difference Operators

Let  $Q_n = \bigcup_{j=1}^n \mathcal{P}_j$  be the collection of causets with cardinality not greater than  $n$  and let  $K_n$  be the (finite-dimensional) Hilbert space  $\mathbb{C}^{Q_n}$  with the standard inner product

$$\langle f, g \rangle = \sum_{x \in Q_n} \overline{f(x)} g(x)$$

Let  $L_n = K_n \otimes K_n$  which we identify with  $\mathbb{C}^{Q_n \times Q_n}$  having the standard inner product. We shall also have need to consider the Hilbert space

$$K = \left\{ f \in \mathbb{C}^{\mathcal{P}} : \sum_{x \in \mathcal{P}} |f(x)|^2 < \infty \right\}$$

with the standard inner product and we define  $L = K \otimes K$ . Notice that  $K_1 \subseteq K_2 \subseteq \dots \subseteq K$  form an increasing sequence of subspaces of  $K$  that generate  $K$  in the natural way.

Let  $\rho_n \in \mathcal{Q}(H_n)$  be a QSGP with corresponding decoherence matrix  $D_n(\omega, \omega')$ ,  $\omega, \omega' \in \Omega_n$ . If  $\omega = \omega_1 \omega_2 \dots \omega_n \in \Omega_n$  and  $\omega_j = x$  for some  $j$ , we say that  $\omega$  goes through  $x$ . For  $x, y \in Q_n$  we define

$$D_n(x, y) = \sum \{ D_n(\omega, \omega') : \omega \text{ goes through } x, \omega' \text{ goes through } y \}$$

Due to the consistency of  $\rho_n$ ,  $D_n(x, y)$  is independent of  $n$ . That is,  $D_n(x, y) = D_m(x, y)$  if  $x, y \in Q_n \cap Q_m$ . Moreover,  $D_n(x, y)$  are the components of a positive semidefinite matrix. We view  $Q_n$  as the analogue of a differentiable manifold and  $D_n(x, y)$  as the analogue of a metric tensor. One might think that the elements of causets should be analogous to points of a differential manifold and not the causets themselves. However, if  $x \in Q_n$ , then  $x$  is intimately related to its producers, each of which determines a unique  $a \in x$ .

Moreover, if  $y \rightarrow x$  we view  $(y, x)$  as a tangent vector at  $x$ . In this way, there are as many tangent vectors at  $x$  as there are producers of  $x$ . Finally, the elements of  $\Omega_n$  are analogues of curves and the elements of  $K_n$  are analogues of smooth functions on a manifold.

For  $x \in Q_n$ ,  $|x|$  denotes the cardinality of  $x$ . Notice if  $\omega = \omega_1\omega_2 \cdots \omega_n \in \Omega_n$  and  $\omega_j = x$ , then  $j = |x|$  and  $\omega$  goes through  $x$  if and only if  $\omega_{|x|} = x$ . We see that a path  $\omega$  through  $x$  determines a tangent vector  $(\omega_{|x|-1}, \omega_{|x|})$  at  $x$  (assuming that  $|x| \geq 2$ ). For  $\omega \in \Omega_n$  we define the *difference operator*  $\Delta_\omega^n$  on  $K_n$  by

$$\Delta_\omega^n f(x) = [f(x) - f(\omega_{|x|-1})] \delta_{x, \omega_{|x|}}$$

for all  $f \in K_n$ , where  $\delta_{x, \omega_{|x|}}$  is the Kronecker delta. Thus,  $\Delta_\omega^n f(x)$  gives the change of  $f$  along the tangent vector  $(\omega_{|x|-1}, \omega_{|x|})$  if  $\omega$  goes through  $x$ . It is clear that  $\Delta_\omega^n$  is a linear operator on  $K_n$ . We now show that  $\Delta_\omega^n$  satisfies a discrete form of Leibnitz's rule. For  $f, g \in K_n$  we have

$$\begin{aligned} \Delta_\omega^n fg(x) &= [f(x)g(x) - f(\omega_{|x|-1})g(\omega_{|x|-1})] \delta_{x, \omega_{|x|}} \\ &= \{ [f(x)g(x) - f(x)g(\omega_{|x|-1})] + [f(x)g(\omega_{|x|-1}) - f(\omega_{|x|-1})g(\omega_{|x|-1})] \} \\ &\quad \cdot \delta_{x, \omega_{|x|}} \\ &= f(x) \Delta_\omega^n g(x) + g(\omega_{|x|-1}) \Delta_\omega^n f(x) \end{aligned} \quad (2.1)$$

Of course, it also follows that

$$\Delta_\omega^n fg(x) = \Delta_\omega^n gf(x) = g(x) \Delta_\omega^n f(x) + f(\omega_{|x|-1}) \Delta_\omega^n g(x) \quad (2.2)$$

Given a function of two variables  $f \in \mathbb{C}^{Q_n \times Q_n} = L_n$  we have a function  $\tilde{f} \in K_n$  of one variable where  $\tilde{f}(x) = f(x, x)$  and given a function  $f \in K_n$  we have the functions of two variables  $f_1, f_2 \in L_n$  where  $f_1(x, y) = f(x)$  and  $f_2(x, y) = f(y)$  for all  $x, y \in Q_n$ . For  $\omega, \omega' \in \Omega_n$  we want an operator  $\Delta_{\omega, \omega'}^n: L_n \rightarrow L_n$  that extends  $\Delta_\omega^n$  and satisfies a discrete Leibnitz's rule. That is,

$$\Delta_{\omega, \omega'}^n f_1(x, y) = \Delta_\omega^n f(x) \delta_{y, \omega'_{|x|}}, \Delta_{\omega, \omega'}^n f_2(x, y) = \Delta_{\omega'}^n f(y) \delta_{x, \omega_{|x|}} \quad (2.3)$$

and

$$\Delta_{\omega, \omega'}^n fg(x, y) = f(x, y) \Delta_{\omega, \omega'}^n g(x, y) + g(\omega_{|x|-1}, \omega'_{|x|-1}) \Delta_{\omega, \omega'}^n f(x, y) \quad (2.4)$$

**Theorem 2.1.** *A linear operator  $\Delta_{\omega, \omega'}^n : L_n \rightarrow L_n$  satisfies (2.3) and (2.4) if and only if  $\Delta_{\omega, \omega'}^n$  has the form*

$$\Delta_{\omega, \omega'}^n f(x, y) = [f(x, y) - f(\omega_{|x|-1}, \omega'_{|y|-1})] \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \quad (2.5)$$

*Proof.* If  $\Delta_{\omega, \omega'}^n$  is defined by (2.5), then for  $f \in K_n$  we have

$$\begin{aligned} \Delta_{\omega, \omega'}^n f_1(x, y) &= [f_1(x, y) - f_1(\omega_{|x|-1}, \omega'_{|y|-1})] \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \\ &= [f(x) - f(\omega_{|x|-1})] \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \\ &= \Delta_{\omega}^n f(x) \delta_{y, \omega'_{|y|}} \end{aligned}$$

In a similar way,  $\Delta_{\omega, \omega'}^n$  satisfies the second equation in (2.3). Moreover, we have

$$\begin{aligned} \Delta_{\omega, \omega'}^n f(x, y) &= [f(x, y)g(x, y) - f(\omega_{|x|-1}\omega'_{|y|-1})g(\omega_{|x|-1}\omega'_{|y|-1})] \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \\ &= [f(x, y)g(x, y) - f(x, y)g(\omega_{|x|-1}, \omega'_{|y|-1})] \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \\ &\quad + [f(x, y)g(\omega_{|x|-1}, \omega'_{|y|-1}) - f(\omega_{|x|-1}\omega'_{|y|-1})g(\omega_{|x|-1}\omega'_{|y|-1})] \\ &\quad \cdot \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \\ &= f(x, y) \Delta_{\omega, \omega'}^n g(x, y) + g(\omega_{|x|-1}, \omega'_{|y|-1}) \Delta_{\omega, \omega'}^n f(x, y) \end{aligned}$$

Conversely, suppose the linear operator  $\Delta_{\omega, \omega'}^n : L_n \rightarrow L_n$  satisfies (2.3) and (2.4). If  $f \in L_n$  has the form  $f(x, y) = g(x)h(y)$ , then

$$\begin{aligned} \Delta_{\omega, \omega'}^n f(x, y) &= \Delta_{\omega, \omega'}^n gh(x, y) = \Delta_{\omega, \omega'}^n g_1 h_2(x, y) \\ &= g_1(x, y) \Delta_{\omega, \omega'}^n h_2(x, y) + h_2(\omega_{|x|-1}\omega'_{|y|-1}) \Delta_{\omega, \omega'}^n g_1(x, y) \\ &= g(x) \Delta_{\omega}^n h(y) \delta_{x, \omega_{|x|}} + h(\omega'_{|y|-1}) \Delta_{\omega}^n g(x) \delta_{y, \omega'_{|y|}} \\ &= \{g(x) [h(y) - h(\omega'_{|y|-1})] + h(\omega'_{|y|-1}) [g(x) - g(\omega_{|y|-1})]\} \\ &\quad \cdot \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \\ &= [g(x)h(y) - g(\omega_{|x|-1})h(\omega'_{|y|-1})] \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \\ &= [f(x, y) - f(\omega_{|x|-1}\omega'_{|y|-1})] \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \end{aligned}$$

Since  $\Delta_{\omega, \omega'}^n$  is linear and every element of  $L_n$  is a linear combination of product functions, the result follows.  $\square$

Of course, Theorem 2.1 is not surprising because (2.5) is the natural extension of  $\Delta_\omega^n$  to  $L_n$ . Also  $\Delta_{\omega,\omega'}^n$  extends  $\Delta_\omega^n$  in the sense that for any  $f \in L_n$  we have

$$\begin{aligned}\Delta_{\omega,\omega}^n f(x, x) &= [f(x, y) - f(\omega_{|x|-1}, \omega_{|x|-1})] \delta_{x, \omega_{|x|}} \\ &= [\tilde{f}(x) - \tilde{f}(\omega_{|x|-1})] \delta_{x, \omega_{|x|}} = \Delta_\omega^n \tilde{f}(x)\end{aligned}$$

As before,  $\Delta_{\omega,\omega'}^n$  satisfies

$$\begin{aligned}\Delta_{\omega,\omega'}^n f g(x, y) &= \Delta_{\omega,\omega'}^n g f(x, y) \\ &= g(x, y) \Delta_{\omega,\omega'}^n f(x, y) + f(\omega_{|x|-1}, \omega'_{|y|-1}) \Delta_{\omega,\omega'}^n g(x, y)\end{aligned}$$

The next result characterizes  $\Delta_\omega^n$  and  $\Delta_{\omega,\omega'}^n$  up to a multiplicative constant.

**Theorem 2.2.** (a) *An operator  $T_\omega: K_n \rightarrow K_n$  satisfies (2.1) and  $T_\omega f(x) = 0$  if  $\omega_{|x|} \neq x$  if and only if there exists a function  $\beta_\omega \in K_n$  such that  $T_\omega = \beta_\omega \Delta_\omega^n$ .*  
(b) *An operator  $T_{\omega,\omega'}: L_n \rightarrow L_n$  satisfies (2.4) and  $T_{\omega,\omega'} f(x, y) = 0$  if  $\omega_{|x|} \neq x$  or  $\omega'_{|y|} \neq y$  if and only if there exists a function  $\beta_{\omega,\omega'} \in L_n$  such that  $T_{\omega,\omega'} = \beta_{\omega,\omega'} \Delta_{\omega,\omega'}^n$ .*

*Proof.* If  $T_\omega$  satisfies (2.1), it follows from (2.2) that

$$f(x)T_\omega g(x) + g(\omega_{|x|-1})T_\omega f(x) = g(x)T_\omega f(\omega) + f(\omega_{|x|-1})T_\omega g(x)$$

Hence,

$$[g(x) - g(\omega_{|x|-1})] T_\omega f(x) = T_\omega g(x) [f(x) - f(\omega_{|x|-1})]$$

Therefore, if  $g(x) - g(\omega_{|x|-1}) \neq 0$ , we have

$$T_\omega f(x) = \frac{T_\omega g(x)}{g(x) - g(\omega_{|x|-1})} [f(x) - f(\omega_{|x|-1})]$$

Letting

$$\beta_\omega(x) = \frac{T_\omega g(x)}{g(x) - g(\omega_{|x|-1})}$$

gives the result. The converse is straightforward. The proof of (b) is similar.  $\square$

It is clear that  $\mu_n(x) = D_n(x, x)$  is not stationary; that is  $\Delta_\omega^n \mu_n(x) \neq 0$  for all  $x \in Q_n$  in general. Is there a function  $\alpha_\omega \in K_n$  such that  $(\Delta_\omega^n + \alpha_\omega) \mu_n(x) = 0$  for all  $x \in Q_n$ ? If  $\alpha_\omega$  exists, we obtain

$$[\mu_n(x) - \mu_n(\omega_{|x|-1})] \delta_{x, \omega_{|x|}} + \alpha_\omega(x) \mu_n(x) = 0$$

If  $\omega_{|x|} = x$  and  $\mu_n(x) = 0$ , this would imply that  $\mu_n(\omega_{|x|-1}) = 0$ . Continuing this process would give

$$\mu_n(\omega_{|x|-2}) = \mu_n(\omega_{|x|-3}) = \cdots = 0$$

which leads to a contradiction. It is entirely possible for  $\mu_n(x)$  to be zero for some  $x \in Q_n$  so we abandon this attempt. How about functions  $\alpha_\omega, \beta_\omega \in K_n$  such that  $(\beta_\omega \Delta_\omega^n + \alpha_\omega) \mu_n(x) = 0$  for all  $x \in Q_n$ ? We then obtain

$$\beta_\omega(x) [\mu_n(x) - \mu_n(\omega_{|x|-1})] \delta_{x, \omega_{|x|}} + \alpha_\omega(x) \mu_n(x) = 0 \quad (2.6)$$

If  $\omega_{|x|} = x$  and  $\mu_n(x) = 0$  but  $\beta_\omega(x) \neq 0$  we obtain the same contradiction as before. We conclude that  $\beta_\omega(x) = 0$  whenever  $\mu_n(x) = 0$ . The simplest choice of such a  $\beta_\omega$  is  $\beta_\omega(x) = \mu_n(x)$ . This choice also has the advantage of being independent of  $\omega$ . Equation (2.6) becomes

$$\mu_n(x) [\mu_n(x) - \mu_n(\omega_{|x|-1})] \delta_{x, \omega_{|x|}} + \alpha_\omega(x) \mu_n(x) = 0 \quad (2.7)$$

If  $\mu_n(x) = 0$ , then (2.7) holds. If  $\mu_n(x) \neq 0$ , then we obtain

$$\alpha_\omega(x) = [\mu_n(\omega_{|x|-1}) - \mu_n(x)] \delta_{x, \omega_{|x|}} = -\Delta_\omega^n \mu_n(x)$$

The numbers  $\alpha_\omega(x)$  are an analogue of the Christoffel symbols. We call  $\nabla_\omega^n = \mu_n \Delta_\omega^n + \alpha_\omega$  the *covariant difference operator*. The operator  $\nabla_\omega^n$  is not a difference operator in the usual sense because  $\nabla_\omega^n 1 \neq 0$ . Instead, we have  $\nabla_\omega^n 1 = \alpha_\omega$ .

Following the previous steps for  $\Delta_{\omega, \omega'}^n$ , we define the *covariant bidifference operator*  $\nabla_{\omega, \omega'}^n = D_n \Delta_{\omega, \omega'}^n + \alpha_{\omega, \omega'}$  where

$$\alpha_{\omega, \omega'}(x, y) = [D_n(\omega_{|x|-1}, \omega'_{|y|-1}) - D_n(x, y)] \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}}$$

and again,  $\alpha_{\omega, \omega'}(x, y)$  are analogous to Christoffel symbols. Notice that  $\nabla_{\omega, \omega'}^n D_n(x, y) = 0$  for all  $x, y \in Q_n$  and  $\nabla_{\omega, \omega'}^n f(x, x) = \nabla_\omega^n \tilde{f}(x)$ . Complete expressions for  $\nabla_\omega^n$  and  $\nabla_{\omega, \omega'}^n$  are

$$\nabla_\omega^n f(x) = [\mu_n(\omega_{|x|-1}) f(x) - \mu_n(x) f(\omega_{|x|-1})] \delta_{x, \omega_{|x|}} \quad (2.8)$$



and

$$\begin{aligned} \nabla_{\omega, \omega'}^n f(x, y) &= [D_n(\omega_{|x|-1}, \omega'_{|y|-1})f(x, y) - D_n(x, y)f(\omega_{|x|-1}, \omega'_{|y|-1})] \\ &\quad \cdot \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \end{aligned} \quad (2.9)$$

The form of (2.8) and (2.9) shows that  $\nabla_{\omega}^n$  and  $\nabla_{\omega, \omega'}^n$  are “weighted” difference operators.

### 3 Curvature Operators

The linear operator  $\mathcal{R}_{\omega, \omega'}^n: L_n \rightarrow L_n$  defined by

$$\mathcal{R}_{\omega, \omega'}^n = \nabla_{\omega, \omega'}^n - \nabla_{\omega', \omega}^n$$

is called the *curvature operator*. Applying (2.9) we have

$$\begin{aligned} \mathcal{R}_{\omega, \omega'}^n f(x, y) &= D_n(x, y) \left[ f(\omega'_{|x|-1}, \omega_{|y|-1}) \delta_{x, \omega'_{|x|}} \delta_{y, \omega_{|y|}} - f(\omega_{|x|-1}, \omega'_{|y|-1}) \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \right] \\ &\quad + \left[ D_n(\omega_{|x|-1}, \omega'_{|y|-1}) \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} - D_n(\omega'_{|x|-1}, \omega_{|y|-1}) \delta_{x, \omega'_{|x|}} \delta_{y, \omega_{|y|}} \right] f(x, y) \end{aligned} \quad (3.1)$$

If  $x = y$ , then (3.1) reduces to

$$\begin{aligned} \mathcal{R}_{\omega, \omega'}^n f(x, x) &= \mu_n(x) [f(\omega'_{|x|-1}, \omega_{|x|-1}) - f(\omega_{|x|-1}, \omega'_{|x|-1})] \delta_{x, \omega_{|x|}} \\ &\quad + 2i \operatorname{Im} D_n(\omega_{|x|-1}, \omega'_{|x|-1}) f(x, x) \delta_{x, \omega_{|x|}} \end{aligned}$$

We call the operator  $\mathcal{D}_{\omega, \omega'}^n: L_n \rightarrow L_n$  given by

$$\begin{aligned} \mathcal{D}_{\omega, \omega'}^n f(x, y) &= D_n(x, y) \left[ f(\omega'_{|x|-1}, \omega_{|y|-1}) \delta_{x, \omega'_{|x|}} \delta_{y, \omega_{|y|}} - f(\omega_{|x|-1}, \omega'_{|y|-1}) \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} \right] \end{aligned}$$

the *metric operator* and the operator  $\mathcal{T}_{\omega, \omega'}^n: L_n \rightarrow L_n$  given by

$$\begin{aligned} \mathcal{T}_{\omega, \omega'}^n f(x, y) &= \left[ D_n(\omega_{|x|-1}, \omega'_{|y|-1}) \delta_{x, \omega_{|x|}} \delta_{y, \omega'_{|y|}} - D_n(\omega'_{|x|-1}, \omega_{|y|-1}) \delta_{x, \omega'_{|x|}} \delta_{y, \omega_{|y|}} \right] f(x, y) \end{aligned}$$

the *mass-energy operator*. Then (3.1) gives

$$\mathcal{R}_{\omega,\omega'}^n = \mathcal{D}_{\omega,\omega'}^n + \mathcal{T}_{\omega,\omega'}^n \quad (3.2)$$

Equation (3.2) is a discrete analogue of Einstein's equation [12]. In this sense, Einstein's equation always holds in the present framework no matter what we have for the quantum dynamics  $\rho_n$ . One might argue that we obtained this discrete analogue of Einstein's equation just by definition. However,  $\mathcal{R}_{\omega,\omega'}^n$  is a reasonable counterpart of the curvature tensor in general relativity [12] and  $\mathcal{D}_{\omega,\omega'}^n$  is certainly a counterpart of the metric tensor.

Equation (3.2) does not give direct information about  $D_n(x, y)$  and  $D_n(\omega, \omega')$  (which are, after all, what we want to find), but it may give useful indirect information. If we can find  $D_n(\omega, \omega')$  such that the classical Einstein equation is an approximation to (3.2), then this would give information about  $D_n(\omega, \omega')$ . Moreover, an important problem in discrete quantum gravity theory is how to test whether general relativity is a close approximation to the theory. Whether Einstein's equation is an approximation to (3.2) would provide such a test. Another variant of a discrete Einstein equation can be obtained by defining the operator  $\mathcal{R}_{x,y}^n$  for  $x, y \in Q_n$  by

$$\mathcal{R}_{x,y}^n = \sum \{ \mathcal{R}_{\omega,\omega'}^n : \omega|_x = x, \omega'|_y = y \}$$

With similar definitions for  $\mathcal{D}_{x,y}^n$  and  $\mathcal{T}_{x,y}^n$  we obtain

$$\mathcal{R}_{x,y}^n = \mathcal{D}_{x,y}^n + \mathcal{T}_{x,y}^n$$

In order to consider approximations by Einstein's equation, it may be necessary to let  $n \rightarrow \infty$  in (3.2). However, the convergence of the operators depends on  $D_n$  and will be left to a later paper. In a similar vein, it may be possible that limit operators  $\mathcal{R}_{\omega,\omega'}$ ,  $\mathcal{D}_{\omega,\omega'}$  and  $\mathcal{T}_{\omega,\omega'}$  can be defined as (possibly unbounded) operators directly on the Hilbert space  $L$ .

## 4 Matrix Elements

We have introduced several operators on  $K_n$  and  $L_n$  in Sections 2 and 3. In order to understand such operators more directly, it is frequently useful to write them in terms of their matrix elements. First we denote the standard basis on  $K_n$  by  $e_x^n, x \in Q_n$ . The matrix that is zero except for a one in

the  $xy$  entry is denoted by  $|e_x^n\rangle\langle e_y^n|$  and we call this the  $xy$  matrix element,  $x, y \in Q_n$ . Of course, in Dirac notation,  $|e_x^n\rangle\langle e_y^n|$  can be considered directly as a linear operator without referring to a matrix. In any case, every linear operator  $T$  on  $K_n$  can be represented uniquely as

$$T = \sum_{x, y \in Q_n} t_{x, y} |e_x^n\rangle\langle e_y^n|$$

for  $t_{x, y} \in \mathbb{C}$ . In a similar way,  $e_x^n \otimes e_y^n$ ,  $x, y \in Q_n$  form an orthonormal basis for  $L_n = K_n \otimes K_n$  and every linear operator  $T$  on  $L_n$  has a unique representation

$$T = \sum \{ t_{x, y; x', y'} |e_x^n \otimes e_y^n\rangle\langle e_{x'}^n \otimes e_{y'}^n| : x, y, x', y' \in Q_n \}$$

**Theorem 4.1.** *If  $\omega = \omega_1 \omega_2 \cdots \omega_n \in \Omega_n$ , then*

$$\Delta_\omega^n = \sum_{j=1}^n |e_{\omega_j}^n\rangle \left( \langle e_{\omega_j}^n| - \langle e_{\omega_{j-1}}^n| \right)$$

and

$$\nabla_\omega^n = \sum_{j=1}^n |e_{\omega_j}^n\rangle \left[ \mu_n(\omega_{j-1}) \langle e_{\omega_j}^n| - \mu_n(\omega_j) \langle e_{\omega_{j-1}}^n| \right]$$

where we use the conventions  $\mu_n(\omega_0) = e_{\omega_0}^n = e_{\omega_{n+1}}^n = 0$ .

*Proof.* We first observe that

$$\sum_{j=1}^n |e_{\omega_j}^n\rangle \left( \langle e_{\omega_n}^n| - \langle e_{\omega_{j-1}}^n| \right) e_{\omega_k}^n = e_{\omega_k}^n - e_{\omega_{k+1}}^n$$

On the other hand

$$\Delta_\omega^n e_{\omega_k}^n(x) = [e_{\omega_k}^n(x) - e_{\omega_k}^n(\omega_{|x|-1})] \delta_{x, \omega_{|x|}} \quad (4.1)$$

Now the right side of (4.1) is zero if  $\omega_{|x|} \neq x$ , 1 if  $\omega_k = x$  and  $-1$  if  $\omega_{k+1} = x$ . The first result now follows. The second result is similar.  $\square$

The proof of the next theorem is similar to that of Theorem 4.1.

**Theorem 4.2.** *If  $\omega = \omega_1\omega_2\cdots\omega_n$ ,  $\omega' = \omega'_1\omega'_2\cdots\omega'_n \in \Omega_n$ , then*

$$\Delta_{\omega,\omega'}^n = \sum_{j,k=1}^n \left| e_{\omega_j}^n \otimes e_{\omega'_k}^n \right\rangle \left[ \left\langle e_{\omega_j}^n \otimes e_{\omega'_k}^n \right| - \left\langle e_{\omega_{j-1}}^n \otimes e_{\omega'_{k-1}}^n \right| \right]$$

and

$$\begin{aligned} \nabla_{\omega,\omega'}^n &= \sum_{j,k=1}^n \left| e_{\omega_j}^n \otimes e_{\omega'_k}^n \right\rangle \left[ D_n(\omega_{j-1}, \omega'_{k-1}) \left\langle e_{\omega_j}^n \otimes e_{\omega'_k}^n \right| - D_n(\omega_j, \omega'_k) \left\langle e_{\omega_{j-1}}^n \otimes e_{\omega'_{k-1}}^n \right| \right] \end{aligned}$$

It follows from Theorem 4.2 that

$$\begin{aligned} \mathcal{R}_{\omega,\omega'}^n &= \nabla_{\omega,\omega'}^n - \nabla_{\omega',\omega}^n \\ &= \sum_{j,k=1}^n \left[ D_n(\omega_{j-1}, \omega'_{k-1}) \left| e_{\omega_j}^n \otimes e_{\omega'_{k-1}}^n \right\rangle \left\langle e_{\omega_j}^n \otimes e_{\omega'_k}^n \right| \right. \\ &\quad \left. - D(\omega'_{j-1}, \omega_{k-1}) \left| e_{\omega'_j}^n \otimes e_{\omega_k}^n \right\rangle \left\langle e_{\omega'_j}^n \otimes e_{\omega_k}^n \right| \right] \\ &\quad + \sum_{j,k=1}^n \left[ D_n(\omega'_k, \omega_j) \left| e_{\omega'_j}^n \otimes e_{\omega_k}^n \right\rangle \left\langle e_{\omega'_{j-1}}^n \otimes e_{\omega_{k-1}}^n \right| \right. \\ &\quad \left. - D(\omega_j, \omega'_k) \left| e_{\omega_j}^n \otimes e_{\omega'_k}^n \right\rangle \left\langle e_{\omega_{j-1}}^n \otimes e_{\omega'_{k-1}}^n \right| \right] \end{aligned} \quad (4.2)$$

The matrix element representations of  $\mathcal{D}_{\omega,\omega'}^n$  and  $\mathcal{T}_{\omega,\omega'}^n$  can now be obtained from (4.2)

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