TORSORS AND TERNARY MOUFANG LOOPS ARISING IN PROJECTIVE GEOMETRY

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Abstract. We give an interpretation of the construction of torsors from [BeKi10a] in terms of classical projective geometry. For the Desarguesian case, this leads to a reformulation of certain results from [BeKi10a], whereas for the Moufang case the result is new. But even in the Desarguesian case it sheds new light on the relation between the lattice structure and the algebraic structures of a projective space.

1. The geometric construction

1.1. **The generic case.** In this first chapter, we describe the general construction of torsors and of ternary loops associated to projective spaces; proofs and computational descriptions are given in the two following chapters. We assume that $\mathcal{X}$ is a projective space of dimension at least two. For projective subspaces a, b of $\mathcal{X}$, let as usual $a \wedge b$ be the meet (intersection) and $a \vee b$ be the join (smallest subspace containing a and b).

Definition 1.1. *Consider a pair (a, b) of hyperplanes in $\mathcal{X}$ and a triple of points (x, y, z), none of them in a or b. Assume that x, y, z are not collinear. Then we define a fourth point $w := xz := (xyz)_{ab}$ as follows: w is the intersection of*

- *the parallel of the line $x \vee y$ through z in the affine space $V_a := \mathcal{X} \setminus a$, with*
- *the parallel of the line $z \vee y$ through x in the affine space $V_b = \mathcal{X} \setminus b$; that is:*

$$w = xz = (xyz)_{ab} = \Big(\big((x \vee y) \wedge a \big) \vee z \Big) \wedge \Big(\big((z \vee y) \wedge b \big) \vee x \Big) .$$

Note that this point of intersection exists since all lines belong to the projective plane spanned by x, y, z. For $a = b$, this is the usual "parallelogram definition" of vector addition in the affine space V_a with origin y, that is, $xz = x + z$ in this case. Hence, for $a \neq b$, $(xyz)_{ab}$ may be seen as a kind of "deformation of vector addition": we have a sort of "fake parallelogram" with vertices y, x, z, w, as shown in the illustrations below. As for "usual" parallelograms, it is easily seen that, with $(xyz) := (xyz)_{ab}$ for fixed (a, b), the conditions

$$w = (xyz)\,, \qquad y = (zwx)\,, \qquad z = (yxw)\,, \qquad x = (wzy) \tag{1.1}$$

are all equivalent. Note also that it is obvious from the definition that

$$(xyz)_{ba} = (zyx)_{ab}\,. \tag{1.2}$$

If we represent a and b by affine lines, intersecting in the affine drawing plane, the construction is visualized like this:

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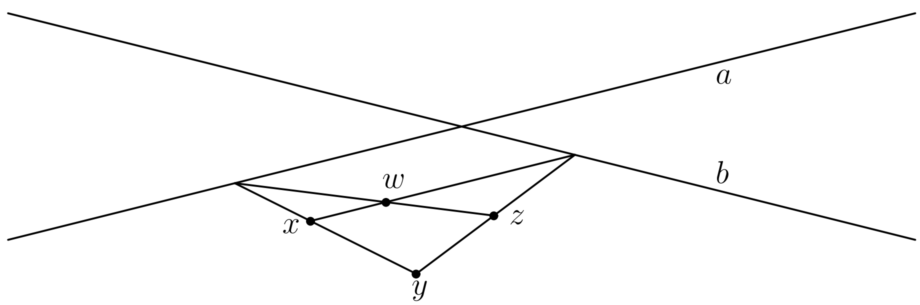
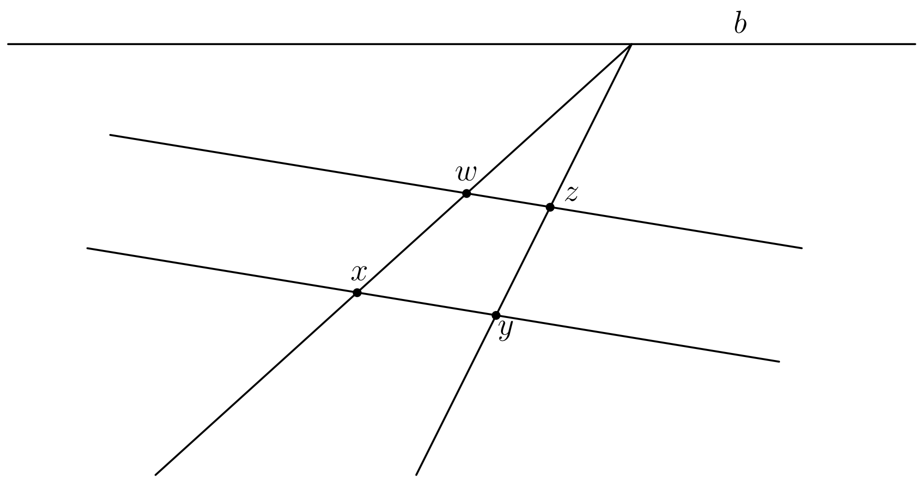


If we choose a as "line at infinity" of our drawing plane, and if we choose to draw b horizontally, then we get the following image:



These images admit a spacial interpretation: we may imagine the observer placed in affine space $\mathbb{R}^3$ inside a plane B which is vizualized only by its "horizon", the line b; then we think of the line $y \vee z$ as lying in a plane B' parallel to B, and of the line $x \vee w$ as lying in another such plane B''; the other two lines $w \vee z$ and $x \vee y$ lie in planes that are parallel to the drawing plane P. This interpretation is not symmetric in x and z: the point z lies "behind" (or "in front of") y, whereas x is considered to be "on the same level" as y.

The product xz is thus in general not commutative, but it is *associative*: we show that, if $\mathcal{X}$ is Desarguesian, then, for any fixed origin y, the binary map $(x, z) \mapsto xz$ gives rise to a *group law* on the intersection of affine parts

$$U_{ab} := \mathcal{X} \setminus (a \cup b) = V_a \cap V_b \; . \tag{1.3}$$

More generally and more conceptually, we show that the ternary law $(x, y, z) \mapsto (xyz)_{ab}$ defines a *torsor structure on* U_{ab} (Theorem 2.1). Naturally, the question arises what we can say for general, non-Desarguesian projective planes, or for still more general lattices.

The most prominent class of non-Desarguesian projective planes are the *Moufang planes*: we show that in this case we get a kind of "alternative version of a torsor" which we call a *ternary Moufang loop* (Theorem 3.1). For $a = b$, these ternary Moufang loops contract to the abelian vector group of an affine plane. For very general projective planes (which need not be "translation planes") it remains an interesting open problem to relate this new algebraic structure to those traditionally considered in the literature: indeed, our definition is closely related to the more traditional ways of coordinatizing projective planes by *ternary rings*. This is related to the following item.

1.2. **The collinear case.** We have not yet defined what $(xyz)_{ab}$ should mean if x, y, z are *collinear*. If $\mathcal{X}$ is a *topological* projective plane, then one would like to complete our definition simply "by continuity", *e.g.*, by taking the limit of $(xyz)_{ab}$ as y, not lying on the line $x \vee z$, converges to a point on $x \vee z$. This is indeed what happens in the classical planes over the division algebras $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. Since we do not know whether in very general cases such a "limit" exists, we restrict ourselves here to the Moufang case, and leave the general case for later work.

Definition 1.2. *Assume that $\mathcal{X}$ is a Moufang plane or a projective space of dimension bigger than 2. Consider a pair (a, b) of hyperplanes and a* collinear *triple (x, y, z) of points, none of them in a or b.*

(1) If $x = y = z$, let $(xyz)_{ab} := x$.

(2) If $x \neq y$, then let us choose a point u not belonging to $x \vee y$ or to a, and we let $w := (xyz)_{ab} :=$

$$(x \vee y) \wedge \Big[\Big((z \vee u) \wedge b \Big) \vee \Big(\Big[\big((x \vee y) \wedge a \big) \vee u \Big] \wedge \Big[\big((u \vee y) \wedge b \big) \vee x \Big] \Big) \Big] .$$

(It will be shown below that w does not depend on the choice of u.)

(3) If $z \neq y$, then we let

$$w := (xyz)_{ab} := (zyx)_{ba} ,$$

where the right hand side is defined by the preceding case.

This definition can be interpreted from two different viewpoints:

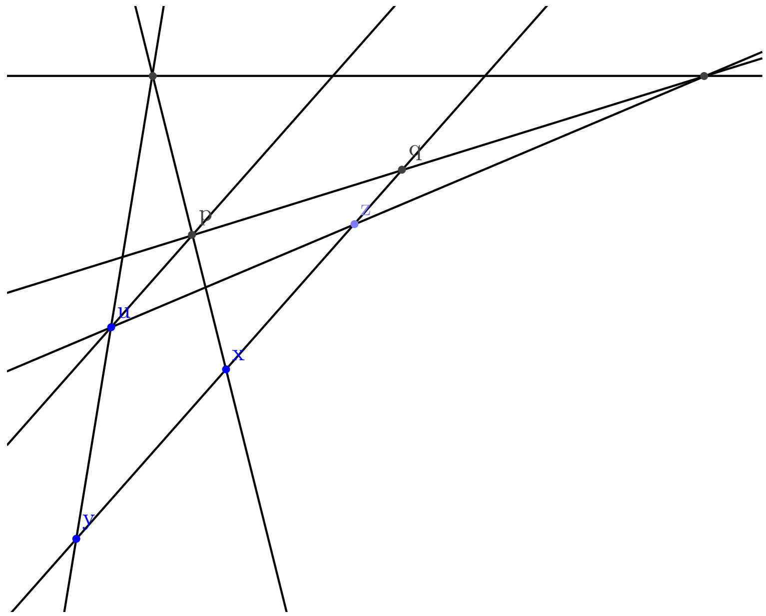
(A) *Algebraic.* In the Desarguesian case, the expression in (2) is derived from our first definition by using para-associativity and idempotency:

$$((xyu)uz) = (xy(uuz)) = (xyz) ,$$

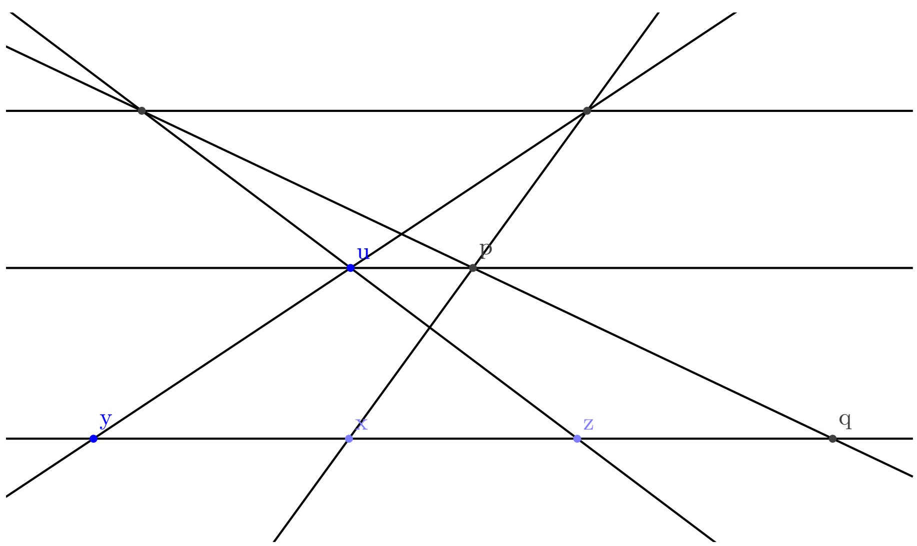
where now the left hand side can be expressed by using twice Definition 1.1 (see Theorem 2.3 below). This is indeed in keeping with idea explained above of "taking a limit" (imagine u tending towards a point on the line L). The argument still goes through in the Moufang case since one does not need for it full para-associativity, but just a special case which remains valid precisely in the Moufang case (but it breaks down as soon as one wants to go further).

(B) *Geometric.* The formula in (2) corresponds to classical "constructions of the field associated to a plane". It is known that in the Moufang case the field does not depend on the "off-line" point u. More specifically, we distinguish two cases in (2):

- generic case: if the points $L \wedge a$ and $L \wedge b$ are different, then $(xyz)_{ab}$ is the product $zy^{-1}x$ on the vector line L with "point at infinity" $L \wedge a$ and "zero point" $L \wedge b$ (in the following illustration, a is the line at infinity and b the horizontal line; in usual textbook drawings, the inverse choice is made. We have marked the points $p = (xyu)$ and $q = (puz) = w$.)



- special case: if the line $L := x \vee y$ intersects $a \wedge b$, then $(xyz)_{ab}$ is the "ternary sum" $x - y + z$ in the affine line L (with $L \wedge (a \wedge b)$ as "point at infinity", see the following illustration, which is the limit case of the preceding one, as L becomes parallell to b).



The main result of the present work can now be stated as follows:

Definition 1.3. *A set G with a map $G^3 \to G$, $(x, y, z) \mapsto (xyz)$ is called a* torsor *if*

(T0) $$(xxy) = y = (yxx)$$

(T1) $$(xy(zuv)) = (x(uzy)v) = ((xyz)uv)$$

and a ternary Moufang loop *if it satisfies* (T0) *and*

$$\text{(MT1)} \qquad (uv(xyx)) = ((uvx)yx)$$

$$\text{(MT2)} \qquad (xy(xyz)) = ((xyx)yz)$$

Theorem 1.4. *If $\mathcal{X}$ is a projective space of dimension bigger than one over a skew-field (i.e., a Desarguesian space), then the preceding constructions define a torsor law on U_{ab}. If $\mathcal{X}$ is a Moufang projective plane, then the constructions define a ternary Moufang loop.*

1.3. **Generalized cross-ratios, and associative geometries.** In the Desarguesian case, a very general theory describing torsors of the kind of U_{ab} has been developed in [BeKi10a]. Comparing with the approach presented here, one may ask for what kinds of lattices there are similar theories – we will, in subsequent work, investigate in more depth the case of Moufang spaces, related to *alternative* algebras, triple systems and pairs. Returning to the Desarguesian case and to classical projective geometry, the link between the lattice and the structure defined in [BeKi10a] is surprisingly close; however, one should not forget that for *projective lines* the lattice structure is completely useless, whereas the structures from [BeKi10a] are at least as strong as the classical *cross-ratio*, and hence are much stronger than the lattice structure. Let us briefly explain this. Given a unital ring $\mathbb{K}$ and $\mathcal{X} := \mathcal{X}(\Omega)$, the full Grassmannian geometry of some $\mathbb{K}$-module Ω (set of all submodules of Ω), we have associated in [BeKi10a] to any 5-tuple $(x, a, y, b, z) \in \mathcal{X}^5$ another element of $\mathcal{X}$ by

$$\Gamma(x,a,y,b,z) := \left\{ \omega \in \Omega \;\middle|\; \begin{array}{c} \exists \xi \in x, \exists \alpha \in a, \exists \eta \in y, \exists \beta \in b, \exists \zeta \in z : \\ \omega = \zeta + \alpha = \alpha + \eta + \beta = \xi + \beta \end{array} \right\}. \tag{1.4}$$

In [BeKi10a], Theorem 2.4, it is shown that the lattice structure is recovered via

$$x \wedge a = \Gamma(x,a,y,x,a), \qquad b \vee a = \Gamma(a,a,y,b,b) \tag{1.5}$$

for any $y \in \mathcal{X}$. On the other hand, in the present work we prove (Theorem 2.3) that, if $\mathbb{K}$ is a field, if a, b are hyperplanes and x, y, z one-dimensional subspaces, then $\Gamma(x, a, y, b, z)$ can be recovered from the lattice structure via

$$\Gamma(x,a,y,b,z) = (xyz)_{ab}\,. \tag{1.6}$$

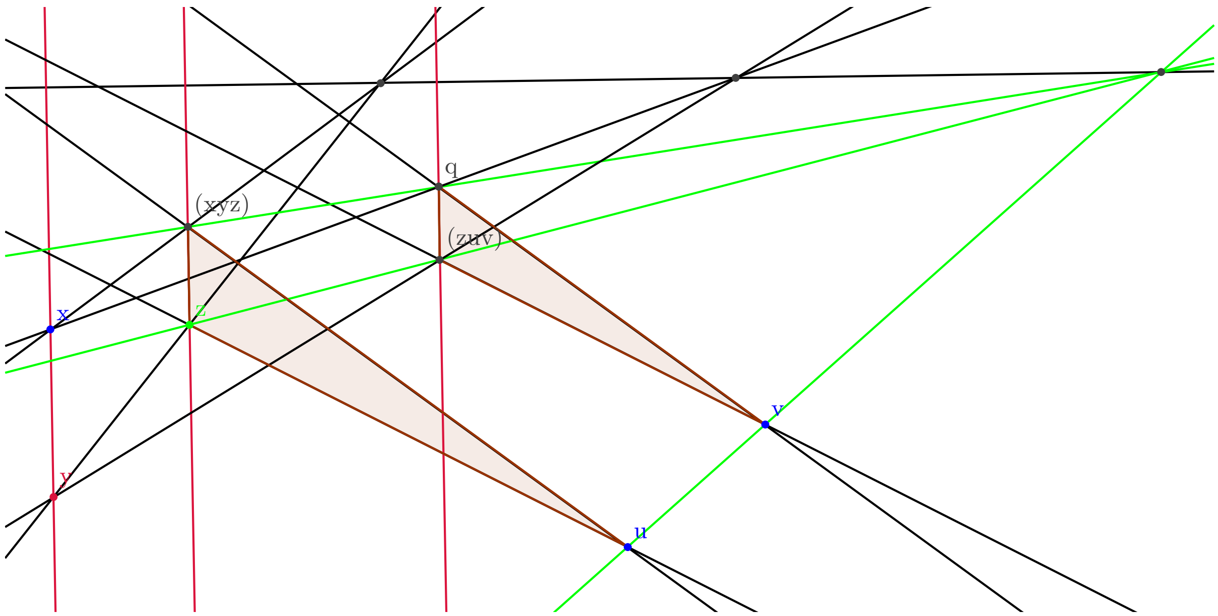
Thus, roughly speaking, for Desarguesian projective spaces of dimension bigger than one, Γ and the lattice structure are essentially equivalent data. Summing up, there are two major approaches to our object: the algebraic approach ([BeKi10a]), based on associative algebras and -pairs and on an underlying group structure of the "background" Ω (cf. [Be12]), and the lattice theoretic approach from the present work, keeping close to classical geometric language, and paving the way to incorporate exceptional geometries into the picture.

2. The Desarguesian case

Theorem 2.1. *Assume $\mathcal{X}$ is a Desarguesian projective space of dimension bigger than one, and fix a pair (a, b) of hyperplanes. Then U_{ab}, together with the ternary product $(xyz) := (xyz)_{ab}$ defined above, is a torsor. In particular, if we fix an "origin" $y \in U_{ab}$, then U_{ab} with product $xz = (xyz)_{ab}$ and origin y becomes a group. If $a \neq b$, then this is group is not commutative.*

Proof. We give three different proofs. In all cases, the main point is to prove "para-associativity" (T1).

(a) The first proof is rather a "drawing exercise": let us show, just by using Desargues' Theorem, that $((xyz)uv) = (xy(zuv))$. We construct first the point $((xyz)uv)$. This is best visualized by choosing a as line at infinity of our drawing plane, and we may draw the lines $y \vee x$ and $z \vee (xyz)$ as vertical lines. Then $((xyz)uv)$ is the point q in the following illustration:



Now, the triangles $u, z, (xyz)$ and $v, (zuv), q$ are in a Desargues configuration, and we conclude that the line $q \vee (zuv)$ is parallel to $z \vee (xyz)$, i.e., it is vertical. But then the triangles $y, z, (zuv)$ and $x, (xyz), q$ are also in Desargues configuration, *i.e.*, the intersection points of corresponding sides lie on a common line, which must be b. It follows that $(x \vee q) \wedge b = (y \vee (zuv)) \wedge b$, from which the desired equality follows.

As a next challenge, one may try to establish the remaining equality defining para-associativity in the same vein.

(b) A computational proof. Let $\mathbb{K}$ be the (skew)field of $\mathcal{X}$, and work in the affine space $V := V_a$. If $a = b$, then (as mentioned above), $(xyz)_{aa} = x - y + z$ is the torsor law of the affine space V_a, and the claim is obviously true. If $a \neq b$, fix some arbitrary origin o in the affine hyperplane $V_a \cap b$. There is a linear form $\beta : V \to \mathbb{K}$ such that $b \cap V = \ker(\beta)$, so that $U_{ab} = \{x \in V_a \mid \beta(x) \in \mathbb{K}^\times\}$.

Lemma 2.2. *For all $x, y, z \in U_{ab}$, in the vector space (V_a, o), we have*

$$(xyz)_{ab} = \beta(z)\beta(y)^{-1}(x - y) + z.$$

Proof. Assume first that x, y, z are not collinear. The parallel of $x \vee y$ through z in V_a is

$$\big((x \vee y) \wedge a\big) \vee z = \{z + s(x - y) \mid \ s \in \mathbb{K}\}.$$

We determine the point $(y \vee z) \wedge b$. If $y \vee z$ is parallel to b, then we get easily from the definition that $(xyz)_{ab} = x - y + z$ is the usual sum, which is in keeping with our claim. Assume that $y \vee z$ is not parallel to b. Then the intersection point $(y \vee z) \wedge$ is obtaining by solving

$\beta\big((1-t)y+tz\big)=0$, whence $t=\beta(y)(\beta(y)-\beta(z))^{-1}$, whence $1-t=-\beta(z)(\beta(y)-\beta(z))^{-1}$ and

$$(y \vee z) \wedge b = -\beta(z)(\beta(y)-\beta(z))^{-1}y + \beta(y)(\beta(y)-\beta(z))^{-1} \, .$$

The intersection of $\big((z\vee y)\wedge b\big)\vee x$ and $\big((x\vee y)\wedge a\big)\vee z = z+\mathbb{K}(x-y)$ is determined by $r,s\in\mathbb{K}$ such that

$$(1-r)x + r\big(-\beta(z)(\beta(y)-\beta(z))^{-1}y + \beta(y)(\beta(y)-\beta(z))^{-1}z\big) = sx - sy + z.$$

Since both sides are barycentric combinations of x,y,z, we may consider y as new origin. Then, if x and z are linearly independent with respect to this origin, this condition is equivalent to

$$1-r=s, \qquad r(\beta(y)\big(\beta(y)-\beta(z)\big)^{-1})=1$$

whence $r=\big(\beta(y)-\beta(z)\big)\beta(y)^{-1}$ and $s=1-r=\beta(z)\beta(y)^{-1}$, and finally

$$(xyz)_{ab} = s(x-y)+z = \beta(z)\beta(y)^{-1}\big(x-y\big)+z\,,$$

proving our claim in the non-collinear case.

Now consider the collinear case. As pointed out after Definition 1.2, in this case the definition of $(xyz)_{ab}$ amounts to the geometric definition of the field operations. If the line L spanned by x,y,z is parallel to b, then $\beta(z)=\beta(y)$, and the formula from the lemma gives the additive torsor law $x-y+x$, as required. Else, choose $o := L\wedge b$ as origin, let $u\in L$ with $\beta(u)=1$ and write $x=\xi u$, $y=\eta u$, $z=\zeta u$ with $\xi,\eta,\zeta\in\mathbb{K}^\times$, and then the formula from the lemma gives $(xyz)_{ab}=\zeta\eta^{-1}(\xi u-\eta u)+\zeta u=\zeta\eta^{-1}\xi u$, which again corresponds to the definition given in this case. Thus the claim holds in all cases. □

Using the lemma, we now prove the torsor laws: first of all, we have

$$\beta\big((xyz)\big) = \beta\Big(\beta(z)\beta(y)^{-1}\big(x-y\big)+z\Big) = \beta(z)\beta(y)^{-1}\beta(x)\ , \tag{2.1}$$

showing that $U_{ab}=V_a\backslash\ker(\beta)$ is stable under the ternary law. The idempotent laws follow by an easy computation from the lemma. For para-associativity, using (2.1), a straightforward computation shows that both $((xyz)uv)$ and $(x(uzy)v)$ are given by

$$\beta(v)\beta(u)^{-1}\beta(z)\beta(y)^{-1}(x-y)+\beta(v)\beta(u)^{-1}(u-z)+v.$$

(b') Remark: there is a slightly different version of (b), having the advantage that the cases $a=b$ and $a\neq b$ can be treated simultaneously, and the drawback that the dependence on y is not visible: choose $o:=y$ as origin in $V=V_a$, and a linear form $\beta : V\to\mathbb{K}$ such that $b\cap V=\{x\in V \mid \beta(x)=1\}$. The case $a=b$ then corresponds to $\beta=0$. A computation similar as above yields

$$xz = (xyz)_{ab} = (1-\beta(z))x+z = x-\beta(z)x+z \tag{2.2}$$

from which associativity of the product xz follows easily. Note that Formula (2.2) is a special case of the formulae given in Section 1.4 of [BeKi10a].

(c) A third proof follows from Theorem 2.3 in [BeKi10a], combined with the following general result: □

Theorem 2.3. *Let $\mathbb{K}$ be a unital ring and $\mathcal{X}=\mathcal{X}(\Omega)$ be the full Grassmannian geometry of some $\mathbb{K}$-module Ω (set of all submodules of Ω), and define, for a 5-tuple $(x,a,y,b,z)\in\mathcal{X}^5$, the submodule $\Gamma(x,a,y,b,z)$ by Equation (1.4).*

(1) Assume that the triple (x,y,z) *is in* general position*, that is,*

$$x \wedge (y \vee z) = 0, \quad \text{or} \quad y \wedge (x \vee z) = 0, \quad \text{or} \quad z \wedge (x \vee y) = 0.$$

Then we have the following equality of submodules of Ω*:*

$$\Gamma(x,a,y,b,z) = \Big(\big((x \vee y) \wedge a \big) \vee z \Big) \wedge \Big(\big((z \vee y) \wedge b \big) \vee x \Big) \, .$$

(2) Assume that z *is contained in* $x \vee y$*, i.e.,* $z \wedge (x \vee y) = z$*. Then, for any choice of* $u \in U_{ab}$ *satisfying* $u \wedge (x \vee y) = 0$*, we have*

$$\begin{aligned}\Gamma(x,a,y,b,z) \quad &= \quad ([([\Big(\big((x \vee y) \wedge a \big) \vee u \Big) \wedge \Big(\big((u \vee y) \wedge b \big) \vee x \Big)] \vee u) \wedge a] \vee z) \\ &\wedge \quad [((z \vee u) \wedge b) \vee \Big(\big((x \vee y) \wedge a \big) \vee u \Big) \wedge \Big(\big((u \vee y) \wedge b \big) \vee x \Big)]\end{aligned}$$

(3) Let a, b *be hyperplanes in a vector space and* x, y, z *lines. Retain assumptions from the preceding item and assume that* $x \neq y$*. Then the expression given there simplifies to*

$$\Gamma(x,a,y,b,z) \quad = \quad (x \vee y) \wedge [((z \vee u) \wedge b) \vee \Big(\big((x \vee y) \wedge a \big) \vee u \Big) \wedge \Big(\big((u \vee y) \wedge b \big) \vee x \Big)] \, .$$

Proof. (1) We prove first the inclusion "$\subset$" (which holds in fact for all triples (x,y,z)): on the one hand, $\Big(\big((x \vee y) \wedge a \big) \vee z \Big) \wedge \Big(\big((z \vee y) \wedge b \big) \vee x \Big)$ is the set of all $\omega \in \Omega$ such that we can write

$$\omega = \alpha + \zeta, \qquad \omega = \beta + \xi$$

with $\alpha \in a$, $\beta \in b$, which in turn can be written

$$\alpha = \xi' + \eta \, , \qquad \beta = \zeta' + \eta'$$

with $\xi' \in x$, etc. This gives us a system (S) of 4 equations.

On the other hand, by definition, $\Gamma(x,a,y,b,z)$ is the set of all $\omega \in \Omega$ such that

$$\exists \xi \in x, \exists \alpha \in a, \exists \eta \in y, \exists \beta \in b, \exists \zeta \in z : \quad \omega = \zeta + \alpha = \alpha + \eta + \beta = \xi + \beta$$

There are several equivalent versions of this system (R) of three equations – see [Be12], Lemma 2.3., from which it is read off that the four conditions from (S) are satisfied for $\omega \in \Gamma(x,a,y,b,z)$ if we choose $\xi' = \xi$, $\eta' = \eta$, $\zeta' = \zeta$. Thus the inclusion "$\subset$" holds always.

The other inclusion does not always hold, but the theorem gives a sufficient condition: indeed, if ω belongs to the set on the right hand side, then (S) implies

$$\omega = \xi' + \eta + \zeta = \zeta' + \eta' + \xi \, ,$$

whence $\xi - \xi' \in y \vee z$. If $x \wedge (y \vee z) = 0$, this implies that $\xi = \xi'$, and three of the four equations from (S) are equivalent to (R). If $y \wedge (x \vee z) = 0$ or $z \wedge (x \vee y) = 0$, then the same argument applies (with respect to another choice of three from the four equations of (S)). In all cases, it follows that $\omega \in \Gamma(x,a,y,b,z)$.

(2) From [BeKi10a], Theorem 2.3, we know that Γ is para-associative and satisfies the idempotent law:

$$\Gamma\big(\Gamma(x,a,y,b,u),a,u,b,z\big) = \Gamma(x,a,y,b,\Gamma(u,a,u,b,z)) = \Gamma(x,a,y,b,z) \, .$$

By assumption, the triple (x,y,u) is in general position, and from this it follows that the triple $((xyu),u,z)$ is also in general position; therefore the left-hand side may be expressed in terms of the lattice structure by applying twice part (1), which leads to the expression from the claim: in a first step, we get

$$(xyu) = \Big(\big((x \vee y) \wedge a \big) \vee u \Big) \wedge \Big(\big((u \vee y) \wedge b \big) \vee x \Big) ,$$

and in a second step

$$((xyu)uz) = (([(\Big(\big((x \vee y) \wedge a \big) \vee u \Big) \wedge \Big(\big((u \vee y) \wedge b \big) \vee x \Big)] \vee u) \wedge a] \vee z)$$
$$\wedge \ \ [((z \vee u) \wedge b) \vee \Big(\big((x \vee y) \wedge a \big) \vee u \Big) \wedge \Big(\big((u \vee y) \wedge b \big) \vee x \Big)] .$$

(3) Under the given assumptions, the first term on the right hand side in (2) reduces to the line $x \vee y$, and hence the claim follows directly from (2). $\square$

Remarks. (a) Not all possible relative positions of (x, y, z) are covered by Theorem 2.3, that is, the lattice theoretic formula for $\Gamma(x, a, y, b, z)$ does not hold for all triples of submodules of Ω. For instance, if $\Omega = \mathbb{K}^{2n}$ and x, y, z are of dimension n, then they cannot be in general position, and in general no u as in (2) exists. This case illustrates the special rôle of "generalized projective lines" (cf. [BeKi10a]) with respect to lattice approaches.

(b) Both for the definitions given here and in [BeKi10a], it is not strictly necessary that x, y, z belong to U_{ab}: they may belong to V_a, or to V_b, or (in [BeKi10a]) be completely arbitrary. We will not enter here into a discussion of the relation of both definitions if x, y or z does not belong to U_{ab}.

(c) Both approaches lead to their own notions of *morphisms*. In the situation of Part (3) of Theorem 2.3, both of these notions must lead to the same result: this is precisely the famous "second fundamental theorem of projective geometry".

3. The Moufang case

Theorem 3.1. *Assume $\mathcal{X}$ is a Moufang projective plane and (a, b) a pair of lines. Then U_{ab}, together with the ternary product $(xyz)_{ab}$ defined in the first section, is a* ternary Moufang loop. *In particular, if we fix an element $y \in U_{ab}$ as origin, then U_{ab} with product $xz = (xyz)_{ab}$ and origin y becomes a (binary) Moufang loop.*

Before proving the theorem, we recall the relevant definitions (cf., e.g., [SB+]):

Definition 3.2. *A projective plane $\mathcal{X}$ is a* Moufang plane *if it satisfies one of the following equivalent conditions*

(1) The group of automorphisms fixing all points of any given line acts transitively on the points not on the line.
(2) The group of automorphisms acts transitively on quadrangles.
(3) Any two ternary rings of the plane are isomorphic.
(4) Some ternary ring of the plane is an alternative division algebra, i.e.*, it is a division algebra satisfying the following identities:*

$$x(xy) = (xx)y , \qquad (yx)x = y(xx) , \qquad (xy)x = x(yx) .$$

(5) $\mathcal{X}$ is isomorphic to the projective plane over an alternative division ring.
(6) The "small Desargues theorem" holds in all affine parts of $\mathcal{X}$.

The set of invertible elements in alternative algebra forms a *Moufang loop*. A basic reference for loops in general and Moufang loops in particular is [Bru58].

Definition 3.3. *A* loop $(Q, \cdot)$ *is a set* Q *with a binary operation* $Q^2 \to Q; (x, y) \mapsto xy$ *such that for each* x*, the maps* $y \mapsto xy$ *and* $y \mapsto yx$ *are bijections of* Q*, and having an element* e *such that* $ex = xe = x$ *for all* $x \in Q$*.*

A Moufang loop *is a loop* Q *that satisfies any, and hence all of the following equivalent identities (the* Moufang identities*):*

$$z(x(zy)) = ((zx)z)y \tag{M1}$$

$$x(z(yz)) = ((xz)y)z \tag{M2}$$

$$(zx)(yz) = (z(xy))z \tag{N1}$$

$$(zx)(yz) = z((xy)z) \tag{N2}$$

The *left* and *right multiplication maps* (sometimes called translations) in a loop are defined, respectively by $L_x y := xy =: R_y x$. The Moufang identities can be written in terms of the left and right multiplication maps. For instance, the first two identities state that

$$L_z L_x L_z = L_{zxz} \text{ and } \qquad R_z R_y R_z = R_{zyz} .$$

Moufang's Theorem implies that Moufang loops are *diassociative*, that is, for any a, b, the subloop $\langle a, b\rangle$ generated by a, b is a group. This can be seen as a loop theoretic analog of Artin's Theorem for alternative algebras. Two particular instances of diassociativity are the *left* and *right inverse properties*

$$x^{-1}(xy) = y \tag{LIP}$$

$$(xy)y^{-1} = x \, , \tag{RIP}$$

where x^{-1} is the unique element satisfying $xx^{-1} = x^{-1}x = e$. The following lemma gives the Moufang analog of the well-known relation between torsors and groups:

Lemma 3.4. *Let* Q *be a Moufang loop, and define a ternary operation* $(\cdot\cdot\cdot) : Q^3 \to Q$ *by* $(xyz) := (xy^{-1})z$*. Then the following three identities hold:*

$$(xxy) = y = (yxx) \tag{MT0}$$

$$(uv(xyx)) = ((uvx)yx) \tag{MT1}$$

$$(xy(xyz)) = ((xyx)yz) \tag{MT2}$$

Conversely, if M *is a set with a ternary operation* $(\cdots) : M^3 \to M$ *satisfying (MT0), (MT1) and (MT2), then, for every choice of "origin"* $e \in M$*, the binary operation* $x \cdot y := (xey)$ *and the unary operation* $x^{-1} := (exe)$ *define the structure of a Moufang loop on* M *with neutral element* e*.*

Proof. Firstly assume Q is a Moufang loop. The leftmost identity in (MT0) is trivial while the rightmost follows immediately from (RIP). For (MT1), we compute

$$\begin{aligned}
(uv(xyx)) &= (uv^{-1})((xy^{-1})x) \\
&= (uv^{-1})(x(y^{-1}x)) && (\langle x, y\rangle \text{ is a group}) \\
&= (((uv^{-1})x)y^{-1})x && (M2) \\
&= ((uvx)yx) \, .
\end{aligned}$$

For (MT2),

$$\begin{aligned}
(xy(xyz)) &= (xy^{-1})((xy^{-1})z) \\
&= ((xy^{-1})(xy^{-1}))z && (\langle xy^{-1}, z\rangle \text{ is a group}) \\
&= (((xy^{-1})x)y^{-1})z && (\langle x, y\rangle \text{ is a group}) \\
&= (((xyx)y)z)\,.
\end{aligned}$$

Conversely, suppose M is a set with a ternary operation $(\cdots) : M^3 \to M$ satisfying (MT0), (MT1) and (MT2). Fix $e \in M$ and define $x \cdot y := (xey)$ and $x^{-1} := (exe)$ for all $x, y \in M$. By (MT0), we see that e is neutral element for the binary operation.

First we establish the following identities:

$$x \cdot y^{-1} = (xye)\,, \tag{3.1}$$

$$(x \cdot y^{-1}) \cdot z^{-1} = (xyz^{-1})\,, \tag{3.2}$$

$$(x^{-1})^{-1} \cdot x = e\,, \tag{3.3}$$

$$((x^{-1})^{-1}xy^{-1}) = y\,. \tag{3.4}$$

For (3.1) we compute $x \cdot y^{-1} = (xe(eye)) = ((xee)ye) = (xye)$ using (MT1) in the second equality and (MT0) in the third. For (3.2), we have $(x \cdot y^{-1}) \cdot z^{-1} = ((xye)ze) = (xy(eze)) = (xyz^{-1})$, using (3.1) (twice) and (MT1). For (3.3), $(x^{-1})^{-1} \cdot x = ((x^{-1})^{-1}xe) = ((ex^{-1}e)xe) = (ex^{-1}(exe)) = (ex^{-1}x^{-1}) = e$, using (3.1) in the first equality, (MT1) in the third and (MT0) in the fourth. Finally, for (3.4), $((x^{-1})^{-1}xy^{-1}) = ((x^{-1})^{-1} \cdot x^{-1}) \cdot y^{-1} = e \cdot y^{-1} = y^{-1}$, using (3.2) followed by (3.3).

Next we prove

$$(xy(y^{-1})^{-1}) = x\,. \tag{3.5}$$

Indeed,

$$\begin{aligned}
(xy(y^{-1})^{-1}) &= ((x(y^{-1})^{-1}(y^{-1})^{-1})y(y^{-1})^{-1}) = (x(y^{-1})^{-1}((y^{-1})^{-1}y(y^{-1})^{-1})) \\
&= (x(y^{-1})^{-1}(y^{-1})y^{-1}) \qquad\qquad = x\,,
\end{aligned}$$

where we have used (MT0), (MT1), (3.4) and (MT0).

Taking $y = x$ in (3.5) and applying (MT0), we obtain

$$(x^{-1})^{-1} = x\,. \tag{3.6}$$

From this it follows that $e^{-1} = e$, since $e^{-1} = e \cdot e^{-1} = (e^{-1})^{-1} \cdot e^{-1} = e$.

Now in (3.2), replace z with z^{-1} and use (3.6) to obtain

$$(x \cdot y^{-1}) \cdot z = (xyz)\,. \tag{3.7}$$

Replacing y with y^{-1} and then setting $z = y^{-1}$ in (3.7), we obtain $(x \cdot y) \cdot y^{-1} = (xy^{-1}y^{-1}) = x$ using (MT0). Thus the right inverse property (RIP) holds.

Next we almost obtain the Moufang identity (M2) as follows:

$$\begin{aligned}((x \cdot y) \cdot z) \cdot y &= (((x \cdot e) \cdot y) \cdot z) \cdot y \\ &= ((xey)z^{-1}y) \\ &= (xe(yz^{-1}y)) \\ &= (x \cdot e) \cdot ((y \cdot z) \cdot y)\,,\end{aligned}$$

using (3.7), (MT1) and (3.7) again. In loop theory, this is known as the *right Bol identity.*

We also have the left alternative law:

$$\begin{aligned}(x \cdot x) \cdot y &= (((x \cdot e) \cdot x) \cdot e) \cdot y \\ &= ((xex)ez) \\ &= (xe(xez)) \\ &= (x \cdot e) \cdot ((x \cdot e) \cdot y) \\ &= x \cdot (x \cdot y)\,,\end{aligned}$$

using (3.7), (MT2) and (3.7) again.

The rest of the argument is standard. A magma satisfying the right Bol identity and (RIP) is a loop, called a *right Bol loop* (see, *e.g.*, [Kie02], Theorem 3.11, suitably dualized). A right Bol loop satisfying the left alternative law is a Moufang loop [Rob66]. □

Definition 3.5. *A set M with a map $(\cdot\cdot\cdot) : M^3 \to M$ satisfying the three identities from Lemma 3.4 will be called a* ternary Moufang loop*.*

Remarks. (1) The axioms (MT1) and (MT2) for ternary Moufang loops are precisely the identities (AP2) and (AP3) in Loos' axiomatization of an alternative pair [Lo75].

(2) For an associative torsor $(\cdot\cdot\cdot) : M^3 \to M$, the groups determined by different choices of "origin," that is, fixed middle slot, are all isomorphic. The analog of this does not hold for ternary Moufang loops. Instead the different Moufang loops are *isotopic* [Bru58]. In fact, it is straightforward to show that for a Moufang loop Q, each isotope of Q is isomorphic to an isotope with multiplication given by $x \circ z = (xy^{-1})z$ for some $y \in Q$. Thus just as alternative triple systems encode all homotopes of an alternative algebra into a single structure, so do ternary Moufang loops encode all isotopes of Moufang loops.

(3) Though we did not bother to state this in the lemma, it is clear from the proof that if we start with a Moufang loop Q with neutral element e, construct the corresponding ternary operation $(\cdot\cdot\cdot)$ and then construct the binary and unary operations induced by $(\cdot\cdot\cdot)$ with origin e, we recover the original loop operations. Similarly, if we start with a ternary Moufang loop M, construct the binary and unary operations with origin e and then construct the corresponding ternary operation induced by the loop structure, we recover the original ternary Moufang loop.

Proof of Theorem 3.1. In principle, the first two strategies of proof of Theorem 2.1 carry over:

(a) A proof in the framework of axiomatic geometry. Instead of the full Desargues theorem we now can only use the Little Desargues theorem. The drawings will become more complicated than above since one has to introduce auxiliary points. We will not pursue this proof here.

(b) A computational proof. Let $\mathbb{K}$ be the alternative division ring belonging to the plane. Then the affine space $V := V_a$ is isomorphic to $\mathbb{K}^2$, and affine lines can be described as in the Desarguesian case, eg. $x \vee y = \{(1-t)x + ty \mid t \in \mathbb{K}\}$, where multiplication by "scalars" in $\mathbb{K}^2$ is componentwise. If $a = b$, then $(xyz)_{aa} = x - y + z$ is the torsor law of the abelian group $V_a \cong (\mathbb{K}^2, +)$, and the claim is true. If $a \neq b$, fix some arbitrary origin o in the affine hyperplane $V_a \cap b$. There is a linear form $\beta : V \to \mathbb{K}$ such that $b \cap V = \ker(\beta)$, so that $U_{ab} = \{x \in V_a \mid \beta(x) \in \mathbb{K}^\times\}$. (To fix things, one may choose coordinates such that $\beta = \mathrm{pr}_1$ is the projection onto the first coordinate of $\mathbb{K}^2$, so b is the vertical axis.)

Lemma 3.6. *Let notation be as above. Then, for all $x, y, z \in U_{ab}$, we have*

$$(xyz)_{ab} = (\beta(z)\beta(y)^{-1}) \cdot (x - y) + z.$$

Proof. The proof of Lemma 2.2 carries over without any changes – associativity of the ring has not been used there, only some elementary properties of inverses which are direct consequences of the left and right inverse properties (LIP) and (RIP). ☐

From the lemma we get, as before, the formula

$$\beta\big((xyz)\big) = [\beta(z)\beta(y)^{-1}]\beta(x) = \big(\beta(z)\beta(y)\beta(x)\big) \tag{3.8}$$

which means that β induces a homomorphism from U_{ab} to the ternary Moufang loop $\mathbb{K}^\times$.

This formula is crucial in the proof of the alternative laws of U_{ab}: essentially, it implies that identities holding in $\mathbb{K}^\times$ will carry over to U_{ab}; but the unit loop of $\mathbb{K}$ is a ternary Moufang loop, and hence so will be U_{ab}. For instance, for the proof of (MT1), $(uv(xyx)) = ((uvx)yx)$, write both sides, using the lemma: one sees that equality holds iff, for the vector $w := u - v \in \mathbb{K}^2$ and for all $x, y, v \in \mathbb{K}^2 \setminus \ker(\beta)$, we have

$$(((\beta(x)\beta(y)^{-1})\beta(x))\beta(v)^{-1})w = (\beta(x)\beta(y)^{-1})((\beta(x)\beta(v)^{-1})w)$$

But this amounts to an identity in $\mathbb{K}$ (or, if one prefers, two identities, one for each component of w), of the same form as the one we want to prove; this identity holds since $\mathbb{K}$ is an alternative algebra. ☐

4. Prospects

In subsequent work, we will investigate more thoroughly the geometry corresponding to alternative algebras and alternative pairs (cf. [Lo75]): "alternative geometries" correspond to such algebras in a similar way as the associative geometries from [BeKi10a] correspond to associative algebras and associative pairs. They play a key rôle in the construction of exceptional spaces corresponding to Jordan algebas and Jordan pairs. In the following, we briefly mention some topics to be discussed in this context.

4.1. **Structure of the torsors and ternary Moufang loops.** First of all, it is easy to understand the structure of the groups U_{ab} in the Desarguesian case: for $a = b$, $U_{ab} = V_a$ is a vector group (this is true even in the Moufang case), and for $a \neq b$, U_{ab} is isomorphic to the *dilation* or *$ax + b$-group*

$$\mathrm{Dil}(E) := \{f : E \to E \mid f(x) = ax + b,\ b \in E, a \in \mathbb{K}^\times\} \tag{4.1}$$

of the affine space $E = a_b = a \setminus b$ (where $a \cap b$ is considered as hyperplane at infinity of a). This dilation group, in turn, is a semidirect product of $\mathbb{K}^\times$ with the translation group of E. The resulting homomorphism $U_{ab} \to \mathbb{K}^\times$ can be described in a purely geometric way (cf. [Be12], Theorem 7.4 for the case of very general Grassmannians). For Moufang planes, partial analogs of this hold: there is a split exact sequence of ternary Moufang loops

$$a_b \quad \to \quad U_{ab} \quad \to \quad \mathbb{K}^\times ,$$

where any line L in $\mathcal{X}$ which intersects $a \cup b$ in exactly two different points provides a splitting. But, if the plane is not Desarguesian, the set $\mathrm{Dil}(E)$ defined by (4.1) is then no longer a group, nor is it contained in the automorphism group of the plane. However, it remains true in the Moufang case that one obtains a *symmetric plane* (defined in [Lö79]; in the rough classification of symmetric planes by H. Löwe [Löwe01], our spaces appear among the *split symmetric planes*.)

4.2. **Duality.** Carrying out our geometric construction from Chapter 1 in the *dual projective space*, by general duality principles of projective geometry, we get again torsors, respectively ternary Moufang loops. Remarkably, the description of the torsors in the Desarguesian case by equation (1.4) does not change, except for a switch in a and b. In other words, up to this switch, the map Γ is "self-dual", which is in keeping with results on anti-automorphisms from [BeKi10b]. For the moment, it is an open problem whether a similar "self-dual description" exists also in the Moufang case.

4.3. **General projective planes.** Our definition of $(xyz)_{ab}$ in the generic case (Definition 1.1) makes sense for any projective plane (and even for any lattice if we admit 0 as possible result). What, then, are its properties? In particular, what is its relation with the "ternary field" associated to a quadruple of points in the plane? Put differently, how do we have to modify the definition in the collinear case (Definition 1.2)? Does the "split exact sequence" $a_b \to U_{ab} \to \mathbb{K}^\times$ survive in some suitable algebraic category? Can one re-interprete the classical Lenz-Barlotti types of projective planes (cf., e.g., [SB+], p. 142) in terms of $(xyz)_{ab}$?

4.4. **Perspective drawing.** Our construction also has aspects that should be interesting for applied sciences: as already pointed out, our two-dimensional drawings have a "spacial interpretation". This can be explained by observing that the torsors U_{AB} living in a three-dimensional space $\mathbb{KP}^3$ can be mapped homomorphically onto torsors U_{ab} living in a projective plane P (by choosing $P \subset \mathbb{KP}^3$ intersecting $A \wedge B$ in a single point and projecting from a point $q \in A \wedge B$, $q \notin P$, onto P; then let $a := P \wedge A$ and $b := P \wedge B$). A careful look shows that the torsor structure thus represented on P is quite often implicitly used in two-dimensional "perspective representations" of three-dimensional space; however, to our knowledge, the underlying algebraic structure has so far not yet been clearly recognized.

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