Multiplication groups and inner mapping groups of Cayley–Dickson loops

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Abstract

The Cayley–Dickson loop Q_n is the multiplicative closure of basic elements of the algebra constructed by n applications of the Cayley–Dickson doubling process (the first few examples of such algebras are real numbers, complex numbers, quaternions, octonions, sedenions). We establish that the inner mapping group $Inn(Q_n)$ is an elementary abelian 2-group of order 2^{2^n-2} and describe the multiplication group $Mlt(Q_n)$ as a semidirect product of $Inn(Q_n) \times \mathbb{Z}_2$ and an elementary abelian 2-group of order 2^n. We prove that one-sided inner mapping groups $Inn_l(Q_n)$ and $Inn_r(Q_n)$ are equal, elementary abelian 2-groups of order $2^{2^{n-1}-1}$. We establish that one-sided multiplication groups $Mlt_l(Q_n)$ and $Mlt_r(Q_n)$ are isomorphic, and show that $Mlt_l(Q_n)$ is a semidirect product of $Inn_l(Q_n) \times \mathbb{Z}_2$ and an elementary abelian 2-group of order 2^n.

1 Introduction

1.1 Cayley–Dickson loops

The *Cayley–Dickson doubling process* produces a sequence of power-associative algebras over a field F (see [15]). Let $\mathbb{A}_0 = F$ with conjugation $a^* = a$ for all $a \in F$. Let $\mathbb{A}_{n+1} = \{(a,b) \mid a,b \in A_n\}$ for $n \in \mathbb{N}$, where multiplication, addition, and conjugation are defined as follows:

$$\begin{aligned} (a,b)(c,d) &= (ac - d^*b, da + bc^*), \\ (a,b) + (c,d) &= (a+c, b+d), \\ (a,b)^* &= (a^*, -b). \end{aligned}$$

The dimension of $\mathbb{A}_n$ over F is 2^n.

We consider multiplicative structures that arise from the Cayley–Dickson process. A *loop* is a nonempty set Q with a binary operation such that there is a neutral element $1 \in Q$ satisfying $1 \cdot x = x \cdot 1 = x$ for all $x \in Q$, and for every $a, b \in Q$ there is a unique x and a unique y satisfying $x \cdot a = b$, $a \cdot y = b$. Define *Cayley–Dickson loops* $(Q_n, \cdot)$ over F inductively as follows:

$$Q_0 = \{1, -1\}, \quad Q_n = \{(x,0), (x,1) \mid x \in Q_{n-1}\},$$

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with multiplication

$$\begin{aligned}(x,0)(y,0) &= (xy,0),\\ (x,0)(y,1) &= (yx,1),\\ (x,1)(y,0) &= (xy^*,1),\\ (x,1)(y,1) &= (-y^*x,0),\end{aligned}$$

and conjugation

$$\begin{aligned}(x,0)^* &= (x^*,0),\\ (x,1)^* &= (-x,1).\end{aligned}$$

Note that the Cayley–Dickson loops are independent of the underlying field F of characteristic not two. The reader can assume $F = \mathbb{R}$ without loss of generality from now on. The first few Cayley–Dickson loops are the real group $\mathbb{R}_2$ (abelian); the complex group $\mathbb{C}_4$ (abelian); the quaternion group $\mathbb{H}_8$ (not abelian); the octonion loop $\mathbb{O}_{16}$ (Moufang); the sedenion loop $\mathbb{S}_{32}$ (not Moufang); the loop $\mathbb{T}_{64}$. Note that the Cayley–Dickson loops are not associative after dimension 8. The study of basic elements provides information about the underlying algebra, and is of interest, for example, in Lie theory, graph theory, quantum physics, functional analysis (see [1], [11]).

The order of Q_n is 2^{n+1}. The loop Q_n embeds into Q_{n+1} by $x \mapsto (x,0)$, so that

$$Q_n \cong \{(x,0) \mid (x,0) \in Q_{n+1}\}.$$

All elements of Q_n have norm one. Denote the opposite of an element $(x_1,x_2,x_3,\ldots,x_{n+1})$ by

$$-(x_1,x_2,x_3,\ldots,x_{n+1}) = (-x_1,x_2,x_3,\ldots,x_{n+1}).$$

The elements $1_{Q_n}, -1_{Q_n} \in Q_n$ are

$$\begin{aligned}1_{Q_n} &= (1,\underbrace{0,\ldots,0}_{n}),\\ -1_{Q_n} &= (-1,\underbrace{0,\ldots,0}_{n}).\end{aligned}$$

We call 1_{Q_n} by 1, and -1_{Q_n} by -1. One can see that 1 and -1 commute and associate with every element of Q_n. We denote the loop generated by elements $x_1,\ldots,x_n$ of a loop L by $\langle x_1,\ldots,x_n\rangle$. Let i_n be the element $(1_{Q_{n-1}},1) = (1,\underbrace{0,\ldots,0}_{n-1},1)$ of Q_n. Such element i_n satisfies $Q_n = Q_{n-1} \cup (Q_{n-1}i_n) = \langle Q_{n-1}, i_n\rangle$. Thus $Q_n = \langle i_1,i_2,\ldots,i_n\rangle$. We call $i_1,i_2,\ldots,i_n$ the *canonical generators* of Q_n. Any $x \in Q_n$ can be written as

$$x = \pm\prod_{j=1}^{n} i_j^{\epsilon_j}, \quad \epsilon_j \in \{0,1\}.$$

For example,

$$\begin{aligned}Q_0 = \mathbb{R}_2 &= \{1,-1\},\\ Q_1 = \mathbb{C}_4 &= \pm\{(1,0),(1,1)\} = \langle i_1\rangle = \{1,-1,i_1,-i_1\},\\ Q_2 = \mathbb{H}_8 &= \pm\{(1,0,0),(1,1,0),(1,0,1),(1,1,1)\} = \langle i_1,i_2\rangle = \pm\{1,i_1,i_2,i_1i_2\},\\ Q_3 = \mathbb{O}_{16} &= \langle i_1,i_2,i_3\rangle = \pm\{1,i_1,i_2,i_1i_2,i_3,i_1i_3,i_2i_3,i_1i_2i_3\},\end{aligned}$$

We use

$$e = i_n$$

for the unit added in the last step of the process. Let us recall some basic properties of the Cayley–Dickson loops, for more details see [9].

Proposition 1. *Let Q_n be a Cayley–Dickson loop, and $x, y \in Q_n$. The following hold:*

1. *the conjugate $x^* = -x$ for $x \neq \pm 1$, $1^* = 1$, $(-1)^* = -1$;*
2. *the inverse $x^{-1} = x^*$;*
3. *the order $|x| = 4$ for $x \neq \pm 1$, $|1| = 1$, $|-1| = 2$;*
4. *the loop Q_n is Hamiltonian (every subloop S is normal in Q_n, i.e., $xS = Sx$, $(xS)y = x(Sy)$, $x(yS) = (xy)S$).*

A loop Q is *diassociative* if every pair of elements of Q generates a group in Q. A loop Q is the *inverse property loop* if there exist bijections $\lambda : x \mapsto x^\lambda$ and $\rho : x \mapsto x^\rho$ on Q such that $x^\lambda(xy) = y$ and $(yx)x^\rho = y$ for every $y \in Q$. If Q is an inverse property loop, then it satisfies the *anti-automorphic inverse property* $(xy)^{-1} = y^{-1}x^{-1}$ for every $x, y \in Q$. One can see that diassociative loops are also inverse property loops.

Theorem 2. *[4] Cayley–Dickson loop Q_n is diassociative. Any pair of elements of Q_n generate a subgroup of the quaternion group. In particular, a pair x, y generates a real group when $x = \pm 1$ and $y = \pm 1$; a complex group when either $x = \pm 1$, or $y = \pm 1$ (but not both), or $x = \pm y \neq \pm 1$; a quaternion group otherwise.*

For a loop Q and $x, y, z \in Q$ define the *commutator* $[x, y]$ by $xy = (yx)[x, y]$ and the *associator* $[x, y, z]$ by $xy \cdot z = (x \cdot yz)[x, y, z]$. The *center* of a loop Q, denoted by $Z(Q)$, is the set of elements that commute and associate with every element of Q, more precisely, $Z(Q) = \{a \in Q \mid ax = xa, a \cdot xy = ax \cdot y, xa \cdot y = x \cdot ay, xy \cdot a = x \cdot ya, \forall x, y \in Q\}$.

Lemma 3. *Let Q_n be a Cayley–Dickson loop, and $x, y, z \in Q_n$. The following hold:*

1. *the commutator $[x, y] \in \{1, -1\}$, in particular, $[x, y] = -1$ when $\langle x, y\rangle \cong \mathbb{H}_8$, and $[x, y] = 1$ when $\langle x, y\rangle < \mathbb{H}_8$;*
2. *the associator $[x, y, z] \in \{1, -1\}$, in particular, $[x, y, z] = 1$ when $\langle x, y, z\rangle \leq \mathbb{H}_8$, and $[x, y, z] = -1$ when $\langle x, y, z\rangle \cong \mathbb{O}_{16}$;*
3. *the center $Z(Q_n) = \{1, -1\}$ when $n \geq 2$, and $Z(Q_n) = Q_n$ when $n < 2$.*

Theorem 4. *If Q_n is a Cayley–Dickson loop, then $Q_n/\{1, -1\} \cong (\mathbb{Z}_2)^n$.*

Lemma 5. *Let S be a subloop of Q_n. The following hold:*

1. *the center $Z(Q_n) \leq S$ for any $S \leq Q_n, S \neq \{1\}, n \geq 2$;*
2. *If $x \in Q_n \backslash S$, then $|\langle S, x\rangle| = 2|S|$ when $S \neq \{1\}$, and $\langle S, x\rangle = \{1, -1, x, -x\}$ when $S = \{1\}$;*
3. *any n elements of a Cayley–Dickson loop generate a subloop of order 2^k, $k \leq n + 1$;*

4. the order of S is 2^m for some $m \le n$.

It follows from Lemma 5 that $|\langle x,y,z\rangle| \le 16$ for $x,y,z \in Q_n$. Subloops of size 16 of the sedenion loop $\mathbb{S}_{32}$ are either isomorphic to the octonion loop $\mathbb{O}_{16}$, or the quasioctonion loop $\tilde{\mathbb{O}}_{16}$ (see [3]). In fact, a stronger statement holds.

Lemma 6. *Let Q_n be a Cayley–Dickson loop. If x,y,z are elements of Q_n such that $|\langle x,y,z\rangle| = 16$, then either*

$$\langle x,y,z\rangle \cong \mathbb{O}_{16} \text{ or } \langle x,y,z\rangle \cong \tilde{\mathbb{O}}_{16}.$$

Any subloop of size 16 of Q_n containing the element e is isomorphic to the octonion loop $\mathbb{O}_{16}$.

Lemma 7. *Let Q_n be a Cayley–Dickson loop. If x and y are elements of Q_n such that $e \notin \langle x,y\rangle \cong \mathbb{H}_8$, then $\langle x,y,e\rangle \cong \mathbb{O}_{16}$.*

Finally, for a nonassociative loop Q_n the automorphism group $Aut(Q_n)$ is a direct product of $(\mathbb{Z}_2)^{n-3}$ and $Aut(\mathbb{O}_{16})$ (it was established in [10] that $Aut(\mathbb{O}_{16})$ has size 1344 and is an extension of the elementary abelian group $(\mathbb{Z}_2)^3$ by the simple group of symmetries of the Fano plane $PSL_2(7)$).

Theorem 8. *Let Q_n be a Cayley–Dickson loop and let $n \ge 3$. Then $Aut(Q_n) \cong Aut(\mathbb{O}_{16}) \times (\mathbb{Z}_2)^{n-3}$. The order of $Aut(Q_n)$ is therefore $1344 \cdot 2^{n-3}$.*

1.2 Inner mapping groups and multiplication groups

Let us recall some basic facts about inner mapping groups and multiplication groups, notions that are of significant interest and importance in loop theory. Let Q be a loop and $x,a \in Q$. Mappings $L_x(a) = xa$ and $R_x(a) = ax$ are called *left* and *right translations*, these mappings are permutations on Q. Define the following subgroups of $Sym(Q)$:

the multiplication group of Q, $Mlt(Q) = \langle L_x, R_x \mid x \in Q\rangle$,
the inner mapping group of Q, $Inn(Q) = Mlt(Q)_1 = \{f \in Mlt(Q) \mid f(1) = 1\}$,
the left multiplication group of Q, $Mlt_l(Q) = \langle L_x \mid x \in Q\rangle$,
the left inner mapping group of Q, $Inn_l(Q) = Mlt_l(Q)_1 = \{f \in Mlt_l(Q) \mid f(1) = 1\}$,
the right multiplication group of Q, $Mlt_r(Q) = \langle R_x \mid x \in Q\rangle$,
the right inner mapping group of Q, $Inn_r(Q) = Mlt_r(Q)_1 = \{f \in Mlt_r(Q) \mid f(1) = 1\}$.

Let $R_Q = \{R_x \mid x \in Q\}$. Then R_Q is a *left transversal* to $Inn(Q)$ in $Mlt(Q)$, and also a *right transversal* to $Inn(Q)$ in $Mlt(Q)$. That is, for every $f \in Mlt(Q)$ there is a unique $x \in Q$ and a unique $y \in Q$ such that $f \in R_x Inn(Q)$, $f \in Inn(Q)R_y$. An analogous statement is true for $L_Q = \{L_x \mid x \in Q\}$.

Define the *middle, left* and *right inner mappings* on Q by $T_x = L_x^{-1}R_x$, $L_{x,y} = L_{yx}^{-1}L_yL_x$, and $R_{x,y} = R_{xy}^{-1}R_yR_x$. Note that the inner mapping T_x plays the role of conjugation, and the mappings $L_{x,y}$, $R_{x,y}$ measure deviations from associativity, just as T_x measures deviations from commutativity. In an inverse property loop we have $R_x^{-1} = R_{x^{-1}}$ and $L_x^{-1} = L_{x^{-1}}$.

Theorem 9. *[13] Let Q be a loop. Then*

$$\begin{aligned} Inn(Q) &= \langle L_{x,y}, R_{x,y}, T_x \mid x,y \in Q\rangle, \\ Inn_l(Q) &= \langle L_{x,y} \mid x,y \in Q\rangle, \\ Inn_r(Q) &= \langle R_{x,y} \mid x,y \in Q\rangle. \end{aligned}$$

Lemma 10. *Let Q be a finite loop. Then $|Mlt(Q)| = |Q|\,|Inn(Q)|$, $|Mlt_l(Q)| = |Q|\,|Inn_l(Q)|$, and $|Mlt_r(Q)| = |Q|\,|Inn_r(Q)|$.*

Lemma 11. *Let G be a group. If G is abelian, then $Mlt(G) \cong G$ and $Inn(G) \cong \{1\}$. If G is not abelian, then $Mlt(G) \cong (G \times G)/\{(g,g) \mid g \in Z(G)\}$ and $Inn(G) \cong G/Z(G)$.*

1.3 Overview

For a Cayley–Dickson loop Q_n we study inner mapping groups $Inn(Q_n)$ and multiplication groups $Mlt(Q_n)$. This work is organized as follows. In Section 2 we establish that elements of $Mlt(Q_n)$ are even permutations and have order $1, 2$ or 4. We prove that $Inn(Q_n)$ is an elementary abelian 2-group of order 2^{2^n-2}, moreover, every $f \in Inn(Q_n)$ is a product of disjoint transpositions of the form $(x, -x)$. This implies that nonassociative Cayley–Dickson loops are not automorphic. We begin Section 3 by proving a number of lemmas that allow to construct an elementary abelian 2-group K based on left translations by the canonical generators of Q_n. We then show that the direct product of $Inn(Q_n)$ and a cyclic group $\mathbb{Z}_2$ is a normal subgroup of $Mlt(Q_n)$, and that $Mlt(Q_n)$ is a semidirect product of $Inn(Q_n) \times \mathbb{Z}_2$ and K. In Section 4 we prove that the groups $Inn_l(Q_n)$ and $Inn_r(Q_n)$ are equal, elementary abelian 2-groups of order $2^{2^{n-1}-1}$, and establish that the groups $Mlt_l(Q_n)$ and $Mlt_r(Q_n)$ are isomorphic. We conclude with Section 5 showing that $Mlt_l(Q_n)$ is a semidirect product of $Inn_l(Q_n) \times \mathbb{Z}_2$ and K. We used GAP system for computational discrete algebra [5], specifically the LOOPS package [12], to perform numerical experiments and verify conjectures, however, the final results do not rely on computational proofs.

2 Inner Mapping Groups

In this section we discuss inner mapping groups and begin to study multiplication groups of the Cayley–Dickson loops Q_n. When $n \le 2$, the loop Q_n is a group, and the structure of $Mlt(Q_n)$ and $Inn(Q_n)$ is known (see Lemma 11). We therefore focus on nonassociative Cayley–Dickson loops Q_n, $n \ge 3$.

Lemma 12. *Let Q_n be a Cayley–Dickson loop. Elements of $Mlt(Q_n)$ are even permutations.*

Proof. Consider L_x. If $|x| = 1$ then $L_x = id$. If $|x| = 2$ then $L_x L_x(y) = xxy = y$ for every y, so L_x is a product of $|Q_n|/2 = 2^n$ transpositions (of the form (y, xy)), and since 2^n is even, L_x is even. If $|x| = 4$ then L_x is a product of 2^{n-1} 4-cycles (of the form $(y, xy, xxy, xxxy)$), and since 2^{n-1} is even, L_x is even. Similarly for right translations. Hence $Mlt(Q_n)$ is generated by even permutations, and it therefore consists of even permutations. □

If $\langle x, y, z\rangle \le \mathbb{H}_8$, then $\langle x, y, z\rangle$ is a group and $[y, x, z] = [z, x, y] = 1$. If $|\langle x, y, z\rangle| = 16$, then

$$\langle x, y, z\rangle = \pm\{1, x, y, xy, z, xz, yz, (xy)z\},$$

where all elements are distinct. This implies that $z \notin \pm\{1, xy\}$, $x \notin \pm\{1, y\}$, $x \notin \pm\{1, zy\}$, $y \notin \pm\{1, z\}$, and by Lemma 3

$$[xy, z] = [x, y] = [x, zy] = [y, z] = -1. \tag{1}$$

Lemma 13. *Let x, y, z be elements of a Cayley–Dickson loop Q_n, then*

$$[x, y, z] = [z, y, x].$$

Proof. We have

$$\begin{aligned} xy\cdot z &= [xy,z]z\cdot xy = [xy,z][x,y]z\cdot yx = [xy,z][x,y][z,y,x]zy\cdot x \\ &= [xy,z][x,y][z,y,x][x,zy]x\cdot zy = [xy,z][x,y][z,y,x][x,zy][y,z]x\cdot yz \\ &= [xy,z][x,y][z,y,x][x,zy][y,z][x,y,z]xy\cdot z. \end{aligned}$$

If $\langle x,y,z\rangle$ is a group then we are done, else $[xy,z]=[x,y]=[x,zy]=[y,z]=-1$ by (1) and we are done again. □

Lemma 14. *Let Q_n be a Cayley–Dickson loop, and let $x,y\neq\pm1$, $x\neq\pm y$ be elements of Q_n. Then*

$$\begin{aligned} T_x &= \prod_{1,x\neq z\in Q_n/\{\pm1\}}(z,-z), \\ T_yT_x &= (x,-x)(y,-y). \qquad (2) \\ L_{x,e} &= \prod_{1,x,e,xe\neq z\in Q_n/\{\pm1\}}(z,-z), \\ L_{y,e}L_{x,e} &= (x,-x)(y,-y)(xe,-xe)(ye,-ye), \quad \textit{for } x,y\neq\pm e. \qquad (3) \end{aligned}$$

Proof. Consider $T_x, R_{x,y}, L_{x,y}$ acting on $z\in Q_n$. Using diassociativity,

$$\begin{aligned} T_x(z) &= x^{-1}(zx) = [x,z]x^{-1}(xz) = [x,z](x^{-1}x)z = [x,z]z, \qquad (4) \\ L_{x,y}(z) &= (yx)^{-1}(y(xz)) = [y,x,z](yx)^{-1}((yx)z) \qquad (5) \\ &= [y,x,z]((yx)^{-1}(yx))z = [y,x,z]z, \\ R_{x,y}(z) &= ((zx)y)(xy)^{-1} = [z,x,y](z(xy))(xy)^{-1} \qquad (6) \\ &= [z,x,y]z((xy)(xy)^{-1}) = [z,x,y]z. \end{aligned}$$

Let $x,y\neq\pm1$, $x\neq\pm y$. If $z\in\pm\{1,x\}$, then $\langle x,z\rangle\cong\langle x\rangle\cong\mathbb{C}_4$, and $[x,z]=1$. Otherwise, $\langle x,z\rangle\cong\mathbb{H}_8$, and $[x,z]=-1$. Using (4),

$$T_x(z)=[x,z]z=\begin{cases} z, & \text{if } z\in\pm\{1,x\},\\ -z & \text{otherwise.}\end{cases}$$

Similarly, if $z\in\pm\{x,y\}$, then $[y,z][x,z]=-1$. Otherwise, if $z\neq\pm1$, then $\langle x,z\rangle\cong\langle y,z\rangle\cong\mathbb{H}_8$, and $[y,z]=[x,z]=-1$, if $z=\pm1$, then $\langle x,z\rangle\cong\langle y,z\rangle\cong\mathbb{C}_4$, and $[y,z]=[x,z]=1$. We get

$$T_yT_x(z)=[y,z][x,z]z=\begin{cases} -z, & \text{if } z\in\pm\{x,y\},\\ z & \text{otherwise.}\end{cases}$$

Let $x,y\neq\pm e$. If $z\in\pm\{1,x,e,xe\}$, then $\langle e,x,z\rangle\cong\langle e,x\rangle\cong\mathbb{H}_8$, and $[e,x,z]=1$. Otherwise, $\langle e,x,z\rangle\cong\mathbb{O}_{16}$ by Lemma 7, and $[e,x,z]=-1$. Using (5),

$$L_{x,e}(z)=[e,x,z]z=\begin{cases} z, & \text{if } z\in\pm\{1,x,e,xe\},\\ -z & \text{otherwise.}\end{cases}$$

Similarly,

$$L_{y,e}L_{x,e}(z)=[e,y,z][e,x,z]z=\begin{cases} -z, & \text{if } z\in\pm\{x,y,xe,ye\},\\ z & \text{otherwise.}\end{cases}$$

□

Corollary 15. *Let Q_n be a Cayley–Dickson loop. Then*

$$L_{x,y} = R_{x,y} \text{ for all } x, y \in Q_n.$$

Proof. Let $x, y, z \in Q_n$. By Lemma 13,

$$[y, x, z] = [z, x, y],$$

$L_{x,y} = R_{x,y}$ follows from (5), (6) in Lemma 14. □

Theorem 16. *Let Q_n be a Cayley–Dickson loop, $n \geq 1$. Then $Inn(Q_n)$ is an elementary abelian 2-group of order 2^{2^n-2}. Moreover, every $f \in Inn(Q_n)$ is a product of disjoint transpositions of the form $(x, -x)$.*

Proof. Recall that $Z(Q_n) = \{1, -1\}$. Inner mappings fix $Z(Q_n)$ pointwise, therefore

$$f(1) = 1, f(-1) = -1.$$

Let $x \in Q_n, x \neq \pm 1$. Then $|x| = 4$ and $S = \langle x \rangle = \{1, x, -1, -x\}$. We know that Q_n is Hamiltonian, therefore $S \trianglelefteq Q_n$. Inner mappings fix normal subloops, thus $f(S) = S$, and it follows that either $f(x) = x$, $f(-x) = -x$, or $f(x) = -x$, $f(-x) = x$. Hence every f has the desired form. In particular, $|f| = 2$. A group of exponent 2 is an elementary abelian 2-group.
Let $e = i_n$ be a canonical generator of Q_n, let $x \in Q_n$, $x \notin \pm\{1, e\}$. Then $T_x T_e = (x, -x)(e, -e)$ by Lemma 14. For every $f \in Inn(Q_n)$, there is $\tilde{f} = T_x T_e f \in Inn(Q_n)$ such that

$$\tilde{f}(z) = \begin{cases} -f(z), & \text{when } z \in \pm\{x, e\}, \\ f(z), & \text{otherwise.} \end{cases}$$

Also, the values of $f(e), f(-e)$ are uniquely determined by the values of $f(z), z \neq \pm e$, since f should remain an even permutation by Lemma 12 (see Figure 1).

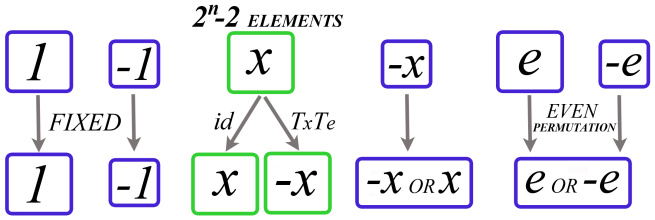

Figure 1: Inner mapping group of Q_n

It follows that

$$|Inn(Q_n)| = 2^{|Q_n|/2-2} = 2^{2^n-2}.$$ □

Lemma 17. *Let $f \in Mlt(Q_n)$, then $|f| \in \{1, 2, 4\}$. In particular, f is a product of disjoint 2-cycles and 4-cycles.*

Proof. Denote by -1 the translation $L_{-1} = R_{-1} \in Mlt(Q_n)$. Let $f \in Mlt(Q_n)$. Let $x \in Q_n$ be such that $f(1) = x$. Then there is $h \in Inn(Q_n)$ such that $f = L_x h$. If $x = 1$ then $f \in Inn(Q_n)$, and we are done. If $x = -1$ then $f = -h$, $f^2 = (-h)(-h) = h^2 = 1$, and we are also done. Assume that $x \neq \pm 1$. We know that $f \in L_x Inn(Q_n)$. There is $y \in Q_n$ such that $f \in Inn(Q_n)L_y$, we want to determine y. Let $f = L_x h = kL_y$ for some $h, k \in Inn(Q_n)$. Since $x \neq \pm 1$, we have $y \neq \pm 1$. Then $f(-y) = kL_y(-y) = k(1) = 1$ and $f(-y) = L_x h(-y) = x(\pm y)$ (since $h(-y)$ is either y or $-y$), so we conclude $y = x$ or $y = -x$. In the former case, we have $f = L_x h = kL_x$, and so $f^2 = kL_x L_x h = k(-1)h = -kh$, which has order at most two, so $f^4 = 1$. In the latter case, we have $f = L_x h = kL_{-x}$, and so $f^2 = kL_{-x}L_x h = kh$, which has order at most two, so $f^4 = 1$. □

A loop Q is *automorphic* if $Inn(Q) \leq Aut(Q)$. Automorphic loops were introduced by Bruck and Paige [2] and received attention in the recent years, with foundational papers [6], [7].

Corollary 18. *Nonassociative Cayley–Dickson loops are not automorphic.*

Proof. Let Q_n be a Cayley–Dickson loop. For $n \leq 2$, Q_n is a group and hence is automorphic. Note that $|Inn(Q_n)| = 2^{2^n-2} > 1344 \cdot 2^{n-3} = |Aut(Q_n)|$ for $n > 3$ (see Theorem 8). Let $n = 3$, and let i_1, i_2, i_3 be canonical generators of Q_n. If $Inn(Q_n) \cap Aut(Q_n) = id$, we are done. Otherwise, let $f \in Inn(Q_n) \cap Aut(Q_n)$ be a nontrivial mapping defined by

$$f(i_k) = f_k, \quad k \in \{1, 2, 3\}.$$

For every $x \in Q_n$, $x \notin \pm\{i_1, i_2, i_3\}$, we know that $x = \prod_{j=1}^{3} i_j^{\epsilon_j}$ (where $\epsilon_j \in \{0, 1\}$), and since f is an automorphism, $f(x)$ is uniquely defined by

$$f(x) = f(\prod_{j=1}^{3} i_j^{\epsilon_j}) = \prod_{j=1}^{3} f(i_j^{\epsilon_j}) = \prod_{j=1}^{3} f_j^{\epsilon_j}.$$

Let $y \notin \pm\{i_1, i_2, i_3, x\}$. Then by (2), $\tilde{f} = T_y T_x f$ satisfies

$$\begin{aligned} \tilde{f} \restriction_{\pm\{i_1,i_2,i_3\}} &= f, \\ \tilde{f}(x) &= -f(x), \end{aligned}$$

so $\tilde{f} \in Inn(Q_n)$ but $\tilde{f} \notin Aut(Q_n)$. □

3 Multiplication Groups

In this section we establish the structure of the multiplication group of Q_n. We prove the auxiliary Lemmas 19, 20, 21, 22 and use them in the construction of Lemma 23 and the proof of Theorem 24.

Lemma 19. *Let G be a finite group, and let $g_1, g_2, \ldots, g_n$ be elements of G of order 2 such that $G = \langle g_1, g_2, \ldots, g_n \rangle$. Then*

$$|g_j g_k| = 2 \text{ iff } g_j g_k = g_k g_j, \quad j, k \in \{1, \ldots, n\}, j \neq k,$$

and if either holds for all j, k, then G is an elementary abelian 2-group.

Proof. Suppose $|g_jg_k| = 2$, then $(g_jg_k)(g_kg_j) = g_jg_k^2g_j = g_j^2 = 1 = (g_jg_k)(g_jg_k)$, and hence $g_kg_j = g_jg_k$. If $g_kg_j = g_jg_k$, then $(g_jg_k)^2 = (g_jg_k)(g_jg_k) = (g_jg_k)(g_kg_j) = g_jg_k^2g_j = g_j^2 = 1$, and $|g_jg_k| = 2$. If $g_kg_j = g_jg_k$ for all $j,k \in \{1,\dots,n\}, j \neq k$, it is straightforward to check that G is an elementary abelian 2-group. □

Lemma 20. *Let Q_n be a Cayley–Dickson loop, i_j, i_k among its canonical generators, and $x \in Q_n$. Let*

$$\begin{aligned} p_{j,k}(x) &= L_{i_j} \restriction_{\pm\{x,i_jx,i_kx,i_j(i_kx)\}} \\ &= (x, i_jx, -x, -i_jx)(i_kx, i_j(i_kx), -i_kx, -i_j(i_kx)), \\ q_{j,k}(x) &= T_{i_kx}T_xp_{j,k}(x), \\ M_{j,k,x,1} &= \{T_{i_jx}T_x, T_{i_j(i_kx)}T_{i_kx}\}, \\ M_{j,k,x,-1} &= \{T_{i_j(i_kx)}T_x, T_{i_kx}T_{i_jx}\}. \end{aligned}$$

Then $\left|q_{j,k}(x)\right| = \left|tp_{k,j}(x)\right| = \left|q_{j,k}(x)(tp_{k,j}(x))\right| = 2$, where $t \in M_{j,k,x,s}$, and $s \in Z(Q_n)$ satisfies $i_j(i_kx) = s(i_k(i_jx))$.

Proof. We write down the corresponding permutations and check that they only contain involutions. Using Lemma 14,

$$q_{j,k}(x) = T_{i_kx}T_xp_{j,k} = (x, i_jx)(-x, -i_jx)(i_kx, i_j(i_kx))(-i_kx, -i_j(i_kx)),$$

hence $\left|q_{j,k}(x)\right| = 2$.
Let s be an element of Q_n such that $i_j(i_kx) = s(i_k(i_jx))$. Note that $s \in Z(Q_n)$ as a product of commutators and associators, therefore $s \in \{1,-1\}$.
If $s = 1$ and $i_j(i_kx) = i_k(i_jx)$, then

$$\begin{aligned} p_{k,j}(x) &= (x, i_kx, -x, -i_kx)(i_jx, i_j(i_kx), -i_jx, -i_j(i_kx)), \\ T_{i_jx}T_xp_{k,j}(x) &= (x, i_kx)(-x, -i_kx)(i_jx, i_j(i_kx))(-i_jx, -i_j(i_kx)), \\ T_{i_j(i_kx)}T_{i_kx}p_{k,j}(x) &= (x, -i_kx)(-x, i_kx)(i_jx, -i_j(i_kx))(-i_jx, i_j(i_kx)). \end{aligned}$$

In this case,

$$\begin{aligned} q_{j,k}(x)\cdot(T_{i_jx}T_xp_{k,j}(x)) &= (x, i_j(i_kx))(-x, -i_j(i_kx))(i_jx, i_kx)(-i_jx, -i_kx), \\ q_{j,k}(x)\cdot(T_{i_j(i_kx)}T_{i_kx}p_{k,j}(x)) &= (x, -i_j(i_kx))(-x, i_j(i_kx))(i_jx, -i_kx)(-i_jx, i_kx). \end{aligned}$$

One can see that $\left|tp_{k,j}(x)\right| = \left|q_{j,k}(x)(tp_{k,j}(x))\right| = 2$, where $t \in \{T_{i_jx}T_x, T_{i_j(i_kx)}T_{i_kx}\}$.
Similarly, if $s = -1$ and $i_j(i_kx) = -i_k(i_jx)$, then

$$\begin{aligned} p_{k,j}(x) &= (x, i_kx, -x, -i_kx)(i_jx, -i_j(i_kx), -i_jx, i_j(i_kx)), \\ T_{i_j(i_kx)}T_xp_{k,j}(x) &= (x, i_kx)(-x, -i_kx)(i_jx, i_j(i_kx))(-i_jx, -i_j(i_kx)), \\ T_{i_kx}T_{i_jx}p_{k,j}(x) &= (x, -i_kx)(-x, i_kx)(i_jx, -i_j(i_kx))(-i_jx, i_j(i_kx)). \end{aligned}$$

In this case,

$$\begin{aligned} q_{j,k}(x)\cdot(T_{i_j(i_kx)}T_xp_{k,j}(x)) &= (x, i_j(i_kx))(-x, -i_j(i_kx))(i_jx, i_kx)(i_jx, i_kx), \\ q_{j,k}(x)\cdot(T_{i_kx}T_{i_jx}p_{k,j}(x)) &= (x, -i_j(i_kx))(-x, i_j(i_kx))(i_jx, -i_kx)(-i_jx, i_kx). \end{aligned}$$

Again, $\left|tp_{k,j}(x)\right| = \left|q_{j,k}(x)(tp_{k,j}(x))\right| = 2$, where $t \in \{T_{i_j(i_kx)}T_x, T_{i_kx}T_{i_jx}\}$. □

We use the following property to prove Lemmas 21 and 22.

Lemma 21. *Let Q_n be a Cayley–Dickson loop, and let $i_1, i_2, \dots, i_n$ be its canonical generators. Then $i_k(i_n x) = -i_n(i_k x)$ for any $x \in \langle i_1, i_2, \dots, i_{n-1}\rangle$, $k < n$.*

Proof. Let $x \in \langle i_1, i_2, \dots, i_{n-1}\rangle$. Then

$$\begin{aligned} i_k(i_n x) &= [i_k, i_n, x](i_k i_n)x = [i_k, i_n][i_k, i_n, x](i_n i_k)x \\ &= [i_k, i_n][i_k, i_n, x][i_n, i_k, x]i_n(i_k x). \end{aligned}$$

Recall that $\langle x, y, i_n\rangle \le \mathbb{O}_{16}$ for any $x, y \in Q_n$, by Lemma 7, and $\langle x, y, i_n\rangle \cong \mathbb{O}_{16}$ implies that $[x, y, i_n] = -1$. Also, $[i_k, i_n] = -1$ as $\langle i_k, i_n\rangle \cong \mathbb{H}_8$. This leads to

$$[i_k, i_n][i_k, i_n, x][i_n, i_k, x] = \begin{cases} -1 \cdot 1 \cdot 1 = -1, & \text{if } x \in \langle i_k, i_n\rangle, \\ -1 \cdot (-1) \cdot (-1) = -1, & \text{otherwise.} \end{cases}$$

We conclude that

$$i_k(i_n x) = -i_n(i_k x). \qquad \square$$

Lemma 22. *Let Q_n be a Cayley–Dickson loop, and let $i_1, i_2, \dots, i_n$ be its canonical generators. For any $x \in \langle i_1, i_2, \dots, i_{n-1}\rangle$, $j < n, k < n, j \ne k$, if $i_j(i_k x) = s(i_k(i_j x))$, then $i_j(i_k(x i_n)) = s(i_k(i_j(x i_n)))$ (where $s \in Z(Q_n)$).*

Proof. Let $x \in \langle i_1, i_2, \dots, i_{n-1}\rangle$, and let $s \in Z(Q_n)$ be such that $i_j(i_k x) = s(i_k(i_j x))$. Then

$$\begin{aligned} i_j(i_k(x i_n)) &= [i_k, x, i_n]i_j((i_k x)i_n) = [i_k, x, i_n][i_j, i_k x, i_n](i_j(i_k x))i_n \\ &= [i_k, x, i_n][i_j, i_k x, i_n]s((i_k(i_j x))i_n) \\ &= [i_k, x, i_n][i_j, i_k x, i_n][i_k, i_j x, i_n]s i_k((i_j x)i_n) \\ &= [i_k, x, i_n][i_j, i_k x, i_n][i_k, i_j x, i_n][i_j, x, i_n]s i_k(i_j(x i_n)). \end{aligned}$$

Recall that $\langle x, y, i_n\rangle \le \mathbb{O}_{16}$ for any $x, y \in Q_n$, by Lemma 7, and $\langle x, y, i_n\rangle \cong \mathbb{O}_{16}$ implies that $[x, y, i_n] = -1$, which leads to

$$[i_k, x, i_n][i_j, i_k x, i_n][i_k, i_j x, i_n][i_j, x, i_n] = \begin{cases} 1 \cdot (-1) \cdot (-1) \cdot 1 = 1, & \text{if } x = \pm 1, \\ -1 \cdot (-1) \cdot 1 \cdot 1 = 1, & \text{if } x = \pm i_j, \\ 1 \cdot 1 \cdot (-1) \cdot (-1) = 1, & \text{if } x = \pm i_k, \\ -1 \cdot 1 \cdot 1 \cdot (-1) = 1, & \text{if } x = \pm i_j i_k, \\ -1 \cdot (-1) \cdot (-1) \cdot (-1) = 1 & \text{otherwise.} \end{cases}$$

We conclude that

$$i_j(i_k(x i_n)) = s(i_k(i_j(x i_n))). \qquad \square$$

In Lemma 23 we present a construction of the subgroup K of $Mlt(Q_n)$ which is used in Theorem 24 to establish that $Mlt(Q_n) \cong (Inn(Q_n) \times Z(Q_n)) \rtimes K$. For every $x \in Q_n/\{1, -1\}$, we want K to contain the element k_x such that $k_x(1) \in x$. This holds when K is generated by $\{L_{i_k} \mid i_k \text{ a canonical generator of } Q_n\}$. We also want K to be sufficiently small to allow $(Inn(Q_n) \times Z(Q_n)) \cap K = id$. To achieve this, we should adjust the left translations L_{i_k} so that they generate a group as small as needed. This is done by multiplying L_{i_k} by $\psi_k \in Inn(Q_n)$ such that

$|\psi_k L_{i_k}| = |\psi_j L_{i_j}| = |(\psi_k L_{i_k}) \cdot (\psi_j L_{i_j})| = 2$ for all $j,k \leq n, j \neq k$. Consider a group $\mathbb{H}_8$, where left translations by canonical generators are

$$\begin{aligned} L_{i_1} &= (1, i_1, -1, -i_1)(i_2, i_1 i_2, -i_2, -i_1 i_2), \\ L_{i_2} &= (1, i_2, -1, -i_2)(i_1, i_2 i_1, -i_1, -i_2 i_1) = (1, i_2, -1, -i_2)(i_1, -i_1 i_2, -i_1, i_1 i_2). \end{aligned}$$

For an inner mapping $\psi_1 \in Inn(\mathbb{H}_8)$ such that $|\psi_1 L_{i_1}| = 2$ we can either take T_{i_2}, or $T_{i_1 i_2}$ (one can check that $|T_{i_1} L_{i_1}| = 4$),

$$\begin{aligned} T_{i_2} L_{i_1} &= (1, -i_1)(-1, i_1)(i_2, -i_1 i_2)(-i_2, i_1 i_2), \\ T_{i_1 i_2} L_{i_1} &= (1, -i_1)(-1, i_1)(i_2, i_1 i_2)(-i_2, -i_1 i_2). \end{aligned}$$

Similarly, for an inner mapping $\psi_2 \in Inn(\mathbb{H}_8)$ such that $|\psi_2 L_{i_2}| = 2$ we can either take T_{i_1}, or $T_{i_1 i_2}$,

$$\begin{aligned} T_{i_1} L_{i_2} &= (1, -i_2)(-1, i_2)(i_1, i_1 i_2)(-i_1, -i_1 i_2), \\ T_{i_1 i_2} L_{i_2} &= (1, -i_2)(-1, i_2)(i_1, -i_1 i_2)(-i_1, i_1 i_2). \end{aligned}$$

For a pair of mappings ψ_1, ψ_2 such that $|(\psi_1 L_{i_1}) \cdot (\psi_2 L_{i_2})| = 2$ we can either take $\psi_1 = T_{i_2}, \psi_2 = T_{i_1 i_2}$, or $\psi_1 = T_{i_1 i_2}, \psi_2 = T_{i_1}$,

$$\begin{aligned} (T_{i_2} L_{i_1}) \cdot (T_{i_1 i_2} L_{i_2}) &= (1, i_1 i_2)(-1, -i_1 i_2)(i_1, i_2)(-i_1, -i_2), \\ (T_{i_1 i_2} L_{i_1}) \cdot (T_{i_1} L_{i_2}) &= (1, -i_1 i_2)(-1, i_1 i_2)(i_1, i_2)(-i_1, -i_2). \end{aligned}$$

Without loss of generality, we choose $\psi_1 = T_{i_2}, \psi_2 = T_{i_1 i_2}$, and $K_2 = \langle g_{1,2}, g_{2,2} \rangle = \langle T_{i_2} L_{i_1}, T_{i_1 i_2} L_{i_2} \rangle$. The group K_2 is not unique, and this particular choice allows to generalize the construction for higher dimensions. The group we present in Lemma 23 is based on this choice and suffices to establish the structure of $Mlt(Q_n)$. Note that the structure of $Mlt(\mathbb{H}_8)$ is known (see Lemma 11), so the construction of K for $\mathbb{H}_8$ is only used as an initial step of the inductive construction for Q_n.

Next, consider $\mathbb{O}_{16}$. We want to construct K_3 based on K_2 by extending the generators of K_2 to form the elements of K_3, and including one more generator based on L_{i_3}. By Lemma 3, we have

$$\begin{aligned} i_1(i_2 i_3) &= -(i_1 i_2) i_3, \\ i_2(i_1 i_3) &= -(i_2 i_1) i_3 = (i_1 i_2) i_3, \\ i_3(i_1 i_2) &= -(i_1 i_2) i_3, \end{aligned}$$

hence

$$\begin{aligned} L_{i_1} &= (1, i_1, -1, -i_1)(i_2, i_1 i_2, -i_2, -i_1 i_2)(i_3, i_1 i_3, -i_3, -i_1 i_3)(i_2 i_3, -(i_1 i_2) i_3, -i_2 i_3, (i_1 i_2) i_3), \\ L_{i_2} &= (1, i_2, -1, -i_2)(i_1, -i_1 i_2, -i_1, i_1 i_2)(i_3, i_2 i_3, -i_3, -i_2 i_3)(i_1 i_3, (i_1 i_2) i_3, -i_1 i_3, -(i_1 i_2) i_3), \\ L_{i_3} &= (1, i_3, -1, -i_3)(i_1, -i_1 i_3, -i_1, i_1 i_3)(i_2, -i_2 i_3, -i_2, i_2 i_3)(i_1 i_2, -(i_1 i_2) i_3, -i_1 i_2, (i_1 i_2) i_3). \end{aligned}$$

For every cycle $(x, i_k x, -x, -i_k x)$ we want ψ_k to include either T_x, or $T_{i_k x}$ (but not both), so that the cycle becomes a product of two 2-cycles, either $(x, i_k x)(-x, -i_k x)$, or $(x, -i_k x)(-x, i_k x)$. Note that a product of an odd number of mappings $T_{x_1} T_{x_2} T_{x_3}$ (where $x_1, x_2, x_3 \in \mathbb{O}_{16}$) fixes $\pm\{1, x_1, x_2, x_3\}$

and moves all other elements (see Lemma 14). Taking $\psi_1 = T_{i_2}T_{i_3}T_{i_2i_3}, \psi_2 = T_{i_1i_2}T_{i_3}T_{i_1i_2i_3}$, we get

$$\begin{aligned}
g_{1,3} &= T_{i_2}T_{i_3}T_{i_2i_3}L_{i_1}\\
&= (1,-i_1)(-1,i_1)(i_2,-i_1i_2)(-i_2,i_1i_2)\\
&\quad (i_3,-i_1i_3)(-i_3,i_1i_3)(i_2i_3,(i_1i_2)i_3)(-i_2i_3,-(i_1i_2)i_3),\\
g_{2,3} &= T_{i_1i_2}T_{i_3}T_{i_1i_2i_3}L_{i_2}\\
&= (1,-i_2)(-1,i_2)(i_1,-i_1i_2)(-i_1,i_1i_2)\\
&\quad (i_3,-i_2i_3)(-i_3,i_2i_3)(i_1i_3,(i_1i_2)i_3)(-i_1i_3,-(i_1i_2)i_3),\\
g_{1,3}g_{2,3} &= (1,i_1i_2)(-1,-i_1i_2)(i_1,i_2)(-i_1,-i_2)\\
&\quad (i_3,-(i_1i_2)i_3)(-i_3,(i_1i_2)i_3)(i_1i_3,i_2i_3)(-i_1i_3,-i_2i_3).
\end{aligned}$$

Again, this is one of several possible choices of $g_{1,3}, g_{2,3}$. Finally, we need to add a generator $g_{3,3}$ such that $|g_{3,3}| = |g_{1,3}g_{3,3}| = |g_{2,3}g_{3,3}| = 2$, one can choose, for example,

$$\begin{aligned}
g_{3,3} &= T_{i_1i_2}T_{i_1i_3}T_{i_2i_3}L_{i_3}\\
&= (1,-i_3)(-1,i_3)(i_1,-i_1i_3)(-i_1,i_1i_3)\\
&\quad (i_2,-i_2i_3)(-i_2,i_2i_3)(i_1i_2,(i_1i_2)i_3)(-i_1i_2,-(i_1i_2)i_3),
\end{aligned}$$

which results in

$$\begin{aligned}
g_{1,3}g_{3,3} &= (1,i_1i_3)(-1,-i_1i_3)(i_1,i_3)(-i_1,-i_3)\\
&\quad (i_2,-(i_1i_2)i_3)(-i_2,(i_1i_2)i_3)(i_1i_2,i_2i_3)(-i_1i_2,-i_2i_3),\\
g_{2,3}g_{3,3} &= (1,i_2i_3)(-1,-i_2i_3)(i_2,i_3)(-i_2,-i_3)\\
&\quad (i_1,-(i_1i_2)i_3)(-i_1,(i_1i_2)i_3)(i_1i_2,i_1i_3)(-i_1i_2,-i_1i_3).
\end{aligned}$$

Below is the description of the construction for Q_n.

Lemma 23. *Let $i_1, i_2, \dots, i_n$ be canonical generators of a Cayley–Dickson loop Q_n, and let K be the group constructed inductively as follows*

$$\begin{aligned}
s_{1,2} &= \{1, i_2\}, \quad s_{2,2} = \{1, i_1i_2\},\\
g_{1,2} &= \Big(\prod_{x\in s_{1,2}} T_x\Big)L_{i_1},\\
g_{2,2} &= \Big(\prod_{x\in s_{2,2}} T_x\Big)L_{i_2},\\
K_2 &= \langle g_{1,2}, g_{2,2}\rangle,\\
s_{k,n} &= \{x, i_nx \mid x \in s_{k,n-1}\}, \quad k \in \{1,\dots,n-1\},\\
s_{n,n} &= \left\{\prod_{j=1}^{n} i_j^{p_j} \mid p_j \in \{0,1\}, \sum_{j=1}^{n} p_j \in 2\mathbb{Z}\right\},\\
g_{k,n} &= \Big(\prod_{x\in s_{k,n}} T_x\Big)L_{i_k} = \Big(\prod_{x\notin s_{k,n}} (x,-x)\Big)L_{i_k}, \quad k \in \{1,\dots,n\},\\
K &= K_n = \langle g_{1,n}, g_{2,n}, \dots, g_{n,n}\rangle.
\end{aligned}$$

Then K is an elementary abelian 2-group of order 2^n.

Proof. We show by induction on n that generators of K have order 2. If $n = 2$, then

$$\begin{aligned} g_{1,2} &= T_{i_2} T_1 L_{i_1} = (1, -i_1)(-1, i_1)(i_2, -i_1 i_2)(-i_2, i_1 i_2), \\ g_{2,2} &= T_{i_1 i_2} T_1 L_{i_2} = (1, -i_2)(-1, i_2)(i_1, -i_1 i_2)(-i_1, i_1 i_2) \end{aligned}$$

are of order 2. Suppose that generators $g_{1,n-1}, g_{2,n-1}, \ldots, g_{n-1,n-1}$ of K_{n-1} have order 2. Note that a product of an odd number of mappings $T_{x_1} \ldots T_{x_{2^{n-1}-1}}$ (where $x_1, \ldots, x_{2^{n-1}-1} \in Q_n$) fixes $\pm\{1, x_1, \ldots, x_{2^{n-1}-1}\}$ and moves all other elements (see Lemma 14). Left translation L_{i_k} consists of 4-cycles of the form

$$(x, i_k x, -x, -i_k x).$$

In order to transform such cycle into two 2-cycles, L_{i_k} is multiplied by either T_x, or $T_{i_k x}$. Then the cycle

$$(i_n x, i_k(i_n x), -i_n x, -i_k(i_n x))$$

that appears in the next step of the inductive construction, is multiplied by either $T_{i_n x}$ or $T_{i_n(i_k x)}$, leading to either

$$\begin{aligned} &(i_n x, -i_k(i_n x))(-i_n x, i_k(i_n x)), \text{ or} \\ &(i_n x, i_k(i_n x))(-i_n x, -i_k(i_n x)), \end{aligned}$$

respectively.
If $x = 1$, the 4-cycle that corresponds to $(1, i_k, -1, -i_k)$ in the next step of the inductive construction is $(i_n, i_k i_n, -i_n, -i_k i_n)$, which is multiplied by T_{i_n} and becomes $(i_n, -i_k i_n)(-i_n, i_k i_n)$. It follows that $g_{k,n}$ consists of 2-cycles and therefore $|g_{k,n}| = 2$.
Consider a generator $g_{n,n}$ added at the n-th step of the inductive construction. Left translation L_{i_n} consists of cycles of the form

$$(x, i_n x, -x, -i_n x)$$

where either x, or $i_n x$ (but not both) is a product of even number of units i_k, for some $k \le n$. In the former case, if $x \neq \pm 1$, then multiplication of L_{i_n} by T_x transforms a cycle $(x, i_n x, -x, -i_n x)$ into

$$(x, -i_n x)(-x, i_n x),$$

otherwise, multiplication of L_{i_n} by $T_{i_n x}$ transforms it into

$$(x, i_n x)(-x, -i_n x).$$

Also, L_{i_n} is multiplied by $T_{x_k}, x_k \neq \pm i_n$, an odd number of times, mapping i_n to $-i_n$, and a cycle $(1, i_n, -1, -i_n)$ becomes

$$(1, -i_n)(-1, i_n).$$

Generator $g_{n,n}$ consists of 2-cycles and therefore $|g_n| = 2$.
Next, use induction on n to show that

$$|g_{j,n} g_{k,n}| = 2, \text{ for all } j, k \in \{1, \ldots, n\}, j \neq k.$$

If $n = 2$, then

$$g_{1,2} g_{2,2} = (1, i_1 i_2)(-1, -i_1 i_2)(i_1, i_2)(-i_1, -i_2)$$

and $|g_{1,2}g_{2,2}| = 2$. Suppose that $\left|g_{j,n-1}g_{k,n-1}\right| = 2$ for any pair of generators $g_{j,n-1}, g_{k,n-1}$ of K_{n-1}. Without loss of generality, let $j < k$. Up to renaming x and $i_j x$, the cycles

$$p_{j,k}(x) = L_{i_j} \restriction_{\pm\{x, i_j x, i_k x, i_j(i_k x)\}} = (x, i_j x, -x, -i_j x)(i_k x, i_j(i_k x), -i_k x, -i_j(i_k x))$$

are acted upon by $T_{i_k x}T_x$ to construct $g_{j,n}$. Then, by Lemma 20, the cycles

$$p_{k,j}(x) = L_{i_k} \restriction_{\pm\{x, i_j x, i_k x, i_j(i_k x)\}} = (x, i_k x, -x, -i_k x)(i_j x, i_k(i_j x), -i_j x, -i_k(i_j x))$$

are acted upon by $t \in M_{j,k,x,s}$, where $t = T_y T_z$ for some $y, z \in Q_n$. The cycles

$$p_{j,k}(xi_n) = (xi_n, i_j(xi_n), -xi_n, -i_j(xi_n))(i_k(xi_n), i_j(i_k(xi_n)), -i_k(xi_n), -i_j(i_k(xi_n)))$$

added at the next step of the inductive construction are multiplied by $T_{i_n(i_k x)}T_{i_n x}$. The cycles

$$\begin{aligned} p_{k,j}(xi_n) &= L_{i_k} \restriction_{\pm\{xi_n, i_j(xi_n), i_k(xi_n), i_j(i_k(xi_n))\}} \\ &= (xi_n, i_k(xi_n), -xi_n, -i_k(xi_n))(i_j(xi_n), i_k(i_j(xi_n)), -i_j(xi_n), -i_k(i_j(xi_n))) \end{aligned}$$

are multiplied by $T_{yi_n}T_{zi_n} \in M_{j,k,xi_n,s}$. By Lemma 22, $i_j(i_k(xi_n)) = s(i_k(i_j(xi_n)))$, and therefore by Lemma 20,

$$\left|(T_{(i_k x)i_n}T_{xi_n}p_{j,k}(xi_n)) \cdot (tp_{k,j}(xi_n))\right| = 2 \text{ for } t \in M_{j,k,xi_n,s}.$$

It is left to show that $|g_{j,n}g_{n,n}| = 2$, where $j < n$. Up to renaming x and $i_j x$, the cycles

$$p_{j,k}(x) = L_{i_j} \restriction_{\pm\{x, i_j x, i_n x, i_j(i_n x)\}} = (x, i_j x, -x, -i_j x)(i_n x, i_j(i_n x), -i_n x, -i_j(i_n x))$$

are acted upon by $T_{i_n x}T_x$. By Lemma 21, $i_j(i_n x) = -i_n(i_j x)$, therefore by Lemma 20,

$$\left|(T_{i_n x}T_x p_{j,k}(x)) \cdot (tp_{n,j}(x))\right| = 2 \text{ where } t \in \{T_x T_{i_j(i_n x)}, T_{i_j x}T_{i_n x}\}.$$

If x is a product of even number of units i_k, for some $k \le n$, then $i_j(i_n x)$ is also a product of even number of units, so $x, i_j(i_n x)$ are in $s_{n,n}$, and $T_x T_{i_j(i_n x)}$ is a part of the construction of $g_{n,n}$. If x is a product of odd number of units, then $i_j x, i_n x$ are products of even number of units, and are included in $s_{n,n}$, so $T_{i_j x}T_{i_n x}$ is a part of the construction of $g_{n,n}$. In both cases this leads to $|g_{j,n}g_{n,n}| = 2$.

Summarizing, K satisfies the assumptions of Lemma 19 and is therefore an elementary abelian 2-group.

To determine the order of K, define a mapping $\phi : Q_n/\{1,-1\} \to K$ by

$$\phi(\{i_k, -i_k\}) = g_{k,n}, k \in \{1, \ldots, n\}.$$

Note that for any

$$\begin{aligned} x &= \pm\{\prod_{j=1}^{n} i_j^{\epsilon_j}\} \in Q_n/\{1,-1\} \quad (\text{ where } \epsilon_j \in \{0,1\}), \text{ there is} \\ g &= \prod_{j=1}^{n} \phi(i_j^{\epsilon_j}) = \prod_{j=1}^{n} g_{j,n}^{\epsilon_j}, \end{aligned} \tag{7}$$

such that $g(1) \in x$. We conclude that

$$|K| \ge |Q_n/\{1,-1\}| = \frac{|Q_n|}{2} = 2^n.$$

Also, K is an elementary abelian 2-group with n generators, so

$$|K| \leq 2^n. \tag{8}$$

We conclude that the order of K is 2^n. □

For any loop Q, Albert showed $Z(Mlt(Q)) = \{L_x \,|\, x \in Z(Q)\} \cong Z(Q)$. To improve legibility, we will identify $Z(Mlt(Q))$ with $Z(Q)$ in what follows.

In Theorem 24 we use the group $N = \langle Inn(Q_n), Z(Q_n)\rangle = Inn(Q_n)Z(Q_n)$ to establish the structure of $Mlt(Q_n)$. Recall that elements of $Inn(Q_n)$ are all even products of 2-cycles $(x, -x)$ (where $1 \neq x \in Q_n/\{1,-1\}$). A group $Inn(Q_n)$ stabilizes 1, therefore $Inn(Q_n) \cap Z(Q_n) = 1$. The index $[N : Inn(Q_n)] = 2$, therefore $Inn(Q_n) \trianglelefteq N$, and $Z(Q_n) \trianglelefteq Mlt(Q_n)$ implies $Z(Q_n) \trianglelefteq N$. It follows that $N = Inn(Q_n) \times Z(Q_n)$, and $N = Inn(Q_n) \cup (-Inn(Q_n))$.
A basis for $Inn(Q_n)$ can be taken to be

$$\{T_xT_e = (x,-x)(e,-e)\,|1, e \neq x \in Q_n/\{1,-1\}\}.$$

Elements of N are all even products of 2-cycles $(x,-x)$, for $x \in Q_n/\{1,-1\}$. A mapping $L_{-1}T_e = (1,-1)(e,-e)$ can be used to construct a basis for N (see Figure 2),

$$N^* = \{L_{-1}T_e, T_xT_e\,|1, e \neq x \in Q_n/\{1,-1\}\}.$$

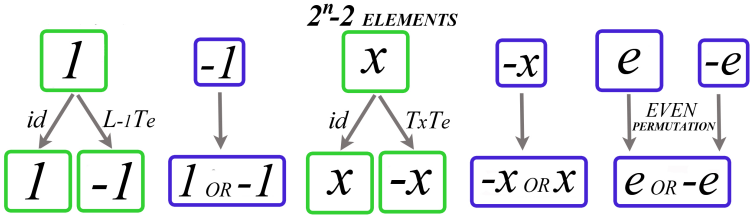

Figure 2: Group $N = Inn(Q_n) \times Z(Q_n)$

Theorem 24. *Let Q_n be a Cayley–Dickson loop, $n \geq 2$. Then $Mlt(Q_n) \cong (Inn(Q_n) \times Z(Q_n)) \rtimes K$, where K is the group constructed in Lemma 23. In particular, $Mlt(Q_n) \cong ((\mathbb{Z}_2)^{2^n-2} \times \mathbb{Z}_2) \rtimes (\mathbb{Z}_2)^n$.*

Proof. Let $G = Mlt(Q_n)$, $N = Inn(Q_n) \times Z(Q_n)$, and K be the group constructed in Lemma 23. We want to show that $G = N \rtimes K$.

1. Let $\alpha \in N, g \in G$. There exist $x \in Q_n, \beta \in Inn(Q_n)$ such that $g = \beta L_x$. Consider $g\alpha g^{-1}$ acting on 1,

$$g\alpha g^{-1}(1) \quad = \quad \beta L_x \alpha (\beta L_x)^{-1}(1) = \beta L_x \alpha L_x^{-1} \underbrace{\beta^{-1}(1)}_{1} = \beta \underbrace{L_x \alpha L_x^{-1}(1)}_{\pm 1} = \pm\beta(1) = \pm 1.$$

   This shows that $g\alpha g^{-1} \in Inn(Q_n) \cup (-Inn(Q_n)) = N$, so N is normal in G.

2. By (7), (8), K contains a unique element g such that $g(1) \in \{1,-1\}$. Since K is a group, $g = id$, thus $N \cap K = id$.

3. We established that $N \trianglelefteq G, K \leq G$, and $N \cap K = id$. We have $N \rtimes K \leq G$. Recall that

$$\begin{aligned} [Mlt(Q_n) : Inn(Q_n)] &= |Q_n|, \text{ thus} \\ [Mlt(Q_n) : (Inn(Q_n) \times Z(Q_n))] &= [Mlt(Q_n) : Inn(Q_n)]/2 = 2^n = |K|, \end{aligned}$$

and $(Inn(Q_n) \times Z(Q_n)) \rtimes K \cong Mlt(Q_n)$ follows. □

We have shown that $Mlt(Q_n)$ is a semidirect product of two permutation groups N, K, both elementary abelian 2-groups. Recall that if N, K are groups and $\phi : K \to Aut(N)$ is a homomorphism, then the external semidirect product is defined on $N \times K$ by

$$(h_1, k_1) \circ (h_2, k_2) = (h_1 * \phi_{k_1}(h_2), k_1 \cdot k_2), \quad h_1, h_2 \in N, k_1, k_2 \in K.$$

In an internal semidirect product $G = N \rtimes_\phi K$, the action $\phi : K \to Aut(N)$ is natural, that is, by conjugation $\phi_{k_1}(h_2) = k_1 h_2 k_1^{-1}$.

Lemma 25. *[14, p.170] Let G, N, K be finite groups such that $N \trianglelefteq G$, $K \leq G$, and $G = N \rtimes_\phi K$. Then K acts on N by conjugation.*

In [8] we construct an isomorphic copy of $Mlt(Q_n)$ as an external semidirect product of two abstract elementary abelian 2-groups.

4 Left and Right Inner Mapping Groups

In this section we discuss one-sided inner mapping groups of Cayley–Dickson loops. It is well known that $Mlt_l(Q) \cong Mlt_r(Q)$ and $Inn_l(Q) \cong Inn_r(Q)$ in any inverse property loop Q. We give the proofs in Theorem 26 and Corollary 28 for completeness.

Theorem 26. *Let Q be an inverse property loop. Then $Mlt_l(Q) \cong Mlt_r(Q)$.*

Proof. Define a partial mapping $f : Mlt_l(Q) \to Mlt_r(Q)$ by $f(L_a) = R_a^{-1}$. We want to extend this mapping to a homomorphism. Let $S \in Mlt_l(Q)$, then

$$S = \prod_{i=1}^{n} L_{a_i}^{\epsilon_i}, \quad a_i \in Q, \epsilon_i \in \{0, 1\}.$$

To verify that a mapping

$$f(S) = f(\prod_{i=1}^{n} L_{a_i}^{\epsilon_i}) = \prod_{i=1}^{n} R_{a_i}^{-\epsilon_i}$$

is well-defined, we show that if

$$S = \prod_{i=1}^{n} L_{a_i}^{\epsilon_i} = \prod_{j=1}^{m} L_{b_j}^{\phi_j},$$

then

$$f(S) = \prod_{i=1}^{n} R_{a_i}^{-\epsilon_i} = \prod_{j=1}^{m} R_{b_j}^{-\phi_j}.$$

Let $x \in Q$, then

$$\begin{aligned} a_n^{\epsilon_n}(\dots a_2^{\epsilon_2}(a_1^{\epsilon_1} x)) &= b_m^{\phi_m}(\dots b_2^{\phi_2}(b_1^{\phi_1} x)), \text{ and} \\ (a_n^{\epsilon_n}(\dots a_2^{\epsilon_2}(a_1^{\epsilon_1} x)))^{-1} &= (b_m^{\phi_m}(\dots b_2^{\phi_2}(b_1^{\phi_1} x)))^{-1}. \end{aligned}$$

Inverse property implies that $(xy)^{-1} = y^{-1}x^{-1}$, thus

$$(a_n^{\epsilon_n}(\dots a_2^{\epsilon_2}(a_1^{\epsilon_1}x)))^{-1} = ((x^{-1}a_1^{-\epsilon_1})a_2^{-\epsilon_2})\dots a_n^{-\epsilon_n} = \prod_{i=1}^{n} R_{a_i}^{-\epsilon_i}(x),$$

and

$$(b_m^{\phi_m}(\dots b_2^{\phi_2}(b_1^{\phi_1}x)))^{-1} = ((x^{-1}b_1^{-\phi_1})b_2^{-\phi_2})\dots b_m^{-\phi_m} = \prod_{j=1}^{m} R_{b_j}^{-\phi_j}(x).$$

We conclude that

$$\prod_{i=1}^{n} R_{a_i}^{-\epsilon_i} = \prod_{j=1}^{m} R_{b_j}^{-\phi_j}$$

and the mapping f is well-defined. The mapping f is a homomorphism since it has an inverse $g : Mlt_r(Q) \to Mlt_l(Q)$ defined by $g(R_a) = L_a^{-1}$. □

Corollary 27. *Let Q_n be a Cayley–Dickson loop. Then $Mlt_l(Q_n) \cong Mlt_r(Q_n)$.*

Corollary 28. *Let Q be an inverse property loop. Then $Inn_l(Q) \cong Inn_r(Q)$.*

Proof. Note that $f \restriction_{Inn_l(Q)}$ is an isomorphism from $Inn_l(Q)$ to $Inn_r(Q)$. If

$$\begin{aligned} S &= \prod_{i=1}^{n} L_{a_i}^{\epsilon_i} \in Inn_l(Q), \text{ then} \\ S(1) &= a_n^{\epsilon_n}(\dots a_2^{\epsilon_2}(a_1^{\epsilon_1}1)) = 1. \end{aligned}$$

Taking the inverse, we have

$$(a_n^{\epsilon_n}(\dots a_2^{\epsilon_2}(a_1^{\epsilon_1}1)))^{-1} = (((1a_1^{-\epsilon_1})a_2^{-\epsilon_2})\dots a_n^{-\epsilon_n}) = 1,$$

thus

$$f(Inn_l(Q)) = \prod_{i=1}^{n} R_{a_i}^{-\epsilon_i} \in Inn_r(Q).$$ □

In fact, a stronger statement holds for the Cayley–Dickson loops. As can be seen in the following Lemma, when Q_n is a Cayley–Dickson loop, the left inner mapping groups $Inn_l(Q_n)$ are equal to the right inner mapping groups $Inn_r(Q_n)$.

Lemma 29. *Let Q_n be a Cayley–Dickson loop. Then $Inn_l(Q_n) = Inn_r(Q_n)$, and $Inn(Q_n) = \langle T_x, L_{x,y} \mid x, y \in Q_n \rangle$.*

Proof. For all $x, y \in Q_n$, we have $L_{x,y} = R_{x,y}$ by Corollary 15. □

Lemma 30 serves a purpose similar to that of Lemma 7, providing information about associators. Lemmas 30 and 33 are used in the proof of Theorem 34.

Lemma 30. *Let Q_n be a Cayley–Dickson loop, i_k its canonical generator, $x \in Q_k$, $y \in Q_k e$, $n \geq 4$, and $k < n$. Then*

$$[i_k, x, y] = [x, i_k, y] = \begin{cases} 1, & \text{when } y \in Q_k e \setminus \pm\{e, i_k e, xe, xi_k e\}, \\ -1, & \text{otherwise.} \end{cases}$$

Moreover, if $x \notin \pm\{1, i_k\}$, then

$$\langle i_k, x, y\rangle \cong \begin{cases} \tilde{\mathbb{O}}_{16}, & \text{when } y \in Q_k e \setminus \pm\{e, i_k e, xe, xi_k e\}, \\ \mathbb{O}_{16}, & \text{otherwise.} \end{cases}$$

Proof. Since $x \in Q_k$ and $y \in Q_k e$, we get $y \notin \langle i_k, x\rangle$. Consider the loop $\langle i_k, x, y\rangle$. If $x \in \pm\{1, i_k\}$, then $\langle i_k, x, y\rangle \cong \mathbb{H}_8$ and $[i_k, x, y] = [x, i_k, y] = 1$. If $x \notin \pm\{1, i_k\}$, then $\langle i_k, x\rangle \cong \mathbb{H}_8$ and $|\langle i_k, x, y\rangle| = 16$ by Lemma 5. In this case, if $y \in \pm\{e, i_k e, xe, xi_k e\}$, then $\langle i_k, x, y\rangle = \pm\{1, x, i_k, xi_k, e, xe, i_k e, xi_k e\} \cong \mathbb{O}_{16}$ and $[i_k, x, y] = [x, i_k, y] = -1$ by Lemmas 7 and 3. It remains to consider $x \in Q_k \setminus \pm\{1, i_k\}$, $y \in Q_k e \setminus \pm\{e, i_k e, xe, xi_k e\}$. We can write $y = ze$ for some $z \in Q_k$, then

$$\begin{aligned} (i_k x)(ze) &= [i_k x, z, e]((i_k x)z)e = [i_k x, z, e][i_k, x, z](i_k(xz))e \\ &= [i_k x, z, e][i_k, x, z][i_k, xz, e]i_k((xz)e) \\ &= [i_k x, z, e][i_k, x, z][i_k, xz, e][x, z, e]i_k(x(ze)) \\ &= i_k(x(ze)), \\ (xi_k)(ze) &= [xi_k, z, e]((xi_k)z)e = [xi_k, z, e][x, i_k, z](x(i_k z))e \\ &= [xi_k, z, e][x, i_k, z][x, i_k z, e]x((i_k z)e) \\ &= [xi_k, z, e][x, i_k, z][x, i_k z, e][i_k, z, e]x(i_k(ze)) = x(i_k(ze)), \end{aligned}$$

since $x, z \in Q_k$, and

$$\begin{aligned} [i_k x, z, e] &= [i_k, x, z] = [i_k, xz, e] = [x, z, e] = -1, \\ [xi_k, z, e] &= [x, i_k, z] = [x, i_k z, e] = [i_k, z, e] = -1 \end{aligned}$$

by Lemmas 7 and 3. Thus

$$\begin{aligned} [i_k, x, ze] &= [i_k, x, y] = 1, \\ [x, i_k, ze] &= [x, i_k, y] = 1. \end{aligned}$$

If $|\langle i_k, x, y\rangle| = 16$ and $[i_k, x, y] = 1$, then $\langle i_k, x, y\rangle \cong \tilde{\mathbb{O}}_{16}$ by Lemmas 7 and 6. □

Lemma 31. *[9] If $x, y, z \in Q_{n-1}$, then in Q_n we have*

(a) $[(x,0),(y,0),(z,1)] = [x,y][z,y,x]$,

(b) $[(x,0),(y,1),(z,0)] = [x,z][y,x,z][y,z,x]$,

(c) $[(x,0),(y,1),(z,1)] = [x,y][x,z][z,x,y][x,z,y]$,

(d) $[(x,1),(y,0),(z,0)] = [y,z][x,y,z]$,

(e) $[(x,1),(y,0),(z,1)] = [y,x][y,z][z,y,x]$,

(f) $[(x,1),(y,1),(z,0)] = [z,x][z,y][y,x,z][y,z,x]$,

(g) $[(x,1),(y,1),(z,1)] = [x,y][x,z][y,z][z,x,y][x,z,y]$.

Lemma 32. *Let Q_n be a Cayley–Dickson loop, and let $x, y \in Q_n$ such that $x = (\bar{x}, x_n)$, $y = (\bar{y}, y_n)$, $\bar{x}, \bar{y} \in Q_{n-1}$, $x_n, y_n \in \{0, 1\}$. Then*

$$\begin{aligned} L_{x,y}(z) &= [\bar{x}, \bar{y}] L_{x,y}(ze), \\ L_{x,e} &= L_{xe,e}. \end{aligned} \tag{9}$$

Proof. By Lemma 13, $[x, y, z] = [z, y, x]$ for any $x, y, z \in Q_{n-1}$. Thus (9) follows from Lemma 31.

If $x \in \pm\{1, e\}$, then $L_{x,e} = L_{xe,e} = id$. Otherwise,

$$L_{x,e}(z) = [e, x, z]z = \begin{cases} z, & \text{if } z \in \pm\{1, x, e, xe\}, \\ -z & \text{otherwise,} \end{cases}$$

and

$$L_{xe,e}(z) = [e, xe, z]z = \begin{cases} z, & \text{if } z \in \pm\{1, x, e, xe\}, \\ -z & \text{otherwise,} \end{cases}$$

thus $L_{x,e} = L_{xe,e}$. □

Lemma 33. *Let Q_n be a Cayley–Dickson loop, $n \geq 4$. Then an inner mapping on Q_n*

$$h = \prod_{x \in Q_{n-2}/\{1,-1\}} L_{x,i_{n-1}}$$

can be written as the following permutation

$$h = \prod_{z \in (Q_n/\{1,-1\}) \setminus (Q_{n-1}/\{1,-1\})} (z, -z).$$

Proof. Let $x \in Q_{n-2}/\{1, -1\}$. By (6),

$$L_{x,i_{n-1}}(z) = [i_{n-1}, x, z]z.$$

If $z \in \langle x, i_{n-1} \rangle = \pm\{1, x, i_{n-1}, xi_{n-1}\}$, then $[i_{n-1}, x, z] = 1$. If $z \in Q_{n-1} \setminus \pm\{1, x, i_{n-1}, xi_{n-1}\}$, then $[i_{n-1}, x, z] = -1$ by Lemmas 7 and 3. If $z \in \{e, xe, i_{n-1}e, xi_{n-1}e\}$, then

$$\langle i_{n-1}, x, z \rangle = \{1, x, i_{n-1}, xi_{n-1}, e, xe, i_{n-1}e, xi_{n-1}e\} \cong \mathbb{O}_{16}$$

and $[i_{n-1}, x, z] = -1$ by Lemmas 7 and 3. If $z \in Q_{n-1}e \setminus \{e, xe, i_{n-1}e, xi_{n-1}e\}$, then $[i_{n-1}, x, z] = 1$ by Lemma 30. Summarizing, we have

$$L_{x,i_{n-1}}(z) = \begin{cases} z, & \text{when } z \in \pm\{1, x, i_{n-1}, xi_{n-1}\} \cup (Q_{n-1}e \setminus \{e, xe, i_{n-1}e, xi_{n-1}e\}), \\ -z, & \text{otherwise.} \end{cases}$$

Next, consider a mapping

$$h = \prod_{x \in Q_{n-2}/\{1,-1\}} L_{x,i_{n-1}}.$$

If $z \in \pm\{1, i_{n-1}\}$, then clearly $h(z) = z$. If $z \in Q_{n-2} \setminus \pm\{1\}$, then $L_{x,i_{n-1}}(z) = -z$ for all $x \neq \pm z$, there is an even number (in fact, $2^{n-2}-2$) of such mappings, and therefore $h(z) = z$. If $z \in Q_{n-2}i_{n-1} \setminus \pm\{i_{n-1}\}$, then $z = yi_{n-1}$ for some $y \in Q_{n-2}$, and $L_{x,i_{n-1}}(z) = -z$ for all $x \neq \pm y$, there is $2^{n-2} - 2$ such mappings, and therefore $h(z) = z$. We get $h(z) = z$ for $z \in Q_{n-1}$. Consider $z \in Q_{n-1}e$. If $z \in \pm\{e, i_{n-1}e\}$, then $L_{x,i_{n-1}}(z) = -z$ for all $x \neq 1$, there is $2^{n-2} - 1$ such mappings, and thus $h(z) = -z$. Finally, if $z \in Q_{n-1}e \setminus \{e, i_{n-1}e\}$, then either $z = ye$, or $z = yi_{n-1}e$ for some $y \in Q_{n-2}$, and $L_{x,i_{n-1}}(z) = -z$ only when $x = \pm y$, again, $h(z) = -z$. We get $h(z) = -z$ for $z \in Q_{n-1}e$. □

Theorem 34. *Let Q_n be a Cayley–Dickson loop. Then $Inn_l(Q_n)$ is an elementary abelian 2-group of order $2^{2^{n-1}-1}$.*

Proof. Let $x \in Q_{n-1}/\{1,-1\}$, $x \neq 1$. Then by Lemma 14

$$L_{x,e}L_{i_{n-1},e} = (x,-x)(i_{n-1},-i_{n-1})(xe,-xe)(i_{n-1}e,-i_{n-1}e).$$

For every $f \in Inn_l(Q_n)$, there is $\tilde{f} = L_{x,e}L_{i_{n-1},e}f \in Inn_l(Q_n)$ such that

$$\tilde{f}(z) = \begin{cases} -f(z), & \text{when } z \in \{x, i_{n-1}, xe, i_{n-1}e\}, \\ f(z), & \text{otherwise.} \end{cases}$$

There are $2^{n-1}-2$ distinct inner mappings $L_{x,e}L_{i_{n-1},e}$, $x \in Q_{n-1}/\{1,-1\}$, $x \neq 1$, they generate a group of order $2^{2^{n-1}-2}$. Let

$$h = \prod_{y \in Q_{n-2}/\{1,-1\}} L_{y,i_{n-1}} = \prod_{z \in (Q_n/\{1,-1\}) \setminus (Q_{n-1}/\{1,-1\})} (z,-z).$$

be the mapping constructed in Lemma 33. For every $f \in Inn_l(Q_n)$, a mapping $\tilde{f} = hf$ satisfies

$$\tilde{f}(z) = \begin{cases} f(z), & \text{when } z \in Q_{n-1}, \\ -f(z), & \text{otherwise.} \end{cases}$$

The group

$$G = \langle L_{x,e}L_{i_{n-1},e}, h \mid 1 \neq x \in Q_{n-1}/\{1,-1\} \rangle$$

therefore has order $2^{2^{n-1}-1}$ and is a subgroup of $Inn_l(Q_n)$.
To show that $Inn_l(Q_n) = G$, recall that $L_{x,y}(z) = [\bar{x},\bar{y}]L_{x,y}(ze)$ for $\bar{x},\bar{y} \in Q_{n-1}$, by (9). The value of $L_{x,y}(ze)$ is therefore uniquely determined by that of $L_{x,y}(z)$, moreover, $L_{x,y}(1) = 1$, thus $Inn_l(Q_n)$ has order at most $2^{\frac{|Q_{n-1}|}{2}-1} = 2^{2^{n-1}-1}$. □

5 Left and Right Multiplication Groups

Let Q_n be a Cayley–Dickson loop. A group $Mlt_l(Q_n)$ is a proper subgroup of $Mlt(Q_n)$ by Theorems 16 and 34,

$$Mlt_l(Q_n)_1 = Inn_l(Q_n) < Inn(Q_n) = Mlt(Q_n)_1.$$

We showed in Corollary 27 that $Mlt_l(Q_n) \cong Mlt_r(Q_n)$. The following theorem establishes the structure of $Mlt_l(Q_n)$.

Theorem 35. *Let Q_n be a Cayley–Dickson loop, $n \geq 2$. Then $Mlt_l(Q_n) \cong (Inn_l(Q_n) \times Z(Q_n)) \rtimes K$, where K is the group constructed in Lemma 23. In particular, $Mlt_l(Q_n) \cong ((\mathbb{Z}_2)^{2^{n-1}-1} \times \mathbb{Z}_2) \rtimes (\mathbb{Z}_2)^n$.*

Proof. Since $Z(Q_n) \leq Mlt_l(Q_n)$, let $N = \langle Inn_l(Q_n), Z(Q_n) \rangle = Inn_l(Q_n)Z(Q_n)$. A group $Inn_l(Q_n)$ stabilizes 1, therefore $Inn_l(Q_n) \cap Z(Q_n) = 1$. The index $[N : Inn_l(Q_n)] = 2$, therefore $Inn_l(Q_n) \trianglelefteq N$, and $Z(Q_n) \trianglelefteq Mlt_l(Q_n)$ implies $Z(Q_n) \trianglelefteq N$. It follows that $N = Inn_l(Q_n) \times Z(Q_n)$. Let $G = Mlt_l(Q_n)$ and K be the group constructed in Lemma 23. We want to show that $G = N \rtimes K$.

1. Let $\alpha \in N, g \in G$. There exist $x \in Q_n, \beta \in Inn_l(Q_n)$ such that $g = \beta L_x$. Consider $g\alpha g^{-1}$ acting on 1,

$$\begin{aligned} g\alpha g^{-1}(1) &= \beta L_x \alpha (\beta L_x)^{-1}(1) = \beta L_x \alpha L_x^{-1} \underbrace{\beta^{-1}(1)}_{1} \\ &= \beta \underbrace{L_x \alpha L_x^{-1}(1)}_{\pm 1} = \pm\beta(1) = \pm 1. \end{aligned}$$

This shows that $g\alpha g^{-1} \in Inn_l(Q_n) \cup (-Inn_l(Q_n)) = N$, so N is normal in G.
Recall a mapping h constructed in Lemma 33,

$$h = \prod_{z \in (Q_n/\{1,-1\}) \setminus (Q_{n-1}/\{1,-1\})} (z, -z).$$

Note that by Lemma 14 we have

$$T_x T_{xe} T_e = \prod_{1,e,x,xe \neq z \in Q_n/\{1,-1\}} (z, -z) = L_{x,e},$$

which allows to rewrite the construction in Lemma 23 as follows

$$\begin{aligned} s_{1,2} &= \{1, i_2\}, \quad s_{2,2} = \{1, i_1 i_2\}, \\ s_{k,n} &= \{x, i_n x \mid x \in s_{k,n-1}\}, \quad k \in \{1, \dots, n-1\}, \\ s_{n,n} &= \left\{ \prod_{j=1}^{n} i_j^{p_j} \mid p_j \in \{0,1\}, \sum_{j=1}^{n} p_j \in 2\mathbb{Z} \right\}, \\ \bar{s}_{n,n} &= \left\{ \prod_{j=1}^{n} i_j^{p_j} \mid p_j \in \{0,1\}, \sum_{j=1}^{n} p_j \notin 2\mathbb{Z} \right\}, \\ g_{k,n} &= (\prod_{x \in s_{k,n}} T_x) L_{i_k} = (\prod_{x \in s_{k,n-1}} T_x T_{xe}) T_e L_{i_k} \\ &= (\prod_{x \in s_{k,n-1}} T_x T_{xe})(\prod_{x \in \{1,\dots,2^{n-2}-1\}} T_e) L_{i_k} \\ &= (\prod_{x \in s_{k,n-1}} T_x T_{xe} T_e) L_{i_k} \\ &= (\prod_{x \in s_{k,n-1}} L_{x,e}) L_{i_k}, \quad k \in \{1, \dots, n-1\}, \end{aligned}$$

$$\begin{aligned} g_{n,n} &= (\prod_{x \in s_{n,n}} T_x) L_{i_k} = (\prod_{x \in s_{n-1,n-1}} T_x \prod_{x \in \bar{s}_{n-1,n-1}} T_{xe}) L_{i_k} \\ &= (\prod_{x \in \bar{s}_{n,n}} (x, -x)) L_{i_k} \\ &= (\prod_{x \in \bar{s}_{n-1,n-1}} (x, -x))(\prod_{x \in s_{n-1,n-1}} (xe, -xe)) L_{i_k} \\ &= (\prod_{x \in \bar{s}_{n-1,n-1}} (x, -x)(xe, -xe))(\prod_{x \in Q_{n-1} e} (x, -x)) L_{i_k} \\ &= (\prod_{x \in s_{n-1,n-1}} L_{x,e}) h L_{i_k}, \\ K &= K_n = \langle g_{1,n}, g_{2,n}, \dots, g_{n,n} \rangle. \end{aligned}$$

Thus $K \leq Mlt_l(Q_n)$.

2. By (7), (8), K contains a unique element g such that $g(1) \in \{1, -1\}$. Since K is a group, $g = id$, thus $N \cap K = id$.

3. We established that $N \trianglelefteq G, K \leq G$, and $N \cap K = id$, therefore $N \rtimes K \leq G$. Recall that

$$\begin{aligned} [Mlt_l(Q_n) : Inn_l(Q_n)] &= |Q_n|, \text{ thus} \\ [Mlt_l(Q_n) : (Inn_l(Q_n) \times Z(Q_n))] &= [Mlt_l(Q_n) : Inn_l(Q_n)] / 2 = 2^n = |K|, \end{aligned}$$

and $(Inn_l(Q_n) \times Z(Q_n)) \rtimes K \cong Mlt_l(Q_n)$ follows. □

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