THE CAUSAL POSET IS DIRECTED BUT NOT LATTICE ORDERED

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Abstract

In the causal set approach to discrete quantum gravity the universe grows one element at a time in discrete steps. At each step the process has the form of a causal set (causet) and the "completed" universe is given by a path through a discretely growing chain of causets. The collection of causets forms a partially ordered set (poset) in a natural way. We first show that this poset is directed. We then give a counterexample which shows it is not lattice ordered.

1 Notation and Definitions

This section sets the notation and definitions that we will need concerning the causal set approach to discrete quantum gravity. For motivation and further details we refer the reader to [1, 2, 3, 4, 5, 6]. In this paper a finite partially ordered set (poset) will be called a *causet*. For a causet $(x, <)$ we denote the cardinality of x by $|x|$. All of our causets are unlabeled and we identify isomorphic causets. Let $\mathcal{P}_n$ be the collection of all causets with cardinality n, $n = 1, 2, \ldots$ and let $\mathcal{P} = \cup\mathcal{P}_n$ be the collection of all causets. An element $a \in x$ for $x \in \mathcal{P}$ is *maximal* if there is no $b \in x$ with $a < b$. If $x \in \mathcal{P}_n$ $y \in \mathcal{P}_{n+1}$ then x *produces* y if y is obtained from x by adjoining a single maximal element a to x. If x produces y we write $x \to y$.

The transitive closure of $\to$ makes $\mathcal{P}$ into a poset itself and we call $(\mathcal{P}, \to)$ the *causal poset*. For $x, y \in \mathcal{P}$ with $|x| < |y|$, a *path* from x to y is a finite sequence $\omega_1\omega_2\cdots\omega_m$ where $\omega_1 = x$, $\omega_m = y$ and $\omega_i \to \omega_{i+1}$, $i = 1, \ldots, m-1$. It is clear that $x < y$ if and only if there is a path from x to y. We view a causet x as a possible universe at time $|x|$. An infinite path starting at the one vertex causet x, is considered to be a "completed" universe that includes its histories.

If $a, b \in x$ with $x \in \mathcal{P}$ we say that a and b are *comparable* if one of the following hold: $a = b$, $a < b$, $b < a$. If a and b are not comparable, we say they are incomparable. Let $(P, <)$ be an arbitrary poset and let $A \subseteq P$. We say that $b \in P$ is an *upper bound* for A if $a \leq b$ for all $a \in A$. The poset P is *directed* if for any pair $a, b \in P$, the set $\{a, b\}$ has an upper bound. We call P *lattice ordered* if for any pair $a, b \in P$, the set $\{a, b\}$ has a least upper bound. That is, there exists an upper bound c for $\{a, b\}$ such that $c \leq d$ for any upper bound d. In the next section we show that the causal poset $(\mathcal{P}, \rightarrow)$ is directed, but is not lattice ordered. We interpret this as saying that for a given time, the paths of any two universes will cross at some later time.

2 Directed But Not Lattice Ordered

Theorem. *$\mathcal{P}$ is directed.*

Proof. Let $x, y \in \mathcal{P}$ with $|x| = |y|$. We can assume that $x \cap y = \emptyset$. Construct a path beginning with x and growing one element at a time with the vertices of y until the path arrives at the causet $z = x \cup y$ where the vertices of y are incomparable with all the vertices of x in z (in technical terms, z is the horizontal sum of x and y). We then have $x \leq z$. In a similar way, we can construct a path from y to z so that $y \leq z$. In the other case $|x| \neq |y|$ we can assume without loss of generality that $|y| < |x|$. Form a path beginning at y and ending at any caused y_1 with $|y_1| = |x|$. Then $y \leq y_1$ and from our previous work there exists a $z \in \mathcal{P}$ with $x, y \leq z$. Since $y \leq y_1$, we have that $x, y \leq z$. □

We now present a counterexample which shows that $\mathcal{P}$ is not lattice ordered. The counterexample is given in Figure 1 which employs the usual Hasse diagram method for displaying causets . In Figure 1, causets x and y both produce the distinct causets u and v. Thus u and v are both upper bounds for x and y. However, there is no smaller upper bound for x and y so $\{x, y\}$ has no least upper bound. We conjecture that this is the smallest counterexample possible.

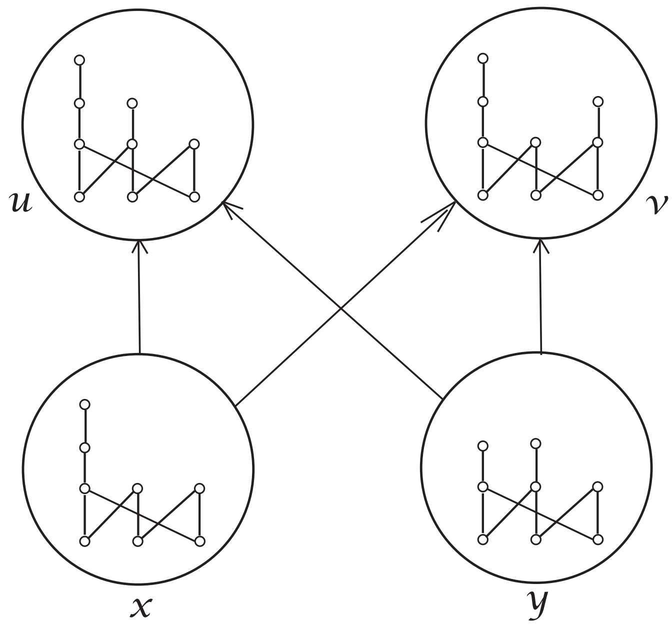

Figure 1

References

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