

# A note on the Relationship between Localization and Norm-1 Property

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**Abstract.** The paper focuses on the problem of localization in quantum mechanics. It is well known that it is not possible to define a localization observable for the photon by means of projection valued measures. Conversely, that is possible by using positive operator valued measures. On the other hand, projection valued measures imply a kind of localization which is stronger than the one implied by positive operator valued measure. It has been claimed that the norm-1 property would in some sense reduce the gap between the two kinds of localization. We give a necessary condition for the norm-1 property and show that it is not satisfied by several important localization observables.

Submitted to: *J. Phys. A: Math. Gen.*

## 1. Introduction

In the standard formulation of quantum mechanics the observables of a quantum system are represented by self-adjoint operators. The spectral theorem [1] assures the existence of a one-to-one correspondence between self-adjoint operators and Projection Valued Measures (PVMs). In particular, for each self-adjoint operator  $A$  there is a map (a PVM)  $E : \mathcal{B}(\mathbb{R}) \rightarrow \mathcal{E}(\mathcal{H})$  from the Borel  $\sigma$ -algebra of the reals to the space of projection operators on the Hilbert space  $\mathcal{H}$  such that  $E(\mathbb{R}) = \mathbf{1}$ , and, for each sequence of disjoint Borel sets  $\{\Delta_i\}_{i \in \mathbb{N}}$ ,  $\cup_{i=1}^{\infty} \Delta_i = \Delta$ ,

$$\sum_{i=1}^{\infty} E(\Delta_i) = E(\Delta)$$

where the convergence is in the weak operator topology.

As it is well known [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14], there are quantum observables (e.g. position observable for the photon, phase observable, time observable) that are not representable by means of self-adjoint operators or projection valued measures. (There are exceptions in the case of the time observable [15].).

A fruitful way to overcome the problem is to generalize the concept of observable by means of Positive Operator Valued Measures (POVMs) [5, 12, 10, 11, 16] of which the PVMs are a special case. In the following  $\mathcal{F}(\mathcal{H})$  denotes the space of positive linear operators less than or equal to the identity  $\mathbf{1}$ .

**Definition 1.1.** Let  $X$  be a topological space and  $\mathcal{B}(X)$  the Borel  $\sigma$ -algebra on  $X$ . A POVM is a map  $F : \mathcal{B}(X) \rightarrow \mathcal{F}(\mathcal{H})$  such that:

$$F\left(\bigcup_{n=1}^{\infty} \Delta_n\right) = \sum_{n=1}^{\infty} F(\Delta_n).$$

where,  $\{\Delta_n\}$  is a countable family of disjoint sets in  $\mathcal{B}(X)$  and the series converges in the weak operator topology. It is said to be normalized if

$$F(X) = \mathbf{1}.$$

**Definition 1.2.** A POVM is said to be commutative if

$$[F(\Delta_1), F(\Delta_2)] = \mathbf{0} \quad \forall \Delta_1, \Delta_2 \in \mathcal{B}(X). \quad (1)$$

**Definition 1.3.** A POVM is said to be orthogonal if

$$F(\Delta_1)F(\Delta_2) = \mathbf{0} \quad \text{if } \Delta_1 \cap \Delta_2 = \emptyset. \quad (2)$$

**Definition 1.4.** A PVM is an orthogonal, normalized POVM.

In the case of a PVM  $E$  we have  $\mathbf{0} = E(\Delta)[1 - E(\Delta)] = E(\Delta) - E^2(\Delta)$ . Therefore,  $E(\Delta)$  is a projection operator for every  $\Delta \in \mathcal{B}(X)$ . We have proved the following proposition.

**Proposition 1.5.** A PVM  $E$  on  $X$  is a map  $E : \mathcal{B}(X) \rightarrow \mathcal{E}(\mathcal{H})$  from the Borel  $\sigma$ -algebra of  $\mathcal{B}(X)$  to the space of projection operators on  $\mathcal{H}$ .

**Definition 1.6.** *The spectrum  $\sigma(F)$  of POVM  $F$  is the set of points  $x \in X$  such that  $F(\Delta) \neq \mathbf{0}$ , for any open set  $\Delta$  containing  $x$ .*

From a general theoretical viewpoint, the introduction of POVMs can be justified by analyzing the statistical description of a measurement [10] but, as we pointed out above, there are important physical motivations that go in the same direction. (See also [12, 16].)

The present note focuses on the problem of localization in quantum mechanics. We start by giving the definition of covariance which plays a key role in the definition of localization.

**Definition 1.7.** *Let  $G$  be a locally compact topological group. Let  $x \mapsto gx$ ,  $g \in G$ , be the action of  $G$  on a topological space  $X$ . Let  $U$  be a strongly continuous unitary representation of  $G$  in Hilbert space  $\mathcal{H}$ . A POVM  $F : \mathcal{B}(X) \rightarrow \mathcal{F}(\mathcal{H})$  is covariant with respect to  $G$  if, for any Borel set  $\Delta \in \mathcal{B}(X)$ ,*

$$U_g^\dagger F(\Delta) U_g = F(g\Delta)$$

where,  $g\Delta = \{x' \in X \mid x' = gx, x \in \Delta\}$ .

Now, we can proceed to give the definition of localization. Localization requires covariance with respect to a group  $G$  describing the kinematics of the system. We start by considering the general case of localization in an abstract topological space  $X$ . Then, we specialize  $X$  to  $\mathbb{R}^3$  for the case of space localization and to the phase space,  $\Gamma$ , if for the case of phase space localization. We have two possible definitions of localization; i.e., sharp localization and unsharp localization. Sharp localization is defined by requiring covariance of a PVM under the group  $G$ .

**Definition 1.8.** *Let  $G$  be a locally compact topological group describing the kinematics of a quantum system. Let  $x \mapsto gx$ ,  $g \in G$ , be the action of  $G$  on a topological space  $X$ . Let  $U$  be a strongly continuous unitary representation of  $G$  in Hilbert space  $\mathcal{H}$ . A PVM  $E : \mathcal{B}(X) \rightarrow \mathcal{F}(\mathcal{H})$  represents a sharp localization observable in  $X$  (with respect to  $(G, U)$ ) if*

$$U_g E(\Delta) U_g^\dagger = E(g\Delta).$$

The previous definition can be weakened by replacing the PVM by a POVM. That corresponds to an unsharp localization.

**Definition 1.9.** *Let  $G$  be a locally compact topological group describing the kinematics of a quantum system. Let  $x \mapsto gx$ ,  $g \in G$ , be the action of  $G$  on a topological space  $X$ . Let  $U$  be a strongly continuous unitary representation of  $G$  in Hilbert space  $\mathcal{H}$ . A POVM  $F : \mathcal{B}(X) \rightarrow \mathcal{F}(\mathcal{H})$  represents an unsharp localization observable in  $X$  (with respect to  $(U, G)$ ) if*

$$U_g E(\Delta) U_g^\dagger = E(g\Delta).$$

Now, we specialize to the case of relativistic localization in  $\mathbb{R}^3$ . In the relativistic case, the relevant group is the Poincaré group and sharp localization is defined as follows

[8, 11]: Let  $W$  be a continuous unitary representation of the universal covering of the Poincaré group. Let  $U$  be the restriction of  $W$  to the universal covering group  $ISU(2) = \{(\mathbf{a}, B), |\mathbf{a} \in \mathbb{R}^3, B \in SU(2)\}$  of the Euclidean group and  $\Lambda : SU(2) \rightarrow SO(3)$  the universal covering homomorphism. A quantum system is said to be Wightman localizable if there is a PVM  $E : \mathcal{B}(\mathbb{R}^3) \rightarrow \mathcal{E}(\mathcal{H})$  such that

$$U(\mathbf{a}, B)E(\Delta)U^\dagger(\mathbf{a}, B) = E(\mathbf{a} + \Lambda(B)\Delta) \quad (\text{sharp localization}).$$

The covariance assures that the results of a localization measurement do not depend on the choice of the origin and the orientation of the reference frame. As we have just remarked, in the case of the photon, sharp localization is impossible [5, 8, 2, 7]. Conversely, localization of the photon in  $\mathbb{R}^3$  might be described by means of POVMs  $F : \mathcal{B}(\mathbb{R}^3) \rightarrow \mathcal{F}(\mathcal{H})$  such that

$$U(\mathbf{a}, B)F(\Delta)U^\dagger(\mathbf{a}, B) = F(\mathbf{a} + \Lambda(B)\Delta) \quad (\text{unsharp localization}).$$

By specializing  $X$  to a phase space  $\Gamma$  we get the concept of unsharp localization in phase space which requires the POVM,  $F$ , to be defined on  $\Gamma = X$  and to be covariant with respect to a group  $G$  which characterizes the symmetries of the system [17]. Examples of symmetry groups are the Galilei group in the non relativistic case and the Poincaré group in the relativistic case. (See section 3 for further details.) For the case of the photons in their phase space the relevant group is the Poincaré group. [17]

Clearly the introduction of the POVMs for the description of localization observables implies a change in the standard concept of localization in quantum mechanics. If a localization observable is described by a covariant PVM  $E$  (sharp localization) then, for any Borel set  $\Delta$  such that  $E(\Delta) \neq \mathbf{0}$ , there exists a unit vector  $\psi$  for which  $\langle \psi, E(\Delta)\psi \rangle = 1$ ; i.e., the probability that a measure of the position of the system in the state  $\psi$  gives a result in  $\Delta$  is one. Conversely, if a localization observable is described by a POVM  $F$ , there are Borel sets  $\Delta$  such that  $0 < \langle \psi, F(\Delta)\psi \rangle < 1$  for any vector  $\psi$  (unsharp localization). There are even covariant POVMs such that the condition  $\mathbf{0} \leq F(\Delta) < \mathbf{1}$  holds for any  $\Delta \in \mathcal{B}(X)$  with  $\mu(X - \Delta) > 0$ , where  $\mu$  is the measure on  $X$ . [18]

Sharp localizability being untenable in relativistic theory, [19, 20] we need to switch to unsharp localizability. It is worth remarking that the relationships between localization and relativistic causality is quite problematic [7, 13].

What is said above raises the following question for the unsharp case: Is it true that, for any Borel set  $\Delta$  with  $F(\Delta) \neq \mathbf{0}$ , there is a family of unit vectors  $\psi_n$  such that  $\lim_{n \rightarrow \infty} \langle \psi_n, F(\Delta)\psi_n \rangle = 1$ ? Whenever such a property holds, we say that the POVM has the norm-1 property. [21, 22] Clearly, the norm-1 property implies a kind of unsharp localization which is closer to the sharp one. Indeed, for each  $\epsilon$ , it is possible to find a unit vector  $\psi$  such that the probability  $\langle \psi, F(\Delta)\psi \rangle$  that a measure gives a result in  $\Delta$  is greater than  $1 - \epsilon$ .

In the present note we analyze some general aspects of the concept of the norm-1 property which are related to the concepts of absolute continuity and uniform continuity

of a POVM (section 2) and derive some consequences for the concept of localization in phase space and configuration space (sections 3 and 4). In particular, we give a necessary condition for the norm-1 property to hold and prove that it is not satisfied for a class of localization observables in phase space as well as for the corresponding marginal observables.

## 2. A necessary condition for the norm-1 property

In the present section, we give a necessary condition for the norm-1 property of a POVM defined on a topological manifold of dimension  $n$ . First, we recall the definition of the norm-1 property.

**Definition 2.1.** *A POVM  $F : \mathcal{B}(X) \rightarrow \mathcal{F}(\mathcal{H})$  has the norm-1-property if  $\|F(\Delta)\| = 1$ , for each  $\Delta \in \mathcal{B}(X)$  such that  $F(\Delta) \neq \mathbf{0}$ .*

The following proposition points out the physical meaning of the norm-1 property.

**Proposition 2.2.**  *$F$  has the norm-1 property if and only if, for each  $\Delta \in \mathcal{B}(X)$  such that  $F(\Delta) \neq \mathbf{0}$ , there is a sequence of unit vectors  $\psi_n$  such that  $\lim_{n \rightarrow \infty} \langle \psi_n, F(\Delta) \psi_n \rangle = 1$ .*

*Proof.* Suppose  $F$  has the norm-1 property and  $\|F(\Delta)\| \neq \mathbf{0}$ . Then,

$$1 = \|F(\Delta)\| = \sup_{\|\psi\|=1} \{\langle \psi, F(\Delta) \psi \rangle\}.$$

Hence, there is a sequence  $|\psi_n\rangle$  such that

$$\lim_{n \rightarrow \infty} \langle \psi_n, F(\Delta) \psi_n \rangle = 1. \quad (3)$$

Conversely, since  $F(\Delta) \leq \mathbf{1}$ , equation (3) implies  $\|F(\Delta)\| = 1$ .  $\square$

In quantum mechanics, an observable is a PVM or a POVM,  $F$ , while the state of a system is a trace class positive operator of trace 1,  $\rho$ . In particular, a pure state is a projector  $\rho = P_\psi = |\psi\rangle\langle\psi|$  where  $\psi$  is a unit vector in a Hilbert space  $\mathcal{H}$ . From an operational viewpoint, the states represent the *preparation instruments* while the observables represent the *measurement instruments*.[23, 24, 16] The connection between the two mathematical terms (states and observables) and the experimental data is given by the expression  $p_\psi^F(\Delta) := \text{Tr}(\rho F(\Delta))$  which is interpreted as the probability that the pointer of the measurement instrument (represented by  $F$ ) gives a result in  $\Delta$  if the state of the system is  $\rho$ . In the case of a pure state  $\rho = P_\psi$ , we have  $p_\psi^F(\Delta) = \langle \psi, F(\Delta) \psi \rangle$  so that the unit vectors,  $\psi$ , are usually identified with the pure states of the system. It is then clear why proposition 3 explains the physical meaning of the norm-1 property; i.e., if  $F$  has the norm-1 property then, for any  $\epsilon > 0$ , there is a pure state  $P_\psi$  such that  $p_\psi^F(\Delta) > 1 - \epsilon$ . That implies a kind of localization very close to the one we can realize with the PVMs. In other words, the norm-1 property implies that, for any  $\Delta$ , there exists a preparation procedure such that the quantum mechanical system can be localized within  $\Delta$  as accurately as desired although not sharply. It is worth noticing that

a slightly different terminology can be found in the literature. For example, the property of a quantum system to be localized as accurately as desired (although not sharply), has been called *concentrability* by Castrigiano [8] who also proved the concentrability (in particular he proved the norm-1 property for any Borel set with non-void interior) of a class of Euclidean covariant POVMs with the property of being dilational covariant [8]. Toller [25] uses the term “definite” for systems having the norm-1 property for any Borel set with non-void interior and gives a sufficient condition for the definiteness of a class of covariant POVMs defined on the Minkowski space-time.

Now, we need to introduce the concept of absolute continuity which will be helpful in the study of localization in phase space.

**Definition 2.3.** Let  $F : \mathcal{B}(X) \rightarrow \mathcal{F}(\mathcal{H})$  be a POVM and  $\nu : X \rightarrow \mathbb{R}$  a regular measure.  $F$  is absolutely continuous with respect to  $\nu$  if there exists a number  $c$  such that

$$\|F(\Delta)\| \leq c\nu(\Delta), \quad \forall \Delta \in \mathcal{B}(X).$$

**Definition 2.4.** Let  $F : \mathcal{B}(X) \rightarrow \mathcal{F}(\mathcal{H})$  be a POVM.  $F$  is said to be uniformly continuous at  $\Delta$  if, for any disjoint decomposition  $\Delta = \bigcup_{i=1}^{\infty} \Delta_i$ ,  $\lim_{n \rightarrow \infty} \sum_{i=1}^n F(\Delta_i) = F(\Delta)$  in the uniform operator topology.  $F$  is said uniformly continuous if it is uniformly continuous at each  $\Delta \in \mathcal{B}(X)$ .

**Proposition 2.5.** A POVM  $F$  is uniformly continuous at  $\Delta$  if and only if, it is uniformly continuous from below at  $\Delta$ , i.e., for any increasing sequence  $\Delta_i \uparrow \Delta$ ,

$$\lim_{n \rightarrow \infty} \|F(\Delta) - F(\Delta_i)\| = 0.$$

$F$  is uniformly continuous if and only if it is uniformly continuous from below at each  $\Delta$ .

*Proof.* Suppose that  $\lim_{n \rightarrow \infty} \|F(\Delta) - F(\Delta_i)\| = 0$  whenever  $\Delta_i \uparrow \Delta$ . Let  $\{\Delta_i\}_{i \in \mathbb{N}}$  be a sequence of disjoint sets such that  $\bigcup_{i=1}^{\infty} \Delta_i = \Delta$ . Then, we can define the family of sets  $\overline{\Delta}_n = \bigcup_{i=1}^n \Delta_i$ . We have  $\overline{\Delta}_i \uparrow \Delta$ . Therefore,

$$\lim_{n \rightarrow \infty} \|F(\Delta) - \sum_{i=1}^n F(\Delta_i)\| = \lim_{n \rightarrow \infty} \|F(\Delta) - F(\overline{\Delta}_n)\| = 0$$

Conversely, suppose that  $F$  is uniformly continuous at  $\Delta$ . Let  $\Delta_i$  be such that  $\Delta_i \uparrow \Delta$ . We can define the family of sets  $\overline{\Delta}_i = \Delta_i - \Delta_{i-1}$  with  $\Delta_0 = \emptyset$ . We have,  $\overline{\Delta}_i \cap \overline{\Delta}_j = \emptyset$ ,  $i \neq j$ . Moreover,  $\Delta_n = \bigcup_{i=1}^n \overline{\Delta}_i$  and  $\bigcup_{i=1}^{\infty} \overline{\Delta}_i = \Delta$ . Therefore,

$$\lim_{n \rightarrow \infty} \|F(\Delta) - F(\Delta_n)\| = \lim_{n \rightarrow \infty} \|F(\Delta) - \sum_{i=1}^n F(\overline{\Delta}_i)\| = 0.$$

If  $F$  is uniformly continuous, the last reasoning is true for any  $\Delta$ . □

**Proposition 2.6.**  $F$  is uniformly continuous if and only if,  $\lim_{i \rightarrow \infty} \|F(\Delta_i)\| = 0$  whenever  $\Delta_i \downarrow \emptyset$ .

*Proof.* Suppose that  $\lim_{i \rightarrow \infty} \|F(\Delta_i)\| = 0$  whenever  $\Delta_i \downarrow \emptyset$ . Let  $\{\Delta_i\}_{i \in \mathbb{N}}$  be a disjoint sequence of sets such that  $\cup_{i=1}^{\infty} \Delta_i = \Delta$ . We have  $\Delta - \cup_{i=1}^n \Delta_i \downarrow \emptyset$ . Therefore,

$$\lim_{n \rightarrow \infty} \|F(\Delta) - \sum_{i=1}^n F(\Delta_i)\| = \lim_{n \rightarrow \infty} \|F(\Delta - \cup_{i=1}^n \Delta_i)\| = 0.$$

Conversely, suppose  $F$  is uniformly continuous and  $\Delta_i \downarrow \emptyset$ . We can define the family of sets  $\overline{\Delta}_i = \Delta_1 - \Delta_i$ . Clearly,  $\overline{\Delta}_i \uparrow \Delta_1$ . Therefore, by proposition 2.5,

$$\lim_{i \rightarrow \infty} \|F(\Delta_i)\| = \lim_{i \rightarrow \infty} \|F(\Delta_i) - F(\Delta_1) + F(\Delta_1)\| = \lim_{i \rightarrow \infty} \|F(\Delta_1) - F(\overline{\Delta}_i)\| = 0.$$

□

Now, we can prove a necessary condition for the norm-1-property.

**Theorem 2.7.** *Let  $F : X \rightarrow \mathcal{F}(\mathcal{H})$  be uniformly continuous and let  $\sigma(F)$  be the spectrum of  $F$ . Then,  $F$  has the norm-1-property only if  $\|F(\{x\})\| \neq 0$  for each  $x \in \sigma(F)$ .*

*Proof.* We proceed by contradiction. Suppose that  $F$  has the norm-1 property and that there exists  $x \in \sigma(F)$ , such that  $\|F(\{x\})\| = 0$ . Since  $F$  is regular, for any  $x \in \sigma(F)$ , there is a decreasing family of open sets  $\{\Delta_i\}_{i \in \mathbb{N}} \subset B(X)$ ,  $x \in \Delta_i$ ,  $\forall i \in \mathbb{N}$ , such that

$$w - \lim_{i \rightarrow \infty} F(\Delta_i) = F(\{x\}).$$

Moreover,

$$F(\{x\}) = w - \lim_{i \rightarrow \infty} F(\Delta_i) = w - \lim_{n \rightarrow \infty} F(\cap_{i=1}^n \Delta_i) = F(\Delta) \quad (4)$$

where  $\Delta = \cap_{i=1}^{\infty} \Delta_i$ .

By the uniform continuity of  $F$ , proposition 2.6, and equation (4),

$$F(\{x\}) = F(\Delta) = u - \lim_{i \rightarrow \infty} F(\Delta_i) \quad (5)$$

By equation (5) and the norm-1 property,

$$\begin{aligned} 1 &= \lim_{i \rightarrow \infty} \|F(\Delta_i)\| = \lim_{i \rightarrow \infty} \|F(\Delta_i) - F(\{x\}) + F(\{x\})\| \\ &\leq \lim_{i \rightarrow \infty} \|F(\Delta_i) - F(\{x\})\| + \|F(\{x\})\| = 0. \end{aligned}$$

□

Notice that the absolute continuity with respect to a finite regular measure implies the uniform continuity.

**Theorem 2.8.** *Let  $F$  be absolutely continuous with respect to a finite measure  $\nu$ . Then,  $F$  is uniformly continuous.*

*Proof.* Suppose  $\Delta_i \uparrow \Delta$ . We have

$$\begin{aligned} \lim_{i \rightarrow \infty} \|F(\Delta) - F(\Delta_i)\| &= \lim_{i \rightarrow \infty} \|F(\Delta - \Delta_i)\| \\ &\leq c \lim_{i \rightarrow \infty} \nu(\Delta - \Delta_i) = 0. \end{aligned}$$

Proposition 2.5 ends the proof. □

Theorems 2.7 and 2.8 imply the following corollary

**Corollary 2.9.** *Let  $F$  be absolutely continuous with respect to a finite measure  $\nu$ . Then  $F$  has the norm-1 property only if  $\|F(\{x\})\| \neq 0$  for each  $x \in \sigma(F)$ .*

In the case that  $F$  is absolutely continuous with respect to an infinite measure, we have the following weak version of theorem 2.8.

**Theorem 2.10.** *Suppose  $F$  is absolutely continuous with respect to a regular measure  $\nu$ . Suppose,  $\Delta$  is such that  $\nu(\Delta) < \infty$ . Then,  $F$  is uniformly continuous at  $\Delta$ .*

*Proof.* By proposition 2.5,  $F$  is uniformly continuous at  $\Delta$  if and only if for any increasing family of sets  $\{\Delta_i\}_{i \in \mathbb{N}}$ ,  $\Delta_i \uparrow \Delta$ ,

$$\lim_{i \rightarrow \infty} \|F(\Delta) - F(\Delta_i)\| = 0.$$

Suppose  $\nu(\Delta) < \infty$ . Suppose  $\Delta_i \uparrow \Delta$ . Since  $\Delta_i \subset \Delta$ ,  $\nu(\Delta_i) \leq \nu(\Delta) < \infty$  for every  $i \in \mathbb{N}$ . By the continuity of  $\nu$ ,  $\lim_{i \rightarrow \infty} \nu(\Delta - \Delta_i) = 0$ . Hence, by the absolute continuity of  $F$ ,

$$\begin{aligned} \lim_{i \rightarrow \infty} \|F(\Delta) - F(\Delta_i)\| &= \lim_{i \rightarrow \infty} \|F(\Delta - \Delta_i)\| \\ &\leq c \lim_{i \rightarrow \infty} \nu(\Delta - \Delta_i) = 0. \end{aligned}$$

□

**Theorem 2.11.** *Suppose  $F$  absolutely continuous with respect to a regular measure  $\nu$ . Suppose,  $\Delta$  is such that  $\nu(\Delta) < \infty$ . Then, for each decreasing family of sets  $\Delta_i \downarrow \Delta$  such that  $\nu(\Delta_i) < \infty$ ,*

$$\lim_{i \rightarrow \infty} \|F(\Delta_i) - F(\Delta)\| = 0.$$

*Proof.* Suppose  $\nu(\Delta) < \infty$ ,  $\Delta_i \downarrow \Delta$  and  $\nu(\Delta_i) < \infty$ . By the continuity of  $\nu$ ,  $\lim_{i \rightarrow \infty} \nu(\Delta_i - \Delta) = 0$ . Hence, by the absolute continuity of  $F$ ,

$$\begin{aligned} \lim_{i \rightarrow \infty} \|F(\Delta_i) - F(\Delta)\| &= \lim_{i \rightarrow \infty} \|F(\Delta_i - \Delta)\| \\ &\leq c \lim_{i \rightarrow \infty} \nu(\Delta_i - \Delta) = 0. \end{aligned}$$

□

Corollary 2.9 can be generalized as follows.

**Theorem 2.12.** *Let  $F : \mathcal{B}(X) \rightarrow \mathcal{F}(\mathcal{H})$  be absolutely continuous with respect to a regular measure  $\nu$ . Then,  $F$  has the norm-1 property only if  $\|F(\{x\})\| \neq 0$  for each  $x \in X$  such that  $\nu(\{x\}) < \infty$ .*

*Proof.* We proceed by contradiction. Suppose that  $F$  has the norm-1 property. Suppose that  $x \in \sigma(F)$  is such that  $\nu(\{x\}) < \infty$  and  $\|F(\{x\})\| = 0$ . Thanks to the regularity of  $\nu$ , there is a decreasing family of open sets  $\{\Delta_i\}_{i \in \mathbb{N}} \subset \mathcal{B}(X)$ ,  $x \in \Delta_i$ ,  $\forall i \in \mathbb{N}$ , such that

$\lim_{i \rightarrow \infty} \nu(\Delta_i) = \nu(\{x\})$ . Notice that  $\nu(\{x\}) < \infty$  implies that there exists an index  $n_0$  such that  $\nu(\Delta_n) < \infty$  for each  $n \geq n_0$ . By the continuity from above of  $\nu$  [26],

$$\nu(\{x\}) = \lim_{i \rightarrow \infty} \nu(\Delta_i) = \lim_{n \rightarrow \infty} \nu(\cap_{i=1}^n \Delta_i) = \nu(\Delta)$$

where  $\Delta = \cap_{i=1}^{\infty} \Delta_i$ . Then,  $\nu(\Delta) < \infty$  and, by the absolute continuity of  $F$  with respect to  $\nu$ ,  $F(\Delta) = F(\{x\})$ . Then, by theorem 2.11,

$$\lim_{i \rightarrow \infty} \|F(\Delta_i) - F(\{x\})\| = \lim_{i \rightarrow \infty} \|F(\Delta_i) - F(\Delta)\| = 0$$

By the norm-1 property,

$$\begin{aligned} 1 &= \lim_{i \rightarrow \infty} \|F(\Delta_i)\| = \lim_{i \rightarrow \infty} \|F(\Delta_i) - F(\{x\}) + F(\{x\})\| \\ &\leq \lim_{i \rightarrow \infty} \|F(\Delta_i - \{x\})\| + \|F(\{x\})\| = 0. \end{aligned}$$

□

The necessary condition in Theorem 2.12 applies also in the case of an absolutely continuous POVM  $F$  such that  $\|F(\Delta)\| = 1$  for each Borel set with non-void interior.

**Definition 2.13.** A POVM  $F : \mathcal{B}(X) \rightarrow \mathcal{F}(\mathcal{H})$  is concentratable or definite if  $\|F(\Delta)\| = 1$  for any Borel set  $\Delta \in \mathcal{B}(X)$  with non-void interior.

The same argument used in order to prove theorem 2.12 can be used to prove the following theorem.

**Theorem 2.14.** Let  $F : \mathcal{B}(X) \rightarrow \mathcal{F}(\mathcal{H})$  be absolutely continuous with respect to a regular measure  $\nu$ . Then,  $F$  is concentratable or definite only if  $\|F(\{x\})\| \neq 0$  for each  $x \in X$  such that  $\nu(\{x\}) < \infty$ .

### 3. Localization in phase space

In the present section, we show that localization observables in phase space cannot satisfy the norm-1 property. In the following the phase space is denoted by the symbol  $\Gamma$ . First, we recall some key elements of the phase space approach to quantum mechanics [5, 28, 6, 16]. We follow References [16, 28].

The main idea is that one can represent the state  $\rho$  of a quantum system (i.e., a trace class positive operator of trace 1) by means of a distribution function  $f_\rho(\mathbf{q}, \mathbf{p})$  on a suitable phase space. At variance with the Wigner approach [29], the distribution functions are positive definite. Wigner's theorem [29] forbids that the marginals of the distribution functions satisfy the following relations:

$$\begin{aligned} \int_{\Delta} d\mathbf{q} \int f_\rho(\mathbf{q}, \mathbf{p}) d\mathbf{p} &= \text{Tr}(\rho Q(\Delta)), \\ \int_{\Delta} d\mathbf{p} \int f_\rho(\mathbf{q}, \mathbf{p}) d\mathbf{q} &= \text{Tr}(\rho P(\Delta)), \end{aligned} \tag{6}$$

where  $Q(\Delta)$  and  $P(\Delta)$  are the spectral measures corresponding to the position and momentum operators respectively. Relations (6) might be replaced by

$$\begin{aligned} \int_{\Delta} d\mathbf{q} \int f_{\rho}(\mathbf{q}, \mathbf{p}) d\mathbf{p} &= \text{Tr}(\rho F^Q(\Delta)), \\ \int_{\Delta} d\mathbf{p} \int f_{\rho}(\mathbf{q}, \mathbf{p}) d\mathbf{q} &= \text{Tr}(\rho F^P(\Delta)). \end{aligned}$$

where  $F^Q(\Delta)$  and  $F^P(\Delta)$  are POVMs corresponding to  $Q$  and  $P$  respectively. In particular,  $F^Q(\Delta)$  and  $F^P(\Delta)$  are the smearing of the position and momentum operators  $Q$  and  $P$

$$\begin{aligned} F^Q(\Delta) &= \int_{\mathbb{R}} \omega_{\Delta}(x) dQ_x, \\ F^P(\Delta) &= \int_{\mathbb{R}} \nu_{\Delta}(x) dP_x, \end{aligned}$$

and are called unsharp position and momentum observables [30, 31, 32, 33, 34, 35]. The maps  $\omega$  and  $\nu$  are such that  $\omega_{\Delta}(\cdot)$  and  $\nu_{\Delta}(\cdot)$  are measurable functions for each  $\Delta$  and  $\omega_{(\cdot)}(x)$  and  $\nu_{(\cdot)}(x)$  are probability measures for each  $x$ . They are usually called Markov kernels and describe a stochastic diffusion of the standard observables  $Q$  and  $P$  [36, 37]. That is why  $F^Q$  and  $F^P$  are usually called the unsharp version of the sharp observables  $Q$  and  $P$  respectively. All that shows that POVMs play a key role in the phase space formulation. Moreover, it is worth remarking that a derivation of classical and quantum mechanics in a unique mathematical framework is possible [38, 39].

One of the main steps in this approach is the construction of the phase space. In brief, we can say that there is a procedure that starting from a Lie group  $G$  allows the classification of all the closed subgroups  $H \subset G$  such that  $G/H$  is a symplectic space (i.e, a phase space). For example, in the case of the Galilei group, a possible choice for  $H$  is the group  $H = SO_3$ . Then,  $\Gamma = G/H = \mathbb{R}^3 \times \mathbb{R}^3$ , which coincides with the phase space of classical mechanics. A different choice of  $H$  generates a different phase space. In other words, the procedure allows the calculation of all the phase spaces corresponding to a locally compact Lie group,  $G$ , with a finite dimensional Lie algebra. Once we have the phase space, we can look for a strongly continuous unitary representation of  $G$  in a Hilbert space  $\mathcal{H}$  and then we can define the localization observable [16].

**Definition 3.1** (See [16].). *Let  $G$  be a locally compact topological group,  $H$  a closed subgroup of  $G$ ,  $U$  a strongly continuous unitary irreducible representation of  $G$  in a complex Hilbert space  $\mathcal{H}$  and  $\mu$  a volume measure on  $G/H$ . A localization observable is represented by a POVM*

$$A^{\eta}(\Delta) = \int_{\Delta} |U(\sigma(x))\eta\rangle\langle U(\sigma(x))\eta| d\mu(x).$$

where,  $\sigma : G/H \mapsto G$  is a measurable map and  $\eta$  is a unit vector such that

$$\int_{G/H} |U(\sigma(x))\eta\rangle\langle U(\sigma(x))\eta| d\mu(x) = \mathbf{1}. \quad (7)$$

As we said above, from an operational viewpoint, the POVM  $A^\eta : \mathcal{B}(G/H) \rightarrow \mathcal{F}(\mathcal{H})$  represents a measurement procedure. The novelty of the phase space approach is that  $A^\eta$  is defined on the space of Borel subsets of the phase space  $\Gamma = G/H$  and therefore the measurement procedure it describes is of a quite general kind. For example, if  $G$  is the Galilei group and  $H = \mathbb{R}^2 \times \mathbb{R}^2 \times SO(3)$ , then  $G/H = \mathbb{R}_q \times \mathbb{R}_p$ ,  $\mu$  is the Lebesgue measure,  $U(\sigma(x)) = U_{q,p} = e^{-iqP}e^{ipQ}$ ,  $\eta \in L^2(\mathbb{R})$  and the localization observable reads

$$A(\Delta_q \times \Delta_p) = \int_{\Delta_q \times \Delta_p} |U_{q,p}\eta\rangle\langle U_{q,p}\eta| d\mathbf{q} d\mathbf{p}.$$

The marginals

$$F^Q(\Delta) := A^\eta(\Delta \times \mathbb{R}_p) = \int_{-\infty}^{\infty} (\mathbf{1}_\Delta * |\eta|^2)(x) dQ_x, \quad \Delta \in \mathcal{B}(\mathbb{R}), \quad (8)$$

$$F^P(\Delta) := A^\eta(\mathbb{R}_q \times \Delta) = \int_{-\infty}^{\infty} (\mathbf{1}_\Delta * |\tilde{\eta}|^2)(x) dP_x, \quad \Delta \in \mathcal{B}(\mathbb{R}) \quad (9)$$

where,  $\tilde{\eta}$  is the Fourier transform of  $\eta$ , are the unsharp position and momentum observables respectively [30]. Notice that the map  $\mu_\Delta(x) := \mathbf{1}_\Delta * |\eta(x)|^2$  defines a Markov kernel.

Therefore,  $A^\eta(\Delta_q \times \Delta_p)$  is a joint measurement for the unsharp position and momentum observables. In other words, the phase space approach allows the description of joint measurements of unsharp position and momentum also if they do not commute; i.e.,  $[F^Q(\Delta_q), F^P(\Delta_p)] \neq \mathbf{0}$ . That is one of the main advantages from the physical viewpoint of using the phase space approach outlined above. A second relevant aspect we would like to note is that Definition 3.1 allows the introduction of a quantization procedure [16, 40]. Indeed, for any real-valued Borel function  $f$ ,

$$A^\eta(f) = \int_{G/H} f(x) |U(\sigma(x))\eta\rangle\langle U(\sigma(x))\eta| d\mu(x)$$

defines a self-adjoint operator which is positive whenever  $f$  is positive. Moreover, for any pure state  $\rho = |\psi\rangle\langle\psi|$ ,

$$\text{Tr}(A^\eta(f)\rho) = \int_{G/H} f(x) f_\rho(x) d\mu(x)$$

where  $f_\rho(x) = |\langle U(\sigma(x))\eta, \psi \rangle|^2$ . The last relevant aspect on which we would like to remark is that in the phase space approach a relativistically conserved positive current can be defined [6, 28, 40].

Before we proceed with the analysis of the norm-1 property of the phase space localization observables, it is worth stressing the dependence of the localization observable  $A^\eta(\cdot)$  on the vector  $\eta$ . The physical relevance of such a dependence has been analyzed in Ref.s [28, 16, 40]. We limit ourselves to remark that  $\{|U(\sigma(x))\eta\rangle\langle U(\sigma(x))\eta|, x \in G/H\}$  defines a coherent state basis and that  $\eta$  can be interpreted as characterizing the measuring device in the sense that the interaction of the measuring device (described by  $A^\eta(\cdot)$ ) with the system prepared in the state  $\rho = P_\psi$  depends on the transition probability  $|\langle U(\sigma(x))\eta, \psi \rangle|^2$  and hence on  $\eta$ .

**Theorem 3.2.** [16] *The POVM  $A^\eta$  defined in Definition 3.1 is covariant with respect to  $U$ .*

A general property of the localization observables in Definition 3.1 is that they are absolutely continuous with respect to the measure  $\mu$ .

**Theorem 3.3.** [16] *The POVM in Definition 3.1 is absolutely continuous with respect to  $\mu$ .*

*Proof.* For each  $\psi \in \mathcal{H}$ ,

$$\begin{aligned}\langle \psi, A^\eta(\Delta)\psi \rangle &= \int_{\Delta} \langle \psi, U(\sigma(x))\eta \rangle \langle U(\sigma(x))\eta, \psi \rangle d\mu(x) \\ &= \int_{\Delta} |\langle \psi, U(\sigma(x))\eta \rangle|^2 d\mu(x) \leq \int_{\Delta} d\mu(x).\end{aligned}$$

□

The localization of the photon in phase space was introduced in Ref. [17] with the same procedure we just described. Therefore, at variance with the usual definition of localization (where the covariance under the Euclidean group is required), localization in phase space requires that  $F$  is covariant with respect to the group which describes the symmetry of the system (the Galilei group in the non-relativistic case and the Poincaré group in the relativistic case).

Before we prove that the norm-1 property is not possible for localization observables in phase space, we want to give a physical motivation which is based on the Uncertainty Relations. Let  $F$  be a phase space localization observable covariant with respect to the Galilei group. In this case the phase space  $\Gamma$  corresponding to the system can be chosen to be  $\Gamma = \mathbb{R}_q^3 \times \mathbb{R}_p^3 = \mathbb{R}^3 \times \mathbb{R}^3$  (See [16], page 425.) Then, suppose that the norm-1 property holds, i.e., for each Borel set  $\Delta_q \times \Delta_p \in \Gamma$  with  $F(\Delta_q \times \Delta_p) \neq \mathbf{0}$ , there exists a family of unit vectors  $\psi_n$  such that

$$\lim_{n \rightarrow \infty} \langle \psi_n, F(\Delta_q \times \Delta_p)\psi_n \rangle = 1 \quad (10)$$

where,  $\langle \psi_n, F(\Delta_q \times \Delta_p)\psi_n \rangle$  is interpreted as the probability that an outcome of a joint measurement of the unsharp position and momentum observables is in  $\Delta_q \times \Delta_p$  when the state is  $P_{\psi_n} = |\psi_n\rangle\langle\psi_n|$ . Therefore, the violation of Uncertainty Relations comes from the fact that (10) holds for any Borel set  $\Delta_q \times \Delta_p$ .

In the following, we apply Theorem 2.12 to the case of the Galilei group  $G = \{(t, \mathbf{q}, \mathbf{p}, R) | t \in \mathbb{R}, \mathbf{q}, \mathbf{p} \in \mathbb{R}^3, R \in SO(3)\}$  with  $H = \{(t, \mathbf{0}, \mathbf{0}, R) | t \in \mathbb{R}, R \in SO(3)\}$ . Therefore,  $G/H = \mathbb{R}_q^3 \times \mathbb{R}_p^3 = \mathbb{R}^3 \times \mathbb{R}^3$ . In that case, the invariant measure is the Lebesgue measure. In the following we set  $\mathbf{x} = (\mathbf{q}, \mathbf{p}) \in \mathbb{R}^3 \times \mathbb{R}^3$ .

Theorem 2.12 implies that the POVM  $A^\eta$  in Definition 3.1 does not have the norm-1 property.

**Theorem 3.4.** *The localization observable represented by the POVM  $A^\eta$  with  $G/H = \mathbb{R}^3 \times \mathbb{R}^3$  and  $\mu$  the Lebesgue measure does not have the norm-1 property.*

*Proof.* Let  $\mathbf{x} \in G/H$ . By Theorem 3.3,

$$\|A^\eta(\{\mathbf{x}\})\| \leq \mu(\{\mathbf{x}\}).$$

Since  $\mu$  is the Lebesgue measure on  $G/H$ ,

$$\|A^\eta(\{\mathbf{x}\})\| \leq \mu(\{\mathbf{x}\}) = 0.$$

Theorem 2.12 completes the proof.  $\square$

An analogous result can be proved in the case of massless relativistic particles. In the general relativistic case  $G$  is the double cover  $T^4 \oslash SL(2, \mathbb{C})$  of the Poincaré group. In particular,  $T^4$  is the Minkowski space and  $SL(2, \mathbb{C})$  is the double cover of the Lorentz group  $L$ ,  $A : SL(2, \mathbb{C}) \rightarrow L$ . The symbol  $\oslash$  denotes the semidirect product. In the massless relativistic case the relevant subgroup is  $H = \mathbb{R} \oslash (\mathbb{R}^2 \oslash \tilde{O}(2))$  where  $\tilde{O}(2)$  is the double cover of the group of rotations in  $\mathbb{R}^2$ . The invariant measure on  $G/H$  is (see equation (344), page 463, in Ref.[16])

$$d\mu = d(\alpha)d(\gamma)d(\delta) \times (p^0)^{-1}dp^1 \wedge dp^2 \wedge dp^3 \quad (11)$$

where  $\alpha = a_\mu(A[p_0])^\mu$ ,  $\gamma = a_\mu(A[u_0])^\mu$ ,  $\delta = a_\mu(A[v_0])^\mu$ , where  $p_0 = (1, 0, 0, 1)$ ,  $u_0 = (0, 1, 0, 0)$ ,  $v_0 = (0, 0, 1, 0)$ , and  $(\mathbf{a}, A)$  is an element of the double cover of the Poincaré group. Thus  $\alpha, \gamma, \delta$  are in  $\mathbb{R}$ . Hence, we have a representation of the zero mass particles. Moreover,  $\mu$  is zero in each single point subset of the phase space so that the reasoning in the proof of theorem 3.4 can be used.

**Theorem 3.5.** *If  $G/H = T^4 \oslash SL(2, \mathbb{C})/\mathbb{R} \oslash (\mathbb{R}^2 \oslash \tilde{O}(2))$  with the measure  $\mu$  in equation (11), the POVM  $A^\eta$  in Definition 3.1 does not have the norm-1 property.*

*Proof.* Let  $\mathbf{x} \in G/H$ . By Theorem 3.3,

$$\|A^\eta(\{\mathbf{x}\})\| \leq \mu(\{\mathbf{x}\}).$$

Since  $\mu(\{\mathbf{x}\}) = 0$ ,

$$\|A^\eta(\{\mathbf{x}\})\| \leq \mu(\{\mathbf{x}\}) = 0.$$

Theorem 2.12 completes the proof.  $\square$

#### 4. Localization in Configuration Space

Now, we study the marginals of  $A^\eta$  in the non-relativistic case and prove that they cannot have the norm-1 property. We limit ourselves to the marginal  $F_\eta^Q(\Delta_{\mathbf{q}}) := A^\eta(\Delta_{\mathbf{q}} \times \mathbb{R}_{\mathbf{p}})$  which represents the unsharp position observable. Clearly what we prove applies also to the marginal  $F_\eta^P(\Delta_{\mathbf{p}}) := A^\eta(\mathbb{R}_{\mathbf{q}} \times \Delta_{\mathbf{p}})$  which represents the unsharp momentum observable.

**Theorem 4.1.** *The POVM  $F_\eta^Q(\Delta_{\mathbf{q}})$  is absolutely continuous with respect to the Lebesgue measure on  $\mathbb{R}_{\mathbf{q}}$ .*

*Proof.*

$$\begin{aligned}
 F_\eta^Q(\Delta_{\mathbf{q}}) &= A^\eta(\Delta \times \mathbb{R}_{\mathbf{p}}) = \int_{\Delta \times \mathbb{R}_{\mathbf{p}}} U(\sigma(\mathbf{q}, \mathbf{p})) |\eta\rangle\langle\eta| U^\dagger(\sigma(\mathbf{q}, \mathbf{p})) d\mathbf{q} d\mathbf{p} \\
 &= \int_{\Delta} d\mathbf{q} \int_{\mathbb{R}_{\mathbf{p}}} U(\sigma(\mathbf{q}, \mathbf{p})) |\eta\rangle\langle\eta| U^\dagger(\sigma(\mathbf{q}, \mathbf{p})) d\mathbf{p} \\
 &= \int_{\Delta} \widehat{Q}_\eta(\mathbf{q}) d\mathbf{q} \leq \int_{\Delta} 1 d\mathbf{q},
 \end{aligned}$$

where

$$\widehat{Q}_\eta(\mathbf{q}) = \int_{\mathbb{R}_{\mathbf{p}}} U(\sigma(\mathbf{q}, \mathbf{p})) |\eta\rangle\langle\eta| U^\dagger(\sigma(\mathbf{q}, \mathbf{p})) d\mathbf{p}$$

and equation (7) in definition 3.1 has been used.  $\square$

Theorems 2.12 implies the following corollary.

**Corollary 4.2.**  $F_\eta^Q$  cannot have the norm-1 property

In Ref. [13] it is shown that in order for a localization observable to satisfy Einstein causality, the localization observable must be commutative. It is worth remarking that, although  $A^\eta$  is not commutative,  $F_\eta^Q$  is commutative and can be characterized as the smearing of the position operator [31]-[37], [41, 42, 43]. Unfortunately, as we have just proved,  $F_\eta^Q$  does not satisfy the norm-1 property. It would be interesting to analyze in general the relationships between causality and norm-1 property. That will be the topic of a future work.

## References

- [1] S.K. Berberian *Notes on Spectral Theory* 1966 Princeton, New Jersey: D. Van Nostrand Company, Inc.)
- [2] Jauch J M and Piron C 1967 *Helv. Phys. Acta*, **40** 559.
- [3] Amrein W O 1969 *Helv. Phys. Acta* **42** 149-190.
- [4] Kraus K. 1977 *Position Observables of the Photon (The Uncertainty Principle and Foundations of Quantum Mechanics* pp. 293-320) ed W.C. Price and S.S. Chissick (London: Wiley)
- [5] Ali S T and Emch G G 1974 *J. Math. Phys.* **15** 176-182
- [6] Ali S T 1985 *La Rivista del Nuovo Cimento* **8** 1-127
- [7] Hegerfeldt G C 1974 *Phys. Rev. D* **10** 3320-3321.
- [8] Castrigiano D P I 1981 *Letters in Mathematical Physics* **5** 303-309.
- [9] Rosewarne D and Sarkar S 1992 *Quant. Optics* **4** 405-413.
- [10] Holevo A S 1982 *Probabilistic and statistical aspects of quantum theory* (Amsterdam: North Holland).
- [11] Holevo A S 1984 *Statistical structure of quantum physics* (*Lecture Notes in Physics* vol. 1055, pp. 153-172) (Berlin: Springer-Verlag)
- [12] Busch P, Grabowski M and Lahti P 1995 *Operational quantum physics* (*Lecture Notes in Physics* vol. **31**) (Berlin: Springer-Verlag)
- [13] Busch P 1999 *J.Phys. A: Math. Gen.* **32** 6535
- [14] Muga J G, Mayato R S and Egusquiza I L 2008 *Time in Quantum Mechanics-Vol. 1 (Lecture Notes in Physics* vol. 734) (Berlin: Springer).
- [15] Galapon E A 2002 *Proc. R. Soc. Lond. A* **458** 451-472

- [16] Schroeck F E Jr. 1996 *Quantum Mechanics on Phase Space* (Dordrecht: Kluwer Academic Publishers)
- [17] Brooke J A and Schroeck F E Jr. 1996 *J. Math. Phys.* **37** 5958
- [18] Schroeck F E Jr. 2005 *Int. J. Theor. Phys.* **44** 2091-2100
- [19] Newton T D and Wigner E P 1949 *Revs. Modern Phys.* **21** 400
- [20] A.S. Wightman, 1962 *Rev. Mod. Phys.* **34** 845-872
- [21] Heinonen T, Lahti P, Pelloppaa J P, Pulmannová S and Ylinen K 2003 *J. Math. Phys.* **44** 1998-2008
- [22] Carmeli C, Heinonen T and Toigo A 2004 *J. Math. Phys.* **45** 2526-2539
- [23] Ludwig G 1983 *Foundations of Quantum Mechanics I*, (New York, Heidelberg, Berlin: Springer-Verlag)
- [24] de Muynck W M 2002 *Foundations of Quantum Mechanics, an Empiricist Approach* ( Dordrecht: Kluwer Academic Publishers)
- [25] Toller M 1999 *Phys. Rev. A* **59** 960-970
- [26] Loéve M 1977 *Probability Theory I*, (New York, Berlin, Heidelberg: Springer-Verlag)
- [27] Munkres J R 2000 *Topology* ( Upper Saddle River, NJ: Prentice Hall)
- [28] Prugovečki E 1984 *Stochastic Quantum Mechanics and Quantum Spacetime* ( Dordrecht: D. Reidel Publishing Company)
- [29] Wigner E P 1971 *Quantum Mechanical Distribution Functions Revisited (Perspectives in Quantum Theory)* ed. W. Yourgrau and A. van der Merwe (Cambridge, Mass: MIT Press)
- [30] Davies E B 1976 *Quantum mechanics of Open Systems* (London, New York, San Francisco: Academic Press)
- [31] Beneduci R and Nisticó G 2003 *J. Math. Phys.* **44** 5461
- [32] Beneduci R 2006 *J. Math. Phys.* **47** 062104
- [33] Beneduci R 2006 *Int. J. Geom. Meth. Mod. Phys.* **3** 1559
- [34] Beneduci R 2007 *J. Math. Phys.* **48** 022102
- [35] Beneduci R 2008 *Il Nuovo Cimento B* **123** 43-62
- [36] Beneduci R 2010 *Bull. Lond. Math. Soc.* **42** 441-451
- [37] Beneduci R 2012 Semispectral Measures and Feller Markov Kernels *Preprint* arXiv:1207.0086v1
- [38] Beneduci R, Brooke J, Curran R and Schroeck F E Jr. 2011 *Int. J. Theor. Phys.* **50** 3682-3696
- [39] Beneduci R, Brooke J, Curran R and Schroeck F E Jr. 2011 *Int. J. Theor. Phys.* **50** 3697-3723
- [40] Schroeck F E Jr. 2012 *J. Phys. A* **45** 065303
- [41] Beneduci R 2010 *Linear Algebra and its Applications* **43** 1224-1239
- [42] Beneduci R 2010 *Int. J. Theor. Phys.* **49** 3030-3038
- [43] Beneduci R 2011 *Int. J. Theor. Phys.* **50** 3724-3736