Amenability of horospheric products of uniformly growing trees

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Abstract: The horospheric product of two uniformly growing trees is amenable if and only if their growth rates match.

Keywords: (Non-)amenability of graphs, horospheric (or horocyclic) products, uniform growth

1 Introduction and Main Results

While horospheric (or 'horocyclic') products of trees have been introduced in the context of the question whether there exists a vertex-transitive graph which is not quasi-isometric to the Cayley-graph of a group [7, 2, 3], they also have the interesting property of usually having exponential growth and, in spite of this, being amenable. Amenability is a rather special property of horospheric products, one which is intimitely connected to the proportion of growth of the associated trees [8]. In [5] and [6], sufficiant criteria for horospheric products of random trees have been given. The results of the present approach, one which makes stricter assumptions on the quality of the trees' growth, allows for sufficient *and* necessary criteria. The characteristic assumption of our setting is that the trees involved have **uniform** exponential growth.

We reprove a Theorem by Daniela Bertacchi [1], and extend it to non-homogeneous trees. Following this paper in the use of notation, the vertex $\langle v_1, v_2 \rangle$ as an element of the vertex set of the product graph, with $v_1 \in V(T_1)$ and $v_2 \in V(T_2)$, is abbreviated by $v_1 v_2$. As we will be concerned with trees with a fixed root o and a fixed end γ (as well as transformations thereof), we will distinguish the tree T itself from the tripple $\widehat{T} = \langle T, o, \gamma \rangle$. Furthermore, the subgraph of a graph G, which is induced by a subset A of the vertex-set $V(G)$ of this graph will be denoted by $G|A$. The graph-distance of a graph G will be denoted by $\mathrm{dist}_G(\cdot, \cdot)$. The symbol X_n^v will denote the cardinality of the punctuated sphere centered at v with radius n, where punctuation refers to the removal of a single vertex.

2 Model and Main Assumptions

Let T be an infinite, undirected tree with vertex-set $V(T)$, edge-set $E(T)$, and set of ends $\partial T = \{\gamma : \mathbb{N} \to V \mid \gamma \text{ injective}, \{\gamma_k, \gamma_{k+1}\} \in E(T), \gamma_0 = o\}$. Let $o \in V$ be a fixed 'root', and $\gamma \in \partial T$ a fixed end. Let the triple $\widehat{T} := \langle T, o, \gamma \rangle$ be called **rooted tree pointed at infinity**[4]. The **confluent** $v \wedge_\gamma o$ is the vertex $\gamma(n)$, with $n \in \mathbb{N}$ minimising $\mathrm{dist}_T(v, \gamma(n))$. The graph $\widehat{T}$ has a **Busemann function** $b : V(T) \to \mathbb{Z}$, which assigns a 'level-structure' to the vertices of the tree [8] by

$$v \mapsto \mathrm{dist}_T(v, v \wedge_\gamma o) - \mathrm{dist}_T(v \wedge_\gamma o, o).$$

Let the n-th **predecessor** of v be the n-th element in the geodesic ray representing γ which starts at $v = \gamma(0)$ (i.e. with strictly decreasing Busemann-function). Then T is called **uniformly growing** (UG) with rate λ, if

$$\forall \epsilon > 0 \;\; \exists n_o \in \mathbb{N} \;\; \forall n > n_o \;\; \forall v \in V(T) \quad e^{-\epsilon n} \leq X_n^v e^{-n\lambda} \leq e^{\epsilon n}, \tag{1}$$

where $X_n^v \;:=\; \{\, w \in V(T) \;:\; v \text{ is } n-\text{th predecessor of } w \,\}$. Note that $\lambda = \lim_n (1/n)\log X_n^v$ does not depend on the choice of v.

The rooted tree pointed at infinity is said to have **strongly uniform growth** (SUG) with rate λ, if

$$\exists C > 0 \quad \forall v \in V \quad C^{-1} \;\leq\; e^{-n\lambda} X_n^v \;\leq\; C. \tag{2}$$

Again, of course this implies $\lambda = \lim\limits_n \frac{1}{n} \log X_n^v$ exists independently of v.

It is ease to show that these assumptions imply the respective property for balls in T, i.e. (1) and (2) hold also with X_n^v replaced by the cardinality of spheres of radius n with center at v.

The **horospheric product** $\widehat{T}_1 \circ \widehat{T}_2$ of two rooted trees pointed at infinity is the graph $\widehat{G} = \langle \widehat{V}, \widehat{E} \rangle$ with vertex set

$$\widehat{V} \;=\; \{\, v_1 v_2 \in V(T_1) \times V(T_2) \;:\; b_1(v_1) \;+\; b_2(v_2) \;=\; 0 \,\},$$

and

$$\widehat{E} \;=\; \{\, \{\, v_1 v_2,\, w_1 w_2 \} \subset \widehat{V} \;:\; \{v_1, w_1\} \in E(T_1),\;\; \{v_2, w_2\} \in E(T_2) \,\}.$$

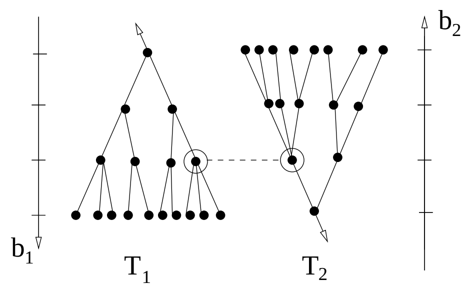

Figure 1: Representation of the horospheric product of two rooted trees $\widehat{T}_1$ and $\widehat{T}_2$ pointed at infinity. The arrowheads in the trees point towards the fixed ends γ_1, and γ_2. Vertices in $\widehat{V}$ are pairs of vertices of the trees represented by circles around them and connect by a 'flexible' horizontal bar (dashed line). This indicates that only such pairs of vertices are allowed for which the sum of the Busemann-functions vanish. The sets vertices in each tree which have equal Busemann level are called horospheres (or horocycles).

A map $\phi : X \to Y$ from the metric space $\langle X, d\rangle$ to the metric space $\langle Y, d'\rangle$ is a **quasi-isometry** if there are constants $C, D > 0$, such that

$$C^{-1} d(x,y) \;-\; D \;\leq\; d'(\phi(x), \phi(y)) \;\leq\; C d(x,y) \;+\; D. \tag{3}$$

If the image of X under ϕ is uniformly d'-close to every point in Y, then X and Y are called a pair of quasi-isometric spaces.

Lemma 1. *$\widehat{T} = \langle T, o, \gamma\rangle$ has SUG with rate λ iff there exists $D > 0$, such that*

$$\frac{1}{D} \;\leq\; X^u_{n+m}/(X^v_n \cdot X^w_m) \;\leq\; D,$$

for all vertices v, u, w, and $n, m \in \mathtt{N}$.

Proof: Assume, that $\widehat{T}$ has SUG with rate λ. Then $D^{-1}e^{n\lambda} \leq X^v_n \leq De^{n\lambda}$, and the inequality follows with $D = C^3$. Conversely, let $g(n) := X^v_n \exp -\lambda n$. We show that $g(n)$ is bounded from above and strictly positive. By assumption, on X^v_n and properties of the exponential function, there is $C > 0$, such that

$$C^{-1}g(n)g(m) \;\leq\; g(n+m) \;\leq\; Cg(n)g(m).$$

Furthermore, by the definition of λ, we also have

$$\lim_{n\to\infty} \frac{\log g(n)}{n} \;=\; 0.$$

This implies, that $A(n) := \log g(n)$ fulfills the following two properties: There is $C' > 0$ (namely $C' := \log C$) with

$$A(n) \;+\; A(m) \;-\; C' \;\;\leq\;\; A(n+m) \;\;\leq\;\; A(n) \;+\; A(m) \;+\; C',$$

and

$$(*) \qquad \frac{A(n)}{n} \;\longrightarrow\; 0, \qquad \text{as } n \to \infty.$$

We show that $n \mapsto A(n)$ is bounded from above and from below (implying that $g = \exp A$ is bounded from above and positive). Indeed, $2C'$ is an upper bound: Suppose, there is an $m \in \mathtt{N}$ with $A(m) \geq 2C'$. Then this would imply

$$A(n \;+\; m) \;\geq\; A(n) \;+\; A(m) \;-\; C' \;\geq\; A(n) \;+\; C'.$$

By induction, it would follow that

$$A(1 \;+\; k \cdot m) \;\geq\; A(1) \;+\; kC'$$

for all $k \in \mathtt{N}$, in contradiction with $(*)$. The same argument gives the lower bound $-2C'$. All these considerations are uniform in v. □.

3 Amenability

Let $\widehat{T_1} = \langle T_1, o_1, \gamma_1 \rangle$, $\widehat{T_2} = \langle T_2, o_2, \gamma_2 \rangle$, and $\widehat{T_1'} = \langle T_1', o_1', \gamma_1' \rangle$, $\widehat{T_2'} = \langle T_2', o_2', \gamma_2' \rangle$ be two pairs of rooted trees pointed at infinity with roots o_1, o_2, and o_1', o_2', fixed ends $\gamma_1 \gamma_2$, and γ_1', γ_2', and Busemannfunctions $b_i : V(T_i) \to \mathbb{Z}$, and $b_i' : V(T_i) \to \mathbb{Z}$, $(i \in \{1,2\})$, respectively. Assume also, that every vertex in each tree has at least degree two, i.e. there are no leaves.

Definition: For a finite vertex set $W \subset V$ of a graph $G = \langle V, E \rangle$, let $\partial_G W = \{ \{x,y\} \in E \mid x \in W,\ y \notin W \}$ be the edge-boundary of W in G. The graph G is called **amenable** if there exists a sequence of subgraphs induced by finite vertex sets $V_n \subset V$, such that the isoperimetric ratios

$$n \quad \mapsto \quad \frac{|\partial_G V_n|}{|V_n|}$$

converges to zero. The sequence (V_n) is called **Følner sequence**.

A graph G is called **strongly amenable** if there is an exhausting Følner sequence.

Definition: (see [7]). A **tetrahedron** of height $n \in \mathbb{N}$ is the finite connected subgraph $G_n := \langle V_n, E_n \rangle$ of a horospheric product $G = \widehat{T_1} \circ \widehat{T_2}$ induced by

$$V_n \quad = \quad \{ v_1 v_2 \in V(G) \ : \ |b_1(v_1)| \le n,\ v_1 v_2 \leftrightarrow_n o_1 o_2 \},$$

where $a \leftrightarrow_n b$ means that the vertices $a_1 a_2$ and $b_1 b_2 \in V_n$ are connected by a path of vertices $u_1(j) u_2(j), j \in \{0, ..., m\}$ in G, each of which has a Busemann-function not exceeding n in absolute value, i.e. $|b_1(u_1(j))| = |b_2(u_2(j))| \le n$ for all j, where m is the length of the path.

Theorem 2. *If $\widehat{T_1}$ and $\widehat{T_2}$ both have SUG with rate λ, then $\widehat{T_1} \circ \widehat{T_2}$ is strongly amenable. More precisely, there exists $\kappa > 0$, such that the sequence of tetrahedrons form a Følner sequence (V_n) with*

$$\frac{|\partial_G V_n|}{|V_n|} \quad \le \quad \frac{\kappa}{n}\ .$$

Proof: Let $X_n^{(i)}$ be the number of leaves of the finite tree rooted at the n'th predeccessor of o_i in $\widehat{T_i}$ growing away from γ_i with height $2 \cdot n$. We calculate the isoperimetric ratio of the tetrahedrons $G_n = \langle V_n, E_n \rangle$ in G to be

$$F_n \quad := \quad \frac{X_{n+1}^{(1)} \quad + \quad X_{n+1}^{(2)}}{\sum\limits_{h=-n}^{n} X_j^{(1)} X_{n-j}^{(2)}}\ .$$

Redistributing F_n, we get

$$F_n \;=\; \frac{1}{\sum\limits_{h=-n}^{n} X_j^{(1)} X_{n-j}^{(2)} / X_{n+1}^{(1)}} \;+\; \frac{1}{\sum\limits_{h=-n}^{n} X_j^{(1)} X_{n-j}^{(2)} / X_{n+1}^{(2)}}. \tag{4}$$

Now, due to the Lemma, because $\widehat{T}^1$ has SUG, there is some $C > 0$, such that

$$\sum_{h=-n}^{n} X_j^{(1)} X_{n-j}^{(2)} \;/\; X_{n+1}^{(1)} \geq C^{-1} \sum_{h=-n}^{n} X_{n-j}^{(2)} \;/\; X_{n+1-j}^{(1)}.$$

Since $\widehat{T}^{(1)}$ and $\widehat{T}^{(2)}$ both have SUG with the same rate λ,

$$X_{n-j}^{(2)} \;/\; X_{n+1-j}^{(1)} \;\geq\; C^{-2} e^{\lambda}.$$

The second term in (4) is treated in the same way. We therefore obtain

$$F_n \;\leq\; \frac{C^3 e^{\lambda}}{2n+1}.$$

So, $(V_n) = O(\frac{1}{n})$ is a Følner sequence in G. □

Remark: If $\widehat{T}_1$ has UG with λ_1 and $\widehat{T}_2$ has UG with rate $\lambda_2 \neq \lambda_1$, then the sequence of tetraedrons in $\widehat{T}_1 \circ \widehat{T}_2$ doesn't form a Følner sequence. Indeed, if, without restricting generality, $\lambda_1 < \lambda_2$, we can show, the sum in the denomenator of the second term in (4) remains bounded: Choosing $\epsilon \;:=\; \frac{1}{4}\,(\lambda_2 \;-\; \lambda_1)$, it follows by the assumption of UG that for all j

$$X_j^{(2)} \;\leq\; \text{const. } \exp\left((\lambda_2 \;+\; \epsilon) j\right).$$

by assumption of UG it holds that

$$X_{N-j}^{(2)} \;/\; X_j^{(1)} \;\;\leq\;\; \text{const. } \exp\left((-\lambda_2 + 2\epsilon) j\right)$$

and

$$X_j^{(1)} \;\leq\; \text{const. } \exp\left((\lambda_1 + \epsilon) j\right).$$

With this, the sum can be bounded from above for large j by $\sum \exp\left((\lambda_1 \;-\; \lambda_1 + 3\epsilon) j\right)$. The above choice of ϵ lets this is be majorised by a geometric series. Therefore, the sum is finite and the corresponding term in F is bounded from below.

4 Horospheric products, contractions, and quasi-isometries

Contractions of trees and quasi-isometries are used in the proof of our non-amenability criterion. Since these concepts play an important role in the general theory, we develop them in a separate section. In what follows, the metric spaces occuring are the vertex-sets of graphs of bounded geometry. The type of quasi-isometry considered here will always be a map from the vertices of a graph to the vertices of another graph. The metric is in each case given by the graph metric.

Let $\widehat{T_1} = \langle T_1, o_1, \gamma_1\rangle$, $\widehat{T_2} = \langle T_2, o_2, \gamma_2\rangle$, and $\widehat{T_1'} = \langle T_1', o_1', \gamma_1'\rangle$, $\widehat{T_2'} = \langle T_2', o_2', \gamma_2'\rangle$ be two pairs of rooted trees pointed at infinity with roots o_1, o_2, o_1', o_2', fixed ends $\gamma_1, \gamma_2, \gamma_1', \gamma_2'$, and Busemannfunctions $b_i : V(T_i) \to \mathtt{Z}$, and $b_i' : V(T_i) \to \mathtt{Z}$, $(i \in \{1,2\})$, respectively. Assume also, that every vertex in each tree has at least degree two. Let there be $\phi_1 :\to V(T_1')$ and $\phi_2 : V(T_2) \to V(T_2')$ with the property $V(T_i') \subseteq V(T_i), i \in \{1,2\}$. We will show that if T_1 and T_2 are uniformly growing, and ϕ_1 and ϕ_2 are quasi-isometries, the map given by

$$\begin{aligned} \Phi : V(\widehat{T_1} \circ \widehat{T_2}) &\to V(\widehat{T_1'} \circ \widehat{T_2'}), \\ \langle v, w\rangle &\mapsto \langle \phi_1(v), \phi_2(w)\rangle \end{aligned}$$

is a quasi-isometry with respect to the graph-metric in $\widehat{T_1} \circ \widehat{T_2}$ and in $\widehat{T_1'} \circ \widehat{T_2'}$.

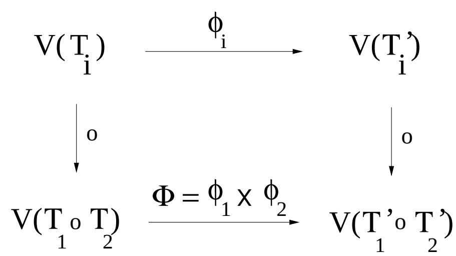

Figure 2: Φ is a quasi-isometry, if ϕ_1 and ϕ_2 are.

We define the maps ϕ_1 and ϕ_2 as maps between the metric spaces $V(T_i)$ and $V(T_i')$ with the respective graph metrics:

$$\phi_i \ : \ V(T_i) \ \to \ V(T_i')$$

and require them to have the following properties:

1.
$$V(T_i') \subset V(T_i)$$

2.
$$\{k, l\} \in E(T_i), \text{ and } \phi_i(k) \neq \phi_i(l) \ \Rightarrow \ \{\phi_i(k), \phi_i(l)\} \in E(T_i').$$

We call maps with these two properties **graph contractions**. Note that the edge-sets $E(T_1'), E(T_2')$ may contain 'loops', i.e. subsets of the vertex-sets containing only one vertex. However, they have no effect on the metric in the image-spaces $V(T_1')$ and $V(T_2')$.

Lemma 4.1. *Let $\phi : V(T) \to V(T')$ be a graph contraction and $\gamma \in \partial T$ a fixed end represented by the ray (x_n). Then the limit $\lim_{n\to\infty} \phi(x_n)$ exists in $V(T') \cup \partial T'$. If the inverse image of any finite set under ϕ is finite then this limit obeyes $\phi(\gamma) \in \partial T'$.*

Proof: The existence of the limit follows by the contractive property

$$\mathrm{dist}_{T'}(\phi(x_n), \phi(x_m)) \quad \leq \quad \mathrm{dist}_T(x_n, x_m)$$

because the image of the converging sequence (x_n) will be a Cauchy sequence in $V(T') \cup \partial T'$. If $\lim \phi(x_n) \in V(T')$, then the inverse image of this limit-point is infinite. □

Remark: The lemma shows: For graph contractions with finite inverse images of finite sets it is natural to identify $\phi(\partial T)$ with $\phi(T')$. The reason for this is that ϕ being a graph contraction implies $\phi(\partial T) \subset \partial T$, and under the assumption the lemma implies $\partial T \subset \phi(\partial T)$. In other words, these mappings conserve ends.

All graph contractions we will be concerned with will have finite inverse images of finite sets. Here are some examples of graph transformations $\Phi : V(\widehat{T}_1 \circ \widehat{T}_2) \to V(\widehat{T}_1' \circ \widehat{T}_2')$ with $\Phi = \phi_1 \times \phi_2$ of which we show that they are quasi-isometries with respect to the graph metrics in $\widehat{T}_1 \circ \widehat{T}_2$ and $\widehat{T}_1{}' \circ \widehat{T}_2{}'$.

Example 1: (*Shift–T_1–by–one*)

The first example refers to horospheric products of leaf-less trees in which the first tree is shifted away from its fixed end by one horospheric. Each element of the vertex-set V of $\widehat{T}_1 \circ \widehat{T}_2$ is mapped to the vertex-set V' of $\widehat{T}_1' \circ \widehat{T}_2$ where $\widehat{T}_1'$ is the same as $\widehat{T}_1$ except that its root $o_1 o_2$ is mapped to $o_1^{-1} o_2$, where o_1^{-1} is the immediate predecessor of o_1. We denote this mapping from the metric-space (V, d) to (V', d'), where d is the graph distance in $\widehat{T}_1 \circ \widehat{T}_2$, and d' the graph distace in $\widehat{T}_1' \circ \widehat{T}_2$, by $\Phi : V \to V', x_1 x_2 \mapsto x_1^{-1} x_2$. Note, that due to the assumption of there being no leaves, the structure of T_1 is left invariant, so that the graph metrics of T_1 coincides with the graph metric in the shifted tree.

Proposition 3. *Let $\widehat{T}_1$ and $\widehat{T}_2$ be leafless rooted trees, pointed at infinity. Shift-T_1-by-one is a quasi-isometry.*

Proof: Let $x_1' y_2' = x' = \Phi(x)$, $y_1' y_2' = y' = \Phi(y)$ with $x = x_1 x_2$ and $y = y_1 y_2$. If x_1 and y_1 are on the same ray representing γ_1 of $\widehat{T}_1$, let $k = 0$. Otherwise, let $k = 2$.

Then, by Bertacci's formular (8), we have for

$$\begin{aligned} d_2(x',y') &= \mathrm{dist}_{T_1}(x_1',y_1') + \mathrm{dist}_{T_2}(x_2',y_2') + |b_1'(x_1') - b_1'(y_1')| \\ &= \mathrm{dist}_{T_1}(x_1,y_1) - k + \mathrm{dist}_{T_2}(x_2,y_2) + |b_1(x_1)+1 - (b_1(y_1)+1)| \\ &= d_1(x,y) - k. \end{aligned}$$

Therefore, Φ is a quasi-isometry, where $C = 1$ in (3), and $D = 2$. □

Example 2: (*Contract–*$\widehat{T}_1 \circ \widehat{T}_2$*–by–bounded*)

If $v \in V(T_i)$, for $i \in \{1,2\}$, let v^{-n} be its n-th predecessor in $\widehat{T}_i$. For $m \in \mathbb{Z}$ and $Q \subset \mathbb{Z}$, let

$$\lceil m \rceil_Q = \sup\{ k \in Q \mid k \leq m \}, \quad \text{and} \quad \lfloor m \rfloor_Q = \sup\{ k \in Q \mid k < m \}.$$

Throughout this paper we assume that Q is an infinite set with the property that there is a uniform upper bound on the difference between two consecutive elements of Q, i.e. there is $M > 0$, such that

$$\forall_{k \in Q} \quad \inf\{ l \in Q \mid k < l \} - k \leq M. \tag{5}$$

Furthermore, we let $V_1' = \{v \in V_1 \mid b_1(v) \in Q \}$, and $V_2' = \{w \in V_2 \mid b_2(w) \in -Q \}$. We call $\widehat{V}' = V_1' \times V_2'$. Now, let $n_1 = b_1(x_1) - \lceil b_1(x_1) \rceil_Q$ and $n_2 = b_2(x_2) - \lfloor b_2(x_2) \rfloor_{-Q}$, and define $\Phi : \widehat{V} \to \widehat{V}'$ by

$$x_1\, x_2 \mapsto x_1^{-n_1}\, x_2^{-n_2}. \tag{6}$$

This sets the vertices of T_1 back to their youngest ancestors with Busemannlevels in Q and the vertices of T_2 always to a *proper* ancestor in $-Q$ (see Figure 3).

Note that Φ has product form, so Φ acts independently in each component. The edges of the image-graphs under the component maps of Φ of the trees T_1, T_2 are those subsets $\{x_1'y_1'\}$ of V_1' and $\{x_2'y_2'\}$ of V_2', for which there are edges $\{x_1y_1\}$ in E_1 and $\{x_2y_2\}$ of E_2 such that $x_1'x_2' = \Phi(x_1x_2)$ and $y_1'y_2' = \Phi(y_1y_2)$. This implies that the image-graphs are trees. We call them $T_1' = \langle V_1', E_1' \rangle$, and $T_2' = \langle V_2', E_2' \rangle$.

In order to identify the image graph of the horospheric product $\widehat{T}_1 \circ \widehat{T}_2$ as a graph similar to a horospheric product, we have to define the Busemannfunctions for T_1' and T_2'.

Note that the vertex set of the image of $\widehat{T}_1 \circ \widehat{T}_2$ under Φ is $\widehat{V}' = \{v' = v_1'v_2' \in V_1' \times V_2' \mid \exists_{v=v_1v_2 \in \widehat{V}}\ v' = \Phi(v) \}$. The edge set of the image graph of $\widehat{T}_1 \circ \widehat{T}_2$ under Φ are the subsets $\{x_1'x_2', y_1'y_2'\}$ of $V_1' \times V_2'$ which are incident to images of $x = x_1x_y$ and $y = y_1y_2$ under Φ where $\{x,y\} \in \widehat{E}$. As also $\widehat{V}' \subset \widehat{V}$, this makes Φ a graph-contraction.

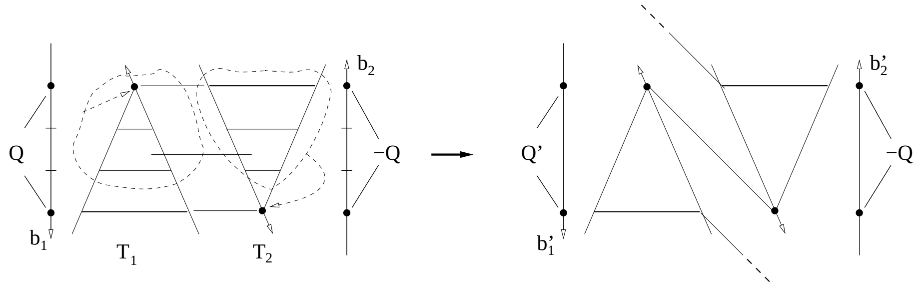

Figure 3: The map Φ shifts vertices x_1x_2 onto vertices $x'_1x'_2$ which have Busemannlevels $\langle b_1(x'_1), b_2(x'_2)\rangle \in Q \times (-Q)$ in $\widehat{T}_1 \circ \widehat{T}_2$. The vertices of the image graph $x'_1x'_2$ have Busemannlevels $b'_1(x'_1)$ and $b'_2(x'_2)$ in the new trees $\widehat{T}'_1$ and $\widehat{T}'_2$ with $b'_1(x'_1) \ + \ b'_2(x'_2) \ = \ 1$.

In $x = x'_1x'_2 = \Phi(x)$ with $x = x_1x_2$, Φ sets the coordinate x_1 either back to the nearest ancestor with Busemannlevel in Q or leaves x_1 invariant (if $b_1(x_1) \in Q$), and it always sets x_2 to the nearest ancestor that has Busemannfunction equal to an element of $-Q$. The reason for this slight asymmetry in the definition of Φ prevents, as we will see, different images of vertices x' to have different values of sums of Busemannfunctions in the image graph. However, this constancy is what characterises horospheric products.

Now, in order to define the Busemannfunctions in the image graph, note that due to condition (5) and the remark under Lemma 4.1, the ends γ_1 and γ_2 are preserved under Φ, i.e. if for $i \in \{1,2\}$ the sequence $x_i(n)$ is a ray in T_i representing γ_i, then $x'_i(n)$ also represents γ'_i in T'_i, and $\gamma'_i = \gamma_i$.

With $o' = o'_1o'_2 = \Phi(o_1o_2)$, the componentwise images of Φ can be found to be rooted trees pointed at infinity: $\widehat{T}'_i = \langle T'_i, \gamma_i, o'_i\rangle$, for $i \in \{1,2\}$. The Busemann-functions $b'_i : V'_i \to \mathbb{Z}$ for $i \in \{1,2\}$ are then defined by

$$b'_i(v) \ = \ d_{T'_i}(v, \ v \wedge_{\gamma_i} o'_i) \ - \ d_{T'_i}(v \wedge_{\gamma_i} o'_i, \ o'_i), \qquad \text{for } v \in V'_i.$$

Since $V'_i \subset V_i$, we can also measure the Busemannlevels of the images x'_1 and x'_2 of x_1 and x_2 under Φ by the original Busemannfunctions b_1 and b_2. In particular, for an element of $x = x_1x_2 \in \widehat{V}$ (for which $b_1(x_1) + b_2(x_2) = 0$), we get

$$b_1(x'_1) = k \in Q, b_2(x'_2) = -l \in -Q \ \Rightarrow \ l < k \text{ and } \forall_{m \in \mathbb{Z}} l < m < k \ \Rightarrow \ m \notin Q.$$

In other words, Φ always maps a vertex $x_1x_2 \in \widehat{V}$ to a pair $x'_1x'_2 \in V_1 \times V_2$ such that $k = b_1(x'_1)$ is the immediate successor in Q of $l = -b_2(x'_2)$. Since the image trees T'_1 and T'_2 have only edges connecting vertices in V'_1 and V'_2, respectively, then it follows that if $k = b_1(x'_1)$ immedeatly succeeds $l = -b_2(x'_2)$ in Q, then $b'_1(x'_1) = -b'_2(x'_2) + 1$, or equivalently

$$b'_1(x'_1) \ + \ b'_2(x'_2) \ = \ 1. \qquad (7)$$

Proposition 4. *Let $\widehat{T}_1$ and $\widehat{T}_2$ be leafless rooted trees, pointed at infinity. Then Contract-$\widehat{T}_1 \circ \widehat{T}_2$-by-bounded is a quasi-isometry.*

Proof: Let $Q \subset \mathbb{Z}$ the set defining Contract-$\widehat{T}_1 \circ \widehat{T}_2$-by-bounded, and let $M > 0$ such that the difference of sucsessive elements in Q are all bounded from above by M. Since there are no leaves on T_1 and T_2, by (8), it holds that for $x' = x'_1 x'_2 = \Phi(x)$ with $x = x_1 x_2$ and $y' = y_1 y2 = \Phi(y)$ with $y = y_1 y_2$.

$$d'_{\widehat{T}'_1 \circ \widehat{T}'_2}(x', y') \;=\; d'_{T'_1}(x'_1, y'_1) \;+\; d'_{T'_2}(x'_2, y'_2) \;-\; |b'_1(x'_1) \;+\; b'_1(y'_1)|.$$

Since for both $i = 1$ and $i = 2$, lengths in T_i are contracted by at most M vertices by Φ, it holds that

$$\frac{1}{M} d_{T_i}(x_i, y_i) \leq d_{T'_i}(x_i, y_i) \leq d_{T_i}(x_i, y_i),$$

so that also

$$\frac{1}{M} d_{\widehat{T}_1 \circ \widehat{T}_2}(x, y) \;\leq\; d'_{\widehat{T}'_1 \circ \widehat{T}'_2}(x', y') \;\leq\; d_{\widehat{T}_1 \circ \widehat{T}_2}(x, y),$$

rendering Φ to be a quasi-isometry. □

Remark: It is obvious that the image of a horospheric product under *Contract-$\widehat{T}_1 \circ \widehat{T}_2$-by-bounded* can be mapped into a horospheric product by application of a map of type *Shift-$\widehat{T}_1$-by-one*.

Definition: By a tree *without leaves* (or *leafless* tree) we understand a tree with the property that each vertex has degree larger or equal to two.

Theorem 4.2. *(Bertacci's Distance Formular)*
Let $\widehat{T}_1, \widehat{T}_2$ be two rooted trees without leaves, pointed at infinity, with Busemann-functions b_1, b_2, respectively. Then the graph distance $dist_{\widehat{T}_1 \circ \widehat{T}_2}(v, w)$ between two vertices $v = v_1 v_2$ and $w = w_1 w_2$ of the horospheric product $\widehat{T}_1 \circ \widehat{T}_2$ is given by

$$dist_{\widehat{T}_1 \circ \widehat{T}_2}(v, w) \;=\; dist_{\widehat{T}_1}(v_1, w_1) \;+\; dist_{\widehat{T}_2}(v_2, w_2) \;-\; |b_1(v_1) \;-\; b_1(w_1)|. \tag{8}$$

Remark: This theorem first appeared in the context of horospheric products of homogeneous trees in [1]. We note that for trees with leaves their horospheric product may be a disconnected graph.

Proof: If $\pi(v, w)$ is a geodesic in $\widehat{T}_1 \circ \widehat{T}_2$ connecting vertices v and w, then since being a connected path its projections π_1 and π_2 onto $\widehat{T}_1$ and $\widehat{T}_2$ are also connected paths. Therefore, they have to include the vertices $v_1 \wedge_{\gamma_1} w_1$ and $v_2 \wedge_{\gamma_2} w_2$, respectively. Consequently, π_1 must contain the unique geodesic λ_1 in T_1 between v_1 and $v_1 \wedge_{\gamma_1} w_1$, as well as the geodesic μ_1 in T_1 between $v_1 \wedge_{\gamma_1} w_1$ and w_1. Assuming, without restriction, $b_1(v_1) \geq b_1(w_1)$, it holds that length$(\lambda_1) \geq$ length(μ_1), and for some $l_1 \in \mathbb{Z}_+$, it holds, with $\delta_1 = b_1(v_1) - b_2(w_1)$ and $l_1 =$length(μ_1) in T_1

$$\text{dist}_{T_1}(v_1, w_1) \;=\; \delta_1 \;+\; 2l_1.$$

Similarily, with $\delta_2 = b_2(w_2) - b_2(v_2)$, and l_2 the length of the geodesic between v_2 and $v_2 \wedge_{\gamma_2} w_2$, we have

$$\text{dist}_{T_2}(v_2, w_2) \;=\; \delta_2 \;+\; 2l_2.$$

Now, notice $\delta_1 = \delta_2 =: \delta$, and $l_1, l_2, \delta \geq 0$.

The length of π is *at least* the length $2l_1 + 2l_2 + \delta$, since it decomposes into three disjoint parts: one which has a projection onto T_1 that includes vertices v with $b_1(v) \leq b_1(w_1)$, one which has a projection onto T_2 that includes vertices w with $b_2(w) \geq b_2(v_2)$, and the remaining part with vertices u of which the first coordinate has Busemannfunction between $b_1(w_1)$ and $b_1(w_1) + \delta$.

Now, $2l_1 + 2l_2 + \delta$ is also an *upper* bound for the length of π, since π can be chosen, such that the part with Busemannfunction $b_1(u) \in \{b_1(w_1), ..., b_1(w_1) + \delta\}$ is traversed only once: this is the case when the first part of π has a projection onto T_2 which bends around $v_2 \wedge_{\gamma_2} w_2$. The fact that

$$2l_1 + 2l_2 + \delta = d_{T_1}(v_1, w_1) + d_{T_2}(v_2, w_2) - \delta$$

concludes the proof. □

5 Non-amenability

Let $\widehat{T}_1 = \langle T_1, o_1, \gamma_1 \rangle$, and $\widehat{T}_2 = \langle T_2, o_2, \gamma_2 \rangle$ be two rooted trees, pointed at infinity, with b_1, and b_2 denoting the corresponding Busemann-functions, and let $G = \langle \widehat{V}, \widehat{E} \rangle$ be the horospheric product $\widehat{T}_1 \circ \widehat{T}_2$.

For $v_i \in V_i := V(T_i)$ where $i \in \{1, 2\}$, let the element v_i^{-n} of V_i be the n-th predecessor of v_i under the hierarchy induced by b_i. As in (6), let the map $\Phi : \widehat{V} \to \widehat{V}$ be given by $\Phi(v) = \phi_1(v_1) \cdot \phi_2(v_2)$, where $\phi_i : V_i \to V_i' \subset V_i$ is defined by

$$v_1 \mapsto v_1^{-n(1)} \qquad \text{and} \qquad v_2 \mapsto v_2^{-n(2)}$$

with $n(1) = h_i(v_i) - [h_i(v_i)]_Q$ and $n(1) = h_2(v_2) - \lfloor h_2(v_i) \rfloor_Q$, while $[m]_Q = \sup\{k \in Q \mid k \leq m\}$ and $\lfloor m \rfloor_Q = \sup\{k \in Q \mid k < m\}$ for all $m \in \mathbb{Z}$.

Call $\widehat{T}_i' = \langle T_i', o_i, \gamma_i \rangle$ with $T_i' = \langle V_i', E_i' \rangle$, with the vertices $V_i' = \phi_i^Q(V_i)$ and the edges $E_i' = \Big\{ \{\phi_i^Q(x), \phi_i^Q(y)\} \mid \{x, y\} \in E_i \Big\}$, the Q-contracted tree associated with $\widehat{T}_i$. Let $G' = \langle \widehat{V}', \widehat{E}' \rangle$ be the horospheric product $\widehat{T}_1' \circ \widehat{T}_2'$.

Theorem 5.1. *For any two rooted trees $\widehat{T}_1$, $\widehat{T}_2$ pointed at infinity satisfying (UG), it holds that the there is an $L \in \mathtt{N}$ such that for $Q = L \cdot \mathtt{Z}$, the Q-contracted trees $\widehat{T}_1'$ and $\widehat{T}_2'$ satisfy*

$$\exists_{\epsilon>0}\forall_{v\in\widehat{V}'} \ \frac{d_1'(v_1) \ - \ 1}{d_1'(v_1) \ + \ d_2'(v_2) \ - 2} \quad \geq \quad \frac{1}{2} \ + \ \epsilon, \tag{9}$$

where $d_i'(v_i)$ is the degree of v_i in T_i', for $i \in \{1,2\}$.

Proof: If $X_n^{(i)}(v_i) \ = \ \{ \ w \in V(T_i') \mid w \text{ is } n-\text{th predecessor of } v_i \ \}$, it is clear that $d_i'(v_i) - 1 = X_L^{(i)}(v_i)$. Due to the assumption of UG, it holds that for every $\epsilon > 0$ there is $L \in \mathtt{N}$, sufficiently large, such that for all $v_i \in V(T_i)$ it holds $e^{-\epsilon L} \leq X_L^{(i)} e^{-\lambda_i L} \leq e^{\epsilon L}$ for suitable $\lambda(1), \lambda(2) > 0$, with $\lambda(1) \neq \lambda(2)$. Without restricting generality, let $\lambda(1) < \lambda(2)$. Then, for $X_L^{(i)} = X_L^{(i)}(v_i)$, we have

$$\frac{X_L^{(1)}}{X_L^{(1)} \ + \ X_L^{(2)}} \ = \ \left(1 \ + \ \frac{X_L^{(2)}}{X_L^{(1)}} \frac{e^{-\lambda(2)L}}{e^{-\lambda(1)L}} \frac{e^{\lambda(2)L}}{e^{\lambda(1)L}}\right)^{-1} \ \geq \ \frac{1}{1 \ + \ e^{2\epsilon L} e^{-(\lambda(2)-\lambda(1))L}}.$$

Choosing $\epsilon = \frac{\lambda(2)-\lambda(1)}{4}$, it follows that there is $L \in \mathtt{N}$, such that for all $\langle v_1, v_2 \rangle \in \widehat{V}'$

$$\frac{d_1'(v_1) \ - \ 1}{d_1'(v_1) \ + \ d_2'(v_2) \ - 2} \ \geq \ \frac{1}{1 \ + \ \exp(-\frac{\lambda(2)-\lambda(1)}{2} L)} \ > \ \frac{1}{2} + \frac{\lambda(2)-\lambda(1)}{8} L. \qquad \square$$

Theorem 5.2. *Let $\widehat{T}_1$ and $\widehat{T}_2$ be leafless rooted trees, pointed at infinity obeying UG with different growth rates $\lambda(1)$ and $\lambda(2)$. Then $\widehat{T}_1 \circ \widehat{T}_2$ is non-amenable.*

Proof: We show: the spectral radius of the simple random walk $\rho = \limsup(p_{o,\bar{o}}^{(n)})^{1/n}$ (see [8], II). To this end, let $\widehat{T}_1'$ and $\widehat{T}_2'$ be the Q-contracted trees such that (9) holds. $v = \langle v_1, v_2 \rangle \in \widehat{V}'$ and $f(v) = \exp(\alpha b_1(v_1))$ for some $\alpha > 0$. We show that there is an $\alpha > 0$ such that $f : \widehat{V}' \to \mathtt{R}$ is a r-harmonic function for the SRW with $r < 1$. Let $P : l^2(\widehat{V}') \to l^2(\widehat{V}')$ be the bounded SRW-transition operator on $\widehat{T}_1' \circ \widehat{T}_2'$, then

$$Pf(v) \ = \ \sum_{w \sim_{G'} v} \frac{f(w)}{d_1'(v_1) + d_2'(v_2) - 2} \ = \ f(v) \frac{e^{\alpha}(d_1'(v_1) - 1) \ + \ e^{-\alpha}(d_2'(v_2) - 1)}{d_1'(v_1) - d_2'(v_2) - 2}.$$

For clarity, we let $p := (d_1'(v_1)-1)/(d_1'(v_1)-d_2'(v_2)-2)$, and $q = 1-p$. If $\lambda(1) < \lambda(2)$, by the foregoing theorem, there is L such that with $Q = L \cdot \mathtt{Z}$, the Q-contracted trees, $\widehat{T}_1'$ and $\widehat{T}_2'$ exclusively have vertecies $v = \langle v_1, v_2 \rangle$ with $p > \frac{1}{2} + \epsilon$, where $\epsilon = \frac{\lambda(2)-\lambda(1)}{4}$. This is equivalent to $q < \frac{1}{2} - \epsilon =: q^*$, and

$$Pf(v) = f(v)(pe^{\alpha} \ + \ qe^{-\alpha}) = f(v)(e^{\alpha} \ - \ q \cdot 2\sinh\alpha) \leq f(v)(e^{\alpha} \ - \ q^* \cdot 2\sinh\alpha).$$

From this inequality, we see that we can choose $\alpha > 0$ sufficiently small, such that

$$Pf(v) \leq f(v) \cdot r \tag{10}$$

for some positive $r < 1$. The function $f(v) = \exp(-\alpha b_1(v_1))$ will then be a positive r-super-harmonic function with $r < 1$, and $\widehat{T}_1' \circ \widehat{T}_2'$ will be non-amenable.

The non-amenability of $\widehat{T}_1 \circ \widehat{T}_2$ follows from the following: Let Φ be the Q-contraction mapping $\widehat{T}_1 \circ \widehat{T}_2$ into a graph with components $\widehat{T}_1'$ and $\widehat{T}_2'$. By the remark after Proposition 4, there is a map ϕ of type *Shift-T_1-by-one*, such that the composition $\phi \circ \Phi$ is a horospheric product $\widehat{T}_1' \circ \widehat{T}_2'$. Since these are leafless trees, and since the composition of quasi-isometries is a quasi-isometry, and since ϕ and Φ are quasi-isometries by Propositions 3 and 4, it holds that if $\widehat{T}_1 \circ \widehat{T}_2$ is amenable, $\phi \circ \Phi(\widehat{T}_1' \circ \widehat{T}_2')$ is also an amenable horospheric product. This last statement is true, since amenability is preserved under quasi-isometries. However, this is in contradiction to the graph $\widehat{T}_1' \circ \widehat{T}_2'$ being non-amenable by (10). $\square$

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