

EXTENDER SETS AND MULTIDIMENSIONAL SUBSHIFTS

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ABSTRACT. In this paper, we consider a \mathbb{Z}^d extension of the well-known fact that subshifts with only finitely many follower sets are sofic. As in [4], we adopt a natural \mathbb{Z}^d analogue of a follower set called an extender set. The extender set of a finite word w in a \mathbb{Z}^d subshift X is the set of all configurations of symbols on the rest of \mathbb{Z}^d which form a point of X when concatenated with w . As our main result, we show that for any $d \geq 1$ and any \mathbb{Z}^d subshift X , if there exists n so that the number of extender sets of words on a d -dimensional hypercube of side length n is less than or equal to n , then X is sofic. We also give an example of a non-sofic system for which this number of extender sets is $n + 1$ for every n .

We prove this theorem in two parts. First we show that if the number of extender sets of words on a d -dimensional hypercube of side length n is less than or equal to n for some n , then there is a uniform bound on the number of extender sets for words on any sufficiently large rectangular prisms; to our knowledge, this result is new even for $d = 1$. We then show that such a uniform bound implies soficity.

Our main result is reminiscent of the classical Morse-Hedlund theorem, which says that if X is a \mathbb{Z} subshift and there exists an n such that the number of words of length n is less than or equal to n , then X consists entirely of periodic points. However, most proofs of that result use the fact that the number of words of length n in a \mathbb{Z} subshift is nondecreasing in n , and we present an example (due to Martin Delacourt) which shows that this monotonicity does not hold for numbers of extender sets (or follower sets) of words of length n .

1. INTRODUCTION

For any \mathbb{Z} subshift X and finite word w appearing in some point of X , the **follower set** of w , written $F_X(w)$, is defined as the set of all one-sided infinite sequences s such that the infinite word ws occurs in some point of X . (In some sources, the follower set is defined as the set of all finite words which can legally follow w , but the former definition may be obtained by taking limits of the latter.) It is well-known that for a \mathbb{Z} subshift X , finiteness of $\{F_X(w) : w \text{ in the language of } X\}$ is equivalent to X being sofic, i.e. the image of a shift of finite type under a continuous shift-commuting map. (For instance, see [5].)

In [4], extender sets were defined and introduced as a natural extension of follower sets to \mathbb{Z}^d subshifts with $d > 1$. The **extender set** of any finite word w in the language of X with shape $S \subset \mathbb{Z}^d$, written $E_X(w)$, is the set of all configurations on $\mathbb{Z}^d \setminus S$ which, when concatenated with w , form a legal point of X . We can no longer speak of a subshift having only finitely many extender sets, since extender sets of patterns with different shapes cannot be compared as in the one-dimensional case. One way to deal with this is examine the growth rate of the number of distinct

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extender sets for words in X with a given shape S (which we denote by $N_S(X)$), as the size of S approaches infinity. This works nicely in the one-dimensional case; our Lemma 3.4 (which is routine) shows that soficity of a one-dimensional subshift is equivalent to boundedness of the number of extender sets of n -letter words as $n \rightarrow \infty$. Interestingly, this sequence need not stabilize; Example 3.5, due to Martin Delacourt ([1]), demonstrates a \mathbb{Z} sofic shift X where $N_{[1,n]}(X)$ oscillates between two values as n increases (In this paper, $[a, b]$ for $a, b \in \mathbb{Z}$ will always represent the set $\{a, a + 1, \dots, b\}$).

There are many relations between properties of X and the behavior of $N_S(X)$. For instance, it is easy to see that when X is a nearest-neighbor shift of finite type, the extender set of a word with shape $[1, n]^d$ is determined entirely by the letters on the boundary. This implies that for such X , $N_{[1,n]^d}(X)$ is bounded from above by $|A_X|^{\partial[1,n]^d} \leq |A_X|^{2dn^{d-1}}$, where A_X denotes the alphabet of X . It was conjectured in [4] that X sofic implies that $\frac{\log N_{[1,n]^d}(X)}{n^d} \rightarrow 0$, but this remains open.

A partial answer was proven in [4], using an argument basically present in [7]. A finite sequence S_n of sets, $1 \leq n \leq N$, was defined to be a union increasing chain if $S_n \not\subseteq \bigcup_{i=1}^{n-1} S_i$ for all $1 \leq n \leq N$. Theorem 2.3 of [4] states that if there exist union increasing chains of size $e^{\omega(n^{d-1})}$ of extender sets of words with shape $[1, n]^d$, then X is not sofic. These results can, broadly speaking, be thought of as showing that a very fast growth rate for extender sets implies that a subshift is not an SFT or sofic. Our main result is in the opposite direction, namely it demonstrates that a slow enough growth rate implies soficity.

Theorem 1.1. *For any $d \geq 1$ and any \mathbb{Z}^d subshift X , if there exists n so that $N_{[1,n]^d}(X) \leq n$, then X is sofic.*

The proof of Theorem 1.1 is broken into two mostly disjoint parts:

Theorem 1.2. *For any $d \geq 1$ and any \mathbb{Z}^d subshift X , if there exists n so that $N_{[1,n]^d}(X) \leq n$, then there exist K, N such that for any rectangular prism R with dimensions at least K , $N_R(X) \leq N$.*

Theorem 1.3. *For any $d \geq 1$ and any \mathbb{Z}^d subshift X , if there exist K, N so that $N_R(X) \leq N$ for all rectangular prisms R with all dimensions at least K , then X is sofic.*

Theorem 1.3 can be thought of as a generalization of the previously mentioned fact that one-dimensional shifts with only finitely many follower sets are sofic. We also show that the upper bound in Theorem 1.1 cannot be improved.

Theorem 1.4. *For any $d \geq 1$, there exists a nonsofic \mathbb{Z}^d subshift X for which $N_{[1,n]^d}(X) = n + 1$ for all n .*

Our results have similarities to the famous Morse-Hedlund theorem.

Theorem 1.5. ([6]) *If X is a \mathbb{Z} subshift and there exists an n such that the number of words of length n is less than or equal to n , then X consists entirely of periodic points. Equivalently, there is a uniform upper bound on the number of words of length n .*

It is well-known that the bound in Theorem 1.5 is also tight. Sturmian subshifts have no periodic points and have so-called minimal complexity, i.e. any Sturmian

subshift has $n + 1$ words of length n for all n . (For an introduction to Sturmian subshifts, see [2].)

There are similarities between Theorems 1.1 and 1.5; in fact Theorem 1.5 is used in our proof of Theorem 1.2. However, there are also some interesting differences. In the usual proof of Theorem 1.5, a key component is that the number of n -letter words is nondecreasing in n . However, Example 3.5 shows that $N_{[1,n]}(X)$ is not necessarily nondecreasing.

2. DEFINITIONS AND PRELIMINARIES

Let A denote a finite set, which we will refer to as our alphabet.

Definition 2.1. A **pattern** over A is a member of A^S for some $S \subset \mathbb{Z}^d$, which is said to have **shape** S . For $d = 1$, patterns are generally called **words**, especially in the case where S is an interval.

For any patterns $v \in A^S$ and $w \in A^T$ with $S \cap T = \emptyset$, we define the concatenation vw to be the pattern in $A^{S \cup T}$ defined by $(vw)|_S = v$ and $(vw)|_T = w$.

Definition 2.2. For any finite alphabet A , the \mathbb{Z}^d -**shift action** on $A^{\mathbb{Z}^d}$, denoted by $\{\sigma_t\}_{t \in \mathbb{Z}^d}$, is defined by $(\sigma_t x)(s) = x(s + t)$ for $s, t \in \mathbb{Z}^d$.

We always think of $A^{\mathbb{Z}^d}$ as being endowed with the product discrete topology, with respect to which it is obviously compact.

Definition 2.3. A \mathbb{Z}^d **subshift** is a closed subset of $A^{\mathbb{Z}^d}$ which is invariant under the \mathbb{Z}^d -shift action.

Definition 2.4. The **language** of a \mathbb{Z}^d subshift X , denoted by $L(X)$, is the set of all patterns which appear in points of X . For any finite $S \subset \mathbb{Z}^d$, $L_S(X) := L(X) \cap A^S$, the set of patterns in the language of X with shape S .

Any subshift inherits a topology from $A^{\mathbb{Z}^d}$, and is compact. Each σ_t is a homeomorphism on any \mathbb{Z}^d subshift, and so any \mathbb{Z}^d subshift, when paired with the \mathbb{Z}^d -shift action, is a topological dynamical system. Any \mathbb{Z}^d subshift can also be defined in terms of disallowed patterns: for any set \mathcal{F} of patterns over A , one can define the set $X(\mathcal{F}) := \{x \in A^{\mathbb{Z}^d} : x|_S \notin \mathcal{F} \text{ for all finite } S \subset \mathbb{Z}^d\}$. It is well known that any $X(\mathcal{F})$ is a \mathbb{Z}^d subshift, and all \mathbb{Z}^d subshifts are representable in this way. All \mathbb{Z}^d subshifts are assumed to be nonempty in this paper.

Definition 2.5. A \mathbb{Z}^d **shift of finite type (SFT)** is a \mathbb{Z}^d subshift equal to $X(\mathcal{F})$ for some finite \mathcal{F} . If \mathcal{F} consists only of patterns consisting of pairs of adjacent letters, then $X(\mathcal{F})$ is called **nearest-neighbor**.

Definition 2.6. A (topological) **factor map** is any continuous shift-commuting map ϕ from a \mathbb{Z}^d subshift X onto a \mathbb{Z}^d subshift Y . A factor map ϕ is **1-block** if $(\phi x)(v)$ depends only on $x(v)$ for $v \in \mathbb{Z}^d$.

Definition 2.7. A \mathbb{Z}^d **sofic shift** is the image of a \mathbb{Z}^d SFT under a factor map. It is well-known that for any \mathbb{Z}^d sofic shift Y , there exists a nearest-neighbor \mathbb{Z}^d SFT X and 1-block factor map ϕ so that $Y = \phi(X)$.

For $d = 1$, any \mathbb{Z} sofic shift can also be defined using graphs; a \mathbb{Z} subshift is sofic if and only if it is the set of labels of biinfinite paths for some (edge-)labeled graph \mathcal{G} (see [5] for a proof.)

Definition 2.8. For any \mathbb{Z}^d subshift X and rectangular prism $R = \prod_{i=1}^d [0, n_i - 1]$, the **R -higher power shift** of X , denoted X^R , is a \mathbb{Z}^d subshift with alphabet $L_R(X)$ defined by the following rule: $x \in (L_R(X))^{\mathbb{Z}^d} \in X^R$ if and only if the point y defined by concatenating the $x(v)$, viewed themselves as patterns with shape X , is in X . Formally,

$$\forall v \in \mathbb{Z}^d, y(v) := (x(\lfloor v_1 n_1^{-1} \rfloor, \dots, \lfloor v_d n_d^{-1} \rfloor)) (v_1 \pmod{n_1}, \dots, v_d \pmod{n_d}).$$

Definition 2.9. For any \mathbb{Z} -subshift X and word $w \in L_{[1,n]}(X)$, the **follower set** of w is $F_X(w) = \{x \in A^{\{n+1, n+2, \dots\}} : wx \in L(X)\}$. For any n , we use $M_{[1,n]}(X)$ to denote $|\{F_X(w) : w \in L_{[1,n]}(X)\}|$, the number of distinct follower sets of words of length n .

Definition 2.10. For any \mathbb{Z}^d -subshift X and pattern $w \in L_S(X)$, the **extender set** of w is $E_X(w) = \{x \in A^{\mathbb{Z}^d \setminus S} : wx \in X\}$. For any S , we use $N_S(X)$ to denote $|\{E_X(w) : w \in L_S(X)\}|$, the number of distinct extender sets of patterns with shape S .

3. PROOFS

For the proof of Theorem 1.2 we need the following finite version of the Morse-Hedlund theorem. We include a proof for completeness, though it is essentially the same proof as that of the original theorem.

Lemma 3.1. *For any word $w \in A^N$ and $n \leq \frac{N}{4}$ so that the number of n -letter subwords of w is less than or equal to n , we can write $w = tuv$ where $|t| = |v| = n$ and u is periodic with some period less than or equal to n .*

Proof. Since there are less than or equal to n subwords of w of length n and there are $N - n + 1 > n$ values of i for which $w(i)w(i+1) \dots w(i+n-1)$ is a subword of w , there exists an n -letter subword of w which appears twice. In fact, by the pigeonhole principle we may fix indices $i < k \in [1, n+1]$ such that $w(i)w(i+1) \dots w(i+n-1) = w(k)w(k+1) \dots w(k+n-1)$. Similarly, we may fix indices $\ell < j \in [N-n, N]$ such that $w(\ell-n+1) \dots w(\ell-1)w(\ell) = w(j-n+1) \dots w(j-1)w(j)$. Set $w' = w(i)w(i+1) \dots w(j-1)w(j)$.

It suffices to show that w' is periodic of period less than or equal to n ; if this is true, then taking $t = w(1) \dots w(n)$, $u = w(n+1) \dots w(N-n)$, and $v = w(N-n+1) \dots w(N)$ completes the proof since u is a subword of w' .

Let us now consider the number of m -letter subwords of w' for values of $m \in [1, n]$. If the number of one-letter subwords of w' is equal to 1, then w' is of the form $ss \dots s$ for some symbol s and we are done. If not, then the number of one-letter subwords of w' is greater than 1, whereas the number of n -letter subwords of w' is less than or equal to n . Therefore, there must be an $m \in [1, n-1]$ for which the number of $(m+1)$ -letter subwords of w' is less than or equal to the number of m -letter subwords of w' . Fix m to be this number for the remainder of the proof.

We now claim that for every m -letter subword t of w' , there exists $a \in A$ so that ta is a subword of w' as well. For any choice of t aside from the m -letter suffix of w' , this is obvious. But it is true for the suffix as well, since by construction of w' , if t is a suffix of w' then t is also the suffix of $w(\ell-n+1) \dots w(\ell-1)w(\ell)$ which means $tw(\ell+1)$ is a subword of w' . A similar argument shows that for every m -letter subword t of w' , there exists $b \in A$ so that bt is a subword of w' as well.

Note that because the number of m -letter subwords is less than or equal to the number of $(m + 1)$ -letter subwords of w' , the a and b constructed in the previous paragraph are always unique.

Let $p = k - i$, and note that $w(i)w(i + 1) \dots w(i + m - 1) = w(i + p)w(i + 1 + p) \dots w(i + m - 1 + p)$. Since there is a unique a which extends the word $w(i)w(i + 1) \dots w(i + m - 1)$ as a subword of w' , we get that $w(i + 1)w(i + 2) \dots w(i + m) = w(i + 1 + p)w(i + 2 + p) \dots w(i + m + p)$. Using the same argument and working inductively, we see that $w(i + r)w(i + r + 1) \dots w(i + r + p) = w(i + r + p)w(i + r + p + 1) \dots w(i + r + p + p)$ for any $0 \leq r \leq j - i - p$. In other words, w' is periodic with period $p \leq n$. \square

We remark that since the word u in the previous lemma is periodic with period less than or equal to n , this clearly implies that u is periodic with period $n!$ (though this may be a meaningless statement if $|u| \leq n!$)

Proof of Theorem 1.2. Consider a \mathbb{Z}^d subshift X and n so that $|N_{[1, n]^d}(X)| \leq n$. Define an equivalence relation on $L_{[1, n]^d}(X)$ by $w \sim w'$ iff $E_X(w) = E_X(w')$. For each of the $k \leq n$ equivalence classes, choose a lexicographically maximal element, and denote the collection of these words by M . Then for every $w \in L_{[1, n]^d}(X)$, there exists $w' \in M$ so that $E_X(w) = E_X(w')$. Equivalently, in any $x \in X$ containing w , w can be replaced by w' to make a new point $x' \in X$.

Now, consider any rectangular prism $R = \prod_{i=1}^d [1, n_i]$ with $n_i > \max 4n, 2n + n!$ for all i , and any finite word $v \in L_R(X)$. Iterate the following procedure: if v contains a subword with shape $[1, n]^d$ which is not in M , then replace it by the element of M in its equivalence class. Since each of these replacements increases the entire word on R in the lexicographic ordering, the procedure will eventually terminate, yielding a word v' in which every subword with shape $[1, n]^d$ is in M . (These replacements could possibly be done in many different ways or orders; simply choose a particular one and call the result v' .) In particular, v' contains less than or equal to n distinct subwords with shape $[1, n]^d$. Since v' was obtained from v by a sequence of replacements with identical extender sets, $E_X(v) = E_X(v')$.

We wish to bound the number of such possible v' for a given R . For any translate of the $(d - 1)$ -dimensional hypercube $t + [1, n]^{d-1} \subset \prod_{i=2}^d [1, n_i]$, consider the subpattern $v'|_{[1, n_1] \times (t + [1, n]^{d-1})}$. This can be viewed as an n_1 -letter word in the x_1 -direction, where each ‘‘letter’’ is a cross-section with shape $t + [1, n]^{d-1}$. When viewed in this way, each n -letter subword of $v'|_{[1, n_1] \times (t + [1, n]^{d-1})}$ is a subpattern of v' with shape $[1, n]^d$, and there are less than or equal to n such subpatterns. Therefore, by Lemma 3.1, $v'|_{[n+1, n_1-n] \times (t + [1, n]^{d-1})}$ is periodic with period $n!e_1$. Since $t + [1, n]^{d-1}$ was arbitrary, in fact $v'|_{[n+1, n_1-n] \times \prod_{i=2}^d [1, n_i]}$ is periodic with period $n!e_1$ as well. In other words, if t and $t + n!e_1$ both have first coordinate between $n + 1$ and $n_1 - n$ inclusive, then $v'(t) = v'(t + n!e_1)$. A similar proof shows that if t and $t + n!e_i$ both have i th coordinate between $n + 1$ and $n_i - n$ inclusive, then $v'(t) = v'(t + n!e_i)$.

The above shows that except for sites within n of the boundary of R , v' is determined by the subpattern occurring within a d -dimensional hypercube of side length $n!$. More specifically, the values of v' on the sites in $\prod_{i=1}^d ([1, n] \cup [n_i - n + 1, n_i] \cup [n + 1, n + n!])$ uniquely determine v' , and there are $(2n + n!)^d$ such sites. So, regardless of our choice for R , there are less than or equal to $|A_X|^{(2n + n!)^d}$ possible v' . Since $E_X(v) = E_X(v')$ for every v , this shows that $|N_R(X)| \leq |A_X|^{(2n + n!)^d}$ for

every R with all dimensions at least n , completing the proof for $K = 2n + n!$ and $N = |A_X|^{(2n+n!)^d}$. \square

We now need a few lemmas for the proof of Theorem 1.3. The first shows that for the purposes of proving X sofic, we may always without loss of generality pass to a higher power shift.

Lemma 3.2. *For any d , for any \mathbb{Z}^d subshift X and rectangular prism $R \subseteq \mathbb{Z}^d$, X is sofic if and only if the higher power shift $X^{[R]}$ is sofic.*

Proof. \implies : Suppose that X is sofic. Then there is a 1-block factor map ϕ and \mathbb{Z}^d nearest-neighbor SFT Y so that $X = \phi(Y)$. But then it is easy to check that $X^{[R]} = \phi^{[R]}(Y^{[R]})$, where $\phi^{[R]}$ acts on patterns in A_Y^R via coordinatewise action of ϕ . Since $Y^{[R]}$ is a \mathbb{Z}^d SFT and $\phi^{[R]}$ is a factor map, clearly $X^{[R]}$ is sofic.

\impliedby : Suppose that $X^{[R]}$ is sofic, and without loss of generality, write $R = \prod_{i=1}^d [0, n_i - 1]$. Then there is a 1-block factor map ψ and \mathbb{Z}^d nearest-neighbor SFT Z so that $X^{[R]} = \psi(Z)$. Define a \mathbb{Z}^d nearest-neighbor SFT Z' with alphabet $A_Z \times R$ by the following rules:

- (1) In the x_i -direction, a letter of the form $(a, (v_1, \dots, v_d))$ must be followed by a letter of the form $(b, (v_1, \dots, v_{i-1}, v_i + 1 \pmod{n_i}, v_{i+1}, \dots, v_d))$.
- (2) In rule (1), if $v_i \neq n_i - 1$, then $b = a$.
- (3) In rule (1), if $v_i = n_i - 1$, then b must be a legal follower of a in the x_i -direction in the nearest-neighbor SFT Z .

The effect of these rules is that in any point of Z' , \mathbb{Z}^d is partitioned into translates of R , each translate of R has a constant “label” from A_Z , and the “labels” of these translates comprise a legal point of Z . We now define a 1-block factor map ϕ' on Z' by the rule $\phi'(a, v) = (\phi(a))(v)$, i.e. the letter of A_Z appearing at location v in $\phi(a)$, which was by definition a pattern in A_Z^R . This has the effect of, in each point of Z' , filling every translate of R with the image under ϕ of the letter of A_Z which was its label. Since these labels comprise a point of Z and since $\phi(Z) = X^{[R]}$, the reader may check that $\phi'(Z') = X$, and so X is sofic. \square

Our next lemma shows that an upper bound for $N_R(X)$ over all large finite rectangular prisms R must also be an upper bound for $N_R(X)$ even when we allow R to have some infinite dimensions.

Lemma 3.3. *For any d and any \mathbb{Z}^d subshift X , if there exist K, N so that $N_R(X) \leq N$ for any rectangular prism R with dimensions at least K , then it is also the case that $N_{R'}(X) \leq N$ for any “infinite rectangular prism” of the form $R' = \prod_{i=1}^d I_i$, where each of the I_i is either an interval of integers with length at least K or \mathbb{Z} .*

Proof. Consider any K, N, X satisfying the hypotheses of the theorem, and any “infinite rectangular prism” R' with all dimensions either finite and greater than K or infinite. Suppose for a contradiction that there exist $N+1$ distinct configurations w_1, \dots, w_{N+1} in $L_{R'}(X)$ and that their extender sets $E_X(w_i)$ are distinct. Then, for each pair $i < j \in [1, N+1]$, there exists a pattern $v_{ij} \in L_{R'^c}(X)$ s.t. $v_{ij}w_i \in X$ and $v_{ij}w_j \notin X$ or vice versa. By compactness, for each v_{ij} , there exists n_{ij} so

that $v_{ij}w_i|_{[-n_{ij}, n_{ij}]^d \cap R'} \in L(X)$ and $v_{ij}w_i|_{[-n_{ij}, n_{ij}]^d \cap R'} \notin L(X)$, or vice versa. This property is clearly preserved by increasing n_{ij} . Therefore, if we define $M = \max(K, \{n_{ij}\}_{i < j})$, then for every $i < j \in [1, N+1]$, either $v_{ij}w_i|_{[-M, M]^d \cap R'} \in L(X)$ and $v_{ij}w_i|_{[-M, M]^d \cap R'} \notin L(X)$ or vice versa. Put another way, $E_X(w_i|_{[-M, M]^d \cap R'})$ contains a pattern which equals v_{ij} on R'^c , and $E_X(w_j|_{[-M, M]^d \cap R'})$ contains no such pattern, or vice versa. Either way, this shows that $E_X(w_i|_{[-M, M]^d \cap R'}) \neq E_X(w_j|_{[-M, M]^d \cap R'})$ and, since i, j were arbitrary, that all $N+1$ of the extender sets $E_X(w_i|_{[-M, M]^d \cap R'})$, $i \in [1, N+1]$, are distinct. Since $[-M, M]^d \cap R'$ is a finite rectangular prism with all dimensions at least K , this contradicts the hypotheses of the theorem. Our original assumption was therefore wrong, and $N_{R'}(X) \leq N$. \square

Our final preliminary lemma shows that for $d = 1$, boundedness of $N_{[1, n]}(X)$ is equivalent to soficity of X .

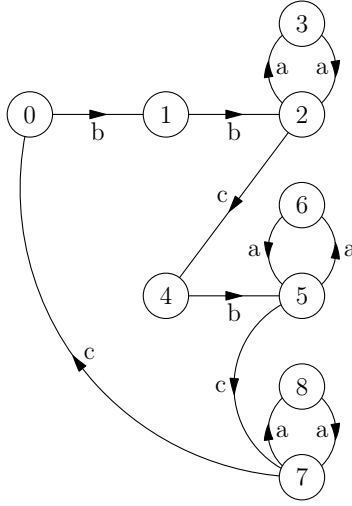
Lemma 3.4. *For a \mathbb{Z} subshift X , X is sofic if and only if $N_{[1, n]}(X)$ is a bounded sequence.*

Proof. \implies : If X is sofic, then there is a 1-block map ϕ and nearest-neighbor SFT Y , with alphabet A_Y , so that $X = \phi(Y)$. Then, for any finite word $w \in L_{[1, n]}(Y)$, clearly $E_X(w) = \bigcup_{y \in \phi^{-1}(w)} \phi(E_Y(y))$. Since Y is a nearest-neighbor SFT, this clearly depends only on the set of pairs of first and last letters of ϕ -preimages of w , and there are fewer than $2^{|A_Y|^2}$ such sets. Therefore, $N_{[1, n]}(X) \leq 2^{|A_Y|^2}$ for all n , and so the sequence $N_{[1, n]}(X)$ is bounded.

\impliedby : We prove the contrapositive, the proof will be similar to Lemma 3.3. Suppose that X is not sofic. Then there are infinitely many follower sets $F(p)$ for infinite pasts $p \in A_X^{\mathbb{Z}^-}$. For any N , choose N pasts p_1, \dots, p_N with distinct follower sets. This means that for every $i < j \in [1, N]$, there exists a future $f_{ij} \in A_X^{\mathbb{N}}$ so that either $p_i f_{ij} \in X$ and $p_j f_{ij} \notin X$, or vice versa. By compactness, there exists N_{ij} so that for any $n > N_{ij}$, either $p_i|_{(-n, 0)} f_{ij} \in X$ and $p_j|_{(-n, 0)} f_{ij} \notin X$ or vice versa. But then if we take $M = \max N_{ij}$, then for every $i < j \in [1, N]$, either $p_i|_{(-M, 0)} f_{ij} \in X$ and $p_j|_{(-M, 0)} f_{ij} \notin X$ or vice versa, meaning that the N extender sets $E_X(p_i|_{(-M, 0)})$, $i \in [1, N]$, are distinct. Therefore, $N_{[1, M]}(X) \geq N$. Since N was arbitrary, $N_{[1, n]}(X)$ is not bounded. \square

As an aside, before proving Theorem 1.3 we present the example mentioned in the introduction, of a \mathbb{Z} sofic shift X where $N_{[1, n]}(X)$ is bounded, but does not stabilize. In fact, the number of distinct follower sets of words of length n also fails to stabilize for this shift, which may be of independent interest.

Example 3.5. ([1]) Define X to be the sofic shift consisting of all labels of biinfinite paths on the labeled graph \mathcal{G} below. Then for all $n > 1$, $M_{[1, 2n]}(X) = 14$, $M_{[1, 2n+1]}(X) = 13$, $N_{[1, 2n]}(X) = 46$, and $N_{[1, 2n+1]}(X) = 44$.



Proof. The reader may check that \mathcal{G} is follower-separated (see [5] for a definition), and so for any $w \in L(X)$, the follower set $F_X(w)$ is determined by the set of terminal vertices for paths in \mathcal{G} with label w , which we'll denote by $T(w)$. We now simply describe the possible sets $T(w)$ for words of even and odd length, with examples of words realizing each set. We use the notation $*$ to indicate that any word may replace the $*$, and n to represent any nonnegative integer.

(even length)		(odd length)	
$T(w)$	w	$T(w)$	w
{0}	$*cc$	{0}	$*cc$
{1}	$*ccb$	{1}	$*ccb$
{2}	$*cbb$	{2}	$*cbb$
{3}	$*bba$	{3}	$*bba$
{4}	$*bbc$	{4}	$*bbc$
{5}	$*bcb$	{5}	$*bcb$
{6}	$*cba$	{6}	$*cba$
{7}	$*cbc$	{7}	$*cbc$
{8}	$*bca$	{8}	$*bca$
{1, 5}	$a^n cb$	{1, 5}	$a^n cb$
{3, 6}	ba^{2n+1}	{2, 5}	$ba^{2(n+1)}$
{4, 7}	$ba^{2n} c$	{0, 4, 7}	$a^n c$
{0, 4, 7}	$a^n c$	{2, 3, 5, 6, 7, 8}	a^{2n+1}
{2, 3, 5, 6, 7, 8}	$a^{2(n+1)}$		

We leave it to the reader to check that there are no follower sets aside from the ones described here, and so $M_{[1,2n]}(X) = 14$ and $M_{[1,2n+1]}(X) = 13$ for all $n > 1$. Informally, the reason that words of even length have an additional follower set is that the word $ba^{2n}c$ has a follower set (given by the set $\{3, 6\}$ of terminating states) which can not be recreated by odd length; every cycle has even length, knowledge of at least one letter on each side of the cycle is required to create a new follower set, and knowledge of two letters on either side makes the word synchronizing (meaning there is only a single terminating state.)

Since listing 46 and 44 extender sets similarly (for even and odd lengths respectively) would be quite cumbersome, we will not give a complete list of these, but will give a sketch of how they appear. First, note that \mathcal{G} is also predecessor separated, and so the extender set of a word $w \in L(X)$ is determined entirely by the set $\{v \rightarrow v'\}$ of possible pairs of initial and terminal vertices of paths in \mathcal{G} with label w , which we denote by $S(w)$. Note that partitioning the vertices into $\{0, 2, 5, 7\}$ and $\{1, 3, 4, 6, 8\}$ shows that \mathcal{G} is bipartite. The reader may check that every possible singleton $\{v \rightarrow v'\}$ for pairs v, v' in the same vertex class occurs as $S(w)$ for a word w of even length, and every possible singleton $\{v \rightarrow v'\}$ for pairs v, v' in opposite vertex classes occurs as $S(w)$ for a word w of odd length. This contributes $5^2 + 4^2 = 41$ extender sets to $N_{[1,2n]}(X)$ and $2 \cdot 5 \cdot 4 = 40$ extender sets to $N_{[1,2n+1]}(X)$ for every $n > 1$. The remaining sets $S(w)$, along with w presenting them, appear in the table below. Note that though we can informally pair up the first through fourth types in each case, again the word $ba^{2n-2}c$ creates an extender set with no analogous extender set for a word of odd length.

(even length)

$S(w)$	w
$\{2 \rightarrow 2, 3 \rightarrow 3, 5 \rightarrow 5, 6 \rightarrow 6, 7 \rightarrow 7, 8 \rightarrow 8\}$	a^{2n}
$\{1 \rightarrow 3, 4 \rightarrow 6\}$	ba^{2n-1}
$\{2 \rightarrow 5, 7 \rightarrow 1\}$	$a^{2n-2}cb$
$\{3 \rightarrow 4, 6 \rightarrow 7, 8 \rightarrow 0\}$	$a^{2n-1}c$
$\{1 \rightarrow 4, 4 \rightarrow 7\}$	$ba^{2n-2}c$

(odd length)

$S(w)$	w
$\{2 \rightarrow 3, 3 \rightarrow 2, 5 \rightarrow 6, 6 \rightarrow 5, 7 \rightarrow 8, 8 \rightarrow 7\}$	a^{2n-1}
$\{1 \rightarrow 2, 4 \rightarrow 5\}$	ba^{2n}
$\{3 \rightarrow 5, 8 \rightarrow 1\}$	$a^{2n-1}cb$
$\{2 \rightarrow 4, 5 \rightarrow 7, 7 \rightarrow 0\}$	$a^{2n}c$

□

Proof of Theorem 1.3. Our proof proceeds by induction on d . The base case $d = 1$ is precisely Lemma 3.4. We now assume that the result holds for \mathbb{Z}^{d-1} subshifts, and will prove it for \mathbb{Z}^d subshifts. To that end, assume that X is a \mathbb{Z}^d subshift and that there exist K, N so that for any rectangular prism R with dimensions at least K , $N_R(X) \leq N$. By Lemma 3.3, the same is true even if R has some infinite dimensions.

Note that by Lemma 3.2, we may without loss of generality replace X by the higher power shift $X^{[[1,K]^d]}$. Since $N_R(X^{[[1,K]^d]}) \leq N$ for all rectangular prisms, with no restrictions on the dimension, we will assume this property for X in the remainder of the proof.

Define $X' = \{x|_{\mathbb{Z}^{d-1} \times \{0\}} : x \in X\}$, the set of restrictions of points of X to hyperplanes spanned by the first $d - 1$ cardinal directions. By the assumption above, there are fewer than N distinct extender sets for $x \in X'$, and so we define equivalence classes C_i , $i \in [1, M]$, $M \leq N$, for the equivalence relation defined by

$x \sim y$ if $E_X(x) = E_X(y)$. In a slight abuse of notation, we denote by $E_X(C_i)$ the common extender set shared by all $x \in C_i$.

Now, consider any $x \in X'$ and $k \in [1, d-1]$. By the pigeonhole principle, there exist $i < j \in [1, M+1]$ so that $\sigma_{ie_k}x \sim \sigma_{je_k}x$. But then for $y \in L_{\mathbb{Z}^{d-1} \times \{0\}^c}(X)$, $y \in E_X(x) \iff \sigma_{ie_k}y \in E_X(\sigma_{ie_k}x) \iff \sigma_{ie_k}y \in E_X(\sigma_{je_k}x) \iff y \in E_X(\sigma_{(j-i)e_k}x)$. Therefore, $x \sim \sigma_{(j-i)e_k}x$, and the same logic shows that $\sigma_{(j-i)me_k}x \sim x$ for any $m \in \mathbb{Z}$. Since $j-i \leq M \leq N$, this shows that the C_i containing x is invariant under shifts by $N!e_k$ for $k \in [1, d-1]$. Since $x \in X'$ was arbitrary, this means that every C_i is invariant under shifts by each $N!e_k$. We may then, again by Lemma 3.2, replace X by its higher power shift $X^{[[1, N!]^{d-1} \times \{0\}]}$, which allows us to assume without loss of generality that all of the C_i are shift-invariant subsets of $A_X^{\mathbb{Z}^{d-1}}$. The classes C_i need not, however, be closed. Their closures are \mathbb{Z}^{d-1} subshifts though, and we will show that they in fact must be sofic.

Claim 1: $\overline{C_i}$ is sofic for each i .

It suffices to show that for any rectangular prism $R \subseteq \mathbb{Z}^{d-1}$ and $w, w' \in L_R(\overline{C_i})$, $E_X(w) = E_X(w') \implies E_{\overline{C_i}}(w) = E_{\overline{C_i}}(w')$, since then $N_{R \times \{0\}}(X) \leq N \implies N_{R \times \{0\}}(\overline{C_i}) \leq N$ for all rectangular prisms R , which will imply the desired conclusion by our inductive hypothesis.

So, assume that $E_X(w) = E_X(w')$ for $w, w' \in L_R(\overline{C_i})$. Suppose also that $vw \in \overline{C_i}$ for some $v \in A_X^{\mathbb{Z}^{d-1} \setminus R}$. Then there exists $v_n \in A_X^{\mathbb{Z}^{d-1} \setminus R}$ so that $v_n \rightarrow v$ and $v_n w \in C_i$ for all n . Then for any $y \in E_X(C_i)$, $yv_n w \in X$, since all $v_n w$ share the same class C_i . Since $E_X(w) = E_X(w')$, $yv_n w' \in X$ as well. Similarly, for any $y \notin E_X(C_i)$, $yv_n w \notin X$, and so $yv_n w' \notin X$. But then $E_X(v_n w') = E_X(C_i)$, and so $v_n w' \in C_i$. By taking limits, $vw' \in \overline{C_i}$. We've then shown that $vw \in \overline{C_i} \implies vw' \in \overline{C_i}$. The converse is true by the same proof, and so $E_{\overline{C_i}}(w) = E_{\overline{C_i}}(w')$, completing the proof of soficity of $\overline{C_i}$ as described above.

Since all elements of any class C_i are interchangeable in points of X , we can define $V \subseteq [1, M]^{\mathbb{Z}}$ which lists legal sequences of classes (in the e_d -direction) within points in X :

$$V := \{(i_n) \in [1, M]^{\mathbb{Z}} : \exists x \in X \text{ such that } \forall n \in \mathbb{Z}, x|_{\mathbb{Z}^{d-1} \times \{n\}} \in C_{i_n}\}.$$

It is obvious that V is shift-invariant since X is shift-invariant. However, it is not immediately clear that V is closed since the C_i are not necessarily closed. We will show that V is closed by proving the following claim.

Claim 2: $X = \{x \in A_X^{\mathbb{Z}^d} : \exists (i_n) \in V \text{ such that } \forall n \in \mathbb{Z}, x|_{\mathbb{Z}^{d-1} \times \{n\}} \in \overline{C_{i_n}}\}.$

In other words, given $v \in V$, not only can you make points of X by substituting in configurations from the classes given by the letters in v , but you may also substitute configurations from the closures of these classes.

$$X \subseteq \{x \in A_X^{\mathbb{Z}^d} : \exists (i_n) \in V \text{ such that } \forall n \in \mathbb{Z}, x|_{\mathbb{Z}^{d-1} \times \{n\}} \in \overline{C_{i_n}}\}:$$

First, we note that for any $x \in X$, by definition $x|_{\mathbb{Z}^{d-1} \times \{n\}} \in X'$ for all $n \in \mathbb{Z}$, and so each of these is in some class C_i . Define $v = (i_n) \in [1, M]^{\mathbb{Z}}$ by saying that the $x|_{\mathbb{Z} \times \{n\}}$ of x is in C_{i_n} . Then by definition of V , $v \in V$. This clearly shows the desired containment.

$$X \supseteq \{x \in A_X^{\mathbb{Z}^d} : \exists (i_n) \in V \text{ such that } \forall n \in \mathbb{Z}, x|_{\mathbb{Z} \times \{n\}} \in \overline{C_{i_n}}\}:$$

Choose any $x \in A_X^{\mathbb{Z}^d}$ so that there is $v = (i_n) \in V$ with the property that $\forall n \in \mathbb{Z}$, $x|_{\mathbb{Z}^{d-1} \times \{n\}} \in \overline{C_{i_n}}$. Then, for each $n \in \mathbb{Z}$, there exists a sequence $x^{(k,n)} \in C_{i_n}$ so that $x^{(k,n)} \xrightarrow[k \rightarrow \infty]{} x|_{\mathbb{Z}^{d-1} \times \{n\}}$ for all n . Also, since $v \in V$, there exists $x' \in X$ so that $\forall n \in \mathbb{Z}$, $x'|_{\mathbb{Z}^{d-1} \times \{n\}} \in C_{i_n}$.

We now define, for every k , the point $x^{(k)} \in A_X^{\mathbb{Z}^d}$ by

$$x^{(k)}|_{\mathbb{Z}^{d-1} \times \{n\}} = \begin{cases} x^{(k,n)} & \text{if } |n| \leq k \\ x'|_{\mathbb{Z}^{d-1} \times \{n\}} & \text{if } k \leq |n| \end{cases}$$

The central $2k+1$ $(d-1)$ -dimensional hyperplanes of $x^{(k)}$ are given by the $x^{(k,n)}$, and the remaining $(d-1)$ -dimensional hyperplanes are unchanged from x' . We note that $x^{(k)}$ can be obtained from $x' \in X$ by making $2k+1$ consecutive replacements of $x'|_{\mathbb{Z}^{d-1} \times \{n\}}$ by $x^{(k,n)}$. Since these replacements involve configurations in the same class C_{i_n} , each of these replacements preserves being in X , and so $x^{(k)} \in X$ for all k . Finally, we note that $x^{(k)} \rightarrow x$, so $x \in X$ as well, showing the desired containment.

Claim 3: V is a sofic subshift.

We first show that V is closed and therefore a subshift. Let $v^{(k)} \in V$ and $v^{(k)} \rightarrow v = (i_n)$. By passing to a subsequence, we may assume that for all $k \geq |n|$, $v_n^{(k)} = i_n$. By definition of V , for every $k \in \mathbb{N}$ there exists $x^{(k)} \in X$ where $x^{(k)}|_{\mathbb{Z}^{d-1} \times \{n\}} \in C_{v_n^{(k)}}$ for every n . For all $n \leq k$, since $C_{v_n^{(k)}} = C_{i_n} = C_{v_n^{(n)}}$ we may replace the pattern $x^{(k)}|_{\mathbb{Z}^{d-1} \times \{n\}}$ in $x^{(k)}$ with $x^{(n)}|_{\mathbb{Z}^{d-1} \times \{n\}}$ to form a legal point in X . In such a way we obtain a new sequence of points $y^{(k)} \in X$ where $y^{(k)}|_{\mathbb{Z}^{d-1} \times \{n\}} \in C_{v_n^{(k)}}$, but with the additional property that $y^{(k)}|_{\mathbb{Z}^{d-1} \times \{n\}} = y^{(n)}|_{\mathbb{Z}^{d-1} \times \{n\}}$ for $k \geq |n|$. The sequence $y^{(k)}$ clearly converges to a point $y \in A_X^{\mathbb{Z}^d}$, and $y \in X$ since X is closed. Since $y|_{\mathbb{Z}^{d-1} \times \{n\}} = y^{(n)}|_{\mathbb{Z}^{d-1} \times \{n\}} \in C_{i_n}$, we have $v \in V$.

We claim that $N_{[1,m]}(V) \leq N$ for all $m \in \mathbb{N}$, which will prove the claim by Lemma 3.4. Suppose for a contradiction that there are $N+1$ words $v^{(1)}, \dots, v^{(N+1)} \in L_m(V)$ s.t. the extender sets $E_V(v^{(i)})$ are all distinct. Then, for every $i < j \in$

$[1, N + 1]$, there exists $w^{(ij)} \in L_{[1,m]^c}(V)$ s.t. either $v^{(i)}w^{(ij)} \in V$ and $v^{(j)}w^{(ij)} \notin V$ or vice versa.

For each $v^{(i)} = v_1^{(1)} \dots v_m^{(1)}$, define $S^{(i)} \in A_X^{\mathbb{Z}^{d-1} \times [1,m]}$ by choosing $S^{(i)}|_{\mathbb{Z}^{d-1} \times \{n\}}$ to be any row in $C_{v_n^{(i)}}$. Similarly, for each $w^{(ij)}$, define a pattern $O^{(ij)} \in A_X^{\mathbb{Z}^{d-1} \times [1,m]^c}$ by choosing $O^{(ij)}|_{\mathbb{Z}^{d-1} \times \{b\}}$ to be any row in $C_{w_n^{(ij)}}$. Then, by Claim 2, $S^{(i)}O^{(ij)} \in X$ and $S^{(j)}O^{(ij)} \notin X$ or vice versa, meaning that all extender sets $E_X(S^{(i)})$, $i \in [1, N + 1]$, are distinct. This is a contradiction to Lemma 3.3, and so our original assumption was wrong, $N_{[1,m]}(V)$ is a bounded sequence, and V is sofic.

We may now finally construct an SFT cover of X to show that it is sofic. Since V is sofic by Claim 3, we may define a 1-block factor ψ and nearest-neighbor SFT W so that $\psi(W) = V$. For each $a \in A_V$, since $\overline{C_a}$ is sofic by Claim 1, there is a 1-block factor ϕ_a and nearest-neighbor \mathbb{Z}^{d-1} SFT Y_a (whose alphabet we denote by A_a) so that $\phi_a(Y_a) = \overline{C_a}$. Now, define a nearest-neighbor \mathbb{Z}^d SFT Y with alphabet $A_Y := \bigcup_{a \in A_W} (\{a\} \times A_{Y_{\psi(a)}})$ by the following adjacency rules:

- (1) Any pair of letters (a, s) , (b, t) which are adjacent in one of the first $d - 1$ cardinal directions must share the same first coordinate, i.e. $a = b$.
- (2) (a, s) may legally precede (a, t) in the e_i -direction for $i \in [1, d - 1]$ if and only if s may legally precede t in the same direction in $Y_{\psi(a)}$.
- (3) (a, s) may legally precede (b, t) in the e_d -direction if and only if a may legally precede b in W . (There is no restriction on the second coordinates s, t .)

Clearly for any $y \in Y$, these rules force the $(d - 1)$ -dimensional hyperplanes $y|_{\mathbb{Z}^{d-1} \times \{n\}}$ to have constant first coordinate (say a_n), force the second coordinates to form a point in $Y_{\psi(a_n)}$, and force the sequence (a_n) to be in W . We now define the 1-block factor map ϕ on Y by $\phi(a, s) = \phi_{\psi(a)}(s)$.

Claim 4: $\phi(Y) = X$.

$\phi(Y) \subseteq X$: Take any $y \in Y$, and define a_n to be the first coordinate shared by all letters in $y|_{\mathbb{Z}^{d-1} \times \{n\}}$. Then by definition of Y , $(a_n) \in W$. Also by definition of Y , the second coordinates of the letters in $y|_{\mathbb{Z}^{d-1} \times \{n\}}$ form a point of $Y_{\psi(a_n)}$, call it $b^{(n)}$. Then, $(\psi(a_n)) \in V$, and for every $n \in \mathbb{Z}$, $(\phi(y))|_{\mathbb{Z}^{d-1} \times \{n\}} = \phi_{\psi(a_n)}(b^{(n)})$ is the $\phi_{\psi(a_n)}$ -image of a point of $Y_{\psi(a_n)}$, and so is in $\overline{C_{a_n}}$. But then, by Claim 2, $\phi(y) \in X$, and since $y \in Y$ was arbitrary, $\phi(Y) \subseteq X$.

$\phi(Y) \supseteq X$: Choose any $x \in X$. For every $n \in \mathbb{Z}$, $x|_{\mathbb{Z}^{d-1} \times \{n\}}$ is in one of the C_i , and if we define a sequence i_n by $x|_{\mathbb{Z}^{d-1} \times \{n\}} \in C_{i_n}$, then $(i_n) \in V$. Choose any $(a_n) \in W$ s.t. $(\psi(a_n)) = (i_n)$. For each $n \in \mathbb{Z}$, since $x|_{\mathbb{Z}^{d-1} \times \{n\}} \in \overline{C_{i_n}} = \overline{C_{\psi(a_n)}}$, there exists $b^{(n)} \in Y_{\psi(a_n)}$ s.t. $\phi_{\psi(a_n)}(b^{(n)}) = x|_{\mathbb{Z}^{d-1} \times \{n\}}$. Define a point $y \in A_Y^{\mathbb{Z}^2}$ by setting, for all $t = (t_1, \dots, t_d) \in \mathbb{Z}^d$, $y(t) = (a_{t_d}, b^{(n)}(t_1, \dots, t_{d-1}))$. Then $y \in Y$ since (a_n) is in W , and for all $n \in \mathbb{Z}$, the second coordinates in each hyperplane

$y|_{\mathbb{Z}^{d-1} \times \{n\}}$ create a point of $Y_{\psi(a_n)}$. It is also clear that $\phi(y) = x$, and since X was arbitrary, $\phi(Y) \supseteq X$.

Since Y was an SFT and ϕ a factor map, this shows that X is sofic, completing the proof of Theorem 1.3. \square

We would like to say a bit more about shifts satisfying the hypotheses of Theorem 1.3, i.e. those with eventually bounded numbers of extender sets, because in fact they satisfy a much stronger (though technical) condition than just being sofic.

Definition 3.6. We say that a \mathbb{Z}^d nearest-neighbor SFT X is **decouplable** if either $d = 1$ (in which case X is automatically called decouplable) or there exist $i \in [1, d]$, a \mathbb{Z} nearest-neighbor SFT W , and decouplable \mathbb{Z}^{d-1} nearest-neighbor SFTs Y_a for each $a \in A_W$, with disjoint alphabets, so that

$$X = \{x \in A_X^{\mathbb{Z}^d} : \exists w = (w_n) \in W \text{ s.t. } \forall n \in \mathbb{Z}, x|_{\mathbb{Z}^{i-1} \times \{n\} \times \mathbb{Z}^{d-i}} \in Y_{w_n}\}.$$

(Here, we have made the obvious identification between $\mathbb{Z}^{i-1} \times \{n\} \times \mathbb{Z}^{d-i}$ and \mathbb{Z}^{d-1} .)

In other words, X is decouplable if one can construct it by starting from a one-dimensional nearest neighbor SFT and then arbitrarily replacing occurrences of each letter in its alphabet by points from a \mathbb{Z}^{d-1} decouplable nearest-neighbor SFT associated to that letter. This definition is obviously recursive; X is decouplable if its $(d-1)$ -dimensional hyperplanes are given by decouplable SFTs, whose $(d-2)$ -dimensional hyperplanes are given by decouplable SFTs, and so on. This means that though X is a \mathbb{Z}^d SFT, its behavior is in some sense one-dimensional.

Remark 3.7. In fact the SFT cover in the proof of Theorem 1.3 can always be chosen to be decouplable. By the inductive hypothesis, the covers Y_a can each be chosen to be decouplable \mathbb{Z}^{d-1} SFTs, and then the construction of X from W and all Y_a clearly yields a decouplable \mathbb{Z}^d SFT.

In order to give an application of Theorem 1.3 and to elucidate the idea of decouplable SFTs, we will present a brief example.

Example 3.8. Define X to be the \mathbb{Z}^2 subshift on $\{0, 1\}$ consisting of all $x \in \{0, 1\}^{\mathbb{Z}^2}$ with either no 1s, a single 1, or two 1s.

Then it is not hard to see that for any $S \subseteq \mathbb{Z}^2$ with $|S| > 1$, $N_S(X) = 3$. It is easily checked that the three possible extender sets for $w \in L_S(X)$ are:

- If w contains no 1s, then $E_X(w)$ consists of all patterns on S^c with either no 1s, a single 1, or two 1s.
- If w contains a single 1, then $E_X(w)$ consists of all patterns on S^c with either no 1 or a single 1.
- If w contains two 1s, then $E_X(w)$ consists of the single pattern on S^c with no 1s, namely 0^{S^c} .

We will now describe how the proof of Theorem 1.3 yields an SFT cover for X . Using the language of the proof of Theorem 1.3, X' consists of all biinfinite 0-1

sequences with either no 1s, a single 1, or two 1s. X' is broken into three classes of rows with the same extender sets in X , which are again classified by number of 1s contained:

- $C_0 = \{x \in \{0, 1\}^{\mathbb{Z}} : x \text{ contains no 1s}\} = \{0^{\mathbb{Z}}\}$.
- $C_1 = \{x \in \{0, 1\}^{\mathbb{Z}} : x \text{ contains exactly one 1}\}$.
- $C_2 = \{x \in \{0, 1\}^{\mathbb{Z}} : x \text{ contains exactly two 1s}\}$.

(We've written the C_i s with subscripts starting from 0 rather than 1 so that V can be more easily described; clearly this has no effect on the proof.) Note that each of the C_i is shift-invariant, but C_1 and C_2 are not closed. However, each closure $\overline{C_i}$ is sofic. This is easily checked, but we will momentarily explicitly describe SFT covers of the $\overline{C_i}$ anyway.

We now wish to find V , the \mathbb{Z} subshift with alphabet $\{0, 1, 2\}$ which describes how the rows in various classes can fit together to make points of X . This is not so hard to see: since points of X must have at most two 1s, and since the classes C_i are partitioned by number of 1s,

$$V = \{v \in \{0, 1, 2\}^{\mathbb{Z}} : v \text{ has only finitely many nonzero digits, and } \sum v_n \leq 2\}.$$

Points of X are then constructed by beginning with a sequence in V , writing it vertically, and replacing each letter v_n with an arbitrary element of C_{v_n} . So, for instance, one could start with $\dots 0002000 \dots \in V$, replace all 0s by the single sequence $\dots 000000 \dots \in C_0$, and replace 2 by any sequence in C_2 , for instance $\dots 0001001000 \dots$. Clearly every point obtained in this way will have at most two 1s and so will be in X . In addition though, as described in Claim 2, one can also replace each v_n by an arbitrary element of the closure $\overline{C_{v_n}}$. For instance, if we chose to replace the 2 in our earlier sequence by $\dots 0001000 \dots$, which is not in C_2 (in fact it's in C_1), but is in $\overline{C_2}$, we would still arrive at a legal point of X .

We now note that V is a sofic shift, with nearest-neighbor SFT cover W defined as follows: $A_W = \{A, B, C, D, E, F\}$, and legal adjacent pairs in W are $AA, AB, BC, CC, CD, DE, EE, AF$, and FE . So,

$$W = \{A^\infty, C^\infty, E^\infty, A^\infty BC^\infty, C^\infty DE^\infty, A^\infty BC^n DE^\infty, A^\infty FE^\infty\}.$$

The factor ψ is defined by $\psi(A) = \psi(C) = \psi(E) = 0$, $\psi(B) = \psi(D) = 1$, and $\psi(F) = 2$, and it is easily checked that $\psi(W) = V$.

Following our proof of Theorem 1.3, the next step is to construct SFT covers of each $\overline{C_i}$, which is straightforward. Define Y_0 to consist of the single fixed point $\{a^\infty\}$, and ϕ_0 by $\phi_0(a) = 0$. Define Y_1 to have alphabet $\{a, b, c\}$ with legal adjacent pairs aa, ab, bc , and cc ; then $Y_1 = \{a^\infty, c^\infty, a^\infty bc^\infty\}$. Define ϕ_1 by $\phi_1(a) = \phi_1(c) = 0$ and $\phi_1(b) = 1$. Finally, define Y_2 to have alphabet $\{a, b, c, d, e\}$ with legal adjacent pairs aa, ab, bc, cd, de , and ee ; then $Y_2 = \{a^\infty, c^\infty, e^\infty, a^\infty bc^\infty, c^\infty de^\infty, a^\infty bc^n de^\infty\}$. Define ϕ_2 by $\phi_2(a) = \phi_2(c) = \phi_2(e) = 0$ and $\phi_2(b) = \phi_2(d) = 1$. The reader may check that $\phi_i(Y_i) = \overline{C_i}$ for each i .

We may now construct an SFT cover Y for X following our proof. The alphabet $A_Y := \bigcup_{a \in A_W} (\{a\} \times A_{\psi(a)}) = \{(A, a), (B, a), (B, b), (B, c), (C, a), (D, a), (D, b), (D, c), (E, a), (F, a), (F, b), (F, c), (F, d), (F, e)\}$. The adjacency rules are that horizontally adjacent letters have the same first (capital) coordinate Π and second (lowercase) coordinates satisfying the adjacency rules given by $Y_{\psi(\Pi)}$, and that vertically adjacent letters have first (capital) coordinates satisfying the adjacency rules of W . So,

for instance, (D, a) cannot appear immediately to the left of (D, c) , since ac is not a legal pair in $Y_{\psi(D)} = Y_1$. On the other hand, (A, a) can appear below (F, d) , since AF is a legal pair in W . The map ϕ is defined, as before by $\phi(a, b) = \phi_{\psi(a)}(b)$, meaning that $\phi(B, b) = \phi(D, b) = \phi(F, b) = \phi(F, d) = 1$, and all other letters of A_Y have ϕ -image 0. It's easy to see that $\phi(Y) = X$; the rules defining Y mean that there are at most two letters with second coordinate b or d , and these are the only letters of A_Y which have ϕ -image 1. □

Finally, we will prove Theorem 1.4, but first need the following definition and theorem from [3].

Definition 3.9. ([3]) A \mathbb{Z}^d subshift X is **effective** if there exists a forbidden list (w_n) for X and a Turing machine which, on input n , outputs w_n .

It is easy to see that not every subshift is effective; there are only countably many Turing machines, and so only countably many effective subshifts.

Theorem 3.10. ([3]) *Any \mathbb{Z}^d sofic shift is effective.*

Proof of Theorem 1.4. For any Sturmian \mathbb{Z} subshift S , we can extend S to a \mathbb{Z}^d subshift \tilde{S} by enforcing constancy along all cardinal directions e_i , $i \in [2, d]$. There are uncountably many Sturmian subshifts, and so there exists one, call it S' , s.t. \tilde{S}' is not effective. (In fact, effectiveness of Sturmian S and/or the shift \tilde{S} is equivalent to computability of the rotation number defining S , but we will not need this fact.) By Theorem 3.10, \tilde{S}' is not sofic. In addition, by the minimal complexity definition of Sturmian subshifts, for every $n \in \mathbb{N}$, $|L_{[1,n]^d}(\tilde{S}')| = |L_{[1,n]}(S')| = n + 1$, and so trivially $N_{[1,n]}(\tilde{S}') \leq n + 1$. Since \tilde{S}' is not sofic, by Theorem 1.1, in fact $N_{[1,n]}(\tilde{S}') = n + 1$ for all n . □

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