

TOPOLOGICAL RAMSEY SPACES AND METRICALLY BAIRE SETS

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ABSTRACT. We characterize a class of topological Ramsey spaces such that each element \mathcal{R} of the class induces a collection $\{\mathcal{R}_k\}_{k<\omega}$ of projected spaces which have the property that every Baire set is Ramsey. Every projected space \mathcal{R}_k is a subspace of the corresponding space of length- k approximation sequences with the Tychonoff, equivalently metric, topology. This answers a question of S. Todorćevic and generalizes the results of Carlson [1], Carlson-Simpson [3], Prömel-Voigt [23], and Voigt [29]. We also present a new family of topological Ramsey spaces contained in the aforementioned class which generalize the spaces of ascending parameter words of Carlson-Simpson [3] and Prömel-Voigt [23] and the spaces $\text{FIN}_m^{[\infty]}$, $0 < m < \omega$, of block sequences defined by Todorćevic [25].

1. INTRODUCTION

There exists a class of topological Ramsey spaces whose members admit, for every $k < \omega$, a set of sequences of k -approximations that can be understood as a topological space where every Baire set satisfies the Ramsey property. Such topological spaces inherit the Tychonoff, (equivalently, metric) topology from the space of approximation sequences associated to the elements of the underlying topological Ramsey space. We shall say that these projected spaces have the property that every metrically Baire subset is Ramsey.

The infinite Dual Ramsey Theorem of Carlson and Simpson in [3] was the first result where this phenomenon was seen. In that paper, it was shown that for Carlson-Simpson's space of equivalence relations on \mathbb{N} with infinitely many equivalence classes, for each $k < \omega$, the projected space of equivalence relations on \mathbb{N} with exactly k equivalence classes has the property that every metrically Baire set is Ramsey. Other examples where this phenomenon occurs are Prömel-Voigt's spaces of parameter words, ascending parameter words and partial G -partitions (where G is a finite group) [23]; and Carlson's space of infinite dimensional vector subspaces of $F^{\mathbb{N}}$ (where F is a finite field) [1], in connection with an extension of the Graham-Leeb-Rothschild Theorem [11] due to Voigt [29]. In this work, answering a question of Todorćevic, we give a characterization of this class of topological Ramsey spaces.

The theory of topological Ramsey spaces has experienced increasing development in recent years. In the book *Introduction to Ramsey spaces* [25] by S. Todorćevic, most of the foundational results, examples and applications are presented within the framework of a more general type of Ramsey space (not necessarily topological). A topological Ramsey space is the main object of the *topological Ramsey theory* (see Section 2 for the definitions). Carlson and Simpson gave the first abstract exposition of this theory in [2]. The first known example of a topological Ramsey space was given in [8], building upon earlier results like [21, 9, 10]. Recent developments in the study of topological Ramsey spaces have been

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made regarding connections to forcing, the theory of ultrafilters, selectivity, Tukey reducibility, parametrized partition theorems, canonization theorems, topological dynamics, structural Ramsey theory, Fraïssé classes and random objects, among others (see for instance [4, 5, 6, 7, 16, 17, 18, 19, 20, 26, 27, 28]).

In this paper we study a feature of some topological Ramsey spaces which had not been fully understood in the abstract setting. More precisely, we establish conditions of sufficiency which characterize those topological Ramsey spaces \mathcal{R} with family of approximations $\mathcal{AR} = \bigcup_{k < \omega} \mathcal{AR}_k$ for which there exist topological spaces $\mathcal{R}_k \subseteq (\mathcal{AR}_k)^\mathbb{N}$, $k < \omega$, such that every Baire subset of \mathcal{R}_k is Ramsey. Each \mathcal{R}_k inherits the Tychonoff topology from $(\mathcal{AR}_k)^\mathbb{N}$, which results when \mathcal{AR}_k is understood as a discrete space.

In Section 2 we give a brief description of the theory of topological Ramsey spaces. Section 3 contains the main result of the paper, the characterization announced in the previous paragraph. In Section 4, we introduce a class of topological Ramsey spaces which generalizes the spaces of ascending parameter words studied by Carlson-Simpson in [3] and Prömel-Voigt in [23] (see subsection 5.2), and which turns out to be also a generalization of the spaces $\text{FIN}_m^{[\infty]}$, $0 < m < \omega$, of block sequences defined by Todorcevic in [25]. We show that each space in this class admits projection spaces where every Baire set is Ramsey, fitting into the abstract setting introduced in Section 3. In Section 5, we show that the classical examples originally introduced in [3, 23, 25, 29] fit the abstract setting given in Section 3. These classical examples motivated this research. At the end of this article we comment about open questions related to the results in Sections 3 and 4.

2. TOPOLOGICAL RAMSEY SPACES

The four axioms which guarantee that a space is a topological Ramsey space (see Definition 2.2 below) can be found at the beginning of Chapter 5 of [25], which we reproduce in this Section.

Consider a triple (\mathcal{R}, \leq, r) of objects with the following properties. \mathcal{R} is a nonempty set, \leq is a quasi-ordering on \mathcal{R} and $r : \mathcal{R} \times \omega \rightarrow \mathcal{AR}$ is a mapping giving us the sequence $(r_n(\cdot) = r(\cdot, n))$ of approximation mappings, where \mathcal{AR} is the collection of all finite approximations to members of \mathcal{R} . For every $B \in \mathcal{R}$, let

$$(1) \quad \mathcal{R}|B = \{A \in \mathcal{R} : A \leq B\}.$$

For $a \in \mathcal{AR}$ and $B \in \mathcal{R}$, let

$$(2) \quad [a, B] = \{A \in \mathcal{R} : A \leq B \text{ and } (\exists n) r_n(A) = a\}.$$

For $a \in \mathcal{AR}$, let $|a|$ denote the length of the sequence a , that is, $|a|$ equals the integer n for which $a = r_n(A)$, for some $A \in \mathcal{R}$. If $m < n$, $a = r_m(A)$ and $b = r_n(A)$ then we will write $a = r_m(b)$. In particular, $a = r_m(a)$, and this is equivalent to $|a| = m$. For $a, b \in \mathcal{AR}$, $a \sqsubseteq b$ if and only if $a = r_m(b)$ for some $m \leq |b|$. $a \sqsubset b$ if and only if $a = r_m(b)$ for some $m < |b|$. For each $n < \omega$, $\mathcal{AR}_n = \{r_n(A) : A \in \mathcal{R}\}$.

A.1 (a) $r_0(A) = \emptyset$ for all $A \in \mathcal{R}$.

(b) $A \neq B$ implies $r_n(A) \neq r_n(B)$ for some n .

(c) $r_n(A) = r_m(B)$ implies $n = m$ and $r_k(A) = r_k(B)$ for all $k < n$.

A.2 There is a quasi-ordering \leq_{fin} on \mathcal{AR} such that

- (a) $\{a \in \mathcal{AR} : a \leq_{\text{fin}} b\}$ is finite for all $b \in \mathcal{AR}$,
- (b) $A \leq B$ iff $(\forall n)(\exists m) r_n(A) \leq_{\text{fin}} r_m(B)$,
- (c) $\forall a, b, c \in \mathcal{AR} [a \sqsubset b \wedge b \leq_{\text{fin}} c \rightarrow \exists d \sqsubset c a \leq_{\text{fin}} d]$.

$\text{depth}_B(a)$ is the least n , if it exists, such that $a \leq_{\text{fin}} r_n(B)$. If such an n does not exist, then we write $\text{depth}_B(a) = \infty$. If $\text{depth}_B(a) = n < \infty$, then $[\text{depth}_B(a), B]$ denotes $[r_n(B), B]$.

A.3 (a) If $\text{depth}_B(a) < \infty$ then $[a, A] \neq \emptyset$ for all $A \in [\text{depth}_B(a), B]$.

- (b) $A \leq B$ and $[a, A] \neq \emptyset$ imply that there is $A' \in [\text{depth}_B(a), B]$ such that $\emptyset \neq [a, A'] \subseteq [a, A]$.

If $n > |a|$, then $r_n[a, A]$ denotes the set $\{r_n(B) : B \in [a, A]\}$. Notice that $a \sqsubset b$ for every $b \in r_n[a, A]$.

A.4 If $\text{depth}_B(a) < \infty$ and $\mathcal{O} \subseteq \mathcal{AR}_{|a|+1}$, then there is $A \in [\text{depth}_B(a), B]$ such that $r_{|a|+1}[a, A] \subseteq \mathcal{O}$ or $r_{|a|+1}[a, A] \subseteq \mathcal{O}^c$.

The family $\{[a, B] : a \in \mathcal{AR}, B \in \mathcal{R}\}$ forms a basis for the *Ellentuck* topology on \mathcal{R} ; it extends the usual metrizable topology on \mathcal{R} when we consider \mathcal{R} as a subspace of the Tychonoff cube $(\mathcal{AR})^{\mathbb{N}}$. Given the Ellentuck topology on \mathcal{R} , the notions of nowhere dense, and hence of meager are defined in the usual way. Thus, we may say that a subset \mathcal{X} of \mathcal{R} has the *property of Baire* if and only if $\mathcal{X} = \mathcal{O} \cap \mathcal{M}$ for some Ellentuck open set $\mathcal{O} \subseteq \mathcal{R}$ and some Ellentuck meager set $\mathcal{M} \subseteq \mathcal{R}$.

Definition 2.1 ([25]). A subset \mathcal{X} of \mathcal{R} is *Ramsey* if for every $\emptyset \neq [a, A]$, there is a $B \in [a, A]$ such that $[a, B] \subseteq \mathcal{X}$ or $[a, B] \cap \mathcal{X} = \emptyset$. $\mathcal{X} \subseteq \mathcal{R}$ is *Ramsey null* if for every $\emptyset \neq [a, A]$, there is a $B \in [a, A]$ such that $[a, B] \cap \mathcal{X} = \emptyset$.

Definition 2.2 ([25]). A triple (\mathcal{R}, \leq, r) is a *topological Ramsey space* if every subset of \mathcal{R} with the property of Baire is Ramsey and every meager subset of \mathcal{R} is Ramsey null.

The following is the generalization of Ellentuck's Theorem to the general framework of topological Ramsey spaces.

Theorem 2.3 (Abstract Ellentuck Theorem – see Theorem 5.4 in [25]). *If (\mathcal{R}, \leq, r) is closed (as a subspace of $(\mathcal{AR})^{\mathbb{N}}$) and satisfies axioms A.1, A.2, A.3, and A.4, then the triple (\mathcal{R}, \leq, r) forms a topological Ramsey space.*

3. MAIN RESULTS

In this section we will characterize those topological Ramsey spaces \mathcal{R} which, for each $k < \omega$, induce a topological space $\mathcal{R}_k \subseteq (\mathcal{AR}_k)^{\mathbb{N}}$ which, with the subspace topology inherited from the Tychonoff topology on $(\mathcal{AR}_k)^{\mathbb{N}}$, has the property that every Baire subset of \mathcal{R}_k is Ramsey. In our context, each \mathcal{AR}_k is understood as a discrete space. Our characterization involves augmenting the structure (\mathcal{R}, \leq, r) of a typical topological Ramsey space with symbols for the projected spaces $(\mathcal{R}_k)_{k < \omega}$, an extra operation symbol \circ , and a new finitization function symbol s .

Consider a structure $(\mathcal{R}, \leq, r, (\mathcal{R}_k)_{k < \omega}, \circ, s)$. Let \mathcal{R}, \leq and r be as in the previous section. For every $k < \omega$, \mathcal{R}_k is a nonempty set. Every \mathcal{R}_k will be understood as a projection of \mathcal{R} to $(\mathcal{AR}_k)^{\mathbb{N}}$, in a sense that will be made clear (see Axiom A.6 and Remark 3.2 below).

The symbol \circ denotes an operation $\circ : \mathcal{R} \times (\mathcal{R} \cup \bigcup_{k < \omega} \mathcal{R}_k) \rightarrow (\mathcal{R} \cup \bigcup_{k < \omega} \mathcal{R}_k)$. The symbol s denotes a function $s : \mathbb{N} \times \bigcup_{k < \omega} \mathcal{R}_k \rightarrow \bigcup_{k < \omega} \mathcal{AR}_k$. We now introduce axioms **A.5–A.7** which we will prove in Theorem 3.3 suffice to obtain the characterization announced at the beginning of this paragraph.

A.5. (Rules for the operation \circ).

- (a) For all $A, B \in \mathcal{R}$, $A \circ B \in \mathcal{R}$.
- (b) For all $(A, X) \in \mathcal{R} \times \mathcal{R}_k$, $A \circ X \in \mathcal{R}_k$.
- (c) For all $A, B, C \in \mathcal{R}$, $A \circ (B \circ C) = (A \circ B) \circ C$.
- (d) For all $A, B \in \mathcal{R}$ and $X \in \mathcal{R}_k$, $A \circ (B \circ X) = (A \circ B) \circ X$.
- (e) For every $A, B \in \mathcal{R}$, if there exists $C \in \mathcal{R}$ such that $B = A \circ C$ then $B \leq A$.

Notation. For every $A \in \mathcal{R}$ and every $k < \omega$, let

$$(3) \quad \mathcal{R}_k|A = \{A \circ X : X \in \mathcal{R}_k\}.$$

Definition 3.1. A set $\mathcal{X} \subseteq \mathcal{R}_k$ is **Ramsey** if for every $B \in \mathcal{R}$ there exists $A \in \mathcal{R}$ with $A \leq B$ such that $\mathcal{R}_k|A \subseteq \mathcal{X}$ or $\mathcal{R}_k|A \cap \mathcal{X} = \emptyset$.

Remark 3.1. Technically, if $k = 0$, then \mathcal{R}_k is a singleton, so every subset is Ramsey.

A.6. (Rules for the function s) For each $k \in \omega$, the following hold:

- (a) Let $X \in \mathcal{R}_k$ be given. If $n \geq k$ then $s(n, X) \in \mathcal{AR}_k$. If $n < k$ then $s(n, X) = r_n(s(k, X))$. For $k > 0$, if $a = s(n, X)$ for some $n \geq k$, then we will assume that $r_{k-1}(a) = s(k-1, X)$.
- (b) For all $X \in \mathcal{R}_k$, $A \in \mathcal{R}$, the following hold: For all $n \geq k$, we have $s(n, A \circ X) \in \mathcal{AR}_k|A$ and $\text{depth}_A s(n, A \circ X) < \text{depth}_A s(n+1, A \circ X)$.
- (c) For $X, Y \in \mathcal{R}_k$, $s(n, X) = s(m, Y)$ implies $n = m$ and $\forall j < n$, $s(j, X) = s(j, Y)$.
- (d) For $X, Y \in \mathcal{R}_k$, $X \neq Y$ if and only if $\exists n$, $s(n, X) \neq s(n, Y)$.

By parts (c) and (d) of **A.6**, each $X \in \mathcal{R}_k$ may be uniquely identified with its sequence $(s(n, X))_{n < \omega}$ of s -approximations. By part (a) of **A.6**, the sequence $(s(n, X))_{n \geq k}$ is an element of the infinite product $(\mathcal{AR}_k)^\mathbb{N}$. Moreover, since $s(k, X)$ determines $s(n, X)$ for all $n < k$, this sequence is uniquely identified with X . Thus, the set \mathcal{R}_k can be identified with a subset of $(\mathcal{AR}_k)^\mathbb{N}$, inheriting the subspace topology from the Tychonoff topology on $(\mathcal{AR}_k)^\mathbb{N}$.

Notation. For $a \in \mathcal{AR}_k$, let $\langle a \rangle$ denote the set $\{X \in \mathcal{R}_k : (\exists n) s(n, X) = a\}$.

The next three facts follow immediately from **A.5** and **A.6**:

Fact 1. The family of $\langle a \rangle$, $a \in \mathcal{AR}_k$, is a base for the Tychonoff topology on \mathcal{R}_k .

Fact 2. For every $A \in \mathcal{R}$, $\mathcal{R}_k|A \subset \bigcup \{\langle a \rangle : a \in \mathcal{AR}_k|A, \text{depth}_A(a) > k\}$.

Fact 3. For every $A, B \in \mathcal{R}$, $\{(A \circ B) \circ X : X \in \mathcal{R}_k\} = \{A \circ (B \circ X) : X \in \mathcal{R}_k\} \subseteq \mathcal{R}_k|A$.

Notation. For $m \leq n$ and $A \in \mathcal{R}$, let $\mathcal{AR}^{(n)}|A = \{a \in \mathcal{AR}_m|A : \text{depth}_A(a) = n\}$, and let $\mathcal{AR}^{(\geq m)}|A$ denote $\bigcup_{j \geq m} \mathcal{AR}^{(j)}|A$. Also, for any $k < \omega$ and $a, b \in \mathcal{AR}_k$, write $a < b$ if there exists $X \in \mathcal{R}_k$ and $m < n \in \omega$ such that $a = s(m, X)$ and $b = s(n, X)$. Write, $a \leq b$ if $a < b$ or $a = b$.

A.7. (Finitization of the operation \circ).

Given $A \in \mathcal{R}$ and $k \leq m \leq n$, the operation \circ can be finitized to a function from $\mathcal{AR} \binom{n}{m} | A \times \mathcal{AR} \binom{m}{k} | A$ onto $\mathcal{AR} \binom{n}{k} | A$, satisfying the following:

- (a) Given $a \in \mathcal{AR} \binom{n}{k} | A$ and $b \in \mathcal{AR}_n | A$, if $b \circ a < c$ for some $c \in \mathcal{AR}_k | A$ then there exists $b' \in \mathcal{AR}_n | A$ such that $b < b'$ and $c = b' \circ a$.
- (b) Given $a \in \mathcal{AR} \binom{n}{k} | A$ and $b, c \in \mathcal{AR}_n | A$, if $b < c$ then $b \circ a < c \circ a$.
- (c) Let $A \in \mathcal{R}$, $a \in \mathcal{AR} \binom{n}{k} | A$, and $X \in \mathcal{R}_k$ with $X \in \langle a \rangle$ be given. If $n > k$, then $s(n, A \circ X) = r_n(A) \circ a$. If $n = k$, then $s(k, A \circ X) = a$.

Remark 3.2. **A.6** allows us to identify each $X \in \mathcal{R}_k$, $k < \omega$, with the sequence $(s(n, X))_{n \geq k}$, and in this way each \mathcal{R}_k can be regarded as a subspace of $(\mathcal{AR}_k)^\mathbb{N}$ with the Tychonoff topology obtained by endowing \mathcal{AR}_k with the discrete topology. Part (b) of **A.6** indicates that for fixed k and $X \in \mathcal{R}_k$, the operation \circ and the function s induce a projection map $\pi(A) = A \circ X$, from \mathcal{R} to \mathcal{R}_k . On the other hand, it is worth mentioning at this point that the space $(\mathcal{AR}_k)^\mathbb{N}$ is a Polish metric space and therefore satisfies the Baire Category Theorem stating that the intersection of countably many open dense sets is dense. We shall say that \mathcal{R}_k is *metrically closed in* $(\mathcal{AR}_k)^\mathbb{N}$ if for each sequence $(a_n)_{n \geq k}$ in $(\mathcal{AR}_k)^\mathbb{N}$ satisfying that $a_m < a_n$, whenever $n > m \geq k$, then $\bigcap_{k \leq n < \omega} \langle a_n \rangle = \{X\}$, for some $X \in \mathcal{R}_k$. The limit of the sequence $(a_n)_{n \geq k}$, denoted $\lim_{n \geq k} a_n$. If \mathcal{R}_k is a closed in $(\mathcal{AR}_k)^\mathbb{N}$, then the subspace topology on \mathcal{R}_k inherited from $(\mathcal{AR}_k)^\mathbb{N}$ is completely metrizable; and hence, \mathcal{R}_k satisfies the Baire Category Theorem. Notice that if \mathcal{R}_k is closed then for every $A \in \mathcal{R}$, $\mathcal{R}_k | A$ is also closed and satisfies the Baire Category Theorem..

The following will be used in the sequel.

Lemma 3.2. $a \leq b$ if and only if $\langle a \rangle \supseteq \langle b \rangle$.

Proof. Suppose $a \leq b$. If $a = b$ then we are done, so assume $a < b$. Fix $X \in \mathcal{R}_k$ and $m < n \in \omega$ such that $a = s(m, X)$ and $b = s(n, X)$. Choose $Y \in \langle b \rangle$. Then there exists $p \in \omega$ such that $b = s(p, Y)$. Therefore, $s(n, X) = s(p, Y)$. By, part (c) of **A.6**, $n = p$ and $\forall j < n$, $s(j, X) = s(j, Y)$. In particular, $s(m, Y) = s(m, X) = a$. Hence $Y \in \langle a \rangle$. Therefore, $\langle a \rangle \supseteq \langle b \rangle$.

Conversely, suppose that $\langle a \rangle \supseteq \langle b \rangle$, and choose $Y \in \langle b \rangle$. By **A.6**, Y can be identified with the sequence $(s(m, Y))_{m < \omega}$. Since $Y \in \langle b \rangle$ and $\langle a \rangle \supseteq \langle b \rangle$, there exist $m, n < \omega$ such that $a = s(m, Y)$ and $b = s(n, Y)$. Notice that $n \geq m$ because otherwise, we would have $\langle a \rangle \not\supseteq \langle b \rangle$. To see this, supposing toward a contradiction that $n < m$, it suffices to define $Z \in \mathcal{R}_k$ such that $s(j, Z) = s(j, Y)$, for $j \leq n$, and $s(j, Z) = s(m + j, Y)$, for $j > n$. Then $Z \in \langle b \rangle$ but $Z \notin \langle a \rangle$, a contradiction. Therefore, it is the case that $n \geq m$, and we conclude $a = b$ or $a < b$. \square

Now we are ready to state the main result of this article.

Theorem 3.3. Suppose $(\mathcal{R}, \leq, r, (\mathcal{R}_k)_{k < \omega}, \circ, s)$ satisfies **A.1** – **A.7**, \mathcal{R} is metrically closed in $\mathcal{AR}^\mathbb{N}$ and \mathcal{R}_k is metrically closed in $\mathcal{AR}_k^\mathbb{N}$, $k < \omega$. For every $B \in \mathcal{R}$, every $k < \omega$ and every finite Baire-measurable coloring of \mathcal{R}_k , there exists $A \in \mathcal{R}$ with $A \leq B$ such that $\{A \circ X : X \in \mathcal{R}_k\}$ is monochromatic.

Thus, Theorem 3.3 implies the following.

Corollary 3.1. Suppose $(\mathcal{R}, \leq, r, (\mathcal{R}_k)_{k < \omega}, \circ, s)$ satisfies **A.1** – **A.7**, \mathcal{R} is metrically closed in $\mathcal{AR}^{\mathbb{N}}$ and for all $k < \omega$, \mathcal{R}_k is metrically closed in $\mathcal{AR}_k^{\mathbb{N}}$. Then for all $k < \omega$, every Baire subset of \mathcal{R}_k is Ramsey.

In order to prove Theorem 3.3 we will use the following lemmas. For the proofs of these lemmas, we will assume that $(\mathcal{R}, \leq, r, (\mathcal{R}_k)_{k < \omega}, \circ, s)$ satisfies **A.1** – **A.7** and \mathcal{R} is metrically closed in $\mathcal{AR}^{\mathbb{N}}$. Notice that Theorem 3.3 is a generalization of the main result in [23]. The following proofs are based on the techniques used in [23].

Lemma 3.4. Let $A \in \mathcal{R}$ and $k, m, n \in \omega$ be given, with $m \geq k$. Let \mathcal{B}_i , $i < n$, be open subsets of \mathcal{R}_k such that $\bigcup_{i < n} \mathcal{B}_i$ is dense. Then for each $b \in \mathcal{AR}_m|A$, there is a $c \in r_{m+1}[b, A]$ satisfying that for every $a \in \mathcal{AR}^{(m+1)}|A$ there is an $i < n$ such that $\langle c \circ a \rangle \subseteq \mathcal{B}_i$.

Proof. Fix $b \in \mathcal{AR}_m|A$ and let a_0, a_1, \dots, a_l be an enumeration of $\mathcal{AR}^{(m+1)}|A$. Let $b' \in r_{m+1}[b, A]$. Since $\langle b' \circ a_0 \rangle$ is open in \mathcal{R}_k and $\bigcup_{i < n} \mathcal{B}_i$ is dense open, their intersection is nonempty. Thus, by Fact 1, there exists $d_0 \in \mathcal{AR}_k|A$ and $j_0 < n$ such that $d_0 > b' \circ a_0$ and $\langle d_0 \rangle \subseteq \mathcal{B}_{j_0}$. By part (a) of **A.7**, there exists $c_0 \in \mathcal{AR}_{m+1}|A$ such that $c_0 > b'$ and $d_0 = c_0 \circ a_0$. Thus, $\langle c_0 \circ a_0 \rangle \subseteq \mathcal{B}_{j_0}$. Using Fact 1 and part (a) of **A.7**, we can inductively build a sequence $b' < c_0 < c_1 < \dots < c_l$ and find integers $j_0, j_1, \dots, j_l < n$ such that for every $p \leq l$, $c_p \in \mathcal{AR}_{m+1}|A$, $\langle c_p \circ a_p \rangle \subseteq \mathcal{B}_{j_p}$, and $c_p > b'$. Let $c = c_l$. Notice that by part (b) of **A.7**, for every $p \leq l$, $b_p < c$. Then by Lemma 3.2, $\langle c \circ a_p \rangle \subseteq \langle b_p \circ a_p \rangle \subseteq \mathcal{B}_{j_p}$.

Claim. $r_m(c) = b$.

Proof of the Claim. There exists $X \in \mathcal{R}_{m+1}$ such that $b' = s(m+1, X)$. Therefore, by **A.6**, b' can be identified with the sequence $\{s(j, X)\}_{j \leq n}$. Notice that $r_m(b') = s(m, X)$. Since $b' < c$, there exists $m+1 \leq p < q < \omega$ and $Y \in \mathcal{R}_{m+1}$ such that $b' = s(p, Y) < s(q, Y) = c$. Again, c can be identified with $\{s(j, Y)\}_{j \leq q}$ and $r_m(c) = s(m, Y)$. It turns out that $s(m+1, X) = b' = s(p, Y)$. By part (c) of **A.6**, $m+1 = p$ and $s(j, X) = s(j, Y)$ for all $j < m+1$. In particular, $b = r_m(b') = s(m, X) = s(m, Y) = r_m(c)$. \square

By the Claim, it follows that $b \sqsubset c$ and c is as required. \square

Lemma 3.5. Let $B \in \mathcal{R}$ and $n \in \omega$ be given. Let \mathcal{M} be a meager subset of $\mathcal{R}_k|B$, and let \mathcal{B}_i , $i < n$, be open subsets of $\mathcal{R}_k|B$ such that $\bigcup_{i < n} \mathcal{B}_i$ is dense in $\mathcal{R}_k|B$. Then there is an $A \in \mathcal{R}$ with $A \leq B$ such that

- (1) For each $a \in \mathcal{AR}_k|A$ with $\text{depth}_A(a) > k$, there exists an $i < n$ such that $\{A \circ X : X \in \langle a \rangle \cap (\mathcal{R}_k|B)\} \subseteq \mathcal{B}_i$; and
- (2) $\{A \circ X : X \in \mathcal{R}_k|B\} \cap \mathcal{M} = \emptyset$.

Proof. We will use Lemma 3.4, relativized to $\mathcal{R}_k|B$. The proof of the relativized version is analogous, passing to the relative topology and using the fact that if \mathcal{R}_k is metrically closed in $\mathcal{AR}_k^{\mathbb{N}}$ then $\mathcal{R}_k|B$ is also metrically closed in $\mathcal{AR}_k^{\mathbb{N}}$. Since $\mathcal{M} \subseteq \mathcal{R}_k|B$ is meager, there exists a sequence $\mathcal{D}_m \subseteq \mathcal{R}_k|B$, $m < \omega$, of dense open sets such that $\mathcal{M} \subseteq (\mathcal{R}_k|B) \setminus \bigcap_{m < \omega} \mathcal{D}_m$. For every $m < \omega$, let $\mathcal{D}_m^* = \bigcap_{l \leq m} \mathcal{D}_l$. Let $b = r_k(B)$.

Since \mathcal{D}_0^* and $\bigcup_{i < n} \mathcal{B}_i$ are dense open in $\mathcal{R}_k|B$, $\langle b \rangle$ is open, and $b \in \mathcal{AR}_k|B$, it follows that there is some $i < n$ for which $\langle b \rangle \cap \mathcal{D}_0^* \cap \mathcal{B}_i \neq \emptyset$. Since $\mathcal{D}_0^* \cap \mathcal{B}_i$ is open in $\mathcal{R}_k|B$, by Fact 1, there is a $b_0 \in \mathcal{AR}_k|B$ such that $b_0 > b$ and $\langle b_0 \rangle \cap (\mathcal{R}_k|B) \subseteq \mathcal{D}_0^* \cap \mathcal{B}_i$.

Let us build a sequence $(b_m)_{m < \omega}$, which in the limit will give us A , as follows. Suppose b_m has been defined. By Lemma 3.4, there is a $b_{m+1} \in \mathcal{AR}_{k+m+1}|B$, with $b_m \sqsubset b_{m+1}$, such that

for every $a \in \mathcal{AR} \binom{k+m+1}{k} | B$, there exists some $i < n$ such that $\langle b_{m+1} \circ a \rangle \cap (\mathcal{R}_k | B) \subseteq \mathcal{B}_i \cap \mathcal{D}_m^*$. Let $A = \lim_m b_m$. Then $A \in \mathcal{R}$.

We claim that A is as required. To see this, let $a \in \mathcal{AR}_k | A$ such that $\text{depth}_A(a) > k$ be given, and let m be such that $k + m + 1 = \text{depth}_B(a)$. By our construction, there is some $i < n$ such that $\langle b_{m+1} \circ a \rangle \cap (\mathcal{R}_k | B) \subseteq \mathcal{B}_i$. Since $b_{m+1} = r_{k+m+1}(A)$ and $a \in \mathcal{AR} \binom{k+m+1}{k} | A$, for each $X \in \langle a \rangle$, it follows from part (c) of **A.7** that $s(k + m + 1, A \circ X) = b_{m+1} \circ a$. Thus,

$$(4) \quad A \circ X \in \langle b_{m+1} \circ a \rangle \cap (\mathcal{R}_k | B) \subseteq \mathcal{B}_i \cap \mathcal{D}_m^*.$$

In particular, (1) holds.

We now check that (2) holds. Let $X \in \mathcal{R}_k | B$ be given. Let $m < \omega$ be given, and let $a = s(k + m + 1, X)$. Then $X \in \langle a \rangle$, and $k + m + 1 = \text{depth}_B(a)$. By Equation (4), $A \circ X \in \mathcal{D}_m^*$. Since this holds for all $m < \omega$, we find that $A \circ X \in \mathcal{M}$. \square

Since \mathcal{R} is a topological Ramsey space the following analog of Ramsey's Theorem is true (see [16, 25]).

Lemma 3.6. *For every $B \in \mathcal{R}$ and every finite coloring of \mathcal{AR}_k , there exists $A \in \mathcal{R}$ with $A \leq B$ such that $\mathcal{AR}_k | A$ is monochromatic.*

Remark 3.3. **A.1 - A.4** and the assumption that \mathcal{R} is metrically closed in $\mathcal{AR}^{\mathbb{N}}$ are sufficient for the proof of Lemma 3.6. In fact, Lemma 3.6 is a special case of the Abstract Nash-Williams Theorem (see [25]), which follows from the Abstract Ellentuck Theorem.

Now, let us prove our main result.

Proof of Theorem 3.3. Fix $B \in \mathcal{R}$. Given $n < \omega$, let $c : \mathcal{R}_k | B \rightarrow n$ be a Baire-measurable coloring. Then there exist open sets $\mathcal{B}_i \subseteq \mathcal{R}_k | B$, $i < n$, such that the sets

$$\mathcal{M}_i := (c^{-1}(\{i\}) \setminus \mathcal{B}_i) \cup (\mathcal{B}_i \setminus c^{-1}(\{i\}))$$

are meager in $\mathcal{R}_k | B$. Let $\mathcal{M} = \bigcup_{i < n} \mathcal{M}_i$. Then $(\mathcal{R}_k | B) \setminus \mathcal{M} \subseteq \bigcup_{i < n} \mathcal{B}_i$. Thus, since $\mathcal{R}_k | B$ satisfies the Baire Category Theorem, $\bigcup_{i < n} \mathcal{B}_i$ is dense in $\mathcal{R}_k | B$. Choose $A \in \mathcal{R}$ as in Lemma 3.5 applied to \mathcal{M} and the \mathcal{B}_i 's; and let $A_0 \leq A$ such that $\text{depth}_A(r_k(A_0)) > k$. It follows that for every $a \in \mathcal{AR}_k | A_0$, there exists $i < n$ such that $\{A_0 \circ X : X \in \langle a \rangle \cap \mathcal{R}_k | B\} \subseteq \mathcal{B}_i \cap c^{-1}(\{i\})$. In particular, for every $a \in \mathcal{AR}_k | A_0$, c is constant on $\{A_0 \circ X : X \in \langle a \rangle \cap \mathcal{R}_k | B\}$.

Define $\hat{c} : \mathcal{AR}_k | A_0 \rightarrow n$ by $\hat{c}(a) = c(A_0 \circ X)$, for any $X \in \langle a \rangle \cap \mathcal{R}_k | B$. By Lemma 3.6, there exists $A_1 \leq A_0$ such that \hat{c} is constant on $\mathcal{AR}_k | A_1$. By the definition of \hat{c} , it follows that for all $X', Y' \in \mathcal{R}_k | A_1 \circ B$, there are some $a, a' \in \mathcal{AR}_k | A_1$ such that $X' \in \langle a \rangle$ and $Y' \in \langle a' \rangle$; and thus $c(A_0 \circ X') = c(A_0 \circ Y')$. By Fact 3, the set $\{(A_0 \circ A_1) \circ X : X \in \mathcal{R}_k | B\} = \{(A_0 \circ A_1) \circ (B \circ X) : X \in \mathcal{R}_k\}$. Letting $A = A_1 \circ A_0 \circ B$, by Fact 2, we see that c is monochromatic on $\mathcal{R}_k | A$. \square

4. GENERALIZED ASCENDING PARAMETER WORDS AND BLOCK SEQUENCES

We introduce a class of topological Ramsey spaces which generalizes the spaces of ascending parameter words studied by Carlson-Simpson [3] and Prömel-Voigt [23] (see Section 5.2), and which turns out to be also a generalization of the spaces $\text{FIN}_m^{[\infty]}$, $0 < m < \omega$, of block sequences defined by Todorćević [25]. We show that each element of this class admits projection spaces where every Baire set is Ramsey, fitting into the abstract setting

introduced in Section 3. In order to show that our space is a topological Ramsey space, we use an infinitary version of the Hales-Jewett Theorem to deduce a pigeon hole principle which generalizes Gowers' Theorem [13].

4.1. Generalized ascending parameter words. Let X and Y be two nonempty sets of integers. Given a set $S \subseteq X \times Y$, let $\text{dom}(S) = \{i \in X : (\exists j \in Y) (i, j) \in S\}$ and $\text{ran}(S) = \{j \in Y : (\exists i \in X) (i, j) \in S\}$. As customary, we will identify each integer $m > 0$ with the set $\{0, 1, \dots, m-1\}$. Let ω be the set of nonnegative integers. Given $t, m < \omega$, with $m > 0$, and $\alpha \leq \beta \leq \omega$, let $\mathcal{S}_t^{<}(\beta, m)$ denote the set of all the surjective functions $A : (t + \beta) \times m \rightarrow t + \alpha$ satisfying

- (1) $A(i, l) = i$ for every $i < t$ and every $l < m$.
- (2) For all $j < \alpha$, $A^{-1}(\{t + j\})$ is a function; that is, for all $i \in \text{dom}(A^{-1}(\{t + j\}))$ there exists a unique $l < m$ such that $(i, l) \in A^{-1}(\{t + j\})$.
- (3) For all $j < \alpha$, $\text{dom } A^{-1}(\{t + j\})$ is a finite set.
- (4) For all $j < t + \alpha$, $m - 1 \in \text{ran}(A^{-1}(\{j\}))$.
- (5) $\min \text{dom } A^{-1}(\{i\}) < \min \text{dom } A^{-1}(\{j\})$ for all $i < j < t + \alpha$.
- (6) $\max \text{dom } A^{-1}(\{t + i\}) < \min \text{dom } A^{-1}(\{t + j\})$ for all $i < j < \alpha$.

The tetris operation. For $S \subseteq (t + \beta) \times m$, let $T(S) = \{(i, \max\{0, j - 1\}) : (i, j) \in S\}$. For $l < \omega$, let us define $T^l(S)$ recursively, as follows. $T^0(S) = S$, $T^1(S) = T(S)$ and $T^{l+1}(S) = T(T^l(S))$.

The composition. For $A \in \mathcal{S}_t^{<}(\gamma, m)$ and $B \in \mathcal{S}_t^{<}(\beta, m)$, the operation $A \cdot B \in \mathcal{S}_t^{<}(\gamma, m)$ is defined by $(A \cdot B)(i, j) = B(A(i, j), m - 1)$.

Remark 4.1. In Theorem 4.1 below we will prove that $\mathcal{S}_t^{<}(\omega, m)$ is a topological Ramsey space. Notice that for $t = 0$, $\mathcal{S}_0^{<}(\omega, 1)$ is essentially the set of infinite subsets of ω , so as a topological Ramsey space $\mathcal{S}_0^{<}(\omega, 1)$ will coincide with Ellentuck's space [8]; and $\mathcal{S}_0^{<}(\omega, m) = \emptyset$, for $m > 1$. So we will assume $t > 0$ throughout the rest of this section.

Remark 4.2. Let $0 < m < \omega$ be given. For a function $p : \omega \rightarrow \{0, 1, \dots, m\}$, let $\text{supp}(p) = \{i \in \omega : p(i) \neq 0\}$. Denote by FIN_m the collection of all the functions $p : \omega \rightarrow \{0, 1, \dots, m\}$ such that $\text{supp}(p)$ is finite and $m \in \text{ran}(p)$. A **block sequence** of elements of FIN_m is a sequence $(p_n)_{n < \omega}$ such that $\max \text{supp}(p_n) < \min \text{supp}(p_{n+1})$, for all $n < \omega$. Let $\text{FIN}_m^{[\infty]}$ be the collection of all such block sequences. Notice that for all $0 < m < \omega$, $\mathcal{S}_1^{<}(\omega, m)$ can be identified with $\text{FIN}_m^{[\infty]}$: A block sequence $P = (p_n)_{n < \omega} \in \text{FIN}_m^{[\infty]}$ determines a function $A_P \in \mathcal{S}_1^{<}(\omega, m)$ defined as follows:

$$A_P(i, j) = \begin{cases} 0 & \text{if } (\forall n < \omega) i \notin \text{supp}(p_n), \\ 0 & \text{if } (\exists n < \omega) i \in \text{supp}(p_n) \wedge p_n(i) \neq j + 1, \\ n + 1 & \text{if } i \in \text{supp}(p_n) \wedge p_n(i) = j + 1. \end{cases}$$

Conversely, a function $A \in \mathcal{S}_1^{<}(\omega, m)$ determines a block sequence $P_A = (p_n)_{n < \omega} \in \text{FIN}_m^{[\infty]}$ where, for each $n < \omega$, p_n is given by

$$p_n(i) = \begin{cases} 0 & \text{if } i \notin \text{dom}(A^{-1}(\{n + 1\})), \\ j + 1 & \text{if } (i, j) \in A^{-1}(\{n + 1\}). \end{cases}$$

4.2. **A topological Ramsey space of generalized ascending parameter words.** The purpose of this section is to prove that $\mathcal{S}_t^{<(\omega, m)}$ is a topological Ramsey space. Define the function r on $\mathbb{N} \times \mathcal{S}_t^{<(\omega, m)}$ as $r(n, A) = \emptyset$, if $n = 0$ and

$$r(n, A) = A \upharpoonright \{(i, l) \in \bigcup_{j < t+n} A^{-1}(\{j\}) : i < \min \text{dom } A^{-1}(\{t+n\})\}, \quad \text{if } n > 0.$$

Let $A \in \mathcal{S}_t^{<(\omega, m)}$ be given. For every $n < \omega$, let $a_n = A^{-1}(\{n\})$, and write $A = \{a_0, a_1, \dots\}$. Let T denote the tetris operation. For $t \leq n < \omega$ and $l < t + m$, define

$$S^l(a_n) = \begin{cases} T^{l-t}(a_n) & \text{if } t \leq l < t + m, \\ a_n \cup a_l & \text{if } l < t. \end{cases}$$

Here $a_n \cup a_l$ is the union of a_n and a_l as subsets of $(t + \omega) \times m$.

Let $[A]$ denote the collection of all the symbols of the form $S^{l_1}(a_{n_1}) + S^{l_2}(a_{n_2}) + \dots + S^{l_q}(a_{n_q})$ such that $n_i \geq t$ and $l_i < t + m$, for all $i \in \{1, \dots, q\}$, and at least one of the l_i 's is equal to t .

We shall identify each $S^{l_1}(a_{n_1}) + S^{l_2}(a_{n_2}) + \dots + S^{l_q}(a_{n_q}) \in [A]$ with a function $f \in \mathcal{S}_t^{<(\frac{e, m}{1})} | A$, for some $0 < e < \omega$, as follows.

Suppose that there exists $i \in \{1, \dots, q\}$ such that $l_i < t$. Let $i_1 < \dots < i_p$ be an increasing enumeration of all such i 's. Let $j_0 \in \{1, \dots, q\}$ be such that $l_{j_0} = t$. Then $S^{l_{j_0}}(a_{n_{j_0}}) = a_{n_{j_0}}$. Let $e = \min \text{dom } A^{-1}(\{t + n_{j_0} + 1\})$. Define the surjective function $f : (t + e) \times m \rightarrow t + 1$ by setting

- (1) $f^{-1}(\{j\}) = (t + e) \times m \cap a_j$ for all $j < t$ with $j \notin \{l_{i_1}, \dots, l_{i_p}\}$.
- (2) $f^{-1}(\{l_{i_1}\}) = (t + e) \times m \cap \bigcup \{a_n : n \geq t, n \notin \{n_1, \dots, n_q\}\} \cup S^{l_1}(a_{n_1}) \cup \dots \cup S^{l_{i_1}}(a_{n_{i_1}}) \setminus a_{n_{j_0}}$.
- (3) $f^{-1}(\{l_{i_{d+1}}\}) = (t + e) \times m \cap S^{l_{i_{d+1}}}(a_{n_{i_{d+1}}}) \cup \dots \cup S^{l_{i_{d+1}}}(a_{n_{i_{d+1}}}) \setminus a_{n_{j_0}}$, $1 \leq d < p$.
- (4) If there exists $i \in \{1, \dots, q\}$ such that $i > i_p$ then let $f^{-1}(\{t\}) = (t + e) \times m \cap S^{l_{i_p+1}}(a_{n_{i_p+1}}) \cup \dots \cup S^{l_i}(a_{n_i}) \cup a_{n_{j_0}}$. Otherwise, let $f^{-1}(\{t\}) = (t + e) \times m \cap a_{n_{j_0}}$.

If for all $i \in \{1, \dots, q\}$ we have $l_i \geq t$ then set

- (1) $f^{-1}(\{0\}) = (t + e) \times m \cap a_0 \cup \bigcup \{a_n : n \geq t, n \notin \{n_1, \dots, n_q\}\}$.
- (2) $f^{-1}(\{j\}) = (t + e) \times m \cap a_j$ for all $0 < j < t$.
- (3) $f^{-1}(\{t\}) = (t + e) \times m \cap S^{l_1}(a_{n_1}) + S^{l_2}(a_{n_2}) + \dots + S^{l_q}(a_{n_q})$.

This finishes the definition of f . The condition that at least one of the l_i 's is equal to t ensures that $f \in \mathcal{S}_t^{<(\frac{e, m}{1})}$. On the other hand, given $0 < e < \omega$, every $f \in \mathcal{S}_t^{<(\frac{e, m}{1})}$ can be represented as an element of $[A]$ for some A .

The quasi-order. If $B \in \mathcal{S}_t^{<(\omega, m)}$ then write $B \leq A$ if $[B] \subseteq [A]$. It can be easily proved that if there exists $C \in \mathcal{S}_t^{<(\omega, m)}$ such that $B = A \cdot C$ then $B \leq A$.

The following is the main result of this Section.

Theorem 4.1. *For every $t > 0$ and every $m > 0$, $(\mathcal{S}_t^{<(\omega, m)}, \leq, r)$ is a topological Ramsey space.*

The case $t = 1$ of Theorem 4.1 was proved by Todorcevic and in virtue of Remark 4.2 it can be restated as follows.

Theorem 4.2 (Todorcevic, Theorem 5.22 in [25]). *For every $m > 0$, $(\mathcal{S}_1^{<(\omega, m)}, \leq, r)$ is a topological Ramsey space.*

Let $\mathcal{S}_t^{<(\omega, m)}$ denote the range of the function r . That is, $\mathcal{S}_t^{<(\omega, m)} = \mathcal{AR}$ for the space $\mathcal{R} = \mathcal{S}_t^{<(\omega, m)}$. Notice that $\mathcal{S}_t^{<(\omega, m)} = \bigcup_{l \leq n < \omega} \mathcal{S}_t^{<(n, m)}$. Before proving Theorem 4.1 for $t > 1$, we will establish a result concerning finite colorings of $\mathcal{S}_t^{<(\omega, m)}$ in Theorem 4.4 below, from which **A.4** for $\mathcal{S}_t^{<(\omega, m)}$ can be easily deduced. In order to prove Theorem 4.4, we will need an infinitary version of the Hales-Jewett Theorem.

Let L be a finite alphabet and let $v \notin L$. Let W_L denote the collection of words over L , and let $W_{L \cup \{v\}}$ denote the collection of words $w(v)$ over the alphabet $L \cup \{v\}$, such that the symbol v appears at least once in $w(v)$. The elements of $W_{L \cup \{v\}}$ are called **variable words**.

Given $w(v) \in W_{L \cup \{v\}}$, the symbol $|w(v)|$ denotes the length of $w(v)$. Also, if $l \in L \cup \{v\}$, let $w(l)$ be the word in $W_L \cup W_{L \cup \{v\}}$ obtained by replacing every occurrence of v in $w(v)$ by l . For an infinite sequence of words $(w_n(v))_{n < \omega}$, let

$$[(w_n(v))_{n < \omega}]_{L \cup \{v\}} = \{w_{n_0}(l_0)w_{n_1}(l_1) \dots w_{n_q}(l_q) \in W_{L \cup \{v\}} : n_1 < \dots < n_q; l_i \in L \cup \{v\} (i \leq q)\}.$$

An infinite sequence $(w_n(v))_{n < \omega}$, with $w_n(v) \in W_{L \cup \{v\}}$, is **rapidly increasing** if for every $n < \omega$, $|w_n(v)| > \sum_{i < n} |w_i(v)|$.

Theorem 4.3 (Infinite Hales-Jewett Theorem. See [25], Theorem 4.21). *Let L be a finite alphabet and let v be a symbol which is not in L . Let $(w_n(v))_{n < \omega}$ be an infinite rapidly increasing sequence of variable words. Then for every finite coloring of $W_{L \cup \{v\}}$ there exists an infinite rapidly increasing sequence $(u_j(v))_{j < \omega}$ with $u_j(v) \in [(w_n(v))_{n < \omega}]_{L \cup \{v\}}$, $j < \omega$, such that $[(u_j(v))_{j < \omega}]_{L \cup \{v\}}$ is monochromatic.*

From Theorem 4.3 we can prove the following.

Theorem 4.4. *Let $A \in \mathcal{S}_t^{<(\omega, m)}$ be given. For every finite coloring of $[A]$ there exists $B \in \mathcal{S}_t^{<(\omega, m)}$ such that $B \leq A$ and $[B]$ is monochromatic.*

Proof. Let us write, $A = \{a_0, a_1, \dots\}$, where $a_n = A^{-1}(\{n\})$. Consider the alphabet $L = t + m \setminus \{t\}$, and let $v = t$. Fix a finite coloring c of $[A]$. Also, let $(w_n(v))_n$ be the infinite rapidly increasing sequence of variable words such that for $n < \omega$ we have $w_n(v) = vv \dots v$, where v appears exactly 2^n times. Now, define the coloring c' on $[(w_n(v))_n]_{L \cup \{v\}}$ by

$$c'(w_{n_0}(l_0)w_{n_1}(l_1) \dots w_{n_q}(l_q)) = c(S^{l_0}(a_{n_0}) + S^{l_1}(a_{n_1}) + \dots + S^{l_q}(a_{n_q}))$$

and let $c'(u) = 0$, if $u \in W_{L \cup \{v\}} \setminus [(w_n(v))_n]_{L \cup \{v\}}$.

Given $w_{n_0}(l_0)w_{n_1}(l_1) \dots w_{n_q}(l_q) \in [(w_n(v))_n]_{L \cup \{v\}}$, at least one l_i must be equal to v (that is, equal to t). Then $S^{l_0}(a_{n_0}) + S^{l_1}(a_{n_1}) + \dots + S^{l_q}(a_{n_q})$ is in fact an element of $[A]$, and it is uniquely determined by $w_{n_1}(l_1)w_{n_2}(l_2) \dots w_{n_q}(l_q)$ because $(w_n(v))_{n < \omega}$ is rapidly increasing. Apply Theorem 4.3, and let $(u_j(v))_{j < \omega}$, with $u_j(v) \in [(w_n(v))_n]_{L \cup \{v\}}$, be an infinite rapidly increasing sequence of variable words such that $[(u_j(v))_j]_{L \cup \{v\}}$ is monochromatic for c' . Let us say that the constant color is p . Define $B = (b_0, b_1, \dots)$ in this way: For every $j < \omega$, if

$$u_j(v) = w_{n_0}(l_0)w_{n_1}(l_1) \dots w_{n_j}(l_{q_j})$$

then

$$b_j = S^{l_0}(a_{n_0}) + S^{l_1}(a_{n_1}) + \cdots + S^{l_{q_j}}(a_{n_{q_j}}).$$

Again, here b_j stands for $B^{-1}(\{j\})$. Thus, $[B]$ is monochromatic for c : If $b \in [B]$ and $b = S^{l_0}(b_{j_0}) + S^{l_1}(b_{j_1}) + \cdots + S^{l_s}(b_{j_s})$ with $b_{j_i} = S^{l_i^0}(a_{n_i^0}) + S^{l_i^1}(a_{n_i^1}) + \cdots + S^{l_i^{q_i}}(a_{n_i^{q_i}})$, $0 \leq i \leq s$, then

$$\begin{aligned} c(b) &= c(S^{l_0}(b_{j_0}) + \cdots + S^{l_s}(b_{j_s})) = c(S^{l_0}(\sum_{0 \leq j \leq q_1} S^{l_0^j}(a_{n_j^0})) + \cdots + S^{l_s}(\sum_{0 \leq j \leq q_s} S^{l_s^j}(a_{n_j^s}))) \\ &= c'((w_{n_0^0}(l_0^0) \dots w_{n_0^{q_1}}(l_0^{q_1}))(l_0) \dots (w_{n_s^0}(l_s^0) \dots w_{n_s^{q_s}}(l_s^{q_s}))(l_s)) = p. \end{aligned}$$

□

Proof of Theorem 4.1. The quasi-order \leq defined on $\mathcal{S}_t^<(\frac{\omega, m}{\omega})$ admits a natural finitization \leq_{fin} defined on $\mathcal{S}_t^<(\frac{<\omega, m}{<\omega})$ as follows. Consider $a, b \in \mathcal{S}_t^<(\frac{<\omega, m}{<\omega})$. Write $a = \{a_0, \dots, a_{n-1}\}$ and $b = \{b_0, \dots, b_{p-1}\}$, where $0 < n, p < \omega$, $a_j = a^{-1}(\{j\})$ and $b_j = b^{-1}(\{j\})$. Let $[b]$ denote the collection of all the sets of the form $S^{l_1}(b_{n_1}) + S^{l_2}(b_{n_2}) + \cdots + S^{l_q}(b_{n_q})$ satisfying that if $n_i \geq t$ for all $i \in \{1, \dots, q\}$, then at least one of the l_i 's is equal to t . Write $a \leq_{\text{fin}} b$ if and only if $\text{dom}(a) = \text{dom}(b)$ and $a_j \in [b]$, $j < n$. It is easy to see that with these definitions, **A.1-A.3** are satisfied, and to prove that $\mathcal{S}_t^<(\frac{\omega, m}{\omega})$ is metrically closed in $(\mathcal{S}_t^<(\frac{<\omega, m}{<\omega}))^{\mathbb{N}}$. Also, **A.4** for $\mathcal{S}_t^<(\frac{\omega, m}{\omega})$ follows easily from Theorem 4.4. This completes the proof of Theorem 4.1. □

4.3. Baire sets of generalized ascending parameter words are Ramsey. Given $k, m, t < \omega$ with $m > 0$. In this Section we will prove that the Baire subsets of $\mathcal{S}_t^<(\frac{\omega, m}{k})$ are Ramsey. Let $\pi : \mathcal{S}_t^<(\frac{\omega, m}{\omega}) \rightarrow \mathcal{S}_t^<(\frac{\omega, m}{k})$ be defined as follows:

$$\pi(A)(i, j) = \begin{cases} 0 & \text{if } A(i, j) \geq t + k \\ A(i, j) & \text{if } 0 \leq A(i, j) < t + k \end{cases}$$

Notice that π is a surjection. For $n > k$, we will extend the function π to $\mathcal{AR}_n = \mathcal{S}_t^<(\frac{<\omega, m}{n})$ as follows. Given $a \in \mathcal{AR}_n$ and (i, j) in the domain of a , let

$$\pi(a)(i, j) = \begin{cases} 0 & \text{if } t + k \leq a(i, j) < t + n \\ a(i, j) & \text{if } 0 \leq a(i, j) < t + k \end{cases}$$

Define the function s on $\mathbb{N} \times \bigcup_k \mathcal{S}_t^<(\frac{\omega, m}{k})$ as follows. Given $n < \omega$, $X \in \mathcal{S}_t^<(\frac{\omega, m}{k})$ and any $A \in \mathcal{S}_t^<(\frac{\omega, m}{\omega})$ such that $\pi(A) = X$, let

$$s(n, X) = \begin{cases} r_n(A) & \text{if } 0 \leq n \leq k \\ \pi(r_n(A)) & \text{if } n > k \end{cases}$$

For $A, B \in \mathcal{S}_t^<(\frac{\omega, m}{\omega})$ and $X \in \mathcal{S}_t^<(\frac{\omega, m}{k})$, $k < \omega$, define $A \circ B = A \cdot B$ and $A \circ X = A \cdot X$. We will prove that the structure $(\mathcal{S}_t^<(\frac{\omega, m}{\omega}), \leq, r, (\mathcal{S}_t^<(\frac{\omega, m}{k}))_k, \circ, s)$ satisfies axioms **A.5 – A.7**. Therefore, by Theorem 3.3, we will obtain the following

Theorem 4.5. *Let $k < \omega$ and $B \in \mathcal{S}_t^<(\frac{\omega, m}{\omega})$ be given. For every finite Baire-measurable coloring of $\mathcal{S}_t^<(\frac{\omega, m}{k})$, there exists $A \in \mathcal{S}_t^<(\frac{\omega, m}{\omega})|B$ such that $\mathcal{S}_t^<(\frac{\omega, m}{k})|A$ is monochromatic.*

Claim. $(\mathcal{S}_t^<(\omega, m), \leq, r, (\mathcal{S}_t^<(\omega, m))_k, \circ, s)$ satisfies axioms **A.5** – **A.7**.

Proof. This follows from the definitions. For instance, to show part (c) of **A.5** notice that

$$\begin{aligned}
& (A \circ B) \circ C (i, j) \\
&= C((A \circ B)(i, j), m) \\
&= C((B(A(i, j), m)), m) \\
&= B \circ C(A(i, j), m) \\
&= A \circ (B \circ C) (i, j).
\end{aligned}$$

And in order to prove part (b) of **A.6**, for instance, notice that if $A \in \mathcal{S}_t^<(\omega, m)$ and $X, Y \in \mathcal{S}_t^<(\omega, m)$ are such that $Y = A \circ X$ then there exists $B \in \mathcal{S}_t^<(\omega, m)$ such that $B \leq A$ and $Y = \pi(B)$. Then, from the definition of s and π we conclude that $\text{depth}_B s(n, Y) < \text{depth}_B s(n+1, Y)$, for all $n < \omega$. Therefore, $\text{depth}_A s(n, Y) < \text{depth}_A s(n+1, Y)$, for all $n < \omega$.

On the other hand, the definition of the operation $A \circ B = A \cdot B$ was done on $\mathcal{S}_t^<(\omega, \alpha)$, for all $\alpha \leq \beta \leq \omega$. Thus, the finitization asked for in **A.7** was defined at the same time. Parts (a) of **A.7** follow from the definition of the operation \circ by an easy extension of functions. Part (b) is straight forward. Let us prove part (c). Let $A \in \mathcal{S}_t^<(\omega, m)$, $a \in \mathcal{S}_t^<(\omega, m)|A$ and $X \in \mathcal{S}_t^<(\omega, m)|A$ be given, with $\text{depth}_A(a) = n$ and $X \in \langle a \rangle$. Define $B \in \mathcal{S}_t^<(\omega, m)$ by

$$B(i, j) = \begin{cases} (A \circ X)(i, j) & \text{if } A(i, j) < t + n \\ A(i, j) & \text{if } A(i, j) \geq t + n \end{cases}$$

Then $B \leq A$ and $\pi(B) = A \circ X$. Notice that

$$s(n, A \circ X) = r_k(B) = B \upharpoonright \min \text{supp } B^{-1}(\{t+k\}) = A \circ X \upharpoonright \min \text{supp } A^{-1}(\{t+n\}) = r_n(A) \circ a.$$

This completes the proof of the Claim and of Theorem 4.5. \square

Remark 4.3. The case $t = 1$ of Theorem 4.5 is just the corresponding version of the infinite Ramsey Theorem [24] for the topological Ramsey space $\text{FIN}_m^{[\infty]}$ proved by Todorcevic [25]. Off course, in the case $t = 1$, Theorem 4.5 holds for all finite colorings (that is, it is not necessary to restrict to Baire-measurable colorings). But for $t > 1$, using the Axiom of Choice, it is possible to define a not Baire-measurable finite coloring of $\mathcal{S}_t^<(\omega, m)$ with no monochromatic set of the form $\mathcal{S}_t^<(\omega, m)|A$.

5. CLASSICAL EXAMPLES

Throughout this Section, we will explore other examples which fit the abstract setting introduced in this Section 3. These classical examples were originally introduced in [3, 23, 25, 29] and motivated this research.

5.1. **Parameter words.** Given $t < \omega$ and ordinals $\alpha \leq \beta \leq \omega$, let $\mathcal{S}_t^{(\beta)}$ denote the set of all surjective functions $A : t + \beta \rightarrow t + \alpha$ satisfying

- (1) $A(i) = i$ for every $i < t$.
- (2) $\min A^{-1}(\{i\}) < \min A^{-1}(\{j\})$ for all $i < j < t + \alpha$.

For $A \in \mathcal{S}_t^{(\gamma)}$ and $B \in \mathcal{S}_t^{(\beta)}$, the composite $A \cdot B \in \mathcal{S}_t^{(\gamma)}$ is defined by $(A \cdot B)(i) = B(A(i))$.

Fix $t < \omega$. Let $\mathcal{R} = \mathcal{S}_t^{(\omega)}$ and for every positive $k < \omega$, let $\mathcal{R}_k = \mathcal{S}_t^{(\omega)}$. Define the operation $\circ : \mathcal{R} \times (\mathcal{R} \cup \bigcup_k \mathcal{R}_k) \rightarrow (\mathcal{R} \cup \bigcup_k \mathcal{R}_k)$ as $A \circ B = A \cdot B$. For $A, B \in \mathcal{R}$, write $A \leq B$ whenever there exists $C \in \mathcal{R}$ such that $A = B \circ C$.

At this point, it is useful to understand $\mathcal{S}_t^{(\omega)}$ as the set of equivalence relations on $t + \omega$ with infinitely many equivalence classes such that for each $A \in \mathcal{S}_t^{(\omega)}$, the restriction $A \upharpoonright t$ is the identity relation on $t = \{0, 1, \dots, t-1\}$ (when $t > 0$). Define the function $r : \mathbb{N} \times \mathcal{R} \rightarrow \mathcal{AR}$ as:

$$r(n, A) = \begin{cases} \emptyset & \text{if } n = 0 \\ A \upharpoonright \min A^{-1}(\{t+n\}) & \text{if } n > 0 \end{cases}$$

The definition of $\mathcal{S}_t^{(\beta)}$, for $t < \omega$ and ordinals $\alpha \leq \beta \leq \omega$, was taken from [23]. But the proof of the following is due to Carlson and Simpson [2]:

Theorem 5.1 (Carlson-Simpson [2]). $(\mathcal{S}_t^{(\omega)}, \leq, r)$ is a topological Ramsey space.

Remark 5.1. Axiom **A.4** for the space $(\mathcal{S}_t^{(\omega)}, \leq, r)$ follows from the infinite version of Ramsey's theorem for parameter words due to Graham and Rothschild [11] (case $k = 1$ of Theorem A in [23], page 191).

Similarly, $\mathcal{S}_t^{(\omega)}$ can be understood as the set of equivalence relations on $t + \omega$ with exactly k classes disjoint from t such that for each $X \in \mathcal{S}_t^{(\omega)}$, the restriction $X \upharpoonright t$ is the identity relation on $t = \{0, 1, \dots, t-1\}$ (when $t > 0$). Let $\pi : \mathcal{S}_t^{(\omega)} \rightarrow \mathcal{S}_t^{(\omega)}$ be defined as follows:

$$\pi(A)(i) = \begin{cases} 0 & \text{if } A(i) \geq t+k \\ A(i) & \text{if } 0 \leq A(i) < t+k \end{cases}$$

Notice that π is a surjection. For $l > k$, we will extend the function π to \mathcal{AR}_l as follows. Given $a \in \mathcal{AR}_l$ and i in the domain of a , let

$$\pi(a)(i) = \begin{cases} 0 & \text{if } t+k \leq a(i) < t+l \\ a(i) & \text{if } 0 \leq a(i) < t+k \end{cases}$$

Define $s : \mathbb{N} \times \bigcup_k \mathcal{S}_t^{(\omega)} \rightarrow \bigcup_{i \leq k} \mathcal{AR}_i$ as follows. Given $n < \omega$, $X \in \mathcal{S}_t^{(\omega)}$ and any $A \in \mathcal{S}_t^{(\omega)}$ such that $\pi(A) = X$, let

$$s(n, X) = \begin{cases} r_n(A) & \text{if } 0 \leq n \leq k \\ \pi(r_n(A)) & \text{if } n > k \end{cases}$$

With these definitions, the structure $(\mathcal{S}_t^{(\omega)}, \leq, r, (\mathcal{S}_t^{(\omega)})_k, \circ, s)$ satisfies axioms **A.1** – **A.7** and $\mathcal{S}_t^{(\omega)}$ is metrically closed. Thus we get the following from Theorem 3.3.

Theorem 5.2 (Carlson-Simpson [2]). *For every $Y \in \mathcal{S}_t^{(\omega)}$ and every finite Borel-measurable coloring of $\mathcal{S}_t^{(\omega)}$, $k < \omega$, there exists $X \in \mathcal{S}_t^{(\omega)}|Y$ such that $\mathcal{S}_t^{(\omega)}|X$ is monochromatic.*

Theorem 5.3 (Prömel-Voigt [23]). *For every $Y \in \mathcal{S}_t^{(\omega)}$ and every finite Baire-measurable coloring of $\mathcal{S}_t^{(\omega)}$, there exists $X \in \mathcal{S}_t^{(\omega)}|Y$ such that $\mathcal{S}_t^{(\omega)}|X$ is monochromatic.*

Remark 5.2. For $t = 0$, Theorem 5.2 is known as the Dual Ramsey Theorem.

5.2. Ascending parameter words. Now, we will explore a special type of parameter words. Let $\mathcal{S}_t^{<(\alpha)}$ denote the set of all $A \in \mathcal{S}_t^{(\beta)}$ satisfying

- (1) $A^{-1}(\{t + j\})$ is finite, for all $j < \alpha$.
- (2) $\max A^{-1}(\{t + i\}) < \min A^{-1}(\{t + j\})$, for all $i < j < \alpha$.

Let $\mathcal{R} = \mathcal{S}_t^{<(\omega)}$ and for every $k < \omega$ let $\mathcal{R}_k = \mathcal{S}_t^{<(\omega)}|_k$. $\mathcal{S}_t^{<(\omega)}$ is a subset of $\mathcal{S}_t^{(\omega)}$. So we can consider in this case the restrictions \leq, \circ, r and s , as defined in Section 5.1, to the corresponding domains within the context of $\mathcal{S}_t^{<(\omega)}$. But by letting $m = 1$ in the definition of the space $\mathcal{S}_t^{<(\omega, m)}$ introduced in Section 4, we can easily verify that $\mathcal{S}_t^{<(\omega)} = \mathcal{S}_t^{<(\omega, 1)}$. Notice that $\mathcal{S}_0^{<(\omega)}$ is essentially Ellentuck's space (see [8]) and, for all $k < \omega$, $\mathcal{S}_0^{<(\omega)}|_k = \emptyset$. On the other hand, $\mathcal{S}_1^{<(\omega)}$ is Milliken's space (see [15]).

Theorem 5.4 (Milliken, [15]). *$(\mathcal{S}_1^{<(\omega)}, \leq, r)$ is a topological Ramsey space.*

Axiom **A.4** for $(\mathcal{S}_1^{<(\omega)}, \leq, r)$ is equivalent to Hindman's theorem [14]. Again, the structure $(\mathcal{S}_t^{<(\omega)}, \leq, r, (\mathcal{S}_t^{<(\omega)})_k, \circ, s)$ satisfies axioms **A.1** – **A.7** and $\mathcal{S}_t^{<(\omega)}$ is metrically closed. Letting $m = 1$ in Theorem 4.5 and Theorem 4.1 we obtain a different proof of the following well-known results.

Theorem 5.5 (Carlson, Theorem 6.9 in [3]). *$(\mathcal{S}_t^{<(\omega)}, \leq, r)$ is a topological Ramsey space.*

Theorem 5.6 (Prömel-Voigt [23]). *For every $Y \in \mathcal{S}_t^{<(\omega)}$ and every finite Baire-measurable coloring of $\mathcal{S}_t^{<(\omega)}$, there exists $A \in \mathcal{S}_t^{<(\omega)}|Y$ such that $\mathcal{S}_t^{<(\omega)}|A$ is monochromatic.*

5.3. Partial \mathcal{G} -partitions. Let \mathcal{G} be a finite group and let $e \in \mathcal{G}$ denote its unit element. Also let ν be a symbol not occurring in \mathcal{G} . Given ordinals $\alpha \leq \beta \leq \omega$, let $\mathcal{S}_{\mathcal{G}}^{(\beta)}$ denote the set of all mappings $A : \beta \rightarrow \{\nu\} \cup (\alpha \times \mathcal{G})$ satisfying

- (1) For every $j < \alpha$ there exists $i < \beta$ such that $A(i) = (j, e)$ and $A(i') \notin \{j\} \times \mathcal{G}$ for all $i' < i$.
- (2) $\min A^{-1}(\{(i, e)\}) < \min A^{-1}(\{(j, e)\})$ for all $i < j < \alpha$.

Elements of $\mathcal{S}_{\mathcal{G}}^{(\beta)}$ are known as *partial \mathcal{G} -partitions of β into α blocks*.

For $A \in \mathcal{S}_{\mathcal{G}}^{(\beta)}$ and $B \in \mathcal{S}_{\mathcal{G}}^{(\alpha)}$, the composite $A \cdot B \in \mathcal{S}_{\mathcal{G}}^{(\beta)}$ is defined by

$$(A \cdot B)(i) = \begin{cases} \nu & \text{if } A(i) = \nu \\ \nu & \text{if } A(i) = (j, b) \text{ and } B(j) = \nu \\ (k, b \cdot c) & \text{if } A(i) = (j, b) \text{ and } B(j) = (k, c) \end{cases}$$

Also for $(n, A) \in \mathbb{N} \times \mathcal{S}_{\mathcal{G}}^{(\omega)}$ let

$$r(n, A) = \begin{cases} \emptyset & \text{if } n = 0 \\ A \upharpoonright \min A^{-1}(\{(n, e)\}) & \text{if } n > 0 \end{cases}$$

Define $A \circ B = A \cdot B$, and as before write $A \leq B$ if there exists C such that $A = B \circ C$.

Theorem 5.7 (Prömel-Voigt [23]). $(\mathcal{S}_{\mathcal{G}}(\omega), \leq, r)$ is a topological Ramsey space.

Remark 5.3. **A.4** for the space $(\mathcal{S}_{\mathcal{G}}(\omega), \leq, r)$ follows from case $k = 1$ of Thorem D in [23].

Given $i < \omega$ and $A \in \mathcal{S}_{\mathcal{G}}(\omega)$, if $A(i) = (n, b)$ then we will write $A(i)_0 = n$. That is, $A(i)_0$ is the first coordinate of $A(i)$. Now, let $\pi : S_t(\omega) \rightarrow S_t(k)$ be defined as follows:

$$\pi(A)(i) = \begin{cases} (0, e) & \text{if } A(i)_0 \geq t + k \\ A(i) & \text{if } 0 \leq A(i)_0 < t + k \end{cases}$$

Notice that π is a surjection. Again, for $l > k$, we will extend the function π to \mathcal{AR}_l as follows. For $a \in \mathcal{AR}_l$ and i in the domain of a , if $a(i) = (n, b)$ then write $a(i)_0 = n$ and let

$$\pi(a)(i) = \begin{cases} (0, e) & \text{if } t + k \leq a(i)_0 < t + l \\ a(i) & \text{if } 0 \leq a(i)_0 < t + k \end{cases}$$

As in the previous Section, define $s : \mathbb{N} \times \bigcup_k \mathcal{S}_{\mathcal{G}}(\omega_k) \rightarrow \bigcup_{i \leq k} \mathcal{AR}_i$ as follows. Given $n < \omega$, $X \in \mathcal{S}_{\mathcal{G}}(\omega_k)$ and any $A \in \mathcal{S}_{\mathcal{G}}(\omega)$ such that $\pi(A) = X$, let

$$s(n, X) = \begin{cases} r_n(A) & \text{if } 0 \leq n \leq k \\ \pi(r_n(A)) & \text{if } n > k \end{cases}$$

With these definitions the following hold:

Theorem 5.8 (Prömel-Voigt, [23]). *For every $Y \in \mathcal{S}_{\mathcal{G}}(\omega)$ and every finite Baire-measurable coloring of $\mathcal{S}_{\mathcal{G}}(\omega)$, there exists $X \in \mathcal{S}_{\mathcal{G}}(\omega) \upharpoonright Y$ such that $\mathcal{S}_{\mathcal{G}}(\omega) \upharpoonright X$ is monochromatic.*

5.4. Infinite dimensional vector subspaces of $\mathbb{F}^{\mathbb{N}}$. Given a finite field \mathbb{F} , let $\mathcal{M}_{\infty} = \mathcal{M}_{\infty}(\mathbb{F})$ denote the set of all reduced echelon $\mathbb{N} \times \mathbb{N}$ -matrices, $A : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{F}$. The i th column of A is the function $A_n : \mathbb{N} \rightarrow \mathbb{F}$ given by $A_n(j) = A(n, j)$. For $A, B \in \mathcal{M}_{\infty}$, $A \leq B$ means that every column of A belongs to the closure (taken in $F^{\mathbb{N}}$ with the product topology) of the linear span of the columns of B . For $n \in \mathbb{N}$ and $A \in \mathcal{M}_{\infty}$ let $p_n(A) = \min \{j : A_n(j) \neq 0\}$ and define $r(0, A) = \emptyset$ and $r(n+1, A) = A \upharpoonright (n \times p_n(A))$.

Theorem 5.9 (Carlson [1]). $(\mathcal{M}_{\infty}, \leq, r)$ is a topological Ramsey space.

Now, for every postive $k \in \mathbb{N}$, let $\mathcal{M}_k = \mathcal{M}_k(\mathbb{F})$ denote the set of all reduced echelon $\mathbb{N} \times k$ -matrices, $A : \mathbb{N} \times k \rightarrow \mathbb{F}$. For $A \in \mathcal{M}_{\infty}$ and $B \in \mathcal{M}_{\infty} \cup \mathcal{M}_k$, let $A \circ B$ be the usual multiplication of matrices. Define

$$s(n, A) = \begin{cases} r(n, A) & \text{if } n \leq k \\ A \upharpoonright (n \times p_k(A)) & \text{if } n > k \end{cases}$$

With these definitions $(\mathcal{M}_{\infty}, \leq, r, (\mathcal{M}_k)_k, \circ, s)$ satisfies **A.1–A.7**, so we get the following:

Theorem 5.10 (Todorćevic, [25]). *For every $B \in \mathcal{M}_\infty$ and every finite Baire-measurable coloring of \mathcal{M}_k , there exists $A \in \mathcal{M}_\infty|B$ such that $\mathcal{M}_k|A$ is monochromatic.*

Now, let $\mathcal{V}_\infty(\mathbb{F})$ denote the collection of all closed infinite-dimensional subspaces of $\mathbb{F}^\mathbb{N}$ and for every k , let $\mathcal{V}_k(\mathbb{F})$ denote the collection of all k -dimensional subspaces of $\mathbb{F}^\mathbb{N}$. If we consider $\mathcal{V}_\infty(\mathbb{F})$ and $\mathcal{V}_k(\mathbb{F})$ with the Vietoris topology then there are natural homeomorphisms between $\mathcal{V}_\infty(\mathbb{F})$ and \mathcal{M}_∞ , and $\mathcal{V}_k(\mathbb{F})$ and \mathcal{M}_k . So Theorem 5.10 can be restated as

Corollary 5.11 (Voigt [29]). *For every $W \in \mathcal{V}_\infty(\mathbb{F})$ and every finite Baire-measurable coloring of $\mathcal{V}_k(\mathbb{F})$, there exists $V \in \mathcal{V}_\infty(\mathbb{F})|W$ such that $\mathcal{V}_k(\mathbb{F})|V$ is monochromatic.*

6. FINAL REMARK

Following with the tradition started in [2, 25], the results contained in Section 3 attempt to serve as part of a unifying framework for the theory of topological Ramsey spaces. The new axioms **A5–A7** are sufficient to capture a particular feature of a class of topological Ramsey spaces which was not revealed by the original 4 axioms proposed in [25]. In Section 3, we showed a characterization of those topological Ramsey spaces \mathcal{R} with family of approximations $\mathcal{AR} = \bigcup_{k < \omega} \mathcal{AR}_k$ for which there exist topological spaces $\mathcal{R}_k \subseteq (\mathcal{AR}_k)^\mathbb{N}$, $k < \omega$, such that every Baire subset of \mathcal{R}_k is Ramsey. While this characterization is essentially based on the axioms **A5–A7**, the question about the necessity of these axioms remains open.

Finally, given $0 < m < \omega$, there is a well-known understanding of the space $\text{FIN}_m^{[\infty]} = \mathcal{S}_1^{<(\omega, m)}$ in terms of Functional Analysis. It has to do with the property of oscillation stability for Lipschitz functions defined on the sphere of the Banach space c_0 (see [13]) and with the solution of the Distortion Problem (see [22]). This leads to the following natural question: For $t > 1$, what is the interpretation (if any) of the space $\mathcal{S}_t^{<(\omega, m)}$ in terms Functional Analysis?

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