A Reduced Informationally Complete Set in the C*-algebra Generated by Phase Space Fuzzy Localization Operators

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Abstract We consider an effect algebra of phase space localization operators for a quantum mechanical Hilbert space that contains no non-trivial projections, and the C*-algebra generated by it. This C*-algebra forms an informationally complete set in the original Hilbert space. Its elements are shown to have singular-value-based decompositions that permit their characterization in terms of limits of linear combinations of products of pairs of the phase space fuzzy localization operators. Through these results, it is shown that the informational completeness of the C*-algebra can be greatly reduced to the informational completeness of the set of products of pairs formed from the elements of the effect algebra.

Keywords C*-algebra · quantum mechanics · phase space · informational completeness · effect algebra

1 Introduction

The phase space formalism of quantum mechanics has been established and studied now for some time [6]. The phase-space localization operators (observables) within this framework are unsharp in that they contain no nontrivial projections. The effect algebra of these phase-space localization operators has been show to be informationally complete in certain Hilbert space representations, such as that of spin representations via the Heisenberg group [6]. *Informational completeness* of a set of self-adjoint operators is an important property that implies that the set uniquely determines the state of the given system.

The effect algebra of phase space localization operators generates a C*-algebra that has been shown in [5] to be informationally complete, independent of the particular phase space representation of quantum mechanics that is chosen. The

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proof proceeds by showing that this C*-algebra must be equal to $\mathcal{B}(\mathcal{H})$. Reductions to much smaller informationally complete sets on the scale of the effect algebra itself would be very useful. Busch [4] gives an equivalent criterion for informational completeness of a set of operators, namely that the span of such a set is weakly dense in $\mathcal{B}(\mathcal{H})$ (that is, in the topology induced by the trace). In this paper, we use this result to prove the informational completeness of a highly reduced subset of the C*-algebra generated by the effect algebra of phase space localization operators (independent of representation). This set consists of the products of pairs of observables from the effect algebra. The fact that the span of the set of such products of pairs is dense in $\mathcal{B}(\mathcal{H})$ arises out of a singular-value based decomposition of products of any finite length.

In Section 2, we give an outline of the phase space formalism for quantum mechanics, and introduce reproducing kernel Hilbert spaces which will be an aid in our development. We show that the quantum mechanical Hilbert spaces under consideration can be framed this way. In Section 3, we define informational completeness and give an equivalent criterion for informational completeness due to Busch. We then present the effect algebra of phase space fuzzy localization operators, and discuss the informational completeness of this set in special cases, and the informational completeness of the C*-algebra generated by this set, independent of the phase space representation chosen, a result due to Schroeck. Section 4 then presents singular-value based decomposition theorems for Hilbert-Schmidt integral operators, and a related important result of Šimša. This is applied to obtain a new singular-value-based decomposition for finite products of elements of the effect algebra of phase space observables under mild conditions. Section 5 utilizes the results of Section 4 in conjunction with the results of Schroeck and of Busch in Section 3 to obtain the informational completeness of the reduced set of products of pairs of the phase space fuzzy localization operators from the effect algebra.

2 The Phase Space Formalism for Quantum Mechanics

2.1 The Construction of Quantum Mechanics on Phase Space

In this section, we present the construction of the phase space formalism for quantum mechanics, which has been covered before at the IQSA conference, so we confine it here to a brief outline. For a more complete treatment, see [6].

We begin with a dynamical group G, which is a Lie group (e.g. Galilei, Poincaré), and H a closed subgroup. We then form the Lie algebra $\mathfrak{g}$, take its dual $\mathfrak{g}^*$, and construct coboundary operator δ between the $\wedge^n(\mathfrak{g}^*)$. We then take $\omega \in Z^2(\mathfrak{g}) \equiv \{\omega \in (\mathfrak{g}\wedge\mathfrak{g})^* \,|\, \delta(\omega) = 0\}$, define the sub-Lie algebra $h_\omega \equiv \{\xi \in \mathfrak{g} \,|\, \omega(\xi, \cdot) = 0\}$, and exponentiate h_ω to obtain the subgroup H_ω and assume it is closed. Let $\Gamma \equiv G/H_\omega$. Then

1. Γ is a transitive symplectic manifold (i.e. a phase space) with 2-form ($\approx$ Poisson bracket) the pullback of ω,
2. Γ has dimension $2m$ and the m-th exterior product of the 2-form is the left-invariant measure μ, and
3. For X any symplectic space under G, X = a union of the Γs. Obtain the canonical variables.

Now we form the separable Hilbert space $L^2_\mu(\Gamma)$. Define the action of the group G via

$$[V^\alpha(g)\Psi](x) = \alpha(h(g^{-1}, x))\Psi(g^{-1}x), \tag{1}$$

where $\Psi \in L^2_\mu(\Gamma)$, h is a generalized co-cycle and α is a one-dimensional representation of H_ω. Note that the representation V^α is highly reducible [6]. Define the multiplication operator by

$$[A(f)\Psi](x) \equiv f(x)\Psi(x) \tag{2}$$

for measurable f. The $A(f)$s are a commuting set, and $L^2_\mu(\Gamma)$ is not a quantum mechanical Hilbert space for a single particle since it is not irreducible. Now, let U be a representation, and $\mathcal{H}$ the irreducible unitary representation space obtained via the "Mackey Machine." Take any Borel section $\sigma : \Gamma \to G$. η in $\mathcal{H}$ is said to be admissible with respect to the section σ if

$$\int_\Gamma |\langle U(\sigma(x))\eta, \eta\rangle|^2 d\mu(x) < \infty.$$

For η admissible with respect to σ, η is said to be α-admissible with respect to σ if $U(h)\eta = \alpha(h)\eta$ for all $h \in H_\omega$, where α is a one-dimensional representation of H_ω. The set of α-admissible vectors with respect to σ is never empty for any of the usual representation spaces of quantum mechanics [3].

Next, define $W^\eta : \mathcal{H} \times \Gamma \to \mathbb{C}$ by

$$[W^\eta(\varphi)](x) = \langle U(\sigma(x))\eta, \varphi\rangle. \tag{3}$$

Then:

1. W^η is a linear map from $\mathcal{H}$ to $L^2_\mu(\Gamma)$ whenever η is α-admissible.
2. W^η intertwines: $W^\eta U(g) = V^\alpha(g)W^\eta$.
3. $W^\eta(\mathcal{H})$ is a closed subspace of $L^2_\mu(\Gamma)$.

Let $P^\eta : L^2_\mu(\Gamma) \to W^\eta(\mathcal{H})$ be the canonical projection. Now we pull back the multiplication operators $A(f)$ from $L^2_\mu(\Gamma)$ to $\mathcal{H}$:

$$A^\eta(f) = [W^\eta]^{-1} P^\eta A(f) W^\eta, \tag{4}$$

Then, we can deduce the following properties of $A^\eta(f)$:

1. $A^\eta(f) : \mathcal{H} \to \mathcal{H}$, and
2. For $||\eta|| = 1$,

   $$A^\eta(f) = \int_\Gamma f(x) T^\eta(x) d\mu(x), \tag{5}$$

   where

   $$T^\eta(x) \equiv P_{U(\sigma(x))\eta} = |U(\sigma(x))\eta\rangle\langle U(\sigma(x))\eta|. \tag{6}$$

For $\psi \in \mathcal{H}$, $Tr(P_\psi A^\eta(f))$ has an interpretation as the "transition probability of the state at P_ψ to (coherent) state $P_{U(\sigma(x))\eta}$ for some x in the fuzzy set determined by f." η is directly related to the instrument with which P_ψ is measured.

2.2 Quantum Mechanical Hilbert Spaces as Reproducing Kernel Hilbert Spaces

It has been shown in [2] that the quantum mechanical Hilbert spaces just derived, namely the subspaces $W^\eta(\mathcal{H})$ of $L^2_\mu(\Gamma)$, are equatable with reproducing kernel Hilbert spaces, which possess many useful properties.

Definition 1 A reproducing kernel is a measurable function $K : \Gamma \times \Gamma \to \mathbb{C}$ such that

1. $K(x,x) > 0$
2. $\overline{K(x,y)} = K(y,x)$

The Riesz Representation Theorem implies that for $K(x,\cdot) \in L^2_\mu(\Gamma)$, there exists a unique reproducing kernel Hilbert space $\mathcal{H}_K \subset L^2_\mu(\Gamma)$ such that for all $\varphi \in \mathcal{H}_K$, $\varphi(x) = \langle \varphi, K(x,\cdot)\rangle$. The map $P_K : L^2_\mu(\Gamma) \to \mathcal{H}_K$ is the projection onto $\mathcal{H}_K$ defined as an integral operator:

$$[P_K(\varphi)](x) = \int_\Gamma K(x,y)\varphi(y)d\mu(y). \tag{7}$$

Following the development above of quantum mechanics on phase space, let

$$K(x,y) \equiv \langle U(\sigma(x))\eta, U(\sigma(y))\eta\rangle. \tag{8}$$

This is a reproducing kernel with projection operator P^η:

$$\int_\Gamma \langle U(\sigma(x))\eta | U(\sigma(y))\eta\rangle \varphi(y)d\mu(y) = [P^\eta \varphi](x). \tag{9}$$

Therefore, $\mathcal{H}_K = W^\eta\mathcal{H}$, and we have the equivalence of our quantum mechanical Hilbert space with a reproducing Kernel Hilbert space.

Reproducing kernel Hilbert spaces have useful properties, one form of which we summarize for our purposes in the following lemma.

Lemma 1 *For $\varphi, \psi \in L^2_\mu(\Gamma)$, $\mathcal{H}_K \subset L^2_\mu(\Gamma)$ a reproducing kernel Hilbert space, and P_K its associated projection operator,*

$$\begin{aligned} \textstyle\int_\Gamma [P_K\varphi](x)\psi(x)d\mu(x) &= \textstyle\int_\Gamma \varphi(x)[P_{\overline{K}}\psi](x)d\mu(x) \\ &= \textstyle\int_\Gamma [P_K\varphi](x)[P_{\overline{K}}\psi](x)d\mu(x). \end{aligned} \tag{10}$$

3 Informational Completeness and the Effect Algebra of Phase Space Fuzzy Localization Operators

3.1 Informational Completeness

Definition 2 A set of bounded, self-adjoint operators $\{A_\beta,\ \beta \in \mathcal{I}\}$ on Hilbert Space $\mathcal{H}$ is **informationally complete** if for all states ρ, ρ', $Tr(\rho A_\beta) = Tr(\rho' A_\beta)$ for all $\beta \in \mathcal{I} \implies \rho = \rho'$.

Informational completeness allows one to recover complete information about the state of a system from the given set of observables. The concept of informational completeness has natural applications in optics, signal processing [4], and even in the remote detection of landmines [1], in which choosing a finite subset of an informationally complete set improved landmine detection significantly over the Fourier transform methods currently in use. It is desirable in these circumstances to obtain as small an informationally complete set as possible. In this vein, Busch proved the following Theorem:

Theorem 1 *[4] I is an informationally complete set in $\mathcal{H}$ if and only if span I is dense in $B(\mathcal{H})$, in the (weak) topology induced by the trace.*

Note that such a weak limit in the topology induced by the trace is a bounded operator A on $\mathcal{H}$ that satisfies that there exists a net $\{A_\lambda\}$ of bounded operators on $\mathcal{H}$ such that $\mathrm{Tr}(\rho A_\lambda) \to \mathrm{Tr}(\rho A)$ for all density operators ρ on $\mathcal{H}$. We will apply Theorem 1 to obtain a highly reduced informationally complete set of self-adjoint operators.

3.2 The Effect Algebra of Phase Space Localization Operators

We now define the effect algebra of phase space fuzzy localization operators. Our starting point is the characterization of the localization operators from Section 2, that is,

$$A^\eta(f) = \int_\Gamma f(x)|U(\sigma(x))\eta\rangle\langle U(\sigma(x))\eta| d\mu(x). \tag{11}$$

The following facts have been shown [6]:

Theorem 2 *Assume $f \in L^1_\mu(\Gamma) \cap L^\infty_\mu(\Gamma)$. Then*

1. *the $A^\eta(f)$s are compact,*
2. *The $A^\eta(f)$s are self-adjoint for f real, positive for $f \geq 0$ μ-a.e., and strictly positive for $f > 0$ μ-a.e.*
3. *The $A^\eta(f)$s are all compatible but generally do not commute.*
4. *If $0 \leq f(x) \leq 1$ μ-a.e., then $A^\eta(f)$ is a contraction operator.*

Define the set, $\mathcal{E}$, by

$$\mathcal{E} \equiv \{A^\eta(f) \,|\, f \in L^1_\mu(\Gamma) \cap L^\infty_\mu(\Gamma),\, 0 \leq f(x) \leq 1\, \mu - a.e.\} \tag{12}$$

In [5], Schroeck shows that $\mathcal{E}$ is an effect algebra containing the phase-space localization operators, which is also an M.V.-algebra and a Heyting algebra, and it contains no non-trivial projections. $\mathcal{E}$ may be thought of the set of phase-space fuzzy localization operators.

3.3 The Informationally Complete C*-Algebra, $\mathcal{F}$, Generated by $\mathcal{E}$

The effect algebra $\mathcal{E}$ is informationally complete only in special cases (e.g. when G is the Heisenberg group), and always under the conditions of α-admissability of η, and that $\langle U(\sigma(x))\eta, \eta\rangle \neq 0$ μ-a.e. However, under these same conditions, Schroeck showed the following to be true, independent of the particular irreducible phase space representation chosen.

Theorem 3 *[5] Suppose that η is α-admissible, and that $\langle U(\sigma(x))\eta, \eta\rangle \neq 0$ μ-a.e. Then, any set whose span is dense in the C^*-algebra, $\mathcal{F}$, generated by*

$$\mathcal{E} \equiv \{A^\eta(f) \mid f \in L^1_\mu(\Gamma) \cap L^\infty_\mu(\Gamma),\, 0 \leq f(x) \leq 1 \ \mu\text{–a.e.,}\} \tag{13}$$

in the topology induced by the trace, is informationally complete in the G-irreducible representation space $\mathcal{H}$, for any G that is a locally compact Lie group with a finite-dimensional Lie algebra, and Γ the associated phase space.

4 A Singular-Value-Based Decomposition of Products of Phase-Space Fuzzy Localization Operators

4.1 Singular Value Decomposition for Hilbert-Schmidt Integral Operators and a Related Result of Šimša

In the late 19th and early 20th centuries, an important analytical development arose in mathematics – the singular value decomposition. Among those responsible for the development are Beltrami, Jordan, Sylvester, Schmidt and Weyl [8]. Erhard Schmidt's major contribution was to extend the theory of singular value decompositions to function spaces.

Now known as the Hilbert-Schmidt Decomposition Theorem, we state a version of it that leads into our investigations.

Theorem 4 *[8] Let $T : L^2_\mu(X) \to L^2_\lambda(Y)$ be defined by*

$$[T\varphi](y) \equiv \int_X h(x, y)\varphi(x)d\mu(x), \tag{14}$$

where $h \in L^2_{\mu\times\lambda}(X \times Y)$. Then,

$$[T\varphi](y) = \sum_{k=1}^{\infty} \beta_k \overline{v}_k(y) \int_X \varphi(y)u_k(x)d\mu(x), \tag{15}$$

where $\{u_k\}$, and $\{v_k\}$ are complete orthonormal systems that form an adjoint pair for h, with a non-increasing sequence β_k of non-negative reals the associated singular values (eigenvalues of T^*T).

Many new results have come out of singular value decomposition theorems, one of which, due to Šimša, we employ in developing a singular value decomposition tailored to quantum mechanics on phase space.

Theorem 5 *[7] Suppose that $h \in L^2_{\mu\times\lambda}(X \times Y)$ is a non-zero function. Then there exist two orthonormal systems $\{u_k\} \subset L^2_\mu(X)$ and $\{v_k\} \subset L^2_\lambda(Y)$ and a non-increasing sequence $\{\beta_k\}$ of nonnegative reals such that*

$$h(x, y) = \sum_{k=1}^{\infty} \beta_k u_k(x)\overline{v}_k(y) \text{ for almost all } (x, y) \in X \times Y. \tag{16}$$

In that case, $\sum_{k=1}^{\infty} \beta_k^2 = ||h||^2$.

Note that if the integral operator in Theorem 4 is trace class, then in addition, $\sum_{k=1}^{\infty} \beta_k < \infty$.

4.2 A Singular Value Decomposition for Quantum Mechanics on Phase Space

For the development that follows, assume that η is α-admissable, $||\eta|| = 1$, and $\langle U(\sigma(x))\eta, \eta\rangle \neq 0$ μ-a.e. We begin with some preliminary notation and observations.

1. Let $U_x\eta \equiv U(\sigma(x))\eta$.
2. Let $K(x, y) \equiv \langle U_x\eta, U_y\eta\rangle$.

Consider the generating elements of the C^*-algebra, $\mathcal{F}$, $\Pi_{i=1}^n A^\eta(f_i)$. Observe that, for a pair,

$$A^\eta(f)A^\eta(g) = \int_\Gamma \int_\Gamma f(x)g(y)K(x, y)|U_x\eta\rangle\langle U_y\eta|d\mu(y)d\mu(x). \tag{17}$$

For a general finite product,

$$\begin{aligned} &\Pi_{i=1}^n A^\eta(f_i) \\ &= \textstyle\int_{\Gamma\times\Gamma} f_1(x)f_n(y)\langle U_x\eta|\Pi_{i=2}^{n-1} A^\eta(f_i)|U_y\eta\rangle|U_x\eta\rangle\langle U_y\eta|d\mu(y)d\mu(x) \\ &= \textstyle\int \Gamma \times \Gamma f_1(x)f_n(y)\dfrac{\langle U_x\eta|\Pi_{i=2}^{n-1} A^\eta(f_i)|U_y\eta\rangle}{K(x, y)}K(x, y)|U_x\eta\rangle\langle U_y\eta|d\mu^2 \\ &= \textstyle\int_{\Gamma\times\Gamma} B_0(x, y)K(x, y)|U_x\eta\rangle\langle U_y\eta|d\mu^2, \end{aligned} \tag{18}$$

where

$$B_0(x, y) \equiv f_1(x)f_n(y)\frac{\langle U_x\eta|\Pi_{i=2}^{n-1} A^\eta(f_i)|U_y\eta\rangle}{K(x, y)}. \tag{19}$$

Note that a finite product of n phase space localization operators has a form similar to a product of a pair of them.

We now develop a related singular value decomposition tailored to the elements of the effect algebra of phase space fuzzy localization operators. We begin with a definition and a lemma.

Definition 3 Define the *trace measure* ν with respect to μ^2 via

$$d\nu \equiv |\langle U(\sigma(x))\eta|U(\sigma(y))\eta\rangle|^2 d\mu^2 = |K(x, y)|^2 d\mu^2, \tag{20}$$

where μ^2 is the standard product measure.

Lemma 2 *Let ν be the trace measure with respect to μ^2,*

$$A_\lambda \equiv \int_{\Gamma\times\Gamma} B_\lambda(x,y)K(x,y)|U(\sigma(x))\eta\rangle\langle U(\sigma(y))\eta| d\mu^2 \tag{21}$$

for $\{B_\lambda\}$ a net in $L^2_\nu(\Gamma\times\Gamma)$. If $B_\lambda \to B_0$ in $L^2_\nu(\Gamma\times\Gamma)$, then $A_\lambda \to A_0$ in the topology induced by the trace.

Proof Let ρ be a density operator on $\mathcal{H}$, $\{\psi_k\}$ an orthonormal basis such that $\rho = \sum_{k=1}^\infty \gamma_k P_{\psi_k}$, $\sum_{k=1}^\infty \gamma_k = 1$, $\gamma_k \geq 0$, $k \in \mathbb{N}$. Then,

$$\begin{aligned}
&|\operatorname{Tr}(\rho A_\lambda) - \operatorname{Tr}(\rho A_0)|^2\\
&= |\textstyle\sum_{k=1}^\infty \gamma_k \langle\psi_k|A_\lambda - A_0|\psi_k\rangle|^2\\
&\leq \textstyle\sum_{k=1}^\infty \gamma_k |\int_{\Gamma\times\Gamma} |B_\lambda(x,y)-B_0(x,y)|^2|K(x,y)|^2|\langle\psi_k|U_x\eta\rangle\langle U_y\eta|\psi_k\rangle|^2 d\mu^2\\
&\leq \textstyle\sum_{k=1}^\infty \gamma_k \int_{\Gamma\times\Gamma} |B_\lambda(x,y)-B_0(x,y)|^2|K(x,y)|^2 d\mu^2\\
&= \textstyle\sum_{k=1}^\infty \gamma_k \int_{\Gamma\times\Gamma} |B_\lambda(x,y)-B_0(x,y)|^2 d\nu,
\end{aligned} \tag{22}$$

which establishes the lemma. □

This brings us to the core decomposition result which will yield the highly reduced informationally complete set of phase space fuzzy localization operators in the next section.

Theorem 6 *For $A^\eta(f_i)$ elements of the effect algebra $\mathcal{E}$, $i \in \mathbb{N}$, if η is α-admissable, and $\langle U(\sigma(x))\eta|\eta\rangle \neq 0$ μ-almost everywhere, then for $n > 2$, $\Pi_{i=1}^n A^\eta(f_i)$ is the limit in the topology induced by the trace, as $\lambda \to 0$, of operators of the form*

$$A^\eta_\lambda \equiv \sum_{k=1}^\infty \beta_{k,\lambda} A^\eta(u_{k,\lambda})A^\eta(\overline{v}_{k,\lambda}), \tag{23}$$

$\{u_{k,\lambda}\}_k$, $\{v_{k,\lambda}\}_k$ are orthonormal systems, and $\beta_{k,\lambda}$ is a non-increasing sequence of non-negative reals as $k \to \infty$.

Proof Let $B_0(x,y) \equiv f_1(x)f_n(y)\dfrac{\langle U_x\eta|\Pi_{i=2}^{n-1}A^\eta(f_i)|U_y\eta\rangle}{K(x,y)}$. B_0 is jointly measurable based on its definition in terms of measurable functions. $B_0 \in L^2_\nu(\Gamma\times\Gamma)$ since

$$\begin{aligned}
\int_{\Gamma\times\Gamma} |B_0(x,y)|^2 d\nu &= \int_{\Gamma\times\Gamma} f_1(x)^2 f_n(y)^2 |\langle U_x\eta|\Pi_{i=2}^{n-1}A^\eta(f_i)|U_y\eta\rangle|^2 d\mu^2\\
&\leq \int_{\Gamma\times\Gamma} f_1(x)^2 f_n(y)^2 d\mu^2\\
&< \infty,
\end{aligned} \tag{24}$$

by Fubini's Theorem and the fact that f_1, $f_n \in L^2_\mu(\Gamma)$ since they are in $L^1_\mu(\Gamma) \cap L^\infty_\mu(\Gamma)$.

At this point, the temptation is to apply Šimša's result in Theorem 5, but to do that we would need B_0 to be in $L^2_{\mu^2}(\Gamma\times\Gamma)$, which it is not in general. We also want the corresponding integral operator to be trace class, so we accomplish this with two approximations. For $\varepsilon > 0$, let

$$B^\varepsilon(x,y) \equiv f_1(x)f_n(y)\frac{\langle U_x\eta|\Pi_{i=2}^{n-1}A^\eta(f_i)|U_y\eta\rangle}{K_\varepsilon(x,y)}, \tag{25}$$

where

$$K_\varepsilon(x,y) \equiv \frac{\varepsilon K(x,y)}{|K(x,y)| \wedge \varepsilon}, \tag{26}$$

Recall that $K(x,y) \equiv \langle U_x\eta|U_y\eta\rangle$. K_ε is just a truncation of K that bounds K away from 0 by a magnitude of ε. B^ε is jointly measurable, and $B^\varepsilon \in L^2_{\mu^2}(\Gamma\times\Gamma)$, since

$$\int_{\Gamma\times\Gamma} |B^\varepsilon(x,y)|^2 d\mu^2 \le \frac{1}{\varepsilon^2}\int_{\Gamma\times\Gamma} f_1(x)^2 f_n(y)^2 d\mu^2 < \infty. \tag{27}$$

Define the Hilbert-Schmidt integral operator $J^\varepsilon : L^2_\mu(\Gamma) \to L^2_\mu(\Gamma)$ by $[J^\varepsilon\varphi](x) = \int_\Gamma B^\varepsilon(x,y)\varphi(y)d\mu(y)$.

Šimša's result implies $B^\varepsilon(x,y) = \sum_{k=1}^\infty \beta_k^\varepsilon u_k^\varepsilon(x) v_k^\varepsilon(y)$ for orthonormal systems $\{u_k^\varepsilon\}$ and $\{v_k^\varepsilon\}$, and nonnegative, non-increasing β_k, with $\sum_{k=1}^\infty (\beta_k^\varepsilon)^2 = ||B^\varepsilon||^2_{2,\mu^2}$.

It can be shown that $\Pi_{i=1}^n A^\eta(f_i)$ is the limit of

$$\int_{\Gamma\times\Gamma} \sum_{k=1}^\infty \beta_k^\varepsilon u_k^\varepsilon(x)\overline{v}_k^\varepsilon(y) K(x,y)|U_x\eta\rangle\langle U_y\eta| d\mu^2 \tag{28}$$

in the topology induced by the trace, as $\varepsilon \searrow 0$. But this is not particularly useful, since we can't interchange the integral and the sum (since J^ε is not trace class). If we could, the kernel in each summand would take the same form as that of a product of a pair. So we perform a second approximation before applying Šimša's result.

For $\mathcal{B}$ any orthonormal basis of $L^2_\mu(\Gamma)$, let P_{V_m} be the projection onto the span of the first m vectors in $\mathcal{B}$. Let $J^{\varepsilon,m} = J^\varepsilon P_{V_m}$. $J^{\varepsilon,m}$ is trace class, since both J^ε and P_{V_m} are Hilbert-Schmidt. By Lemma 1,

$$[J^{\varepsilon,m}\varphi](x) = \int_\Gamma B^\varepsilon(x,y)[P_{V_m}\varphi](y)d\mu(y) = \int_\Gamma [P_{\overline{V}_m} B^\varepsilon(x,\cdot)](y)\varphi(y)d\mu(y) \tag{29}$$

Now using Theorem 5 (Šimša) we obtain

$$B^{\varepsilon,m}(x,y) \equiv [P_{\overline{V}_m} B^{\varepsilon,m}(x,\cdot)](y) = \sum_{k=1}^\infty \beta_k^{\varepsilon,m} u_k^{\varepsilon,m}(x)\overline{v}_k^{\varepsilon,m}(y) \tag{30}$$

for orthonormal systems $\{u_k^{\varepsilon,m}\}$ and $\{v_k^{\varepsilon,m}\}$, and nonnegative, non-increasing $\beta_k^{\varepsilon,m}$ as $k\to\infty$, with $\sum_{k=1}^\infty(\beta_k^{\varepsilon,m})^2 = ||B^{\varepsilon,m}||^2_{2,\mu^2}$. Note that since $J^{\varepsilon,m}$ is trace class, $\sum_{k=1}^\infty \beta_k^{\varepsilon,m} < \infty$.

Observe that for all $x \in \Gamma$, $B^{\varepsilon,m}(x,\cdot)$ converges to $B^\varepsilon(x,\cdot)$ in $L^2_{\mu^2}(\Gamma)$ as $m\to\infty$. From this we show that $B^{\varepsilon,m}$ converges to B^ε in $L^2_\nu(\Gamma\times\Gamma)$ as $m\to\infty$. Fix ε.

$$\begin{aligned} ||B^{\varepsilon,m} - B^\varepsilon||^2_\nu &= \textstyle\int_{\Gamma\times\Gamma} |B^{\varepsilon,m}(x,y) - B^\varepsilon(x,y)|^2 |\langle U_x\eta|U_y\eta\rangle|^2 d\mu^2 \\ &\le \textstyle\int_\Gamma ||B^{\varepsilon,m}(x,\cdot) - B^\varepsilon(x,\cdot)||^2 d\mu(x) \\ &= \textstyle\sum_{l=1}^\infty \int_{D_l} ||B^{\varepsilon,m}(x,\cdot) - B^\varepsilon(x,\cdot)||^2 d\mu(x), \end{aligned} \tag{31}$$

where the D_l are compact and partition Γ, $\mu(D_l) < \infty$, $l \in \mathbb{N}$. For each l, m,

$$\int_{D_l} ||B^{\varepsilon,m}(x,\cdot) - B^\varepsilon(x,\cdot)||^2 d\mu(x) \tag{32}$$

is nonnegative, and for each l decreasing to 0 as $m \to \infty$. Let $\gamma > 0$. For each l, choose m_l such that

$$\int_{D_l} ||B^{\varepsilon,m_l}(x,\cdot) - B^{\varepsilon}(x,\cdot)||^2 d\mu(x) < \frac{\gamma}{2^l \mu(D_l)}. \tag{33}$$

By the Monotone Convergence Theorem,

$$\begin{aligned} &\lim_{m\to\infty} \sum_{l=1}^{\infty} \int_{D_l} ||B^{\varepsilon,m}(x,\cdot) - B^{\varepsilon}(x,\cdot)||^2 d\mu(x) \\ &= \sum_{l=1}^{\infty} \lim_{m\to\infty} \int_{D_l} ||B^{\varepsilon,m}(x,\cdot) - B^{\varepsilon}(x,\cdot)||^2 d\mu(x) \\ &\leq \sum_{l=1}^{\infty} \int_{D_l} ||B^{\varepsilon,m_l}(x,\cdot) - B^{\varepsilon}(x,\cdot)||^2 d\mu(x) \\ &\leq \sum_{l=1}^{\infty} \frac{\gamma}{2^l \mu(D_l)} \mu(D_l) \\ &= \gamma, \end{aligned} \tag{34}$$

whereby $B^{\varepsilon,m}$ converges to B^{ε} in $L^2_\nu(\Gamma \times \Gamma)$ as $m \to \infty$.

In a similar vein, we show that B^{ε} converges to B_0 in $L^2_\nu(\Gamma \times \Gamma)$ as $\varepsilon \to 0$. Let $R_\varepsilon \equiv \{(x,y) \in \Gamma \times \Gamma ||K(x,y)| < \varepsilon\}$, and $\gamma_\varepsilon(x) \equiv \int_\Gamma f_n(y)^2 \chi_{R_\varepsilon}(x,y) d\mu(y)$. Note that on R_ε, $K_\varepsilon(x,y) = \dfrac{\varepsilon K(x,y)}{|K(x,y)|}$, $|K_\varepsilon(x,y)| = \varepsilon$, and

$$\begin{aligned} \Big|\frac{1}{K_\varepsilon(x,y)} - \frac{1}{K(x,y)}\Big|^2 &= \frac{|K(x,y) - K_\varepsilon(x,y)|^2}{|K(x,y)|^2 \varepsilon^2} \\ &= \frac{\Big|K(x,y) - \dfrac{\varepsilon K(x,y)}{K(x,y)}\Big|^2}{|K(x,y)|^2 \varepsilon^2} \\ &= \frac{(K(x,y) - \varepsilon)^2}{|K(x,y)|^2 \varepsilon^2}. \end{aligned} \tag{35}$$

From this, we obtain

$$\begin{aligned} &||B^{\varepsilon} - B_0||_\nu^2 = \int_{\Gamma\times\Gamma} |B^{\varepsilon}(x,y) - B_0(x,y)|^2 |K(x,y)|^2 d\mu^2 \\ &= \int_{R_\varepsilon} f_1(x)^2 f_n(y)^2 |\langle U_x \eta | \Pi_{i=2}^{n-1} A^\eta(f_i) | U_y \eta\rangle|^2 \frac{(|K(x,y)| - \varepsilon)^2}{|K(x,y)|^2 \varepsilon^2} |K(x,y)|^2 d\mu^2 \\ &= \int_{R_\varepsilon} f_1(x)^2 f_n(y)^2 |\langle U_x \eta | \Pi_{i=2}^{n-1} A^\eta(f_i) | U_y \eta\rangle|^2 \frac{(|K(x,y)| - \varepsilon)^2}{\varepsilon^2} d\mu^2 \\ &\leq \int_{R_\varepsilon} f_1(x)^2 f_n(y)^2 d\mu^2 \\ &= \int_\Gamma f_1(x)^2 \int_\Gamma f_n(y)^2 \chi_{R_\varepsilon}(x,y) d\mu(y) d\mu(x) \\ &= \int_\Gamma f_1(x) \gamma_\varepsilon(x) d\mu(x). \end{aligned} \tag{36}$$

Note that for almost all x, by the assumption that $\langle U(\sigma(x))\eta|\eta\rangle \neq 0$ μ-a.e., $\gamma_\varepsilon(x) \searrow 0$ as $\varepsilon \searrow 0$, whereby $f_1(x)\gamma_\varepsilon(x) \searrow 0$ as $\varepsilon \searrow 0$ as well. By the Monotone Convergence Theorem,

$$\int_\Gamma f_1(x)\gamma_\varepsilon(x) d\mu(x) \to 0 \tag{37}$$

as $\varepsilon \searrow 0$, which establishes the convergence of B^{ε} to B_0 in $L^2_\nu(\Gamma \times \Gamma)$ as $\varepsilon \to 0$.

Now fix $\lambda > 0$. Find ε_λ such that for all $\varepsilon < \varepsilon_\lambda$, $||B^{\varepsilon} - B_0||_\nu < \dfrac{\lambda}{2}$. For each ε_λ, find m_λ such that $||B^{\varepsilon_\lambda,m_\lambda} - B^{\varepsilon_\lambda}||_\nu < \dfrac{\lambda}{2}$. For each λ, let $B_\lambda \equiv B^{\varepsilon_\lambda,m_\lambda}$. By

Theorem 5 of Šimša,

$$B_\lambda(x,y)=\sum_{k=1}^{\infty}\beta_{k,\lambda}u_{k,\lambda}(x)\overline{v}_{k,\lambda}(y). \tag{38}$$

By Lemma 2, since B_λ converges to B_0 in $L^2_\nu(\Gamma\times\Gamma)$ by construction, we have a sequence of operators A_λ converging to $\Pi_{i=1}^{n}A^\eta(f_i)$ (see equation 18) in the topology induced by the trace, where

$$\begin{aligned} A_\lambda &= \int_{\Gamma\times\Gamma} B_\lambda(x,y)K(x,y)|U_x\eta\rangle\langle U_y\eta|d\mu^2 \\ &= \int_{\Gamma\times\Gamma}\sum_{k=1}^{\infty}\beta_{k,\lambda}u_{k,\lambda}(x)\overline{v}_{k,\lambda}(y)K(x,y)|U_x\eta\rangle\langle U_y\eta|d\mu^2 \\ &= \sum_{k=1}^{\infty}\beta_{k,\lambda}\int_{\Gamma\times\Gamma}u_{k,\lambda}(x)\overline{v}_{k,\lambda}(y)K(x,y)|U_x\eta\rangle\langle U_y\eta|d\mu^2 \\ &= \sum_{k=1}^{\infty}\beta_{k,\lambda}A^\eta(u_{k,\lambda})A^\eta(\overline{v}_{k,\lambda}), \end{aligned} \tag{39}$$

which completes the proof. □

5 Informational Completeness of the set of Products of Pairs of Phase-Space Fuzzy Localization Operators

Following immediately from Theorem 6 of the last section, we have the following lemma.

Lemma 3 *If η is α-admissable, $||\eta||=1$, and $\langle U(\sigma(x))\eta,\eta\rangle\neq 0$ μ-a.e., the span of $\{A^\eta(f)A^\eta(g)\,|\,0\leq f,g\leq 1\,\mu\text{-a.e.},\ f,g\in L^1(\Gamma)\cap L^\infty(\Gamma)\}$ is dense in the C^*-algebra, $\mathcal{F}$ generated by $\mathcal{E}$ (the effect algebra), in the topology induced by the trace.*

Thus, we arrive at the highly reduced informationally complete set by applying Theorem 3 to obtain

Theorem 7 *If η is α-admissable, $||\eta||=1$, and $\langle U(\sigma(x))\eta,\eta\rangle\neq 0$ μ-a.e., then*

$$\begin{aligned}\{\mathrm{Re}\{A^\eta(f)A^\eta(g)\},\ \mathrm{Im}\{A^\eta(f)A^\eta(g)\}\ |\ & 0\leq f,g\leq 1\,\mu\text{-a.e.},\\ & f,g\in L^1(\Gamma)\cap L^\infty(\Gamma)\}\end{aligned} \tag{40}$$

is informationally complete.

Note that we take the real and imaginary parts of $A^\eta(f)A^\eta(g)$ to ensure that they are self-adjoint (as required in the definition of informational completeness, and in order to obtain real values for the trace calculations), but that the original products $A^\eta(f)A^\eta(g)$ are simply recovered as the sum:

$$\begin{aligned} A^\eta(f)A^\eta(g) &= \mathrm{Re}\{A^\eta(f)A^\eta(g)\}+i\mathrm{Im}\{A^\eta(f)A^\eta(g)\} \\ &= \tfrac{1}{2}\{A^\eta(f),A^\eta(g)\}+\tfrac{i}{2}[A^\eta(f),A^\eta(g)], \end{aligned} \tag{41}$$

where $\{A,B\}\equiv AB+BA$.

6 Conclusion

We have shown that the set of products of pairs of phase space fuzzy localization operators $A^\eta(f)$ given by

$$\{\mathrm{Re}\{A^\eta(f)A^\eta(g)\},\ \mathrm{Im}\{A^\eta(f)A^\eta(g)\} \mid \begin{array}{l} 0 \le f,g \le 1\,\mu\text{–a.e.}, \\ f,g \in L^1(\Gamma)\cap L^\infty(\Gamma)\}, \end{array} \tag{42}$$

is informationally complete, independent of the particular phase space representation chosen. They generate a C*-algebra, which is foundational to the C*-algebraic formalism for physics in the free case [5]. We can generalize this to any physical system for which these phase space fuzzy localization operators can be constructed, to obtain a C*-algebra and the informational completeness of the set of products of pairs obtained from these operators.

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