Quantum mechanics on phase space and teleportation

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Abstract The formalism of quantum mechanics on phase space is used to describe the standard protocol of quantum teleportation with continuous variables in order to partially investigate the interplay between this formalism and quantum information. Instead of the Wigner quasi-probability distributions used in the standard protocol, we use positive definite true probability densities which account for unsharp measurements through a proper wave function representing a non-ideal quantum measuring device. This is based on a result of Schroeck and may be taken on any relativistic or non-relativistic phase space. The obtained formula is similar to a known formula in quantum optics but contains the effect of the measuring device. It has been applied in three cases. In the first case, the two measuring devices, corresponding to the two entangled parts shared by Alice and Bob, are not entangled and described by two identical Gaussian wave functions with respect to the Heisenberg group. They lead to a probability density identical to the Q function which is analyzed and compared with the Wigner formalism. A new expression of the teleportation fidelity F for a coherent state in terms of the quadrature variances is obtained. In the second case, these two measuring devices are entangled in a two-mode squeezed vacuum state. In the third case, two Gaussian states are combined in an entangled squeezed state. The overall observation is that the state of the measuring devices shared by Alice and Bob influences the fidelity of teleportation through their unsharpness and entanglement.

Keywords Quantum mechanics on phase space · positive operator valued measures · phase space representation · quantum teleportation

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1 Introduction

Quantum theory succeeded in describing a large number of phenomena exhibiting a non-classical behavior. Nevertheless, as noticed by many researchers, it suffers from some inconsistencies. A new formulation of quantum mechanics on phase space [1], also called stochastic quantum mechanics [2], aims at solving many of these paradoxes and problems [3]. It uses Positive Operator Valued (POV) measures to describe the observables of a physical system, instead of the usual Projective Valued (PV) measures associated with some self-adjoint operator via the spectral decomposition theorem. In fact, it turns out that the PV measures formalism constitutes a special case of the more general POV measures formalism. The POV measures are encountered in many areas of research, both in the foundations of quantum theory dealing with the quantum measurement problem [4], and in the theoretical analysis of experiments dealing with individual quantum objects such as atoms and photons[5]. They allow for a generalized notion of properties of a system that account for both sharp and unsharp properties, and are useful in the description of joint measurement of complementary observables, such as the position and momentum of a particle or the quadratures of an electromagnetic field mode, by means of unsharp joint observables called phase space observables.

One problem solved by stochastic quantum mechanics is the nonexistence of probability distributions on phase space. For instance, the Wigner quasi-probability distribution is not positive definite for all quantum states, but only for Gaussian states [6]. In fact, stochastic quantum mechanics is formulated using wave functions defined on the phase space, so that the positive definiteness of the corresponding probability densities is guaranteed from the outset. To build such functions in the spinless non-relativistic case, one uses a unitary transform $\mathcal{W}^{\eta} : L^2(\mathbb{R}^n_{\mathrm{Conf}}) \to L^2(\mathbf{\Gamma})$, which connects wave functions $\psi(x)$ in the configuration space $L^2(\mathbb{R}^n_{\mathrm{Conf}})$ with functions $\Psi(q,p)$ belonging to a subspace $\mathcal{W}^{\eta} L^2(\mathbf{\Gamma}) \subset L^2(\mathbf{\Gamma})$, where $\mathbf{\Gamma}$ is the usual phase space. The transform W^{η} is determined by the wave function η which describes the measuring instrument [1]. The subspace $\mathcal{W}^{\eta} L^2(\mathbf{\Gamma})$ is constructed so as to obtain an irreducible representation of the symmetry group of the system under consideration. In the general case of non-relativistic spinless particles, that symmetry corresponds to the Galilei group $\mathcal{G}_{10}$.

The present work aims at investigating, in the particular case of quantum teleportation with continuous variables, the interplay between quantum information theory and the (stochastic) phase space representation of quantum mechanics. In fact, on the one hand, the latter claims to be consistent and fully quantum in the sense of using a quantum description of non-ideal measuring devices. On the other hand, quantum information is expected to shed some light on the foundations of quantum mechanics through experiments which use fewer and fewer particles. Hence, at some stage, consideration of the quantum nature of the measuring device will be necessary and its effect on quantum information processes will have to be studied. In this context, we shall present a theoretical study of quantum teleportation in a phase space representation of the one-dimensional Galilei group $\mathcal{G}(1,1)$.

Quantum teleportation was discovered by Bennett *et al* [7] for quantum states (qubits) in Hilbert spaces of finite dimensions. It consists in the transfer of an unknown quantum state between two separate stations, Alice and Bob, using a classical channel and an entangled state shared between the two parties. The teleportation protocol was then reformulated for systems with continuous variables (position and momentum) by Vaidman [8], using maximally entangled states originally introduced by Einstein, Podolsky and Rosen [9]. Later, Braunstein and Kimble [10] proposed to extend the protocol using entangled states with a finite degree of correlation, taking as continuous variables the quadratures of the electromagnetic field. In the latter case, the entangled states are created unconditionally [11] using a two-mode squeezed state. The standard protocol for this teleportation is based on the description of quantum states in phase space by their Wigner functions. It is this standard protocol that will be investigated in the stochastic phase space representation by using its probability density rather than the Wigner function.

The paper is organized as follows : In section 2, we will review the main results of quantum mechanics on phase space in a general symmetry group setting. This will define our notations and will be useful in the calculation and physical interpretation of the probability distributions for the different states involved in the teleportation protocol. Section 3 is mainly devoted to the reformulation of the teleportation protocol in the stochastic phase space, by replacing the Wigner functions with the stochastic phase space probability distributions in the case of one dimension Galilei symmetry $\mathcal{G}(1,1)$. We derive the teleportation and Fidelity formulas. In section 4, we study a simple case where the measurement devices shared by Alice and Bob are not entangled. Moreover, they are described by identical Gaussian wave functions yielding a stochastic probability density which is identical to the Husimi Q function. Comparison of our results with those obtained in the Wigner representation is carried out and the effect of the measurement unsharpness on the fidelity is analyzed. In section 5, we consider the case where these two measurement devices are in a two-mode squeezed vacuum state. The effect of this entanglement is studied, particularly maximal entanglement recovers the Wigner representation results. In section 7, we briefly discuss the fidelity in the case where the two measuring devices are in an entangled state constructed from a squeezed state and an anti-squeezed state. This case contains the two preceding ones and enables comparison between the effect of apparatuses entanglement against their unsharpness. We recapitulate the main results in the conclusion.

2 Probability distributions in stochastic quantum mechanics

We will concentrate on teleportation with continuous observables. This is crucial as the theory of quantum mechanics on phase space fits well with this teleportation scheme as we consider the teleportation of position and momentum which are just continuous observables. We will take $\hbar = \frac{1}{2}$ and unit mass, this will enable easy comparison with the standard scheme of quantum teleportation with continuous variables. Given a locally compact symmetry group G with a finite-dimensional Lie algebra, the phase space can be defined as a coset space G/H with respect to a closed subgroup H which is to be determined according to the procedure outlined in [3] and detailed in [1]. For the particular case of G being the Poincarï¿œ or Galilei group, this phase space may be relativistic or non-relativistic with points (q,p,s), where q and p are the position and momentum, and s stands for the spin. The phase space representation can be obtained from the configuration or momentum representations by a unitary transformation $\mathcal{W}^{\eta}$ [3,2]:

$$\left[\mathcal{W}^{\eta}\psi\right](q,p,s) = \Psi(q,p,s) = \int_X \eta^{*}_{q,p,s}(y)\,\psi(y)\,\mathrm{d}\nu(y), \tag{1}$$

where X is the configuration $(y = x)$ or momentum space $(y = k)$ with invariant measure ν, and

$$\eta_{q,p,s}(y) = \left[\mathbf{U}_{q,p,s}\eta\right](y). \tag{2}$$

$\mathbf{U}_{q,p,s}$ is the configuration or momentum representation of a translation q, a boost p, and a rotation through an s vector of the sphere S^2. The function η represents the measuring apparatus and has to satisfy the α-admissibility criterion which states that,

1. η must be admissible

$$\int_{G/H} \left|\langle \eta_{q,p,s}|\eta\rangle\right|^2 \mathrm{d}\mu(q,p,s) < \infty \tag{3}$$

   where μ is the invariant measure on the phase space G/H.
2. and there exists a mapping $\alpha : H \to \mathbb{C}$, such that for all $h \in H$, we have

$$\left[U(h)\eta\right](y) = \alpha(h)\eta(y) \tag{4}$$

This criterion insures that the transformation $\mathcal{W}^{\eta}$ is an isometry between the configuration or momentum representation and a subspace of $L^2(G/H)$.

The above considerations enable one to associate with every quantum state $\hat{\rho}$ a positive definite probability density $\rho^{\eta}(q,p,s)$ [3,2]

$$\rho^{\eta}(q,p,s) = \mathrm{Tr}\left(\hat{\rho}\,|\eta_{q,p,s}\rangle\,\langle\eta_{q,p,s}|\right), \tag{5}$$

which simplifies to

$$\rho^{\eta}(q,p,s) = |\Psi(q,p,s)|^2 \tag{6}$$

for pure states $\hat{\rho} = |\psi\rangle\,\langle\psi|$.

Now, we consider the non-relativistic spinless case in a one dimensional setting corresponding to the one-dimensional Galilei group $\mathcal{G}(1,1)$:

$$\mathcal{G}(1,1) = \{g = (b,a,v)\,|b,a,v \in \mathbb{R}\}, \tag{7}$$

where b and a represent translations in time and space, respectively, and v a velocity boost [12]. We have

$$\eta_{q,p}(x) = \left[\mathbf{U}_{q,p}\eta\right](x) = \exp\left[2ip(x-q)\right]\eta(x-q). \tag{8}$$

and

$$\|\eta\| = \langle \eta\,|\,\eta\rangle^{\frac{1}{2}} = (\pi)^{-\frac{1}{2}}, \tag{9}$$

The states $|\eta_{q,p}\rangle$ constitute an overcomplete family yielding a resolution of the identity [2]

$$\int_{\boldsymbol{\Gamma}} |\eta_{q,p}\rangle\,\langle\eta_{q,p}|\,\mathrm{d}q\mathrm{d}p = \mathbf{I}, \tag{10}$$

where $\mathbf{I}$ represents the identity in $L^2(\mathbb{R}_{\mathrm{Conf}})$, $L^2(\mathbb{R}_{\mathrm{Mom}})$, or $L^2(\mathbf{\Gamma}^\eta)$ according to the representation of $|\eta_{q,p}\rangle$. Then, we note that $\rho^\eta(q,p)$ is not a quasi-probability but a true physical probability density that a simultaneous measurement of position and momentum yields the stochastic (or unsharp) outcome $(q,p,\chi_{q,p})$. The confidence function $\chi_{q,p}(x,k)$ is the probability density that the real values be (x,k) when the outcome (q,p) is obtained, or inversely, that the outcome (q,p) be obtained when the real values are (x,k). This interpretation stems from the marginal relation in the configuration space

$$\rho^\eta(q) = \int_{\mathbb{R}} \mathrm{d}p \, |\Psi(q,p)|^2 = \int_{\mathbb{R}} \mathrm{d}x \chi_q(x) \, |\psi(x)|^2 , \tag{11}$$

$$\chi_q(x) = \pi \, |\eta(x-q)|^2 . \tag{12}$$

The confidence function $\chi_q(x)$ has the same interpretation as $\chi_{q,p}(x,k)$ so that its square root η represents the state of the quantum non-ideal measuring device. Hence, $\rho^\eta(q)$ is the probability density that a quantum measuring device yields the outcome q irrespective of the position x. Moreover, $\rho^\eta(q)$ tends to the usual probability density in the limit of sharp measurement with a perfect quantum apparatus ($\chi_q(x) \to \delta(x-q)$). Analogous relations and interpretations hold in the momentum representation with confidence function $\tilde{\chi}_p(k) = \pi \, |\tilde{\eta}(k-p)|^2$, where $\tilde{\eta}$ is the Fourier transform of η. Accordingly, the variables (q,p) and (x,k) are physically different. The former are the readouts of the measurement while the latter are statistically related to them through the χ distributions (they may be mean values for instance). This distinction will turn out to be important in discussing teleportation in a particularly simple instance where the functions η are Gaussian states with minimum uncertainties in position and momentum [2,3].

Again, working in the spinless case, an other important point is the calculation of expectation values and variances in determining the fidelity of teleportation. For a spinless particle, the operators acting in $W^\eta L^2(\mathbf{\Gamma})$ for a given η are defined by [3]

$$\mathbf{A}^\eta(f) = \int f(q,p) \, |\eta_{q,p}\rangle \langle \eta_{q,p}| \, \mathrm{d}q\mathrm{d}p, \tag{13}$$

where the real function $f(q,p)$ of the phase space variables represents a classical observable. We adopt the convention of Ref. [3], putting $\mathbf{A}^\eta(f) = \mathbf{Q}$ for $f=q$ and $\mathbf{A}^\eta(f) = \mathbf{P}$ for $f=p$, with $\mathbf{Q}$ and $\mathbf{P}$ being the position and momentum operators. The expectation value of the operator $\mathbf{A}^\eta(f)$, in the state $\hat{\rho}$, is defined by

$$\begin{aligned} \mathrm{Exp}_{\hat{\rho}} &= \mathrm{Tr}\left(\mathbf{A}^\eta(f)\,\hat{\rho}\right) \\ &= \int_{\mathbf{\Gamma}} f(q,p) \rho^\eta(q,p) \, \mathrm{d}q\mathrm{d}p. \end{aligned} \tag{14}$$

The second equality is the same as that used in the Wigner function formalism. However, the variance

$$\mathrm{Var}_{\hat{\rho}}\left(\mathbf{A}^\eta(f)\right) = \mathrm{Tr}\left(\left[\mathbf{A}^\eta(f)\right]^2 \hat{\rho}\right) - \left[\mathrm{Tr}\left(\mathbf{A}^\eta(f)\,\hat{\rho}\right)\right]^2 , \tag{15}$$

contains the squared operator whose expectation has an expression which is different from that of the Wigner formalism. In fact,

$$\begin{aligned} \mathrm{Tr}\left(\left[\mathbf{A}^\eta(f)\right]^2 \hat{\rho}\right) = \int_{\mathbf{\Gamma}} \int_{\mathbf{\Gamma}} & f(q,p) \, f(q',p') \left\langle \eta_{q',p'} \,\middle|\, \eta_{q,p} \right\rangle \\ & \times \rho^\eta(q,p;q',p') \, \mathrm{d}q\mathrm{d}p \, \mathrm{d}q'\mathrm{d}p' \end{aligned} \tag{16}$$

where

$$\rho^{\eta}\left(q,p;q',p'\right)=\left\langle \eta_{q,p}\right|\hat{\rho}\left|\eta_{q',p'}\right\rangle . \tag{17}$$

The term $\left\langle \eta_{q',p'} \mid \eta_{q,p}\right\rangle$ is a reproducing kernel for the Heisenberg group. It can be generalized to the spinning case and everything in relations (13)-(17) remains the same with the replacement $(q,p)\rightarrow(q,p,s)$.

3 Quantum teleportation in the quantum mechanics on phase space formalism

The standard scheme of teleportation [10] can be reformulated in terms of the probability distributions (5) for the spinless case, by simply replacing the Wigner function description of states by the new phase space representation. For an entangled bipartite system, with parts 1 and 2 shared by Alice and Bob, the mapping (1) leads to the following phase space representation

$$\Psi_{\mathrm{ent}}\left(q_1,p_1,q_2,p_2\right)=\int_{\mathbb{R}^2}\eta^{*}_{q_1,p_1,q_2,p_2}\left(x_1,x_2\right)\psi_{\mathrm{ent}}(x_1,x_2)\mathrm{d}x_1\mathrm{d}x_2, \tag{18}$$

where the corresponding bipartite measurement device, composed of apparatuses 1 and 2, is described by a function $\eta(x_1,x_2)\in L^2(\mathbb{R}^2)$ with norm $\|\eta\|=\pi^{-1}$. The displaced function

$$\begin{aligned}\eta_{q_1,p_1,q_2,p_2}\left(x_1,x_2\right)=&\exp\left[2\mathrm{i}\left(p_1x_1-p_1q_1\right)\right]\\&\times\exp\left[2\mathrm{i}\left(p_2x_2-p_2q_2\right)\right]\eta\left(x_1-q_1,x_2-q_2\right)\end{aligned} \tag{19}$$

is obtained through a tensor product transformation $\mathbf{U}_{q_1,p_1}\otimes\mathbf{U}_{q_2,p_2}$. The probability density associated with the whole system formed by the unknown state ρ_{in} to be teleported and the entangled state $\rho^{\eta}_{\mathrm{ent}}\left(q_1,p_1,q_2,p_2\right)=\left|\Psi_{\mathrm{ent}}\left(q_1,p_1,q_2,p_2\right)\right|^2$ can be written as

$$\rho^{\eta}_{\mathrm{tot}}\left(q_{\mathrm{in}},p_{\mathrm{in}},q_1,p_1,q_2,p_2\right)=\rho_{\mathrm{in}}\left(q_{\mathrm{in}},p_{\mathrm{in}}\right)\rho^{\eta}_{\mathrm{ent}}\left(q_1,p_1,q_2,p_2\right). \tag{20}$$

In accordance with [10], teleportation proceeds through the following steps.

(i) Alice performs the following change of variables

$$\begin{aligned}q_{\mathrm{u}}&=\tfrac{1}{\sqrt{2}}\left(q_{\mathrm{in}}-q_1\right)\quad,\quad p_{\mathrm{u}}=\tfrac{1}{\sqrt{2}}\left(p_{\mathrm{in}}-p_1\right),\\ q_{\mathrm{v}}&=\tfrac{1}{\sqrt{2}}\left(q_{\mathrm{in}}+q_1\right)\ \text{and}\ p_{\mathrm{v}}=\tfrac{1}{\sqrt{2}}\left(p_{\mathrm{in}}+p_1\right).\end{aligned} \tag{21}$$

Then, she measures the observables corresponding to q_{u} and p_{v} (in principle, an infinite number of times) in order to obtain a distribution $D(q_{\mathrm{u}},p_{\mathrm{v}})$. Knowing that a single measurement outcome is $(q_{\mathrm{u}},p_{\mathrm{v}})$, the total conditional probability density is

$$\rho^{\eta}_{\mathrm{tot}}\left(p_{\mathrm{u}},q_{\mathrm{v}},q_2,p_2|q_{\mathrm{u}},p_{\mathrm{v}}\right)=\frac{1}{D(q_u,p_v)}\rho^{\eta}_{\mathrm{tot}}\left(q_{\mathrm{in}},p_{\mathrm{in}},q_1,p_1,q_2,p_2\right), \tag{22}$$

where $(q_{\mathrm{in}},p_{\mathrm{in}},q_1,p_1)$ must be replaced by their expressions in terms of $(q_{\mathrm{u}},p_{\mathrm{u}},q_{\mathrm{v}},p_{\mathrm{v}})$. The corresponding probability density for system (2) is obtained by integrating over q_{v} and p_{u}

$$\rho^{\eta}_{\mathrm{Bob}}\left(q_2,p_2|q_{\mathrm{u}},p_{\mathrm{v}}\right)=\int_{\mathbb{R}^2}\mathrm{d}q_{\mathrm{v}}\mathrm{d}p_{\mathrm{u}}\rho^{\eta}_{\mathrm{tot}}\left(p_{\mathrm{u}},q_{\mathrm{v}},q_2,p_2|q_{\mathrm{u}},p_{\mathrm{v}}\right). \tag{23}$$

Using (21) along with the change of variable $q = \sqrt{2}q_{\mathrm{v}} - q_1 \equiv q_{\mathrm{in}}$ and $p = \sqrt{2}q_{\mathrm{v}} + p_1 \equiv p_{\mathrm{in}}$ so that $\mathrm{d}q_{\mathrm{v}}\mathrm{d}p_{\mathrm{u}} = 2\mathrm{d}q\mathrm{d}p$, this probability density becomes

$$\rho^{\eta}_{\mathrm{Bob}}\left(q_2, p_2|q_{\mathrm{u}}, p_{\mathrm{v}}\right) = \tfrac{2}{D(q_{\mathrm{u}},p_{\mathrm{v}})} \int_{\mathbb{R}^2} \mathrm{d}q\mathrm{d}p\rho^{\eta}_{\mathrm{ent}}\left(q - \sqrt{2}q_{\mathrm{u}}, \sqrt{2}p_{\mathrm{v}} - p, q_2, p_2\right) \rho_{\mathrm{in}}\left(q, p\right) \tag{24}$$

The probability density is obtained by averaging over all possible results $(q_{\mathrm{u}}, p_{\mathrm{v}})$,

$$\begin{aligned}\rho^{\eta}_{\mathrm{Bob}}\left(q_2, p_2\right) &= \int_{\mathbb{R}^2} \mathrm{d}q_{\mathrm{u}}\mathrm{d}p_{\mathrm{v}} D(q_{\mathrm{u}}, p_{\mathrm{v}})\rho^{\eta}_{\mathrm{Bob}}\left(q_2, p_2|q_{\mathrm{u}}, p_{\mathrm{v}}\right) \\ &= 2\int_{\mathbb{R}^2} \mathrm{d}q_{\mathrm{u}}\mathrm{d}p_{\mathrm{v}} \int_{\mathbb{R}^2} \mathrm{d}q_{\mathrm{u}}\mathrm{d}p_{\mathrm{v}}\rho_{\mathrm{in}}\left(q, p\right) \\ &\quad \rho^{\eta}_{\mathrm{ent}}\left(q - \sqrt{2}q_{\mathrm{u}}, \sqrt{2}p_{\mathrm{v}} - p, q_2, p_2\right). \end{aligned} \tag{25}$$

(ii) Alice communicates the results of the measurement to Bob via a classical channel.

(iii) Finally, Bob displacement leads to the following output variables

$$q_{\mathrm{out}} = q_2 + \sqrt{2}q_{\mathrm{u}} \tag{26}$$

$$p_{\mathrm{out}} = p_2 + \sqrt{2}p_{\mathrm{v}}. \tag{27}$$

Substituting in (24) and integrating with respect to q_{u} and p_{v}, we obtain the probability density describing an ensemble of teleported states

$$\begin{aligned}\rho^{\eta}_{\mathrm{tel}}\left(q_{\mathrm{out}}, p_{\mathrm{out}}\right) &= \int_{\mathbb{R}^2} \mathrm{d}q\mathrm{d}p\rho_{\mathrm{in}}\left(q, p\right) P\left(q - q_{out}, p - p_{out}\right) \\ &\equiv \left[\rho_{\mathrm{in}} \circ P\right]\left(q_{\mathrm{out}}, p_{\mathrm{out}}\right), \end{aligned} \tag{28}$$

where the positive function P

$$P\left(q_{\mathrm{out}} - q, p_{\mathrm{out}} - p\right) = 2\int_{\mathbb{R}^2} \mathrm{d}q_{\mathrm{u}}\mathrm{d}p_{\mathrm{v}} \rho^{\eta}_{\mathrm{ent}}\left(q - \sqrt{2}q_{\mathrm{u}}, \sqrt{2}p_{\mathrm{v}} - p, q_{\mathrm{out}} - \sqrt{2}q_{\mathrm{u}}, p_{\mathrm{out}} - \sqrt{2}p_{\mathrm{v}}\right), \tag{29}$$

depends on the measuring apparatus wave function η and the entangled state shared by Alice and Bob.

In order to determine the fidelity, we first rewrite the general relation (28) in an operator form. For this, we write the probability density in the following form

$$\rho_{\mathrm{in}}(q, p) = \mathrm{Tr}\left(\left[\mathbf{U}_{q,p}\mathbf{T}^{\eta}(q_{\mathrm{out}}, p_{\mathrm{out}})\mathbf{U}^{\dagger}_{q,p}\right]\left[\mathbf{U}_{q_{\mathrm{out}},p_{\mathrm{out}}}\hat{\rho}_{\mathrm{in}}\mathbf{U}^{\dagger}_{q_{\mathrm{out}},p_{\mathrm{out}}}\right]\right) \tag{30}$$

where we define the operator $\mathbf{T}^{\eta}$ according to the notation of [3]

$$\mathbf{T}^{\eta}\left(q_{\mathrm{out}}, p_{\mathrm{out}}\right) \equiv \left|\eta_{q_{\mathrm{out}},p_{\mathrm{out}}}\right\rangle \left\langle\eta_{q_{\mathrm{out}},p_{\mathrm{out}}}\right|. \tag{31}$$

We then obtain

$$\rho_{\rm in}(q,p) = {\rm Tr}\left[\mathbf{U}^{\dagger}_{q-q_{\rm out},p-p_{\rm out}}\hat{\rho}_{\rm in} \times \mathbf{U}_{q-q_{\rm out},p-p_{\rm out}}\mathbf{T}^{\eta}\left(q_{\rm out},p_{\rm out}\right)\right], \tag{32}$$

On the other hand, the left hand side of (28) can be written in a similar manner as

$$\rho^{\eta}_{\rm tel}\left(q_{\rm out},p_{\rm out}\right) = {\rm Tr}\left[\hat{\rho}_{\rm tel}\mathbf{T}^{\eta}\left(q_{\rm out},p_{\rm out}\right)\right]. \tag{33}$$

Here we have assumed that the state $\mathbf{T}^{\eta}$ of the measuring device for the input system has been chosen to be identical to that of the output system. This choice can be understood as we intend to measure the overlap between two states via their associated probability densities. It is then convenient to measure these densities in the same conditions and then with identical measuring devices.

Since the operators $\mathbf{T}^{\eta}$ are informationally complete (that is if $\hat{\rho}$ and $\hat{\rho}'$ are two states, then ${\rm Tr}\left[\hat{\rho}\mathbf{T}^{\eta}\right] = {\rm Tr}\left[\hat{\rho}'\mathbf{T}^{\eta}\right]$ if and only if $\hat{\rho} = \hat{\rho}'$) and $\rho_{\rm tel}(q_{\rm out},p_{\rm out}) = {\rm Tr}\left[\hat{\rho}_{\rm tel}\mathbf{T}^{\eta}(q_{\rm out},p_{\rm out})\right]$, relations (32) and (28) lead to

$$\hat{\rho}_{\rm tel} = \int_{\mathbb{R}^2} {\rm d}q{\rm d}p\mathbf{U}^{\dagger}_{(q-q_{\rm out}),(p-p_{\rm out})}\hat{\rho}_{\rm in}\mathbf{U}_{(q-q_{\rm out}),(p-p_{\rm out})} \times P\left(q-q_{out},p-p_{out}\right), \tag{34}$$

By making an appropriate change of variables, we finally obtain

$$\hat{\rho}_{\rm tel} = \int_{\mathbb{R}^2} {\rm d}q{\rm d}pP\left(q,p\right)\,\mathbf{U}^{\dagger}_{q,p}\hat{\rho}_{\rm in}\mathbf{U}_{q,p}. \tag{35}$$

A similar relation is known in quantum optics [13] with a positive P function which, in contrast to our relation, does not depend on the state of the quantum measuring device.

Now, the teleportation fidelity of a pure state $|\psi_{\rm in}\rangle$ is defined by [14]

$$F \equiv \langle\psi_{\rm in}|\,\hat{\rho}_{\rm tel}\,|\psi_{\rm in}\rangle\,. \tag{36}$$

Replacing $\hat{\rho}_{\rm tel}$ by its operator relation, we get

$$F = \int_{\mathbb{R}^2} {\rm d}q{\rm d}pP\left(q,p\right)\left|\langle\psi_{\rm in}|\,\mathbf{U}_{q,p}\,|\psi_{\rm in}\rangle\right|^2. \tag{37}$$

Note that the measure of the probability densities defined in 5 requires, in principle, an infinite number of trials as they represent the average number of times one gets the state $\hat{\rho}$ in the observable defined via the POV measure $\mathbf{T}^{\eta}$ in general. In particular, the probability density defined in 20 requires a triply infinite number of trials, as it represents a state in the product space $L^2\left(\Gamma_{in}\right)\otimes L^2\left(\Gamma_1\right)\otimes L^2\left(\Gamma_2\right)$.

All the steps of the present formulation of teleportation could have been done for a system of spinning particles with relativistic or non-relativistic symmetry. The symmetry groups and corresponding subgroups have been thoroughly studied in Ref.[1] and informationally complete functions η have been shown to exist. However, the entanglement resource $\hat{\rho}_{\rm ent}$ needs to be determined or suitably chosen, especially in the relativistic case.

In the following sections, we shall apply the above considerations in three situations using the same two mode squeezed vacuum state as an entanglement resource, as in the standard protocol [10]. Hence, the wave function of this entanglement resource is given by [11]

$$\psi_{\text{ent}}(x_1, x_2) = \sqrt{\frac{2}{\pi}} \exp\left[-\frac{e^{-2r}}{2}(x_1 + x_2)^2 - \frac{e^{2r}}{2}(x_1 - x_2)^2\right] \tag{38}$$

in the configuration representation, and will be the same in all cases. However, the states of the two apparatuses used in the measurement of this entangled state will change in each case. The measurement device corresponding to the system to be teleported will be non-entangled with these two apparatuses in all three cases.

4 Teleportation with non entangled apparatuses

In order to have a first impression of the physical consequence of teleportation in stochastic phase space, let us apply the general formula 28 when the two apparatuses used in the measurement of the entangled state are identical and not entangled. Then, their total wave function

$$\eta(x_1, x_2) = \eta(x_1)\,\eta(x_2), \tag{39}$$

corresponds to the unitary tensor product transformation $(\mathcal{W}^\eta \otimes \mathcal{W}^\eta)$. Moreover, the state of each apparatus is represented in the configuration space by the following function

$$\eta(x) = \left(\frac{1}{2\pi^3 l^2}\right)^{\frac{1}{4}} \exp\left(-\frac{x^2}{4l^2}\right). \tag{40}$$

This function is called optimal since the variances $\langle \Delta \mathbf{X}^2 \rangle_\eta = l^2$ and $\langle \Delta \mathbf{K}^2 \rangle_\eta = \frac{1}{16l^2}$, in the normalized state $\pi^{\frac{1}{2}}|\eta\rangle$, saturate the uncertainty relation

$$\Delta x \Delta k \geq \frac{1}{4}. \tag{41}$$

The confidence functions being

$$\chi_q(x) = \left(\frac{1}{2\pi l^2}\right)^{\frac{1}{2}} \exp\left(-\frac{(x-q)^2}{2l^2}\right), \tag{42}$$

$$\tilde{\chi}_p(k) = \left(\frac{8l^2}{\pi}\right)^{\frac{1}{2}} \exp\left(-8l^2(k-p)^2\right), \tag{43}$$

the length parameter l acquires the physical interpretation of accounting for the unsharpness of the measuring device in a statistical point of view. The sharp measurement limits in configuration or momentum spaces are obtained by the limits $\lim_{l\to 0}\chi_q(x) = \delta(x-q)$ or $\lim_{l\to\infty}\tilde{\chi}_p(k) = \delta(k-p)$ [2]. Obviously, these two limits cannot be taken both together.

Before proceeding with teleportation, let us note that in this particular instance, the phase space distribution can be directly related to the Wigner function [2]

$$\rho^\eta(q, p) = \int_{\mathbb{R}^2} dx dk\, \chi_{q,p}(x, k))W(x, k), \tag{44}$$

$$\chi_{q,p}(x, k) = \chi_q(x)\tilde{\chi}_p(k) = \frac{2}{\pi}\exp\left(-\frac{(x-q)^2}{2l^2} - 8l^2(k-p)^2\right), \tag{45}$$

and is identified with the Husimi Q function[3]. This shows that the Q function can be endowed with the physical interpretation of a true probability density rather than being a mere positive quasi-probability distribution [3]. In order to outline the steps of calculation, we shall not use (44) which holds for this particular case only. We instead use (40) and (19), to get

$$\eta_{q_1,p_1,q_2,p_2}(x_1,x_2) = \left(8\pi^3 l^2\right)^{-\frac{1}{2}} \exp\left[-\frac{1}{4l^2}(x_1-q_1)^2 - \mathrm{i}p_1(x_1-q_1)\right] \times \exp\left[-\frac{1}{4l^2}(x_2-q_2)^2 - \mathrm{i}p_2(x_2-q_2)\right]. \quad (46)$$

The probability density $\rho_{\text{ent}}^{\eta}(q_1,p_1,q_2,p_2)$ for the entangled state is obtained by use of the expressions (38), (18) and (46) in (6). After some calculation, we get

$$\rho_{\text{ent}}^{\eta}(q_1,p_1,q_2,p_2) = \frac{16}{\pi^2\sigma} \exp\left\{-\frac{1}{\sigma}\left(aq_1^2 + aq_2^2 + bq_1q_2 + \alpha p_1^2 + \alpha p_2^2 + \beta p_1 p_2\right)\right\}, \quad (47)$$

where the coefficients σ, a, b,α and β are functions of the two parameters r and l

$$\begin{aligned} \sigma &= 8\cosh(2r) + 16l^2 + \frac{1}{l^2}, \\ a &= \frac{2}{l^2}\cosh(2r) + 8, \\ b &= -\frac{4}{l^2}\sinh(2r), \\ \alpha &= 32l^2\cosh(2r) + 8, \\ \beta &= 64l^2\sinh(2r). \end{aligned} \quad (48)$$

Now, using the expression (47) of the probability density in (28), we get

$$\begin{aligned} \rho_{\text{tel}}^{\eta}(q_{\text{out}},p_{\text{out}}) &= \frac{1}{\pi\sqrt{\mu\nu}} \int \mathrm{d}q\mathrm{d}p\rho_{\text{in}}^{\eta}(q,p) \exp\left[-\frac{(q_{\text{out}}-q)^2}{\mu} - \frac{(p_{\text{out}}-p)^2}{\nu}\right] \\ &\equiv \left[\rho_{\text{in}}^{\eta} \circ G_{\mu\nu}\right](q_{\text{out}},p_{\text{out}}), \end{aligned} \quad (49)$$

where the function P has been denoted $G_{\mu\nu}$ since it is a Gaussian with the variances

$$\mu = \mathrm{e}^{-2r} + 4l^2, \quad (50)$$

$$\nu = \mathrm{e}^{-2r} + \frac{1}{4l^2}. \quad (51)$$

As for the teleportation based on Wigner functions, which yields[11]

$$W_{\text{tel}}(x_{\text{out}},k_{\text{out}}) = \left[W_{\text{in}} \circ G_\sigma\right](x_{\text{out}},k_{\text{out}}), \quad (52)$$

$$G_\sigma(x,k) = \frac{1}{\pi\sigma} \exp\left[\frac{-(x^2+k^2)}{\sigma}\right], \quad (53)$$

with variance $\sigma = e^{-2r}$, the teleported stochastic probability density is the convolution of the input probability density and a Gaussian $G_{\mu\nu}$. The difference is that the variances in position and momentum depend on the parameter l of the measuring device and are different in general, except for the value $l = \frac{1}{2}$. Also, the limit $r \to \infty$ does not lead to a perfect teleportation.

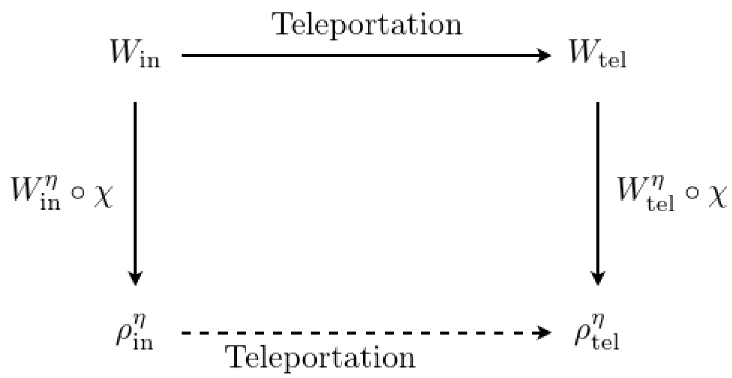
Now, let us stress again that the entangled state (47) could have been obtained directly from the Wigner function

$$W_{\mathrm{ent}}(x_1, p_1, x_2, p_2) = \frac{4}{\pi^2} \exp -e^{-2r} \left[(x_1 + x_2)^2 + (p_1 - p_2)^2 \right] \\ \times \exp -e^{2r} \left[(x_1 - x_2)^2 + (p_1 + p_2)^2 \right] \tag{54}$$

of the two mode squeezed state (38) . Following this line of reasoning, the stochastic teleported density would have been

$$\rho^{\eta}_{\mathrm{tel}}(q_{\mathrm{out}}, p_{\mathrm{out}}) = \left[W^{\eta}_{\mathrm{tel}} \circ \chi \right] (q_{\mathrm{out}}, p_{\mathrm{out}}) = \left[W_{\mathrm{in}} \circ G_{\sigma} \circ \chi \right] (q_{\mathrm{out}}, p_{\mathrm{out}}) \\ = \left[\rho^{\eta}_{\mathrm{in}} \circ G_{\sigma} \right] (q_{\mathrm{out}}, p_{\mathrm{out}}) \tag{55}$$

in apparent contradiction with the result (49). This shows that the following diagram is not commutative



This can be explained by the fact that, in the Wigner functions formalism, the measurement outcomes are supposed to be sharp values (x, k) while in the stochastic case the outcomes are the unsharp variables (q, p). These are totally different physical situations leading to different results as soon as a measurement comes into play such as in the teleportation process. Hence, when the measurement apparatus is considered to be ideal, the Wigner formalism has to be used. But when it is a non-ideal quantum device, the stochastic phase space formalism has to be applied.

In order to show that our results are really a generalization of the Wigner formalism approach to the case where a non-ideal quantum measurement device is considered into the formalism, we consider the sharp measurement limits. For this, we note that the output density (49) satisfies the following marginal properties

$$\int \rho^{\eta}_{\mathrm{tel}} (q_{\mathrm{out}}, p_{\mathrm{out}}) \, \mathrm{d}p_{\mathrm{out}} = \frac{1}{\sqrt{\pi \mu}} \int \mathrm{d}q \mathrm{d}p \rho^{\eta}_{\mathrm{in}} (q, p) \exp \left[-\frac{(q_{\mathrm{out}} - q)^2}{\mu} \right], \tag{56}$$

$$\int \rho^{\eta}_{\mathrm{tel}} (q_{\mathrm{out}}, p_{\mathrm{out}}) \, \mathrm{d}q_{\mathrm{out}} = \frac{1}{\sqrt{\pi \nu}} \int \mathrm{d}q \mathrm{d}p \rho^{\eta}_{\mathrm{in}} (q, p) \exp \left[-\frac{(p_{\mathrm{out}} - p)^2}{\nu} \right]. \tag{57}$$

Then, taking the limit $l \to 0$ in (56) and the limit $l \to \infty$ in (57), we obtain

$$\langle q_{\mathrm{out}} | \hat{\rho}_{\mathrm{tel}} | q_{\mathrm{out}} \rangle = \frac{1}{\sqrt{\pi} \mathrm{e}^{-r}} \int \mathrm{d}q \exp \left[-\frac{(q_{\mathrm{out}} - q)^2}{\mathrm{e}^{-2r}} \right] \langle q | \hat{\rho}_{\mathrm{in}} | q \rangle , \tag{58}$$

$$\langle p_{\mathrm{out}} | \hat{\rho}_{\mathrm{tel}} | p_{\mathrm{out}} \rangle = \frac{1}{\sqrt{\pi} \mathrm{e}^{-r}} \int \mathrm{d}p \exp \left[-\frac{(p_{\mathrm{out}} - p)^2}{\mathrm{e}^{-2r}} \right] \langle p | \hat{\rho}_{\mathrm{in}} | p \rangle . \tag{59}$$

These are the same marginal integrals that one would obtain using a Wigner function description of the teleportation protocol. However, since the functions η (after normalization) represent true wave functions satisfying the uncertainty principle (41), it is impossible to recover the Wigner function of the teleported system, from limits on the probability distribution $\rho^{\eta}_{\rm tel}(q_{\rm out}, p_{\rm out})$, without contradicting the uncertainty relation (41).

Now, we consider the fidelity and take as input state $|\psi_{\rm in}\rangle$ a coherent state defined via the complex amplitude $\beta_{\rm in} = q_{\beta_{\rm in}} + {\rm i}p_{\beta_{\rm in}}$. Noticing that the effect of a displacement in phase space $\mathbf{U}_{q,p}$ on a coherent state with a complex amplitude $\beta_{\rm in}$ leads to a coherent state with complex amplitude $\beta_{\rm in} + \alpha$, with $\alpha = q + {\rm i}p$, the fidelity of teleportation (37) for a coherent state becomes

$$F = \frac{1}{\sqrt{(\mu+1)(\nu+1)}}, \tag{60}$$

with μ and ν given by (50) and (51), respectively. The dependence of the fidelity with respect to the parameter l is illustrated in figure (1), for three values of the parameter r.

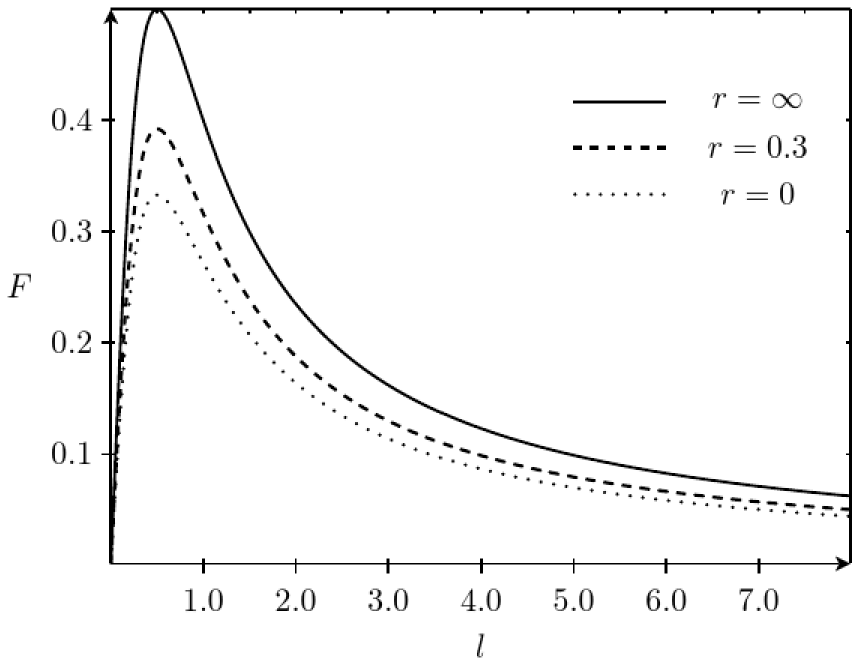


Fig. 1 The fidelity F given by (60) as a function of l for given values of the parameter r.

We note that the maximum fidelity for any value of r is obtained when $l = \frac{1}{2}$. This corresponds to $\Delta x = \Delta k = \frac{1}{2}$, i.e., to a tradeoff between the sharp measurement limits in configuration ($l \to 0$) and momentum ($l \to \infty$) spaces. In either limit, the fidelity falls down to zero confirming the incompatibility of perfect accuracy in either configuration or momentum representation with the phase space representation when optimal functions are chosen. The boundary $F = 0.5$ between classical and quantum domains of coherent state teleportation is the maximum value that can be obtained in the present case. This may be traced to the fact that the measurement apparatuses shared by Alice and Bob are not entangled.

Hence, not only perfect teleportation of a coherent state is impossible when a non-ideal quantum measurement device described by non-entangled optimal function is used, but it appears as a classical teleportation even when the resource is a maximally entangled state.

To account for these facts in relation to some recent experiments attaining a high fidelity, as far as we know, this fidelity has not been directly measured but deduced from an expression in terms of the experimentally accessible variances $\left\langle (\Delta\mathbf{Q}_{\text{out}})^2 \right\rangle$ and $\left\langle (\Delta\mathbf{P}_{\text{out}})^2 \right\rangle$ [15]. Our fidelity (60) can also be written as a function of the variances $\left\langle (\Delta\mathbf{Q}_{\text{out}})^2 \right\rangle$ and $\left\langle (\Delta\mathbf{P}_{\text{out}})^2 \right\rangle$ for the output state. Using the definitions (14) and (15), with a vacuum input, we find that

$$\left\langle (\Delta\mathbf{Q}_{\text{out}})^2 \right\rangle = \frac{1}{2}\left(\mu + \frac{1}{2}\right) = \frac{\mathrm{e}^{-2r}}{2} + 2l^2 + \frac{1}{4}, \tag{61}$$

$$\left\langle (\Delta\mathbf{P}_{\text{out}})^2 \right\rangle = \frac{1}{2}\left(\nu + \frac{1}{2}\right) = \frac{\mathrm{e}^{-2r}}{2} + \frac{1}{8l^2} + \frac{1}{4}. \tag{62}$$

The fidelity then reads

$$F = \frac{1}{2\sqrt{\left(\left\langle (\Delta\mathbf{Q}_{\text{out}})^2 \right\rangle + \frac{1}{4}\right)\left(\left\langle (\Delta\mathbf{P}_{\text{out}})^2 \right\rangle + \frac{1}{4}\right)}}. \tag{63}$$

In the above experiment, the measuring device is treated classically so that the usual formula has been used obtaining thereby a high fidelity such as in Ref.[15]. However, in an experiment with an non-ideal quantum measurement apparatus which can be suitably represented by the optimal function η, one should use our result (63) which equals 1 only when both variances take their minimal values $\left\langle (\Delta\mathbf{Q}_{\text{out}})^2 \right\rangle = \left\langle (\Delta\mathbf{P}_{\text{out}})^2 \right\rangle = \frac{1}{4}$. However, their expressions show that this is impossible since l must simultaneously equal 0 and ∞. Hence, using quantum non-ideal measuring instruments results in constraining these variances so that the fidelity cannot exceed one half. In the next sections, we shall bypass this issue by entangling the measuring apparatuses.

5 Teleportation with apparatuses in a two-mode squeezed vacuum state

Now, we study the case when the apparatuses shared by Alice and Bob, are in an entangled two mode squeezed vacuum state. The apparatus corresponding to the state to be teleported is not entangled with them. In this case, we have

$$\eta\left(x_1, x_2\right) = \sqrt{\frac{2}{\pi^3}}\exp\left(-\frac{e^{-2a}}{2}\left(x_1 + x_2\right)^2 - \frac{e^{2a}}{2}\left(x_1 - x_2\right)^2\right), \tag{64}$$

where a is the squeezing parameter. The variances in the normalized states $\pi\eta$ are the same for all variables

$$\left\langle \Delta\mathbf{X}_i^2 \right\rangle_\eta = \left\langle \Delta\mathbf{P}_i^2 \right\rangle_\eta = \frac{\cosh(2a)}{4}, \quad i = 1, 2, \tag{65}$$

so that the uncertainty principle is observed, but saturated only when the two apparatuses are not entangled ($a = 0$). These apparatuses can never yield sharp measurements since Δx_i and Δp_i never vanish. The parameter a accounts for unsharpness of the measurement and for entanglement. The displaced state

$$\eta_{q_1,p_1,q_2,p_2}\left(x_1,x_2\right) = \sqrt{\frac{2}{\pi^3}}\exp\left(2ip_1\left(x_1-q_1\right)+2ip_2\left(x_2-q_2\right)\right) \times \exp\left(-\frac{e^{-2a}}{2}\left(x_1-q_1+x_2-q_2\right)^2-\frac{e^{2a}}{2}\left(x_1-q_1-x_2+q_2\right)^2\right), \tag{66}$$

leads to the following probability density representing the entangled state (38)

$$\rho^{\eta}(q_1,p_1,q_2,p_2) = \frac{4}{\pi^2(e^{2r}+e^{2a})^2}\exp\left(-\frac{\alpha(q_1^2+p_1^2+q_2^2+p_2^2)+\beta(q_1q_2-p_1p_2)+\gamma}{e^{2r}+e^{2a}}\right), \tag{67}$$

$$\alpha = 1+e^{2(r+a)}, \tag{68}$$

$$\beta = 2\left(1-e^{2(r+a)}\right), \tag{69}$$

$$\gamma = -2\left(r+a\right)\left(e^{2r}+e^{2a}\right). \tag{70}$$

Then, the function $P\left(q_{\mathrm{out}}-q,p_{\mathrm{out}}-p\right)$ becomes a Gaussian

$$G_{\sigma_{r,a}}\left(q_{\mathrm{out}}-q,p_{\mathrm{out}}-p\right) = \frac{1}{\pi\lambda}\exp\left(-\frac{(q_{\mathrm{out}}-q)^2+(p_{\mathrm{out}}-p)^2}{\lambda}\right) \tag{71}$$

with variance

$$\lambda = e^{-2r}+e^{-2a}. \tag{72}$$

Teleportation of a coherent state can be realized with fidelity

$$F(r,a) = \frac{1}{1+e^{-2r}+e^{-2a}} = \frac{1}{1+\lambda}, \tag{73}$$

which is symmetric with respect to the squeezing parameters r and a. It is plotted in figure (2) and leads to the following cases:

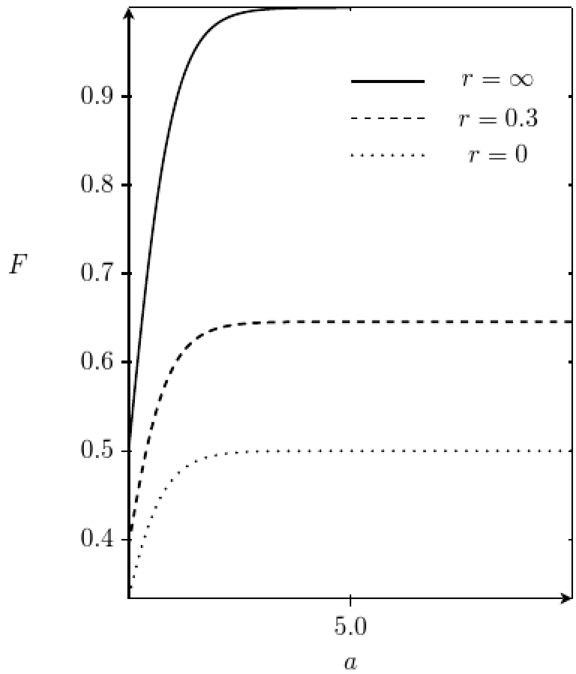


Fig. 2 The fidelity F given by (73) as a function of l for given values of the parameter r.

- When both the systems shared by Alice and Bob and the corresponding apparatuses are maximally entangled ($r = a = \infty$), perfect teleportation is realized $F = 1$.
- When only the apparatuses, or systems, are maximally entangled, the Wigner representation fidelity is recovered

$$\lim_{a\to\infty} F(r,a) = \frac{1}{1+e^{-2r}}, \ \lim_{r\to\infty} F(r,a) = \frac{1}{1+e^{-2a}}.$$

- When only the apparatuses, or systems, are non entangled, the fidelity

$$\lim_{a\to 0} F(r,a) = \frac{1}{2+e^{-2r}}, \ \lim_{r\to 0} F(r,a) = \frac{1}{2+e^{-2a}},$$

  satisfies $F \leq \frac{1}{2}$. The equality is satisfied when the apparatuses are not entangled ($a = 0$) and the systems are maximally entangled ($r \to \infty$) as in the previous section, or when the apparatuses are maximally entangled ($a \to \infty$) and the systems are not entangled ($r = 0$).
- When both the apparatuses and systems are partly entangled ($0 < a < \infty$ and $0 < r < \infty$), fidelity may be less, equal, or greater than $\frac{1}{2}$ depending on whether $a < \ln \frac{1}{\sqrt{(1-e^{-2r})}}$, $a = \ln \frac{1}{\sqrt{(1-e^{-2r})}}$, or $a > \ln \frac{1}{\sqrt{(1-e^{-2r})}}$, respectively. Equivalently, these cases correspond to $r < \ln \frac{1}{\sqrt{(1-e^{-2a})}}$, $r = \ln \frac{1}{\sqrt{(1-e^{-2a})}}$, or $r > \ln \frac{1}{\sqrt{(1-e^{-2a})}}$.
- When both the apparatuses and systems are non entangled, $F = \frac{1}{3}$.

6 Teleportation with apparatuses in a two-mode squeezed state

Let us now consider the shared apparatuses as being in the following state

$$\eta(x_1,x_2)=\left(\frac{1}{2\pi^3 l^2}\right)^{1/2}\exp\left[-\frac{e^{-2a}}{8l^2}(x_1+x_2)^2-\frac{e^{2a}}{8l^2}(x_1-x_2)^2\right], \tag{74}$$

which can be obtained by combining, via an $SU(2)$ transform, a squeezed state and an anti-squeezed state given by

$$\xi(x_i)=\left(\frac{e^{(-1)^i 2a}}{2\pi^2 l^2}\right)^{1/4}\exp\left[-e^{(-1)^i 2a}\frac{x_i^2}{4l^2}\right], \quad i=1,2. \tag{75}$$

The variances are

$$\left\langle \Delta\mathbf{X}_i^2\right\rangle_\eta=\cosh(2a)\frac{l^2}{2}\left\langle \Delta\mathbf{P}_i^2\right\rangle_\eta=\frac{\cosh(2a)}{32l^2}, \quad i=1,2, \tag{76}$$

so that the sharp position and momentum measurement limits correspond to $l\to 0$ and $l\to\infty$, respectively. The non-entangled limit $a\to 0$ corresponds to the first example treated in Sect.4 while $l=\frac{1}{2}$ corresponds to the second example treated in Sect.5.

Following the same steps as before, we obtain the following fidelity:

$$\mathrm{F}(r,a,l)=\frac{1}{\sqrt{\left(e^{-2r}+4l^2e^{-2a}+1\right)\left(e^{-2r}+\frac{e^{-2a}}{4l^2}+1\right)}}. \tag{77}$$

We observe that:

- Perfect teleportation is realized when both entanglement resources, systems and apparatuses, are maximally entangled ($r\to\infty$,$a\to\infty$).
- For maximal entanglement of the measurement apparatuses only ($a\to\infty$), the Wigner formalism result is recovered.
- For maximal entanglement of the systems ($r\to\infty$), the fidelity becomes

$$\mathrm{F}(a,l)=\frac{1}{\sqrt{\left(4l^2e^{-2a}+1\right)\left(\frac{e^{-2a}}{4l^2}+1\right)}}. \tag{78}$$

  Plotting against l for some values of a, as in figure (3), shows that when entanglement of the apparatuses is weak (small a), fidelity follows the same pattern as when they are not entangled, attaining its maximum at $l=\frac{1}{2}$ and falling down to zero in the limits $l\to 0,\infty$, as may be deduced analytically. This pattern is distorted gradually as a increases so that teleportation becomes gradually less sensitive to the value of l, i.e. to the measurement unsharpness.
- When the systems and apparatuses are both unentangled, the fidelity varies according to the bottom curve of figure (1) reaching a maximum value $\frac{1}{3}$ at $l=\frac{1}{2}$.

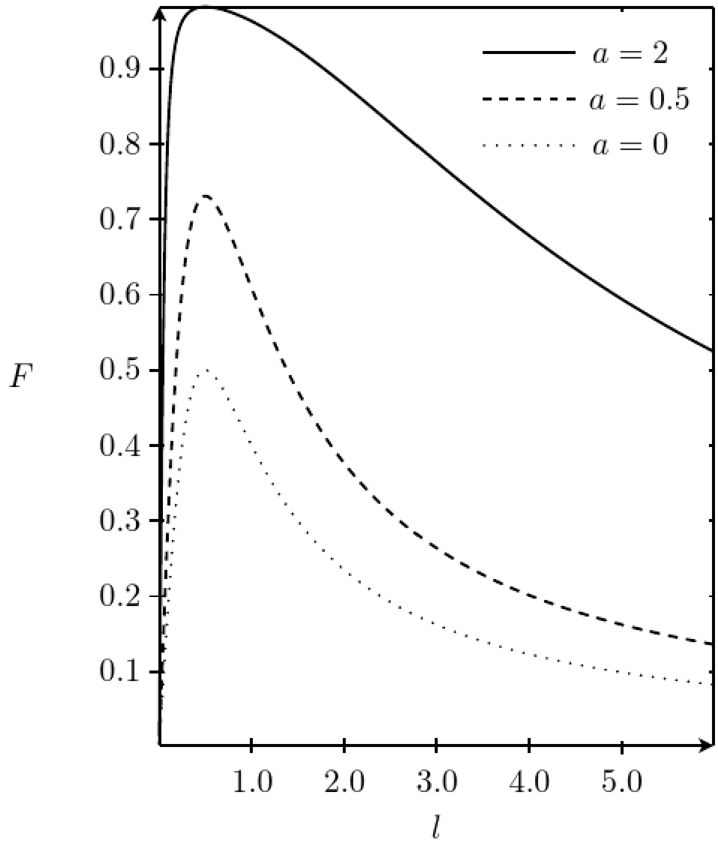


Fig. 3 Fidelity against l for $r \to \infty$ and $a = 0$, 0.5 and 2.

7 Conclusion

We have reformulated the continuous variables quantum teleportation in the formalism of quantum mechanics on phase space. For this, we followed the standard protocol of continuous variables teleportation and easily obtained a generalized formula which holds for any measuring device state η and any entangled state since the formalism provides true positive probability densities. One fundamental concept in this formalism are the informationally complete operators $\mathbf{T}^{\eta}$which provide the probability densities and which have been proven to exist in the spinning and relativistic cases enabling thereby a further generalization of our results. The main physical consequence of this work is that teleportation depends not only on the state of the physical systems shared by Alice and Bob, but on the state of quantum measurement devices associated with these systems as well.

To explicitly show this, we considered three cases where the same entangled state is shared by Alice and Bob, the two mode squeezed vacuum state and changed the state of the associated measurement devices. In the first case, these apparatuses were not entangled and were described by the optimal function η. The teleported state is a convolution which has the same form as that obtained with the Wigner functions. However, in our result the variances of the function $G_{\mu\nu}$ are different and contain the length parameter l which accounts for the unsharpness of the measuring instruments. Its appearance is mainly due to the fact that, when non-ideal quantum apparatuses are used, the steps of the teleportation protocol have to be applied to the stochastic probability density rather than to the Wigner functions.

We have shown that the limits $l \to 0$ and $l \to \infty$ recover the marginal results of the Wigner function formalism confirming the claim that quantum mechanics on (stochastic) phase space generalizes conventional quantum mechanics. Calculation of the teleportation fidelity for a coherent state led to a new expression in terms of the parameter l, and in terms of the experimentally accessible variances. The classical coherent state teleportation boundary $F = 0.5$ is an upper bound corresponding to the particular choice of the optimal function η with $l = \frac{1}{2}$. Expecting this fact to be due to the non-entanglement of the measuring apparatuses, we considered two situations where they are entangled.

We first considered a two-mode squeezed vacuum state described by a parameter a. This case clearly showed that the state of the apparatuses influences teleportation. The latter is perfect only when both the systems and apparatuses shared by Alice and Bob are maximally entangled. The Wigner formalism fidelity is recovered when only the apparatuses are maximally entangled. When the systems or apparatuses are not entangled, fidelity can be no higher than 0.5. When both the systems and apparatuses are partly entangled, fidelity can take any value depending on the relation between their squeezing parameters r and a, respectively. When both are not entangled, fidelity reaches a minimum $\frac{1}{3}$ which can be interpreted as the new classical limit, since no entanglement was used.

We then considered a two-mode squeezed state containing two parameters a and l. Even though a can account for both entanglement and unsharpness, the latter can be fine tuned with l. The values $l = \frac{1}{2}$ and $a = 0$, recover the preceding two cases. The main new result of this case is that unsharpness affects the fidelity of teleportation but this effect can be washed out by maximally entangling the apparatuses. In our opinion, this washout is due to our choice of the function η and may not be observed with another choice.

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