COSETS OF AFFINE VERTEX ALGEBRAS INSIDE LARGER STRUCTURES

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ABSTRACT. Given a finite-dimensional reductive Lie algebra $\mathfrak{g}$ equipped with a nondegenerate, invariant, symmetric bilinear form B, let $V_k(\mathfrak{g}, B)$ denote the universal affine vertex algebra associated to $\mathfrak{g}$ and B at level k. Let $\mathcal{A}_k$ be a vertex (super)algebra admitting a homomorphism $V_k(\mathfrak{g}, B) \to \mathcal{A}_k$. Under some technical conditions on $\mathcal{A}_k$, we characterize the commutant $\mathcal{C}_k = \text{Com}(V_k(\mathfrak{g}, B), \mathcal{A}_k)$ for generic values of k. We establish the strong finite generation of $\mathcal{C}_k$ in full generality in the following cases: $\mathcal{A}_k = V_k(\mathfrak{g}', B')$, $\mathcal{A}_k = V_{k-l}(\mathfrak{g}', B') \otimes \mathcal{F}$, and $\mathcal{A}_k = V_{k-l}(\mathfrak{g}', B') \otimes V_l(\mathfrak{g}'', B'')$. Here $\mathfrak{g}'$ and $\mathfrak{g}''$ are finite-dimensional Lie (super)algebras containing $\mathfrak{g}$, equipped with nondegenerate, invariant, (super)symmetric bilinear forms B' and B'' which extend B, $l \in \mathbb{C}$ is fixed, and $\mathcal{F}$ is a free field algebra admitting a homomorphism $V_l(\mathfrak{g}, B) \to \mathcal{F}$. Our approach is essentially constructive and leads to minimal strong finite generating sets for many interesting examples. As an application, we give a new proof of the rationality of the simple $N = 2$ superconformal algebra with $c = \frac{3k}{k+2}$ for all positive integers k.

1. INTRODUCTION

Vertex algebra are a fundamental class of algebraic structures that arose out of conformal field theory and have applications in a diverse range of subjects. The *coset* or *commutant* construction is a standard way to construct new vertex algebras from old ones. Given a vertex algebra $\mathcal{V}$ and a subalgebra $\mathcal{A} \subset \mathcal{V}$, $\text{Com}(\mathcal{A}, \mathcal{V})$ is the subalgebra of $\mathcal{V}$ which commutes with $\mathcal{A}$. This was introduced by Frenkel and Zhu in [FZ], generalizing earlier constructions in representation theory [KP] and physics [GKO], where it was used to construct the unitary discrete series representations of the Virasoro algebra. Many examples have been studied in both the physics and mathematics literature; for a partial list see [AP, B-H, BFH, DJX, DLY, HLY, JLI, JLII]. Although it is widely believed that $\text{Com}(\mathcal{A}, \mathcal{V})$ will inherit properties of $\mathcal{A}$ and $\mathcal{V}$ such as rationality, C_2-cofiniteness, and strong finite generation, no general results of this kind are known.

Many interesting vertex algebras are known or expected to have coset realizations. For example, given a simple, finite-dimensional Lie algebra $\mathfrak{g}$, let $L_k(\mathfrak{g})$ denote the rational affine vertex algebra of $\mathfrak{g}$ at positive integer level k. There is a diagonal map $L_{k+1}(\mathfrak{g}) \to L_k(\mathfrak{g}) \otimes L_1(\mathfrak{g})$, and a famous conjecture [BS] asserts that when $\mathfrak{g}$ is simply laced,

$$\text{Com}(L_{k+1}(\mathfrak{g}), L_k(\mathfrak{g}) \otimes L_1(\mathfrak{g})) \tag{1.1}$$

coincides with a simple rational $\mathcal{W}$-algebra of type $\mathfrak{g}$ given by the Drinfeld-Sokolov reduction associated to the principal embedding of $\mathfrak{sl}_2$ in $\mathfrak{g}$ [FF, FKW]. The rationality of

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this series of $\mathcal{W}$-algebras was recently established by Arakawa [Ara]. Another interesting family is the coset

$$\text{Com}(L_k(\mathfrak{gl}_n), L_{k-1}(\mathfrak{sl}_{n+1}) \otimes \mathcal{E}(n)). \tag{1.2}$$

In this notation, $L_k(\mathfrak{gl}_n) := \mathcal{H} \otimes L_k(\mathfrak{sl}_n)$ where $\mathcal{H}$ is the Heisenberg algebra, and $\mathcal{E}(n)$ is the rank n bc-system, which admits an action of $L_1(\mathfrak{sl}_n)$. This is conjectured [I] to be a rational super $\mathcal{W}$-algebra of $\mathfrak{sl}(n+1|n)$, which in the case $n=1$ coincides with the $N=2$ superconformal algebra.

We propose that in order to study such discrete series of cosets in a uniform manner, we should first consider the corresponding cosets of the *universal* affine vertex algebra $V_k(\mathfrak{g})$ of $\mathfrak{g}$ at level k. For example, to study the cosets (1.1) and (1.2) we should begin by studying $\text{Com}(V_{k+1}(\mathfrak{g}), V_k(\mathfrak{g}) \otimes L_1(\mathfrak{g}))$ and $\text{Com}(V_{k+1}(\mathfrak{gl}_n), V_{k-1}(\mathfrak{sl}_{n+1}) \otimes \mathcal{E}(n))$, respectively. These are expected to coincide with the universal $\mathcal{W}$-algebras of $\mathfrak{g}$ and $\mathfrak{sl}(n+1|n)$, respectively, for generic values of k. If we could find a strong generating set for the universal coset, it would in many cases descend to a strong generating set for the coset of interest. In general, having a strong generating set for a vertex algebra is an important step for studying its representation theory and establishing properties such as C_2-cofiniteness and rationality. A class of examples for which this approach has been fruitful is the *parafermion algebras*. Given a simple Lie algebra $\mathfrak{g}$ of rank d and a positive integer k, the parafermion algebra $N_k(\mathfrak{g})$ is the coset $\text{Com}(\mathcal{H}(d), L_k(\mathfrak{g}))$, where $\mathcal{H}(d)$ is the rank d Heisenberg algebra corresponding to the Cartan subalgebra of $\mathfrak{g}$. The C_2-cofiniteness of $N_k(\mathfrak{g})$ was recently proven in [ALY] and the rationality was proven in [DR], using results of [DWI, DWII, DWIII]. A key ingredient in proving these important theorems was the explicit strong generating set for the universal parafermion algebra of $\mathfrak{sl}_2$, namely $\text{Com}(\mathcal{H}(1), V_k(\mathfrak{sl}_2))$, which was achieved in [DLY, DLWY].

Let $\mathfrak{g}$ be a finite-dimensional, reductive Lie algebra (i.e., a sum of simple and abelian Lie algebras), equipped with a nondegenerate, symmetric, invariant bilinear form B. We denote by $V_k(\mathfrak{g}, B)$ the universal affine vertex algebra of $\mathfrak{g}$ at level k, relative to B. In this paper, we shall study cosets of $V_k(\mathfrak{g}, B)$ inside a general class of vertex algebras $\mathcal{A}_k$ whose structure constants depend continuously on k. The goal in studying

$$\mathcal{C}_k = \text{Com}(V_k(\mathfrak{g}, B), \mathcal{A}_k)$$

is to understand the behavior of $\text{Com}(L_k(\mathfrak{g}, B), \bar{\mathcal{A}}_k)$, where $L_k(\mathfrak{g}, B)$ denotes the simple quotient of $V_k(\mathfrak{g}, B)$ and $\bar{\mathcal{A}}_k$ is a quotient of $\mathcal{A}_k$. In particular, we are interested in special values of k for which $V_k(\mathfrak{g}, B)$ is reducible and $L_k(\mathfrak{g}, B)$ is rational or admissible. The main examples we have in mind are the following.

(1) $\mathcal{A}_k = V_k(\mathfrak{g}', B')$ where $\mathfrak{g}'$ is a finite-dimensional Lie (super)algebra containing $\mathfrak{g}$, and B' is a nondegenerate, invariant (super)symmetric bilinear form on $\mathfrak{g}'$ extending B.
(2) $\mathcal{A}_k = V_{k-l}(\mathfrak{g}', B') \otimes \mathcal{F}$ where $\mathcal{F}$ is a free field algebra admitting a homomorphism $V_l(\mathfrak{g}, B) \to \mathcal{F}$ for some fixed $l \in \mathbb{C}$ satisfying some mild restrictions. By a free field algebra, we mean any vertex algebra obtained as a tensor product of a Heisenberg algebra $\mathcal{H}(n)$, a free fermion algebra $\mathcal{F}(n)$, a $\beta\gamma$-system $\mathcal{S}(n)$ or a symplectic fermion algebra $\mathcal{A}(n)$.
(3) $\mathcal{A}_k = V_{k-l}(\mathfrak{g}', B') \otimes V_l(\mathfrak{g}'', B'')$. Here $\mathfrak{g}''$ is another finite-dimensional Lie (super)algebra containing $\mathfrak{g}$, equipped with a nondegenerate, invariant, (super)symmetric bilinear form B'' extending B. If $V_l(\mathfrak{g}'', B'')$ is not simple, we may replace $V_l(\mathfrak{g}'', B'')$ with its quotient by any ideal; of particular interest is $L_l(\mathfrak{g}'', B'')$.

In Section 6, we will prove in full generality that $\mathcal{C}_k$ is strongly finitely generated in cases (1) and (2) above for generic values of k, and in case (3) when $\mathcal{A}_k = V_{k-l}(\mathfrak{g}', B') \otimes V_l(\mathfrak{g}'', B'')$, and both k and l are generic. We will also prove this when $\mathcal{A}_k = V_{k-l}(\mathfrak{g}', B') \otimes L_l(\mathfrak{g}'', B'')$ for certain nongeneric values of l in some interesting examples. These are the first general results on the structure of cosets, and our proof is essentially constructive. The key ingredient is a notion of *deformable family* of vertex algebras that was introduced by the authors in [CLI]. A deformable family $\mathcal{B}$ is a vertex algebra defined over a certain ring of rational functions in a formal variable κ, and $\mathcal{B}_\infty = \lim_{\kappa\to\infty}$ has a well-defined vertex algebra structure. This notion fits into the framework of *vertex rings* introduced by Mason [M], and it is useful because a strong generating set for $\mathcal{B}_\infty$ gives rise to a strong generating set for $\mathcal{B}$ with the same cardinality. In the above examples, $\mathcal{C}_k$ is a quotient of a deformable family $\mathcal{C}$, and a strong generating set for $\mathcal{C}$ gives rise to a strong generating set for $\mathcal{C}_k$ for generic values of k. We will show that

$$\mathcal{C}_\infty = \lim_{k\to\infty} \mathrm{Com}(V_k(\mathfrak{g}, B), \mathcal{A}_k) \cong \mathcal{V}^G, \qquad \mathcal{V} = \mathrm{Com}\big(\lim_{k\to\infty} V_k(\mathfrak{g}, B), \lim_{k\to\infty} \mathcal{A}_k\big),$$

where G is a connected Lie group with Lie algebra $\mathfrak{g}$. Moreover, $\mathcal{V}$ is a tensor product of free field and affine vertex algebras and G preserves each tensor factor of $\mathcal{V}$. The description of $\mathcal{C}_k$ for generic values of k therefore boils down to a description of the orbifold $\mathcal{V}^G$. This is an easier problem because $\mathcal{V}$ decomposes into a direct sum of finite-dimensional G-modules, whereas $\mathcal{C}_k$ is generally not completely reducible as a $V_k(\mathfrak{g}, B)$-module.

Building on our previous work on orbifolds of free field and affine vertex algebras [LI, LII, LIII, LIV, CLII], we will prove in Sections 4 and 5 that for any vertex algebra $\mathcal{V}$ which is a tensor product of free field and affine vertex algebras and any reductive group $G \subset \mathrm{Aut}(\mathcal{V})$ preserving the tensor factors, $\mathcal{V}^G$ is strongly finitely generated. The proof depends on a classical theorem of Weyl (Theorem 2.5A of [W]), a result on infinite-dimensional dual reductive pairs (see Section 1 of [KR] as well as related results in [DLM, WaI, WaII]), and the structure and representation theory of the vertex algebras $\mathcal{B}^{\mathrm{Aut}(\mathcal{B})}$ for $\mathcal{B} = \mathcal{H}(n), \mathcal{F}(n), \mathcal{S}(n), \mathcal{A}(n)$.

In Section 7, we shall apply our general result to find minimal strong finite generating sets for $\mathcal{C}_k$ in some concrete examples which have been studied previously in the physics literature. In physics language, the tensor product of two copies of $\mathcal{C}_k$ is the symmetry algebra of a two-dimensional coset conformal field theory of a Wess-Zumino-Novikov-Witten model. Minimal strong generating sets for many examples of coset theories have been suggested in the physics literature; see especially [B-H], and we provide rigorous proofs of a number of these conjectures.

Let $\mathfrak{g}$ be simple, $\mathcal{A}_k$ a vertex algebra admitting a homomorphism $V_k(\mathfrak{g}) \to \mathcal{A}_k$, and $\mathcal{C}_k = \mathrm{Com}(V_k(\mathfrak{g}), \mathcal{A}_k)$ as above. Suppose that k is a parameter value for which $\mathcal{A}_k$ is not simple, and let $\mathcal{I}$ be the maximal proper ideal of $\mathcal{A}_k$, so that $\bar{\mathcal{A}}_k = \mathcal{A}_k/\mathcal{I}$ is simple. Let $\mathcal{J}$ denote the kernel of the map $V_k(\mathfrak{g}) \to \bar{\mathcal{A}}_k$, and suppose that $\mathcal{J}$ is maximal so that $V_k(\mathfrak{g})/\mathcal{J} = L_k(\mathfrak{g})$. Let $\bar{\mathcal{C}}_k = \mathrm{Com}(L_k(\mathfrak{g}), \bar{\mathcal{A}}_k)$ denote the corresponding coset. There is always a vertex algebra homomorphism

$$\pi_k : \mathcal{C}_k \to \bar{\mathcal{C}}_k,$$

which in general need not be surjective. In order to apply our results on the generic behavior of $\mathcal{C}_k$ to the structure of $\bar{\mathcal{C}}_k$, two problems must be solved. First, we need to find conditions for which π_k is surjective, since in this case a strong generating set for $\mathcal{C}_k$

descends to a strong generating set for $\bar{\mathcal{C}}_k$. Second, let S be a strong generating set for $\mathcal{C}_k$ for generic values of k, which corresponds to a strong generating set for $\lim_{k\to\infty}\mathcal{C}_k$. We call a value $k\in\mathbb{C}$ *nongeneric* if $\mathcal{C}_k$ is not strongly generated by S, and we need to find an algorithm for determining which values of k are generic.

In Section 8, we shall prove that π_k is surjective whenever $k+h^\vee$ is a positive real number; in particular, this holds when k is a positive integer. The problem of determining which values of k are generic is more difficult, and we will describe some examples where the nongeneric set can be determined explicitly. In these example, the nongeneric set consists of rational numbers and has compact closure, and all positive real values are generic. Although we do not prove it, we expect these qualitative features to hold in some generality. We will use this approach to give a new proof of the rationality of the simple $N=2$ superconformal algebra with $c=\frac{3k}{k+2}$ for all positive integers k. This algebra is realized as the coset of the Heisenberg algebra inside $L_k(\mathfrak{sl}_2)\otimes\mathcal{E}$, where $\mathcal{E}$ denotes the rank one bc-system. The rationality and regularity of these $N=2$ superconformal algebras were first established by Adamovic [AII], and in that paper the irreducible modules were classified and the fusion rules were computed. In the physics literature, these algebras are known as $N=2$ *superconformal unitary minimal models* [DPYZ]. They are famous in string theory as extended algebras of tensor products of $N=2$ superconformal unitary minimal models at total central charge $3d$ are the so-called Gepner models [G] of sigma models on d-dimensional compact Calabi-Yau manifolds.

Finally, we also mention that our work relates to another current problem in physics. The problem of finding minimal strong generators is presently of interest in the conjectured duality of families of two-dimensional conformal field theories with higher spin gravity on three-dimensional Anti-de-Sitter space. Strong generators of the symmetry algebra of the conformal field theory correspond to higher spin fields, where the conformal dimension becomes the spin. The original higher spin duality [GG] involves cosets $\text{Com}\,(V_k(\mathfrak{sl}_n),\mathcal{A}_k)$, where $\mathcal{A}_k=V_{k-1}(\mathfrak{sl}_n)\otimes L_1(\mathfrak{sl}_n)$. This case is discussed in Example 7.13. Example 7.16 is the algebra appearing in the $N=1$ superconformal version of the higher spin duality [CHRI], and Example 7.5 proves a conjecture of that article on the structure of $\text{Com}(V_k(\mathfrak{sp}_{2n}),V_{k-1/2}(\mathfrak{osp}(1|2n)\otimes\mathcal{S}(n))$. Example 7.11 is the symmetry algebra of the $N=2$ supersymmetric Kazama-Suzuki coset theory on complex projective space [KS]. This family of coset theories is the conjectured dual to the full $N=2$ higher spin supergravity [CHRII]. All these examples are important since they illustrate consistency of the higher spin/CFT conjecture on the level of strong generators of the symmetry algebra. Our results have recently been used in this direction in [FG]. The physics picture is actually that a two-parameter family of CFTs corresponds to higher spin gravity where the parameters relate to the 't Hooft coupling of the gravity. This idea is similar to our idea of a deformable family of vertex algebras.

2. VERTEX ALGEBRAS

We will assume that the reader is familiar with the basic notions in vertex algebra theory, which has been discussed from various points of view in the literature (see for example [B, FBZ, FLM, FHL, LZ, K]). We will follow the notation in [CLII]. Let $\mathfrak{g}$ be a finite-dimensional, Lie (super)algebra, equipped with a (super)symmetric, invariant bilinear form B. The *universal affine vertex (super)algebra* $V_k(\mathfrak{g},B)$ associated to $\mathfrak{g}$ and B is freely generated by elements X^ξ, $\xi \in \mathfrak{g}$, satisfying the operator product expansions

$$X^\xi(z)X^\eta(w) \sim kB(\xi,\eta)(z-w)^{-2} + X^{[\xi,\eta]}(w)(z-w)^{-1}.$$

The automorphism group $\text{Aut}(V_k(\mathfrak{g},B))$ is the same as $\text{Aut}(\mathfrak{g})$; each automorphism acts linearly on the generators X^ξ. If B is the standardly normalized supertrace in the adjoint representation of $\mathfrak{g}$, and B is nondegenerate, we denote $V_k(\mathfrak{g},B)$ by $V_k(\mathfrak{g})$. We recall the *Sugawara construction*, following [KRW]. If $\mathfrak{g}$ is simple and B is nondegenerate, we may choose dual bases $\{\xi\}$ and $\{\xi'\}$ of $\mathfrak{g}$, satisfying $B(\xi',\eta) = \delta_{\xi,\eta}$. The Casimir operator is $C_2 = \sum_\xi \xi\xi'$, and the dual Coxeter number $h^\vee$ with respect to B is one-half the eigenvalue of C_2 in the adjoint representation of $\mathfrak{g}$. If $k + h^\vee \neq 0$, there is a Virasoro element

$$L^{\mathfrak{g}} = \frac{1}{2(k+h^\vee)} \sum_\xi :X^\xi X^{\xi'}: \tag{2.1}$$

of central charge $c = \frac{k \cdot \text{sdim}\mathfrak{g}}{k+h^\vee}$. This is known as the *Sugawara conformal vector*, and each X^ξ is primary of weight one.

The *Heisenberg algebra* $\mathcal{H}(n)$ has even generators α^i, $i = 1,\dots,n$, satisfying

$$\alpha^i(z)\alpha^j(w) \sim \delta_{i,j}(z-w)^{-2}. \tag{2.2}$$

It has the Virasoro element $L^{\mathcal{H}} = \frac{1}{2}\sum_{i=1}^n :\alpha^i\alpha^i$ of central charge n, under which α^i is primary of weight one. The automorphism group $\text{Aut}(\mathcal{H}(n))$ is isomorphic to the orthogonal group $O(n)$ and acts linearly on the generators.

The *free fermion algebra* $\mathcal{F}(n)$ has odd generators ϕ_i, $i = 1,\dots,n$, satisfying

$$\phi^i(z)\phi^j(w) \sim \delta_{i,j}(z-w)^{-1}. \tag{2.3}$$

It has the Virasoro element $L^{\mathcal{F}} = -\frac{1}{2}\sum_{i=1}^n :\phi^i\partial\phi^i:$ of central charge $\frac{n}{2}$, under which ϕ^i is primary of weight $\frac{1}{2}$. We have $\text{Aut}(\mathcal{F}(n)) \cong O(n)$, and it acts linearly on the generators. Note that $\mathcal{F}(2n)$ is isomorphic to the bc-system $\mathcal{E}(n)$, which has odd generators b^i, c^i, $i = 1,\dots,n$, satisfying

$$\begin{aligned} &b^i(z)c^j(w) \sim \delta_{i,j}(z-w)^{-1}, \qquad &c^i(z)b^j(w) \sim \delta_{i,j}(z-w)^{-1}, \\ &b^i(z)b^j(w) \sim 0, \qquad &c^i(z)c^j(w) \sim 0. \end{aligned} \tag{2.4}$$

The *$\beta\gamma$-system* $\mathcal{S}(n)$ has even generators β^i, γ^i, $i = 1,\dots,n$, satisfying

$$\begin{aligned} &\beta^i(z)\gamma^j(w) \sim \delta_{i,j}(z-w)^{-1}, \qquad &\gamma^i(z)\beta^j(w) \sim -\delta_{i,j}(z-w)^{-1}, \\ &\beta^i(z)\beta^j(w) \sim 0, \qquad &\gamma^i(z)\gamma^j(w) \sim 0. \end{aligned} \tag{2.5}$$

It has the Virasoro element $L^{\mathcal{S}} = \frac{1}{2}\sum_{i=1}^n \big(:\beta^i\partial\gamma^i: - :\partial\beta^i\gamma^i:\big)$ of central charge $-n$, under which β^i, γ^i are primary of weight $\frac{1}{2}$. The automorphism group $\text{Aut}(\mathcal{S}(n))$ is isomorphic to the symplectic group $Sp(2n)$ and acts linearly on the generators.

The *symplectic fermion algebra* $\mathcal{A}(n)$ has odd generators e^i, f^i, $i = 1,\dots,n$, satisfying

$$\begin{aligned} &e^i(z)f^j(w) \sim \delta_{i,j}(z-w)^{-2}, \qquad &f^j(z)e^i(w) \sim -\delta_{i,j}(z-w)^{-2}, \\ &e^i(z)e^j(w) \sim 0, \qquad &f^i(z)f^j(w) \sim 0. \end{aligned} \tag{2.6}$$

It has the Virasoro element $L^{\mathcal{A}} = -\sum_{i=1}^n :e^if^i:$ of central charge $-2n$, under which e^i, f^i are primary of weight one. We have $\text{Aut}(\mathcal{A}(n)) \cong Sp(2n)$, and it acts linearly on the generators.

As a matter of terminology, we say that a vertex algebra $\mathcal{A}$ is of type $\mathcal{W}(d_1,\dots,d_k)$ if $\mathcal{A}$ has a minimal strong generating set consisting of an element in each weight $d_1,\dots,d_k$.

Filtrations. A filtration $\mathcal{A}_{(0)} \subset \mathcal{A}_{(1)} \subset \mathcal{A}_{(2)} \subset \cdots$ on a vertex algebra $\mathcal{A}$ such that $\mathcal{A} = \bigcup_{k\geq 0} \mathcal{A}_{(k)}$ is a called a *good increasing filtration* [LiI] if for all $a \in \mathcal{A}_{(k)}$, $b \in \mathcal{A}_{(l)}$, we have

$$a \circ_n b \in \begin{cases} \mathcal{A}_{(k+l)} & n < 0 \\ \mathcal{A}_{(k+l-1)} & n \geq 0 \end{cases}. \tag{2.7}$$

Setting $\mathcal{A}_{(-1)} = \{0\}$, the associated graded object $\text{gr}(\mathcal{A}) = \bigoplus_{k\geq 0} \mathcal{A}_{(k)}/\mathcal{A}_{(k-1)}$ is a $\mathbb{Z}_{\geq 0}$-graded associative, (super)commutative algebra with a unit 1 under a product induced by the Wick product on $\mathcal{A}$. Moreover, $\text{gr}(\mathcal{A})$ has a derivation ∂ of degree zero, and we call such a ring a ∂-ring. For each $r \geq 1$ we have the projection

$$\phi_r : \mathcal{A}_{(r)} \to \mathcal{A}_{(r)}/\mathcal{A}_{(r-1)} \subset \text{gr}(\mathcal{A}). \tag{2.8}$$

The key feature of $\mathcal{R}$ is the following reconstruction property [LL]. Let $\{a_i |\ i \in I\}$ be a set of generators for $\text{gr}(\mathcal{A})$ as a ∂-ring, where a_i is homogeneous of degree d_i. In other words, $\{\partial^k a_i |\ i \in I,\ k \geq 0\}$ generates $\text{gr}(\mathcal{A})$ as a ring. If $a_i(z) \in \mathcal{A}_{(d_i)}$ satisfies $\phi_{d_i}(a_i(z)) = a_i$ for each i, then $\mathcal{A}$ is strongly generated as a vertex algebra by $\{a_i(z) |\ i \in I\}$.

For any Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ and bilinear form B, $V_k(\mathfrak{g}, B)$ admits a good increasing filtration

$$V_k(\mathfrak{g}, B)_{(0)} \subset V_k(\mathfrak{g}, B)_{(1)} \subset \cdots, \qquad V_k(\mathfrak{g}, B) = \bigcup_{j\geq 0} V_k(\mathfrak{g}, B)_{(j)}, \tag{2.9}$$

where $V_k(\mathfrak{g}, B)_{(j)}$ is spanned by iterated Wick products of the generators X^{ξ_i} and their derivatives, of length at most j. We have a linear isomorphism $V_k(\mathfrak{g}, B) \cong \text{gr}(V_k(\mathfrak{g}, B))$, and an isomorphism of graded ∂-rings

$$\text{gr}(V_k(\mathfrak{g}, B)) \cong \big(\text{Sym} \bigoplus_{j\geq 0} V_j\big) \bigotimes \big(\bigwedge_{j\geq 0} \bigoplus W_j\big), \qquad V_j \cong \mathfrak{g}_0, \qquad W_j \cong \mathfrak{g}_1. \tag{2.10}$$

The ∂-ring structure on $\big(\text{Sym} \bigoplus_{j\geq 0} V_j\big) \otimes \big(\bigwedge_{j\geq 0} \bigoplus W_j\big)$ is given by $\partial x_j = x_{j+1}$ for $x \in V_j$ or $x \in W_j$, and the weight grading on $V_k(\mathfrak{g}, B)$ is inherited by $\text{gr}(V_k(\mathfrak{g}, B))$.

For $\mathcal{V} = \mathcal{H}(n), \mathcal{F}(n), \mathcal{S}(n), \mathcal{A}(n)$ we have good increasing filtrations $\mathcal{V}_{(0)} \subset \mathcal{V}_{(1)} \subset \cdots$, where $\mathcal{V}_{(j)}$ is spanned by iterated Wick products of the generators and their derivatives of length at most j. We have linear isomorphisms

$$\mathcal{H}(n) \cong \text{gr}(\mathcal{H}(n)) \qquad \mathcal{F}(n) \cong \text{gr}(\mathcal{F}(n)), \qquad \mathcal{S}(n) \cong \text{gr}(\mathcal{S}(n)), \qquad \mathcal{A}(n) \cong \text{gr}(\mathcal{A}(n)),$$

and isomorphism of ∂-rings

$$\begin{aligned} &\text{gr}(\mathcal{H}(n)) \cong \text{Sym} \bigoplus_{j\geq 0} V_j, \qquad \text{gr}(\mathcal{F}(n)) \cong \bigwedge \bigoplus_{j\geq 0} V_j \\ &\text{gr}(\mathcal{S}(n)) \cong \text{Sym} \bigoplus_{j\geq 0} (V_j \oplus V_j^*), \qquad \text{gr}(\mathcal{A}(n)) \cong \bigwedge \bigoplus_{j\geq 0} (V_j \oplus V_j^*), \end{aligned} \tag{2.11}$$

where $V_j \cong \mathbb{C}^n$ and $V_j^* \cong (\mathbb{C}^n)^*$. As above, the ∂-ring structure is given by $\partial x_j = x_{j+1}$ for $x \in V_j$ or V_j^*, and $\text{gr}(\mathcal{V})$ inherits the weight grading on $\mathcal{V}$.

Finally, for all the vertex algebras $\mathcal{V} = V_k(\mathfrak{g}, B), \mathcal{H}(n), \mathcal{F}(n), \mathcal{S}(n), \mathcal{A}(n)$ these filtrations are $\text{Aut}(\mathcal{V})$-invariant. For any reductive group $G \subset \text{Aut}(\mathcal{V})$, we have linear isomorphisms $\mathcal{V}^G \cong \text{gr}(\mathcal{V}^G)$ and isomorphisms of ∂-rings $\text{gr}(\mathcal{V})^G \cong \text{gr}(\mathcal{V}^G)$.

3. Deformable families

Following [CLI], we recall the notion of a *deformable family* of vertex algebras. Let $K \subset \mathbb{C}$ be a subset which is at most countable, and let F_K denote the $\mathbb{C}$-algebra of rational functions in a formal variable κ of the form $\frac{p(\kappa)}{q(\kappa)}$ where $\deg(p) \leq \deg(q)$ and the roots of q lie in K. A *deformable family* will be a free F_K-module $\mathcal{B}$ with the structure of a vertex algebra with coefficients in F_K. Vertex algebras over F_K are defined in the same way as ordinary vertex algebras over $\mathbb{C}$. We assume that $\mathcal{B}$ possesses a $\mathbb{Z}_{\geq 0}$-grading $\mathcal{B} = \bigoplus_{m\geq 0} \mathcal{B}[m]$ by conformal weight where each $\mathcal{B}[m]$ is free F_K-module of finite rank. For $k \notin K$, we have a vertex algebra

$$\mathcal{B}_k = \mathcal{B}/(\kappa - k),$$

where $(\kappa - k)$ is the ideal generated by $\kappa - k$. Clearly $\dim_{\mathbb{C}}(\mathcal{B}_k[m]) = \mathrm{rank}_{F_K}(\mathcal{B}[m])$ for all $k \notin K$ and $m \geq 0$. We have a vertex algebra $\mathcal{B}_\infty = \lim_{\kappa\to\infty} \mathcal{B}$ with basis $\{\alpha_i |\ i \in I\}$, where $\{a_i |\ i \in I\}$ is any basis of $\mathcal{B}$ over F_K, and $\alpha_i = \lim_{\kappa\to\infty} a_i$. By construction, $\dim_{\mathbb{C}}(\mathcal{B}_\infty[m]) = \mathrm{rank}_{F_K}(\mathcal{B}[m])$ for all $m \geq 0$. The vertex algebra structure on $\mathcal{B}_\infty$ is defined by

$$\alpha_i \circ_n \alpha_j = \lim_{\kappa\to\infty} a_i \circ_n a_j, \qquad i, j \in I, \qquad n \in \mathbb{Z}. \tag{3.1}$$

The F_K-linear map $\phi : \mathcal{B} \to \mathcal{B}_\infty$ sending $a_i \mapsto \alpha_i$ satisfies

$$\phi(\omega \circ_n \nu) = \phi(\omega) \circ_n \phi(\nu), \qquad \omega, \nu \in \mathcal{B}, \qquad n \in \mathbb{Z}. \tag{3.2}$$

Moreover, all normally ordered polynomial relations $P(\alpha_i)$ among the generators α_i and their derivatives are of the form

$$\lim_{\kappa\to\infty} \tilde{P}(a_i),$$

where $\tilde{P}(a_i)$ is a normally ordered polynomial relation among the a_i's and their derivatives, which converges termwise to $P(\alpha_i)$. In other words, suppose that

$$P(\alpha_i) = \sum_j c_j m_j(\alpha_i)$$

is a normally ordered relation of weight d, where the sum runs over all normally ordered monomials $m_j(\alpha_i)$ of weight d, and the coefficients c_j lie in $\mathbb{C}$. Then there exists a relation

$$\tilde{P}(a_i) = \sum_j c_j(\kappa) m_j(a_i)$$

where $\lim_{\kappa\to\infty} c_j(\kappa) = c_j$ and $m_j(a_i)$ is obtained from $m_j(\alpha_i)$ by replacing α_i with a_i.

Example 3.1 (Affine vertex superalgebras)**.** Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite-dimensional Lie superalgebra over $\mathbb{C}$, where $\dim(\mathfrak{g}_0) = n$ and $\dim(\mathfrak{g}_1) = 2m$. Suppose that $\mathfrak{g}$ is equipped with a nondegenerate, invariant, supersymmetric bilinear form B. Fix a basis $\{\xi_1, \dots, \xi_n\}$ for $\mathfrak{g}_0$ and $\{\eta_1^\pm, \dots, \eta_m^\pm\}$ for $\mathfrak{g}_1$, so the generators $X^{\xi_i}, X^{\eta_j^\pm}$ of $V_k(\mathfrak{g}, B)$ satisfy

$$\begin{aligned}
&X^{\xi_i}(z)X^{\xi_j}(w) \sim \delta_{i,j} k(z-w)^{-2} + X^{[\xi_i,\xi_j]}(w)(z-w)^{-1},\\
&X^{\eta_i^+}(z)X^{\eta_j^-}(w) \sim \delta_{i,j} k(z-w)^{-2} + X^{[\eta_i^+,\eta_j^-]}(w)(z-w)^{-1},\\
&X^{\xi_i}(z)X^{\eta_j^\pm}(w) \sim X^{[\xi_i,\eta_j^\pm]}(w)(z-w)^{-1},\\
&X^{\eta_i^\pm}(z)X^{\eta_j^\pm}(w) \sim X^{[\eta_i^\pm,\eta_j^\pm]}(w)(z-w)^{-1}.
\end{aligned} \tag{3.3}$$

Let κ be a formal variable satisfying $\kappa^2 = k$, and let $F = F_K$ for $K = \{0\}$. Let $\mathcal{V}$ be the vertex algebra with coefficients in F which is freely generated by $\{a^{\xi_i}, a^{\eta_j^{\pm}} |\ i = 1,\dots,n,\ j = 1,\dots,m\}$, satisfying

$$\begin{aligned}
&a^{\xi_i}(z)a^{\xi_j}(w) \sim \delta_{i,j}(z-w)^{-2} + \frac{1}{\kappa}a^{[\xi_i,\xi_j]}(w)(z-w)^{-1},\\
&a^{\eta_i^+}(z)a^{\eta_j^-}(w) \sim \delta_{i,j}(z-w)^{-2} + \frac{1}{\kappa}a^{[\eta_i^+,\eta_j^-]}(w)(z-w)^{-1},\\
&a^{\xi_i}(z)a^{\eta_j^{\pm}}(w) \sim +\frac{1}{\kappa}a^{[\xi_i,\eta_j^{\pm}]}(w)(z-w)^{-1},\\
&a^{\eta_i^{\pm}}(z)a^{\eta_j^{\pm}}(w) \sim +\frac{1}{\kappa}a^{[\eta_i^{\pm},\eta_j^{\pm}]}(w)(z-w)^{-1}.
\end{aligned} \tag{3.4}$$

For $k \neq 0$, we have a surjective vertex algebra homomorphism

$$\mathcal{V} \to V_k(\mathfrak{g}, B), \qquad a^{\xi_i} \mapsto \frac{1}{\sqrt{k}}X^{\xi_i}, \qquad a^{\eta_j^{\pm}} \mapsto \frac{1}{\sqrt{k}}a^{\eta_j^{\pm}},$$

whose kernel is the ideal $(\kappa - \sqrt{k})$, so $V_k(\mathfrak{g}, B) \cong \mathcal{V}/(\kappa - \sqrt{k})$. Then

$$\mathcal{V}_\infty = \lim_{\kappa\to\infty} \mathcal{V} \cong \mathcal{H}(n) \otimes \mathcal{F}(m), \tag{3.5}$$

and has even generators α^{ξ_i} for $i = 1,\dots,n$, and odd generators $e^{\eta_j^+}, e^{\eta_j^-}$ for $j = 1,\dots,m$, satisfying

$$\begin{aligned}
&\alpha^{\xi_i}(z)\alpha^{\xi_j}(w) \sim \delta_{i,j}(z-w)^{-2},\\
&e^{\eta_i^+}(z)e^{\eta_j^-}(w) \sim \delta_{i,j}(z-w)^{-2}.
\end{aligned} \tag{3.6}$$

Lemma 3.2 ([CLI], Lemma 8.1)**.** *Let $K \subset \mathbb{C}$ be at most countable, and let $\mathcal{B}$ be a vertex algebra over F_K as above. Let $U = \{\alpha_i|\ i \in I\}$ be a strong generating set for $\mathcal{B}_\infty$, and let $T = \{a_i|\ i \in I\}$ be the corresponding subset of $\mathcal{B}$, so that $\phi(a_i) = \alpha_i$. There exists a subset $S \subset \mathbb{C}$ containing K which is at most countable, such that $F_S \otimes_{F_K} \mathcal{B}$ is strongly generated by T. Here we have identified T with the set $\{1 \otimes a_i|\ i \in I\} \subset F_S \otimes_{F_K} \mathcal{B}$.*

Corollary 3.3. *For $k \notin S$, the vertex algebra $\mathcal{B}_k = \mathcal{B}/(\kappa - k)$ is strongly generated by the image of T under the map $\mathcal{B} \to \mathcal{B}_k$.*

If U is a *minimal* strong generating set for $\mathcal{B}_\infty$ it is not clear in general that T is a minimal strong generating set for $\mathcal{B}$, since there may exist relations of the form $\lambda(k)\alpha_j = P$, where P is a normally ordered polynomial in $\{\alpha_i|\ i \in I,\ i \neq k\}$ and $\lim_{k\to\infty} \lambda(k) = 0$, although $\lim_{k\to\infty} P$ is a nontrivial. However, there is one condition which holds in many examples, under which T is a minimal strong generating set for $\mathcal{B}$.

Proposition 3.4. *Suppose that $U = \{\alpha_i|\ i \in I\}$ is a minimal strong generating set for $\mathcal{B}_\infty$ such that wt$(\alpha_i) < N$ for all $i \in I$. If there are no normally ordered polynomial relations among $\{\alpha_i|\ i \in I\}$ and their derivatives of weight less than N, the corresponding set $T = \{a_i|\ i \in I\}$ is a minimal strong generating set for $\mathcal{B}$.*

Proof. If T is not minimal, there exists a decoupling relation $\lambda(k)a_j = P$ for some $j \in I$ of weight wt$(a_j) < N$. By rescaling if necessary, we can assume that either $\lambda(k)$ or P is nontrivial in the limit $k \to \infty$. We therefore obtain a nontrivial relation among $\{\alpha_i|\ i \in I\}$ and their derivarives of the same weight, which is impossible. □

In our main examples, the fact that relations among the elements of U and their derivatives do not exist below a certain weight is a consequence of Weyl's second fundamental theorem of invariant theory for the classical groups [W].

4. Orbifolds of free field algebras

By a *free field algebra*, we mean any vertex algebra $\mathcal{V} = \mathcal{H}(n) \otimes \mathcal{F}(m) \otimes \mathcal{S}(r) \otimes \mathcal{A}(s)$ for integers $m, n, r, s \geq 0$, where $\mathcal{B}(0)$ is declared to be $\mathbb{C}$ for $\mathcal{B} = \mathcal{H}, \mathcal{S}, \mathcal{F}, \mathcal{A}$. Building on our previous work, we establish the strong finite generation of $\mathcal{V}^G$ for any reductive group $G \subset \text{Aut}(\mathcal{V})$ which preserves the tensor factors of $\mathcal{V}$. Our description of these orbifolds is ultimately based on a classical theorem of Weyl (Theorem 2.5A of [W]). Let $V_k \cong \mathbb{C}^n$ for $k \geq 1$, and let $G \subset GL_n$, which acts on the ring $\text{Sym} \bigoplus_{k \geq 1} V_k$. For all $p \geq 1$, $GL(p)$ acts on $\bigoplus_{k=1}^p V_k$ and commutes with the action of G. There is an induced action of $GL(\infty) = \lim_{p \to \infty} GL(p)$ on $\bigoplus_{k \geq 1} V_k$, so $GL(\infty)$ acts on $\text{Sym} \bigoplus_{k \geq 1} V_k$ and commutes with the action of G. Therefore $GL(\infty)$ acts on $R = (\text{Sym} \bigoplus_{k \geq 1} V_k)^G$ as well. Elements $\sigma \in GL(\infty)$ are known as *polarization operators*, and given $f \in R$, σf is known as a polarization of f.

Theorem 4.1. *R is generated by the polarizations of any set of generators for $(\text{Sym} \bigoplus_{k=1}^n V_k)^G$. Since G is reductive, $(\text{Sym} \bigoplus_{k=1}^n V_k)^G$ is finitely generated, so there exists a finite set $\{f_1, \dots, f_r\}$, whose polarizations generate R.*

As shown in [CLII] (see Theorem 6.4) there is an analogue of this result for exterior algebras. Let $S = (\bigwedge \bigoplus_{k \geq 1} V_k)^G$ and let d be the maximal degree of the generators of $(\text{Sym} \bigoplus_{k=1}^n V_k)^G$. Then S is generated by the polarizations of any set of generators for $(\bigwedge \bigoplus_{k=1}^d V_k)^G$. In particular, S is generated by a finite number of elements together with their polarizations. By a similar argument, the same holds for rings of the form

$$T = \big((\text{Sym} \bigoplus_{k \geq 1} V_k) \otimes (\bigwedge \bigoplus_{k \geq 1} W_k)\big)^G,$$

where $V_k = \mathbb{C}^n$, $W_k = \mathbb{C}^m$, and $G \subset GL_n \times GL_m$ is any reductive group.

Theorem 4.2. *Let $\mathcal{V} = \mathcal{H}(m) \otimes \mathcal{F}(n) \otimes \mathcal{S}(r) \otimes \mathcal{A}(s)$ for integers $m, n, r, s \geq 0$, and let $G \subset O(m) \times O(n) \times Sp(2r) \times Sp(2s)$ be a reductive group of automorphisms of $\mathcal{V}$ that preserves the factors $\mathcal{H}(m)$, $\mathcal{F}(n)$, $\mathcal{S}(r)$, and $\mathcal{A}(s)$. Then $\mathcal{V}^G$ is strongly finitely generated.*

Proof. Note that $\mathcal{V} \cong \text{gr}(\mathcal{V})$ as G-modules, and

$$\text{gr}(\mathcal{V}^G) \cong \text{gr}(\mathcal{V})^G \cong \big((\text{Sym} \bigoplus_{j \geq 0} V_j) \otimes (\bigwedge \bigoplus_{j \geq 0} \bar{V}_j) \otimes (\text{Sym} \bigoplus_{j \geq 0} W_j) \otimes (\bigwedge \bigoplus_{j \geq 0} \bar{W}_j)\big)^G,$$

as supercommutative rings. Here $V_j \cong \mathbb{C}^m$, $\bar{V}_j \cong \mathbb{C}^n$, $W_j \cong \mathbb{C}^{2r}$, $\bar{W}_j \cong \mathbb{C}^{2s}$.

By a general theorem of Kac and Radul [KR] (see also [DLM] for the case of compact G), for each of the vertex algebras $\mathcal{B} = \mathcal{H}(m), \mathcal{S}(n), \mathcal{F}(r), \mathcal{A}(s)$, we have a dual reductive pair decomposition

$$\mathcal{B} \cong \bigoplus_{\nu \in H} L(\nu) \otimes M^{\nu},$$

where H indexes the irreducible, finite-dimensional representations $L(\nu)$ of $\text{Aut}(\mathcal{B})$, and the M^ν's are inequivalent, irreducible, highest-weight $\mathcal{B}^{\text{Aut}(\mathcal{B})}$-modules. Therefore

$$\mathcal{V} \cong \bigoplus_{\nu,\mu,\gamma,\delta} L(\nu)\otimes L(\mu)\otimes L(\gamma)\otimes L(\delta)\otimes M^\nu \otimes M^\mu \otimes M^\gamma \otimes M^\delta,$$

where $L(\nu)$, $L(\mu)$, $L(\gamma)$, and $L(\delta)$ are irreducible, finite-dimensional modules over $O(m)$, $O(n)$, $Sp(2r)$ and $Sp(2s)$, respectively, and M^ν, N^μ, M^γ and M^δ are irreducible, highest-weight modules over $\mathcal{H}(m)^{O(m)}$, $\mathcal{F}(n)^{O(n)}$, $\mathcal{S}(r)^{Sp(2r)}$, and $\mathcal{A}(s)^{Sp(2s)}$, respectively. An immediate consequence whose proof is the same as the proof of Lemma 14.2 of [LV] is that $\mathcal{V}^G$ has a strong generating set which lies in the direct sum of finitely many irreducible modules over $\mathcal{H}(m)^{O(m)}\otimes \mathcal{F}(n)^{O(n)}\otimes \mathcal{S}(r)^{Sp(2r)}\otimes \mathcal{A}(s)^{Sp(2s)}$.

By Theorem 9.4 of [LV], $\mathcal{S}(r)$ is of type $\mathcal{W}(2,4,\dots,2r^2+4r)$ and has strong generators

$$\tilde{w}^{2k+1} = \frac{1}{2}\sum_{i=1}^{r}\big(:\beta^i\partial^{2k+1}\gamma^i: - :(\partial^{2k+1}\beta^i)\gamma^i:\big), \qquad k=0,1,\dots,r^2+2r-1. \tag{4.1}$$

By Theorem 11.1 of [LV], $\mathcal{F}(n)$ is of type $\mathcal{W}(2,4,\dots,2n)$ and has strong generators

$$\tilde{j}^{2k+1} = -\frac{1}{2}\sum_{i=1}^{n} :\phi^i\partial^{2k+1}\phi^i:, \qquad k=0,1,\dots,n-1.$$

By Theorem 3.11 of [CLII], $\mathcal{A}(s)$ is of type $\mathcal{W}(2,4,\dots,2s)$ and has strong generators

$$w^{2k} = \frac{1}{2}\sum_{i=1}^{s}\big(:e^i\partial^{2k}f^i: + :(\partial^{2k}e^i)f^i:\big), \qquad k=0,1,\dots,s-1.$$

In [LIII], it was conjectured that $\mathcal{H}(m)$ is of type $\mathcal{W}(2,4,\dots,m^2+3m)$, and has strong generators

$$j^{2k} = \sum_{i=1}^{m} a^i\partial^{2k}a^i:, \qquad k=0,1,\dots,\frac{1}{2}(m^2+3m-2).$$

In [LIV] we proved this for $m\leq 6$, and although we did not prove it for $m>6$, we showed that $\mathcal{H}(m)^{O(m)}$ has strong generators $\{j^{2k}|\ 0\leq k\leq K\}$ for some K.

For any irreducible $\mathcal{H}(m)^{O(m)}\otimes\mathcal{F}(n)^{O(n)}\otimes\mathcal{S}(r)^{Sp(2r)}\otimes\mathcal{A}(s)^{Sp(2s)}$-submodule $\mathcal{M}$ of $\mathcal{V}$ with highest-weight vector $f=f(z)$, and any subset $S\subset\mathcal{M}$, define $\mathcal{M}_S$ to be the subspace spanned by the elements

$$\begin{aligned} &:\omega_1\cdots\omega_a\nu_1\cdots\nu_b\mu_1\cdots\mu_c\zeta_1\cdots\zeta_d\alpha:,\\ &\omega_i\in\mathcal{H}(m)^{O(m)},\quad \nu_i\in\mathcal{F}(n)^{O(n)},\quad \mu_i\in\mathcal{S}(r)^{Sp(2r)},\quad \zeta_i\in\mathcal{A}(s)^{Sp(2s)},\quad \alpha\in S.\end{aligned} \tag{4.2}$$

By the same argument as Lemma 9 of [LII], there is a finite set S of vertex operators of the form

$$j^{2a_1}(h_1)\cdots j^{2a_t}(h_t)\tilde{j}^{2b_1+1}(j_1)\cdots\tilde{j}^{2b_u+1}(j_u)\tilde{w}^{2c_1+1}(k_1)\cdots\tilde{w}^{2c_v+1}(k_v)w^{2d_1}(l_1)\cdots w^{2d_w}(l_w)f,$$

such that $\mathcal{M}=\mathcal{M}_S$. In this notation

$$\begin{aligned} &j^{2a_i}\in\mathcal{H}(m)^{O(m)}, && 0\leq h_i\leq 2a_i\leq K,\\ &\tilde{j}^{2b_i+1}\in\mathcal{F}(n)^{O(n)}, && 0\leq j_i<2b_i+1\leq 2n-1,\\ &\tilde{w}^{2c_i+1}\in\mathcal{S}(r)^{Sp(2r)}, && 0\leq k_i<2c_i+1\leq 2r^2+4r-1,\\ &w^{2d_i}\in\mathcal{A}(s)^{Sp(2s)}, && 0\leq l_i\leq 2d_i\leq 2s-2.\end{aligned} \tag{4.3}$$

Combining this with the strong finite generation of each of the vertex algebras $\mathcal{H}(m)^{O(m)}$, $\mathcal{F}(n)^{O(n)}$, $\mathcal{S}(r)^{Sp(2r)}$, and $\mathcal{A}(s)^{Sp(2s)}$, completes the proof. $\square$

5. Orbifolds of affine vertex superalgebras

In [LIII] it was shown that for any Lie algebra $\mathfrak{g}$ with a nondegenerate form B, and any reductive group G of automorphisms of $V_k(\mathfrak{g},B)$, $V_k(\mathfrak{g},B)^G$ is strongly finitely generated for generic values of k. Here we extend this result to the case of affine vertex superalgebras. Let $\mathfrak{g}=\mathfrak{g}_0\oplus\mathfrak{g}_1$ be a finite-dimensional Lie superalgebra over $\mathbb{C}$, where $\dim(\mathfrak{g}_0)=n$ and $\dim(\mathfrak{g}_1)=2m$, and let B be a nondegenerate form on $\mathfrak{g}$. Let $\mathcal{V}$ be the deformable vertex algebra from Example 3.1, such that $V_k(\mathfrak{g},B)\cong\mathcal{V}/(\kappa-\sqrt{k})$, and $\mathcal{V}_\infty=\lim_{k\to\infty}\mathcal{V}\cong\mathcal{H}(n)\otimes\mathcal{F}(m)$. Define an F-linear map $\psi:\mathcal{V}\to\mathcal{V}_\infty$ by

$$\psi\big(\sum_r c_r(\kappa)m_r(a^{\xi_i})\big)=\sum_r c_r m_r(\alpha^{\xi_i}),\qquad c_r=\lim_{\kappa\to\infty}c_r(\kappa). \tag{5.1}$$

In this notation, $m_r(a^{\xi_i})$ is a normally ordered monomial in $\partial^j a^{\xi_i}$, and $m_r(\alpha^{\xi_i})$ is obtained from $m_r(a^{\xi_i})$ by replacing each a^{ξ_i} with α^{ξ_i}. This map is easily seen to satisfy $\psi(\omega\circ_n\nu)=\psi(\omega)\circ_n\psi(\nu)$ for all $\omega,\nu\in\mathcal{V}$ and $n\in\mathbb{Z}$.

Note that $\mathcal{V}$ has a good increasing filtration, where $\mathcal{V}_{(d)}$ is spanned by normally ordered monomials in $\partial^l a^{\xi_i}$ and $\partial^l a^{\eta_j^{\pm}}$ of degree at most d. We have isomorphisms of ∂-rings

$$\mathrm{gr}(\mathcal{V})\cong F\otimes_{\mathbb{C}}\big(\mathrm{Sym}\bigoplus_{j\geq0}V_j\big)\bigotimes\big(\bigwedge\bigoplus_{j\geq0}W_j\big)\cong F\otimes_{\mathbb{C}}\mathrm{gr}(\mathcal{V}_\infty),\qquad V_j\cong\mathfrak{g}_0,\qquad W_j\cong\mathfrak{g}_1.$$

The action of G on $\mathcal{V}$ preserves the formal variable κ, and we have

$$\mathrm{gr}(\mathcal{V}^G)\cong\mathrm{gr}(\mathcal{V})^G\cong F\otimes_{\mathbb{C}}R\cong F\otimes_{\mathbb{C}}\mathrm{gr}(\mathcal{V}_\infty)^G\cong F\otimes_{\mathbb{C}}\mathrm{gr}((\mathcal{V}_\infty)^G),$$

where $R=\big((\mathrm{Sym}\bigoplus_{j\geq0}V_j)\bigotimes(\bigwedge\bigoplus_{j\geq0}W_j)\big)^G$. Finally, $\mathcal{V}^G[w]$ is a free F-module and

$$\mathrm{rank}_F(\mathcal{V}^G[w])=\dim_{\mathbb{C}}((\mathcal{V}_\infty)^G[w])=\dim_{\mathbb{C}}(V_k(\mathfrak{g},B)^G[w])$$

for all $w\geq0$ and $k\in\mathbb{C}$.

Fix a basis $\{\xi_{1,l},\dots,\xi_{n,l}\}$ for V_l, which corresponds to

$$\{\partial^l a^{\xi_1},\dots,\partial^l a^{\xi_n}\}\subset\mathcal{V},\qquad\{\partial^l\alpha^{\xi_1},\dots,\partial^l\alpha^{\xi_n}\}\subset\mathcal{V}_\infty,$$

respectively. Similarly, fix a basis $\{\eta^{\pm}_{1,l},\dots,\eta^{\pm}_{m,l}\}$ for W_l corresponding to

$$\{\partial^l a^{\eta_1^{\pm}},\dots,\partial^l a^{\eta_m^{\pm}}\}\subset\mathcal{V},\qquad\{\partial^l\alpha^{\eta_1^{\pm}},\dots,\partial^l\alpha^{\eta_m^{\pm}}\}\subset\mathcal{V}_\infty,$$

respectively. The ring R is graded by degree and weight, where $\xi_{1,l},\dots,\xi_{n,l},\eta^{\pm}_{1,l},\dots,\eta^{\pm}_{m,l}$ have degree 1 and weight $l+1$. Choose a generating set $S=\{s_i|\ i\in I\}$ for R as a ∂-ring, where s_i is homogeneous of degree d_i and weight w_i. We may assume that S contains finitely many generators in each weight. We can find a corresponding strong generating set $T=\{t_i|\ i\in I\}$ for $\mathcal{V}^G$, where

$$t_i\in(\mathcal{V}^G)_{(d_i)},\qquad\phi_{d_i}(t_i)=1\otimes s_i\in F\otimes_{\mathbb{C}}R.$$

Here $\phi_{d_i}:(\mathcal{V}^G)_{(d_i)}\to(\mathcal{V}^G)_{(d_i)}/(\mathcal{V}^G)_{(d_i-1)}\subset\mathrm{gr}(\mathcal{V}^G)$ is the usual projection. In particular, the leading term of t_i is a sum of normally ordered monomials of degree d_i in the variables $a^{\xi_1}, \dots, a^{\xi_n}$ and their derivatives, and the coefficient of each such monomial is independent of κ. Let $u_i = \psi(t_i)$ where ψ is given by (5.1), and define

$$(\mathcal{V}^G)_\infty = \langle U \rangle \subset (\mathcal{V}_\infty)^G, \tag{5.2}$$

where $\langle U \rangle$ is the vertex algebra generated by $\{u_i |\ i \in I\}$.

Fix $w \geq 0$, and let $\{m_1, \dots, m_r\}$ be a set of normally ordered monomials in t_i and their derivatives, which spans the subspace $\mathcal{V}^G[w]$ of weight w. Then $(\mathcal{V}^G)_\infty[w]$ is spanned by the corresponding monomials $\mu_l = \psi(m_l)$ for $l = 1, \dots, r$, which are obtained from m_l by replacing t_i with u_i. Given normally ordered polynomials

$$P(u_i) = \sum_{l=1}^{r} c_l \mu_l \in (\mathcal{V}^G)_\infty[w], \qquad \tilde{P}(t_i) = \sum_{l=1}^{r} c_l(\kappa) m_l \in \mathcal{V}^G[w],$$

with $c_l \in \mathbb{C}$ and $c_l(\kappa) \in F$, we say that $\tilde{P}(t_i)$ *converges termwise* to $P(u_i)$ if

$$\lim_{\kappa \to \infty} c_l(\kappa) = c_l, \qquad l = 1, \dots, r.$$

In particular, $\tilde{P}(t_i)$ converges termwise to zero if and only if $\lim_{\kappa \to \infty} c_l(\kappa) = 0$ for $l = 1, \dots, r$.

Lemma 5.1. *For each normally ordered polynomial relation $P(u_i)$ in $(\mathcal{V}^G)_\infty$ of weight m and leading degree d, there exists a relation $\tilde{P}(t_i) \in \mathcal{V}^G$ of weight m and leading degree d which converges termwise to $P(u_i)$.*

Proof. We may write $P(u_i) = \sum_{a=1}^{d} P^a(u_i)$, where $P^a(u_i)$ is a sum of normally ordered monomials $\mu =: \partial^{j_1} u_{i_1} \cdots \partial^{j_t} u_{i_t}$: of degree $a = d_{i_1} + \cdots + d_{i_t}$. The leading term $P^d(u_i)$ corresponds to a relation in R among the generators s_i and their derivatives, i.e., $P^d(s_i) = 0$. It follows that $P^d(t_i) \in (\mathcal{V}^G)_{(d-1)}$. Since $P^a(u_i) \in ((\mathcal{V}^G)_\infty)_{(a)}$ for $a = 1, \dots, d-1$, we have $P(t_i) \in (\mathcal{V}^G)_{(d-1)}$. Since $\{t_i |\ i \in I\}$ strongly generates $\mathcal{V}^G$, we can express $P(t_i)$ as a normally ordered polynomial $P_0(t_i)$ of degree at most $d-1$. Let $Q(t_i) = P(t_i) - P_0(t_i)$, which is therefore a relation in $\mathcal{V}^G$ with leading term $P^d(t_i)$.

If $P_0(t_i)$ converges termwise to zero, we can take $\tilde{P}(t_i) = Q(t_i)$ since $P(t_i)$ converges termwise to $P(u_i)$. Otherwise, $P_0(t_i)$ converges termwise to a nontrivial relation $P_1(u_i)$ in $(\mathcal{V}^G)_\infty$ of degree at most $d-1$. By induction on the degree, there is a relation $\tilde{P}_1(t_i)$ of leading degree at most $d-1$, which converges termwise to $P_1(u_i)$. Finally, $\tilde{P}(t_i) = P(t_i) - P_0(t_i) - \tilde{P}_1(t_i)$ has the desired properties. □

Corollary 5.2. $(\mathcal{V}^G)_\infty = (\mathcal{V}_\infty)^G = (\mathcal{H}(n) \otimes \mathcal{F}(m))^G$.

Proof. Recall that $\text{rank}_F(\mathcal{V}^G[w]) = \dim_{\mathbb{C}}((\mathcal{V}_\infty)^G[w])$ for all $w \geq 0$. Since $(\mathcal{V}^G)_\infty \subset (\mathcal{V}_\infty)^G$, it suffices to show that $\text{rank}_F(\mathcal{V}^G[w]) = \dim_{\mathbb{C}}((\mathcal{V}^G)_\infty[w])$ for all $w \geq 0$. Let $\{m_1, \dots, m_r\}$ be a basis for $\mathcal{V}^G[w]$ as an F-module, consisting of normally ordered monomials in t_i and their derivatives. The corresponding elements $\mu_l = \psi(m_l)$ for $l = 1, \dots, r$ span $(\mathcal{V}^G)_\infty[w]$, and by Lemma 5.1 they are linearly independent. Otherwise, a nontrivial relation among $\mu_1, \dots, \mu_r$ would give rise to a nontrivial relation among $m_1, \dots, m_r$. □

Theorem 5.3. *$V_k(\mathfrak{g}, B)^G$ is strongly finitely generated for generic values of k.*

Proof. This is immediate from Theorem 4.2 applied to $\mathcal{V} = \mathcal{H}(n) \otimes \mathcal{F}(m)$ and Corollaries 3.3 and 5.2. □

Theorem 5.4. *Let $\mathcal{V} = \mathcal{H}(m) \otimes \mathcal{F}(n) \otimes \mathcal{S}(r) \otimes \mathcal{A}(s)$ be a free field algebra and let $\mathfrak{g}$ be a Lie superalgebra equipped with a nondegenerate form B. Let G be a reductive group of automorphisms of $\mathcal{V} \otimes V_k(\mathfrak{g}, B)$ which preserves each tensor factor. Then $(\mathcal{V} \otimes V_k(\mathfrak{g}, B))^G$ is strongly finitely generated for generic values of k.*

Proof. We have $\lim_{k\to\infty} \mathcal{V} \otimes V_k(\mathfrak{g}, B) \cong \mathcal{V} \otimes \mathcal{H}(n) \otimes \mathcal{A}(m)$ where $n = \text{dim}(\mathfrak{g}_0)$ and $m = \frac{1}{2}\text{dim}(\mathfrak{g}_1)$, and $\lim_{k\to\infty}(\mathcal{V} \otimes V_k(\mathfrak{g}, B))^G \cong (\mathcal{V} \otimes \mathcal{H}(n) \otimes \mathcal{A}(m))^G$. Clearly G preserves the tensor factors, so the claim follows from Theorem 4.2 and Corollary 3.3. □

6. Cosets of $V_k(\mathfrak{g}, B)$ inside larger structures: generic behavior

Let $\mathfrak{g}$ be a finite-dimensional reductive Lie algebra, equipped with a nondegenerate symmetric, invariant bilinear form B. Let $\mathcal{A}_k$ be a vertex (super)algebra whose structure constants depend continuously on k, admitting a homomorphism $V_k(\mathfrak{g}, B) \to \mathcal{A}_k$, and let $\mathcal{C}_k = \text{Com}(V_k(\mathfrak{g}, B), \mathcal{A}_k)$. Many cosets of this form have been studied in both the physics and mathematics literature. One class of examples is $\mathcal{A}_k = V_k(\mathfrak{g}', B')$ where $\mathfrak{g}'$ is a Lie (super)algebra containing $\mathfrak{g}$, and B is a nondegenerate, (super)symmetric invariant form on $\mathfrak{g}'$ extending B. Another class of examples is $\mathcal{A}_k = V_{k-l}(\mathfrak{g}', B') \otimes \mathcal{F}$ where $\mathfrak{g}'$ and B' are as above and $\mathcal{F}$ is a free field algebra admitting a map $\phi : V_l(\mathfrak{g}, B) \to \mathcal{F}$ for some fixed $l \in \mathbb{C}$. We require that the action of $\mathfrak{g}$ on $\mathcal{F}$ integrates to an action of a group G whose Lie algebra is $\mathfrak{g}$, and that G preserves the tensor factors of $\mathcal{F}$. The map $V_k(\mathfrak{g}, B) \to \mathcal{A}_k$ is just the diagonal map $X^{\xi_i} \mapsto X^{\xi_i} \otimes 1 + 1 \otimes \phi(X^{\xi_i})$.

To construct examples of this kind, we recall a well-known vertex algebra homomorphism $\tau : V_{-1/2}(\mathfrak{sp}_{2n}) \to \mathcal{S}(n)$. In terms of the basis $\{e_{i,j} | 1 \leq i \leq 2n,\ 1 \leq j \leq 2n\}$ for $\mathfrak{gl}_{2n}$, a standard basis for $\mathfrak{sp}_{2n}$ consists of

$$e_{j,k+n} + e_{k,j+n}, \qquad -e_{j+n,k} - e_{k+n,j}, \qquad e_{j,k} - e_{n+k,n+j}, \qquad 1 \leq j,k \leq n.$$

Define τ by

$$X^{e_{j,k+n}+e_{k,j+n}} \mapsto :\gamma^j \gamma^k:, \qquad X^{-e_{j+n,k}-e_{k+n,j}} \mapsto :\beta^j \beta^k:, \qquad X^{e_{j,k}-e_{n+k,n+j}} \mapsto :\gamma^j \beta^k:. \tag{6.1}$$

It is easily checked that the $\mathfrak{sp}_{2n}$-action coming from the zero modes $\{X^\xi(0) |\ \xi \in \mathfrak{sp}_{2n}\}$ integrates to the usual action of $Sp(2n)$ on $\mathcal{S}(n)$. There is a similar homomorphism $\sigma : V_1(\mathfrak{so}_m) \to \mathcal{F}(m)$ whose zero mode action integrates to $SO(m)$. If $\mathfrak{g}$ is any reductive Lie algebra which embeds in $\mathfrak{sp}_{2n}$, and B_1 is the restriction of the form on $\mathfrak{sp}_{2n}$ to $\mathfrak{g}$, we obtain a restriction map $\tau_{\mathfrak{g}} : V_1(\mathfrak{g}, B_1) \to \mathcal{S}(n)$. Similarly, if $\mathfrak{g}$ embeds in $\mathfrak{so}_m$ we obtain a restriction map $\sigma_{\mathfrak{g}} : V_1(\mathfrak{g}, B_2) \to \mathcal{F}(m)$, where B_2 is the restriction of the form on $\mathfrak{so}_m$ to $\mathfrak{g}$. Finally, we have the diagonal embedding

$$V_1(\mathfrak{g}, B_1 + B_2) \to \mathcal{S}(n) \otimes \mathcal{F}(m), \qquad X^\xi \mapsto \tau_{\mathfrak{g}}(X^\xi) \otimes 1 + 1 \otimes \sigma_{\mathfrak{g}}(X^\xi).$$

The action of $\mathfrak{g}$ coming from the zero modes integrates to an action of a connected Lie group G with Lie algebra $\mathfrak{g}$, which preserves both $\mathcal{S}(n)$ and $\mathcal{F}(m)$.

Finally, we mention one more class of examples $\mathcal{A}_k = V_{k-l}(\mathfrak{g}', B') \otimes V_l(\mathfrak{g}'', B'')$. Here $\mathfrak{g}''$ is another finite-dimensional Lie (super)algebra containing $\mathfrak{g}$, equipped with a nondegenerate, invariant, (super)symmetric bilinear form B'' extending B. As usual, the map $V_k(\mathfrak{g}, B) \to \mathcal{A}_k$ is the diagonal map $X^{\xi_i} \mapsto X^{\xi_i} \otimes 1 + 1 \otimes X^{\xi_i}$. If $V_l(\mathfrak{g}'', B'')$ is not simple, we may replace $V_l(\mathfrak{g}'', B'')$ with its quotient by any nontrivial ideal in the above definition.

In order to study all the above cosets $\mathcal{C}_k$ from a unified point of view it is useful to axiomatize $\mathcal{A}_k$. A vertex algebra $\mathcal{A}_k$ with structure constants depending continuously on k, which admits a map $V_k(\mathfrak{g}, B) \to \mathcal{A}_k$ will be called *good* if the following conditions hold.

(1) There exists a deformable family $\mathcal{A}$ defined over F_K for some (at most countable) subset $K \subset \mathbb{C}$ containing zero, such that $\mathcal{A}_k = \mathcal{A}/(\kappa - \sqrt{k})$. Letting $\mathcal{V}$ be as in Example 3.1, there is a homomorphism $\mathcal{V} \to \mathcal{A}$ inducing the map $V_k(\mathfrak{g}, B) \to \mathcal{A}_k$ for each k with $\sqrt{k} \notin K$.

(2) For generic values of k, $\mathcal{A}_k$ admits a Virasoro element $L^{\mathcal{A}}$ and a conformal weight grading $\mathcal{A}_k = \bigoplus_{d\in\mathbb{N}} \mathcal{A}_k[d]$. For all d, $\dim(\mathcal{A}_k[d])$ is finite and independent of k.

(3) For generic values of k, $\mathcal{A}_k$ decomposes into finite-dimensional $\mathfrak{g}$-modules, so the action of $\mathfrak{g}$ integrates to an action of a connected Lie group G having $\mathfrak{g}$ as Lie algebra.

(4) We have a vertex algebra isomorphism

$$\mathcal{A}_\infty = \lim_{k\to\infty} \mathcal{A}_k \cong \mathcal{H}(d) \otimes \tilde{\mathcal{A}}, \qquad d = \dim(\mathfrak{g}).$$

Here $\tilde{\mathcal{A}}$ is a vertex subalgebra of $\lim_{k\to\infty} \mathcal{A}_k$ with Virasoro element $L^{\tilde{\mathcal{A}}}$ and $\mathbb{N}$-grading by conformal weight, with finite-dimensional graded components.

(5) Although $L_0^{\mathfrak{g}}$ need not act diagonalizably on $\mathcal{A}_k$, it induces a grading on $\mathcal{A}_k$ into generalized eigenspaces corresponding to the Jordan blocks of each eigenvalue. In general, these generalized eigenspaces can be infinite-dimensional. However, any highest-weight $V_k(\mathfrak{g}, B)$-submodule of $\mathcal{A}_k$ has finite-dimensional components with respect to this grading for generic values of k.

Note that for generic values of k, $\mathcal{C}_k$ has the Virasoro element $L^{\mathcal{C}} = L^{\mathcal{A}} - L^{\mathfrak{g}}$, where $L^{\mathfrak{g}}$ is the Virasoro element in $V_k(\mathfrak{g}, B)$. Note that $\lim_{k\to\infty} L^{\mathfrak{g}} = L^{\mathcal{H}} = \frac{1}{2}\sum_{i=1}^{d} :\alpha^{\xi_i}\alpha^{\xi_i}:$, and that $\lim_{k\to\infty} L^{\mathcal{A}} = L^{\mathcal{H}} + L^{\tilde{\mathcal{A}}}$. It is not difficult to check that all the above examples $\mathcal{A}_k = V_k(\mathfrak{g}', B')$, $\mathcal{A}_k = V_{k-l}(\mathfrak{g}', B') \otimes \mathcal{F}$, and $\mathcal{A}_k = V_{k-l}(\mathfrak{g}', B') \otimes V_k(\mathfrak{g}'', B'')$ are good. Also, $V_{k-l}(\mathfrak{g}', B') \otimes V_k(\mathfrak{g}'', B'')$ remains good if we replace $V_k(\mathfrak{g}'', B'')$ by its quotient by any ideal. Suppose that $\dim(\mathfrak{g}) = d$ and $\mathfrak{g}' = \mathfrak{g}'_0 \oplus \mathfrak{g}'_1$ where $\dim(\mathfrak{g}'_0) = n$ and $\dim(\mathfrak{g}'_1) = 2m$. For $\mathcal{A}_k = V_k(\mathfrak{g}', B')$, we have $\tilde{\mathcal{A}} \cong \mathcal{H}(n-d) \otimes \mathcal{A}(m)$. Similarly, for $\mathcal{A}_k = V_{k-l}(\mathfrak{g}', B') \otimes \mathcal{F}$, we have $\tilde{\mathcal{A}} \cong \mathcal{H}(n-d) \otimes \mathcal{A}(m) \otimes \mathcal{F}$. Finally, for $\mathcal{A}_k = V_{k-l}(\mathfrak{g}', B') \otimes V_l(\mathfrak{g}'', B'')$, we have $\tilde{\mathcal{A}} \cong \mathcal{H}(n-d) \otimes \mathcal{A}(m) \otimes V_l(\mathfrak{g}'', B'')$. The same holds if we replace $V_l(\mathfrak{g}'', B'')$ with its quotient by any ideal.

Lemma 6.1. *Let $\mathfrak{g}$ be reductive and B nondegenerate, and suppose that $\mathcal{A}_k$ is good. For generic values of k, $\mathcal{C}_k = Ker(L_0^{\mathfrak{g}}) \cap (\mathcal{A}_k)^G$.*

Proof. Suppose first that $\mathfrak{g}$ is simple. Clearly any $\omega \in \mathcal{C}_k$ is annihilated by $L_0^{\mathfrak{g}}$ and is G-invariant. Conversely, suppose that $\omega \in \mathrm{Ker}(L_0^{\mathfrak{g}}) \cap (\mathcal{A}_k)^G$. If $\omega \notin \mathcal{C}_k$, $X^{\xi_i} \circ_1 \omega \neq 0$ for each $i = 1, \dots, d$. We show that such an ω cannot exist for generic level k.

Recall that $\mathcal{A}_k$ is $\mathbb{N}$-graded by conformal weight (i.e., $L_0^{\mathcal{A}}$-eigenvalue). Write ω as a sum of terms of homogeneous weight, and let m be the maximum value which appears. Let $\mathfrak{g}_+ \subset \hat{\mathfrak{g}}$ be the Lie subalgebra generated by the positive modes $\{X^{\xi}(k) |\, \xi \in \mathfrak{g},\ k > 0\}$. Note that each element of $U(\mathfrak{g}_+)$ lowers the weight by some $k > 0$, and the conformal weight grading on $U(\mathfrak{g}_+)$ is the same as the grading by $L_0^{\mathfrak{g}}$-eigenvalue. An element $x \in U(\mathfrak{g}_+)$ of weight $-k$ satisfies $x(\omega) \in \mathcal{A}_{m-k}$. Also, $x(\omega)$ lies in the generalized eigenspace of $L_0^{\mathfrak{g}}$ of eigenvalue $-k$, and $x(\omega) = 0$ if $k > m$.

It follows that $U(\mathfrak{g}_+)\omega$ is a finite-dimensional vector space graded by conformal weight. In particular, the subspace $M \subset U(\mathfrak{g}_+)\omega$ of minimal weight m is finite-dimensional. Hence it is a finite-dimensional $\mathfrak{g}$-module, and is thus a direct sum of finite-dimensional highest-weight $\mathfrak{g}$-modules. Moreover, $U(\mathfrak{g}_+)$ acts trivially on M. Since $\mathfrak{g}$ is simple and $L^{\mathfrak{g}}$ is the Sugawara vector at level k, the eigenvalue of $L^{\mathfrak{g}}_0$ on M is given by

$$L^{\mathfrak{g}}_0|_M = \frac{\text{Cas}(M)}{k+h^\vee}, \tag{6.2}$$

where Cas(M) is the Casimir eigenvalue on M. In fact, each irreducible summand of M must have the same $L^{\mathfrak{g}}_0$ eigenvalue and hence the same Casimir eigenvalue. This is a rational number. The $L^{\mathfrak{g}}_0$ eigenvalue on M must actually be a negative integer r, with $-m \leq r \leq -1$. This statement and (6.2) can only be true for special rational values of k.

Similarly, if $\mathfrak{g} = \mathbb{C}$, so that $G = \mathbb{C}^*$ and $V_k(\mathfrak{g}, B) \cong \mathcal{H}(1)$ for $k \neq 0$, we have $\mathcal{C}_k = \text{Ker}(L^{\mathcal{H}}_0) \cap (\mathcal{A}_k)^{\mathbb{C}^*}$ for generic values of k. This is immediate from the fact that $\mathcal{H}(1)$ acts completely reducibly on $\mathcal{A}_k$, so that $(\mathcal{A}_k)^{\mathbb{C}^*} \cong \mathcal{H}(1) \otimes \mathcal{C}_k$. Now it follows by induction on the number of simple and abelian summands that the lemma holds for any reductive Lie algebra $\mathfrak{g}$. □

Theorem 6.2. *Let $\mathfrak{g}$ be reductive and B nondegenerate, and suppose that $\mathcal{A}_k$ is good. Then we have a vertex algebra isomorphism*

$$\lim_{k\to\infty} \mathcal{C}_k \cong \tilde{\mathcal{A}}^G.$$

Proof. The operator $L^{\mathfrak{g}}_0$ acts on the (finite-dimensional) spaces $\mathcal{A}_k[n]$ of weight n and commutes with G, so it maps $\mathcal{A}^G_k[n]$ to itself. By the preceding lemma, $\mathcal{C}_k[n]$ is the kernel of this map. Let $\phi : \mathcal{A}_k[n] \to \mathcal{A}_\infty[n] = (\mathcal{H}(d) \otimes \tilde{\mathcal{A}})[n]$ be the map sending $\omega \mapsto \lim_{k\to\infty} \omega$, which is injective and maps $\mathcal{C}_k[n]$ into $\tilde{\mathcal{A}}[n]$. Then $\Phi = \phi \circ L^{\mathfrak{g}}_0 : \mathcal{A}^G_k[n] \to (\mathcal{H}(d) \otimes \tilde{\mathcal{A}})^G[n]$ also has kernel equal to $\mathcal{C}_k[n]$. It is enough to show that dim(Ker$(\Phi)) \geq$ dim$(\tilde{\mathcal{A}}^G[n])$. Equivalently, we need to show that dim(Coker$(\Phi)) \geq$ dim$(\tilde{\mathcal{A}}^G[n])$. To see this, note that any element in the image of $L^{\mathfrak{g}}_0$ is a linear combination of elements of the form $:(\partial^i a^{\xi_i})\nu:$ for $\xi_i \in \mathfrak{g}$ and $i \geq 0$. Under ϕ these get mapped to $:(\partial^i \alpha^{\xi_i})\phi(\nu):$. In particular, each term has weight at least one under $L^{\mathcal{H}}_0$, so $\tilde{\mathcal{A}}^G[n]$ injects into Coker(Φ). □

Corollary 6.3. *Let $\mathfrak{g}$ be reductive and B nondegenerate, and suppose that $\mathcal{A}_k$ is good. Suppose that $\tilde{\mathcal{A}}$ is a tensor product of free field and affine vertex algebras, and the induced action of G preserves each tensor factor. Then $\mathcal{C}_k$ is strongly finitely generated for generic values of k.*

Corollary 6.4. *Let $\mathfrak{g}$ be reductive and B nondegenerate. Let $\mathfrak{g}'$ be a Lie (super)algebra containing $\mathfrak{g}$, equipped with a nondegenerate (super)symmetric form B' extending B, and let $\mathcal{A}_k = V_k(\mathfrak{g}', B')$. Then*

$$\mathcal{C}_k = \textit{Com}(V_k(\mathfrak{g}, B), V_k(\mathfrak{g}', B'))$$

is strongly finitely generated for generic values of k.

Let $\mathcal{F}$ be a free field algebra admitting a map $V_l(\mathfrak{g}, B) \to \mathcal{F}$ for some fixed l, and let $\mathcal{A}_k = V_{k-l}(\mathfrak{g}', B') \otimes \mathcal{F}$. If the induced action of G preserves the tensor factors of $\mathcal{F}$,

$$\mathcal{C}_k = \textit{Com}(V_k(\mathfrak{g}, B), V_{k-l}(\mathfrak{g}', B') \otimes \mathcal{F})$$

is strongly finitely generated for generic values of k.

Finally, let $\mathfrak{g}''$ be another Lie (super)algebra containing $\mathfrak{g}$, equipped with a nondegenerate (super)symmetric form B'' extending B, and let $\mathcal{A}_k = V_{k-l}(\mathfrak{g}', B') \otimes V_l(\mathfrak{g}'', B'')$. Then

$$\mathcal{C}_k = \mathit{Com}(V_k(\mathfrak{g}, B), V_{k-l}(\mathfrak{g}', B') \otimes V_l(\mathfrak{g}'', B''))$$

is strongly finitely generated for generic values of both k and l.

7. Some examples

To illustrate the constructive nature of our results, this section is devoted to finding minimal strong finite generating sets for $\mathcal{C}_k$ in some concrete examples.

Example 7.1. Let $\mathfrak{g} = \mathfrak{sp}_{2n}$ and let $\mathcal{A}_k = V_{k+1/2}(\mathfrak{sp}_{2n}) \otimes \mathcal{S}(n)$. Using the map $V_{-1/2}(\mathfrak{sp}_{2n}) \to \mathcal{S}(n)$ given by (6.1), we have the diagonal map $V_k(\mathfrak{sp}_{2n}) \to \mathcal{A}_k$. Clearly

$$\mathcal{C}_k = \text{Com}(V_k(\mathfrak{sp}_{2n}), \mathcal{A}_k)$$

satisfies $\mathcal{C}_\infty \cong \mathcal{S}(n)^{Sp(2n)}$ which is of type $\mathcal{W}(2, 4, \dots, 2n^2 + 4n)$ by Theorem 9.4 of [LV]. It follows from Corollary 6.4 that for generic values of k, $\mathcal{C}_k$ is of type $\mathcal{W}(2, 4, \dots, 2n^2 + 4n)$.

It is well known [KWY] that $\mathcal{S}(n)^{Sp(2n)} \cong L_{-1/2}(\mathfrak{sp}_{2n})^{Sp(2n)}$ where $L_{-1/2}(\mathfrak{sp}_{2n})$ denotes the irreducible quotient of $V_{-1/2}(\mathfrak{sp}_{2n})$. We obtain the following result, which was conjectured by Blumenhagen, Eholzer, Honecker, Hornfeck, and Hubel (see Table 7 of [B-H]).

Corollary 7.2. *For $\mathcal{A}_k = V_{k+1/2}(\mathfrak{sp}_{2n}) \otimes L_{-1/2}(\mathfrak{sp}_{2n})$, $\mathcal{C}_k = \mathit{Com}(V_k(\mathfrak{sp}_{2n}), \mathcal{A}_k)$ is of type $\mathcal{W}(2, 4, \dots, 2n^2 + 4n)$ for generic values of k.*

Example 7.3. Let $\mathfrak{g} = \mathfrak{sp}_{2n}$ and $\mathcal{A}_k = V_k(\mathfrak{osp}(1|2n))$. Then $\mathcal{C}_k = \text{Com}(V_k(\mathfrak{sp}_{2n}), V_k(\mathfrak{osp}(1|2n))$ satisfies $\lim_{k\to\infty} \mathcal{C}_k \cong \mathcal{A}(n)^{Sp(2n)}$. Since $\mathcal{A}(n)^{Sp(2n)}$ is of type $\mathcal{W}(2, 4, \dots, 2n)$ by Theorem 3.11 of [CLII], we obtain

Corollary 7.4. *$\mathcal{C}_k = \mathit{Com}(V_k(\mathfrak{sp}_{2n}), V_k(\mathfrak{osp}(1|2n))$ is of type $\mathcal{W}(2, 4, \dots, 2n)$ for generic k.*

In fact, by Corollary 5.7 of [CLII], $\mathcal{A}(n)^{Sp(2n)}$ is *freely generated*; there are no nontrivial normally ordered polynomial relations among the generators and their derivatives. It follows that $\mathcal{C}_k$ is freely generated for generic values of k.

Example 7.5. Next, let $\mathfrak{g} = \mathfrak{sp}_{2n}$ and $\mathcal{A} = V_{k+1/2}(\mathfrak{osp}(1|2n) \otimes \mathcal{S}(n)$. Then

$$\mathcal{C}_k = \text{Com}(V_k(\mathfrak{sp}_{2n}), V_{k-1/2}(\mathfrak{osp}(1|2n) \otimes \mathcal{S}(n))$$

satisfies $\lim_{k\to\infty} \mathcal{C}_k \cong (\mathcal{A}(n) \otimes \mathcal{S}(n))^{Sp(2n)}$.

Lemma 7.6. *$(\mathcal{A}(n) \otimes \mathcal{S}(n))^{Sp(2n)}$ has the following minimal strong generating set:*

$$j^{2k} = \frac{1}{2}\Big(\sum_{i=1}^{n} :e^i \partial^{2k} f^i : + : (\partial^{2k} e^i) f^i :\Big), \qquad 0 \leq k \leq n-1,$$

$$w^{2k+1} = \frac{1}{2}\Big(\sum_{i=1}^{n} :\beta^i \partial^{2k+1} \gamma^i - : (\partial^{2k+1} \beta^i) \gamma^i :\Big), \qquad 0 \leq k \leq n-1, \tag{7.1}$$

$$\mu^k = \frac{1}{2}\Big(\sum_{i=1}^{n} :\beta^i \partial^k f^i - : \gamma^i \partial^k e^i :\Big), \qquad 0 \leq k \leq 2n-1.$$

In particular, $(\mathcal{A}(n) \otimes \mathcal{S}(n))^{Sp(2n)}$ has a minimal strong generating set consisting of even generators in weights $2, 2, 4, 4, \dots, 2n, 2n$ and odd generators in weights $\frac{3}{2}, \frac{5}{2}, \dots, \frac{4n+1}{2}$.

Proof. The argument similar to the proof of Theorem 7.1 of [CLI], and some details are omitted. By passing to the associated graded algebra and applying Weyl's first fundamental theorem of invariant theory for Sp_{2n}, we obtain the following strong generating set for $(\mathcal{A}(n)\otimes\mathcal{S}(n))^{Sp(2n)}$:

$$\frac{1}{2}\big(\sum_{i=1}^n :\partial^a e^i \partial^b f^i: + :(\partial^b\partial^a e^i)f^i:\big),\qquad a,b\geq 0,$$

$$\frac{1}{2}\big(\sum_{i=1}^n :\partial^a \beta^i \partial^b \gamma^i - :(\partial^b\beta^i)\partial^a\gamma^i:\big),\qquad a,b\geq 0,$$

$$\frac{1}{2}\big(\sum_{i=1}^n :\partial^a \beta^i \partial^b f^i - :\partial^a\gamma^i\partial^b e^i:\big),\qquad a,b\geq 0.$$

As in Theorem 7.1 of [CLI], we use the relation of minimal weight to construct decoupling relations eliminating all but the set (7.1). □

Corollary 7.7. *For generic values of k, $\mathcal{C}_k$ has a minimal strong generating set consisting of even generators in weights $2,2,4,4,\dots,2n,2n$ and odd generators in weights $\frac{3}{2},\frac{5}{2},\dots,\frac{4n+1}{2}$.*

Let $L=-j^0+w^1$ denote the Virasoro element of $(\mathcal{A}(n)\otimes\mathcal{S}(n))^{Sp(2n)}$, which has central charge $-3n$. Then L and μ^0 generate a copy of the $N=1$ superconformal algebra with $c=-3n$. Similarly, for noncritical values of k, $L=L^{\mathfrak{osp}(1|2n)}-L^{\mathfrak{sp}_{2n}}+w^1$ and μ^0 generate a copy of the $N=1$ algebra inside $\mathcal{C}_k$.

Example 7.8. Let $\mathfrak{g}=\mathfrak{gl}_n$ and $\mathcal{A}_k=V_k(\mathfrak{gl}(n|1))$. In this case, $\mathcal{C}_k=\text{Com}(V_k(\mathfrak{gl}_n),\mathcal{A}_k)$ satisfies $\lim_{k\to\infty}\mathcal{C}_k\cong\mathcal{H}(1)\otimes(\mathcal{A}(n)^{GL(n)})$. By Theorem 4.3 of [CLII], $\mathcal{A}(n)^{GL(n)}$ is of type $\mathcal{W}(2,3,\dots,2n+1)$ so $\mathcal{C}_k$ is generically of type $\mathcal{W}(1,2,3,\dots,2n+1)$.

Example 7.9. Let $\mathfrak{g}=\mathfrak{gl}_n$ and $\mathcal{A}_k=V_k(\mathfrak{gl}(n|1))\otimes\mathcal{S}(n)$. In this case, $\mathcal{C}_k=\text{Com}(V_k(\mathfrak{gl}_n),\mathcal{A}_k)$ satisfies $\lim_{k\to\infty}\mathcal{C}_k\cong\mathcal{H}(1)\otimes(\mathcal{A}(n)\otimes\mathcal{S}(n))^{GL(n)}$.

Lemma 7.10. $(\mathcal{A}(n)\otimes\mathcal{S}(n))^{GL(n)}$ *has the following minimal strong generating set:*

$$w^k=\sum_{i=1}^n :e^i\partial^k f^i:,\qquad j^k=\sum_{i=1}^n :\beta^i\partial^k\gamma^i:,\qquad 0\leq k\leq 2n-1,$$

$$\nu^k=\sum_{i=1}^n :e^i\partial^k \gamma^i:,\qquad \mu^k=\sum_{i=1}^n :\beta^i\partial^k f^i:,\qquad 0\leq k\leq 2n-1.$$

The even generators are in weights $1,2,2,3,3,\dots,2n,2n,2n+1$, and the odd generators are in weights $\frac{3}{2},\frac{3}{2},\frac{5}{2},\frac{5}{2},\dots,\frac{4n+1}{2},\frac{4n+1}{2}$.

Proof. The argument is the same as the proof of Lemma 7.6. □

Therefore $\mathcal{C}_k$ has a minimal strong generating set with even generators in weights $1,1,2,2,3,3,\dots,2n,2n,2n+1$, and odd generators in weights $\frac{3}{2},\frac{3}{2},\frac{5}{2},\frac{5}{2},\dots,\frac{4n+1}{2},\frac{4n+1}{2}$, for generic values of k. Note that $(\mathcal{A}(n)\otimes\mathcal{S}(n))^{GL(n)}$ has the following $N=2$ superconformal structure of central charge $-3n$.

$$L=j^1-\frac{1}{2}\partial j^0-w^0,\qquad F=j^0,\qquad G^+=\nu^0,\qquad G^-=\mu^0. \tag{7.2}$$

For noncritical values of k, this deforms to an $N=2$ superconformal structure on $\mathcal{C}_k$ given by

$$L = j^1 - \frac{1}{2}\partial j^0 + L^{\mathfrak{gl}(n|1)} - L^{\mathfrak{gl}_n}, \qquad F = j^0, \qquad G^+ = \sum_{i=1}^n :X^{\eta_i^-}\gamma^i:, \qquad G^- = \sum_{i=1}^n :\beta^i X^{\eta_i^+}: .$$

Example 7.11. Let $\mathfrak{g} = \mathfrak{gl}_n$ and $\mathcal{A}_k = V_{k-1}(\mathfrak{sl}_{n+1}) \otimes \mathcal{E}(n)$. There is a map $V_k(\mathfrak{gl}_n) \to V_{k-1}(\mathfrak{sl}_{n+1})$ corresponding to the natural embedding $\mathfrak{gl}_n \to \mathfrak{sl}_{n+1}$, and a homomorphism $V_1(\mathfrak{gl}_n) \to \mathcal{E}(n)$ appearing in [FKRW], so we have a diagonal homomorphism $V_k(\mathfrak{gl}_n) \to \mathcal{A}_k$. Then $\mathcal{C}_k = \mathrm{Com}(V_k(\mathfrak{gl}_n), \mathcal{A}_k)$ satisfies $\lim_{k\to\infty} \mathcal{C}_k = (\mathcal{H}(2n) \times \mathcal{E}(n))^{GL(n)}$, where $\mathcal{H}(2n)$ is the Heisenberg algebra with generators $a^1, \dots, a^n$ and $\bar{a}^1, \dots, \bar{a}^n$ satisfying

$$a^i(z)\bar{a}^j(w) \sim \delta_{i,j}(z-w)^{-2}, \qquad a^i(z)a^j(w) \sim 0, \qquad \bar{a}^i(z)\bar{a}^j(w) \sim 0.$$

Lemma 7.12. *$(\mathcal{H}(2n) \times \mathcal{E}(n))^{GL(n)}$ has a minimal strong generating set*

$$j^k = \sum_{i=1}^n :b^i\partial^k c^i:, \qquad w^k = \sum_{i=1}^n :a^i\partial^k \bar{a}^i: \qquad 0 \le k \le n-1,$$

$$\nu^k = \sum_{i=1}^n :b^i\partial^k \bar{a}^i:, \qquad \mu^k = \sum_{i=1}^n :a^i\partial^k c^i:, \qquad 0 \le k \le n-1.$$

In particular, $(\mathcal{H}(2n) \times \mathcal{E}(n))^{GL(n)}$ has even generators in weights $1,2,2,3,3,\dots,n,n,n+1$, and odd generators in weights $\frac{3}{2}, \frac{3}{2}, \frac{5}{2}, \frac{5}{2}, \dots, \frac{2n+1}{2}, \frac{2n+1}{2}$.

Proof. This is the same as the proof of Lemma 7.6. □

Therefore $\mathcal{C}_k$ has a minimal strong generating set in the same weights for generic values of k. Finally, $(\mathcal{H}(2n) \times \mathcal{E}(n))^{GL(n)}$ has an $N=2$ superconformal structure given by

$$L = -j^1 + \frac{1}{2}\partial j^0 - w^0, \qquad F = j^0, \qquad G^+ = \nu^0, \qquad G^- = \mu^0, \tag{7.3}$$

which deforms to an $N=2$ superconformal structure on $\mathcal{C}_k$.

This example is called the Kazama-Suzuki coset [KS] of complex projective space in the physics literature. It is conjectured [I] to be a super $\mathcal{W}$-algebra of $\mathfrak{sl}(n+1|n)$ corresponding to the principal nilpotent embedding of $\mathfrak{sl}_2$. In Section 8, we will explicitly determine the set of nongeneric values of k in the case $n=1$. As a consequence, we will describe $\mathrm{Com}(\mathcal{H}(1), L_k(\mathfrak{sl}_2) \otimes \mathcal{E}(1))$ for all positive integers k, and prove its rationality.

Example 7.13. Let $\mathfrak{g} = \mathfrak{sl}_n$ and $\mathcal{A}_k = V_{k-1}(\mathfrak{sl}_n) \otimes L_1(\mathfrak{sl}_n)$. Note that $L_1(\mathfrak{sl}_n) \cong \mathrm{Com}(\mathcal{H}, \mathcal{E}(n))$ where $\mathcal{H}$ is the copy of the rank one Heisenberg algebra generated by $\sum_{i=1}^n :b^i c^i:$ and $\mathcal{E}(n)$ is the rank n bc-system. Then $\mathcal{C}_k = \mathrm{Com}(V_k(\mathfrak{sl}_n), \mathcal{A}_k)$ satisfies

$$\lim_{k\to\infty} \mathcal{C}_k \cong L_1(\mathfrak{sl}_n)^{SL(n)} = \mathrm{Com}(\mathcal{H}, \mathcal{E}(n))^{SL(n)} \cong \mathrm{Com}(\mathcal{H}, \mathcal{E}(n)^{SL(n)}) = \mathrm{Com}(\mathcal{H}, \mathcal{E}(n)^{GL(n)}).$$

By [FKRW], $\mathcal{E}(n)^{GL(n)} \cong \mathcal{W}_{1+\infty,n} \cong \mathcal{W}(\mathfrak{gl}_n)$ so $\lim_{k\to\infty} \mathcal{C}_k \cong \mathcal{W}(\mathfrak{sl}_n)$ and hence is of type $\mathcal{W}(2,3,\dots,n)$. It follows that $\mathcal{C}_k$ is of type $\mathcal{W}(2,3,\dots,n)$ for generic values of k.

More generally, let $\mathfrak{g}$ be any simple, finite-dimensional, simply laced Lie algebra. The action of $\mathfrak{g}$ on $L_1(\mathfrak{g})$ integrates to an action of a connected Lie group G with Lie algebra $\mathfrak{g}$, and it is known that $L_1(\mathfrak{g})^G$ is isomorphic to the $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{g})$ associated to the principal embedding of $\mathfrak{sl}_2$ in $\mathfrak{g}$, with central charge $c = \mathrm{rank}(\mathfrak{g})$ [BBSS, BS, F].

Let $\mathcal{A}_k = V_{k-1}(\mathfrak{g}) \otimes L_1(\mathfrak{g})$, equipped with the diagonal embedding $V_k(\mathfrak{g}) \to \mathcal{A}_k$. Then $\mathcal{C}_k = \text{Com}(V_k(\mathfrak{g}), \mathcal{A}_k)$ satisfies

$$\lim_{k\to\infty} \mathcal{C}_k \cong L_1(\mathfrak{g})^G.$$

Therefore $\mathcal{C}_k$ has strong generators in the same weights as $\mathcal{W}(\mathfrak{g})$ for generic values of k.

Example 7.14. Let $\mathfrak{g} = \mathfrak{so}_n$ and let $\mathcal{A}_k = V_{k-1}(\mathfrak{so}_n) \otimes L_1(\mathfrak{so}_n)$. We have a projection $V_1(\mathfrak{so}_n) \to L_1(\mathfrak{so}_n)$, and a diagonal map $V_k(\mathfrak{so}_n) \to \mathcal{A}_k$. In this case we are interested not in $\mathcal{C}_k = \text{Com}(V_k(\mathfrak{so}_n), \mathcal{A}_k)$ but in the orbifold $(\mathcal{C}_k)^{\mathbb{Z}/2\mathbb{Z}}$. Note that $\mathbb{Z}/2\mathbb{Z}$ acts on each of the vertex algebras $V_k(\mathfrak{so}_n)$, $V_{k-1}(\mathfrak{so}_n)$ and $L_1(\mathfrak{so}_n)$; the action is defined on generators by multiplication by -1. There is an induced action of $\mathbb{Z}/2\mathbb{Z}$ on $\mathcal{C}_k$. We have isomorphisms

$$\lim_{k\to\infty}((\mathcal{C}_k)^{\mathbb{Z}/2\mathbb{Z}}) \cong \lim_{k\to\infty}(\mathcal{C}_k)^{\mathbb{Z}/2\mathbb{Z}} \cong \left(L_1(\mathfrak{so}_n)^{SO(n)}\right)^{\mathbb{Z}/2\mathbb{Z}} \cong L_1(\mathfrak{so}_n)^{O(n)} \cong \mathcal{F}(n)^{O(n)}.$$

This appears as Theorems 14.2 and 14.3 of [KWY] in the cases where n is even and odd, respectively; in both cases, $L_1(\mathfrak{so}_n)^{SO(n)}$ decomposes as the direct sum of $\mathcal{F}(n)^{O(n)}$ and an irreducible, highest-weight $\mathcal{F}(n)^{O(n)}$-module. Since $\mathcal{F}(n)^{O(n)}$ is of type $\mathcal{W}(2,4,\dots,2n)$, the following result, which was conjectured in Table 7 of [B-H], is an immediate consequence.

Corollary 7.15. $\lim_{k\to\infty}(\mathcal{C}_k)^{\mathbb{Z}/2\mathbb{Z}}$ *is of type* $\mathcal{W}(2,4,\dots,2n)$*, so* $(\mathcal{C}_k)^{\mathbb{Z}/2\mathbb{Z}}$ *is of type* $\mathcal{W}(2,4,\dots,2n)$ *for generic values of* k*.*

Example 7.16. Let $\mathfrak{g} = \mathfrak{so}_n$ and let $\mathcal{A}_k = V_{k-1}(\mathfrak{so}_{n+1}) \otimes \mathcal{F}(n)$. Recall that we have a map $V_1(\mathfrak{so}_n) \to \mathcal{F}(n)$, so we have a diagonal map $V_k(\mathfrak{so}_n) \to \mathcal{A}_k$. As above, there is an action of $\mathbb{Z}/2\mathbb{Z}$ acts on each of the vertex algebras $V_k(\mathfrak{so}_n)$, $V_{k+1}(\mathfrak{so}_{n+1})$ and $\mathcal{F}(n)$, and therefore on $\mathcal{C}_k$, and we are interested in the orbifold $(\mathcal{C}_k)^{\mathbb{Z}/2\mathbb{Z}}$. We have $\mathcal{C}_\infty = (\mathcal{H}(n) \otimes \mathcal{F}(n))^{SO(n)}$, and $(\mathcal{C}_\infty)^{\mathbb{Z}/2\mathbb{Z}} = (\mathcal{H}(n) \otimes \mathcal{F}(n))^{O(n)}$.

Lemma 7.17. $(\mathcal{H}(n) \otimes \mathcal{F}(n))^{O(n)}$ *has the following minimal strong generating set.*

$$w^{2k+1} = \sum_{i=1}^{n} :\phi^i \partial^{2k+1}\phi^i:, \qquad j^{2k} = \sum_{i=1}^{n} :\alpha^i\partial^{2k}\alpha^i:, \qquad 0 \leq k \leq n-1,$$

$$\mu^k = \sum_{i=1}^{n} :\alpha^i\partial^k\phi^i:, \qquad 0 \leq k \leq 2n-1.$$

In particular, $(\mathcal{H}(n) \otimes \mathcal{F}(n))^{O(n)}$ *has even generators in weights* $2,2,4,4,\dots,2n,2n$ *and odd generators in weights* $\frac{3}{2}, \frac{5}{2}, \dots, \frac{4n+1}{2}$.

The proof is the same as the proof of Lemma 7.6, and it implies that $(\mathcal{C}_k)^{\mathbb{Z}/2\mathbb{Z}}$ has strong generators in the same weights for generic values of k. Moreover, $(\mathcal{H}(n) \otimes \mathcal{F}(n))^{O(n)}$ has an $N=1$ superconformal structure with generators $L = -w^0 + \frac{1}{2}j^0$ and μ^0, which deforms to an $N=1$ structure on $(\mathcal{C}_k)^{\mathbb{Z}/2\mathbb{Z}}$.

Example 7.18. Let $\mathfrak{g}$ be a simple, finite-dimensional Lie algebra of rank d with Cartan subalgebra $\mathfrak{h}$. For a positive integer k, the *parafermion algebra* $N_k(\mathfrak{g})$ is defined to be $\text{Com}(\mathcal{H}(d), L_k(\mathfrak{g}))$, where $\mathcal{H}(d)$ is the Heisenberg algebra corresponding to $\mathfrak{h}$ and $L_k(\mathfrak{g})$ is the irreducible affine vertex algebra at level k. The coset $\mathcal{C}_k(\mathfrak{g}) = \text{Com}(\mathcal{H}(d), V_k(\mathfrak{g}))$ is defined for all $k \in \mathbb{C}$, and for a positive integer k, $N_k(\mathfrak{g})$ is the irreducible quotient of $\mathcal{C}_k(\mathfrak{g})$ by its maximal proper ideal. In the case $\mathfrak{g} = \mathfrak{sl}_2$, it follows from Theorems 2.1 and 3.1 of [DLWY] that $\mathcal{C}_k(\mathfrak{sl}_2)$ is of type $\mathcal{W}(2,3,4,5)$ for all $k \neq 0$. This was used to establish the

C_2-cofiniteness of $N_k(\mathfrak{sl}_2)$ for positive integer values of k, and plays an important role in the structure of $N_k(\mathfrak{g})$ for a general simple $\mathfrak{g}$ [ALY].

For any simple $\mathfrak{g}$ of rank d, a choice of simple roots give rise to d copies of $\mathfrak{sl}_2$ inside $\mathfrak{g}$ which generate $\mathfrak{g}$. We have corresponding embeddings of $\mathcal{W}(2,3,4,5)$ into $\mathcal{C}_k(\mathfrak{g})$, whose images generate $\mathcal{C}_k(\mathfrak{g})$ [DWI]. However, these do not *strongly* generate $\mathcal{C}_k(\mathfrak{g})$. Corollary 6.4 implies that $\mathcal{C}_k(\mathfrak{g})$ is strongly finitely generated for generic values of k for any simple $\mathfrak{g}$. We shall construct a minimal strong generating set for $\mathcal{C}_k(\mathfrak{sl}_3)$ consisting of 30 elements.

We work in the usual basis for $\mathfrak{sl}_3$ consisting of $\{\xi_{ij}|\ i\neq j\}$ together with $\{\xi_{ii}-\xi_{i+1,i+1}\}$ for $i=1,2$. We have $\lim_{k\to\infty} V_k(\mathfrak{sl}_3)\cong \mathcal{H}(2)\otimes\tilde{\mathcal{A}}$ where $\tilde{\mathcal{A}}\cong\mathcal{H}(6)$ with generators $\alpha^{12},\alpha^{23},\alpha^{13},\alpha^{21},\alpha^{32},\alpha^{31}$. After suitably rescaling, these generators satisfy

$$\alpha^{12}(z)\alpha^{21}(w)\sim(z-w)^{-2},\qquad \alpha^{23}(z)\alpha^{32}(w)\sim(z-w)^{-2},\qquad \alpha^{13}(z)\alpha^{31}(w)\sim(z-w)^{-2}.$$

Note that $\mathcal{H}(6)$ carries an action of $G=\mathbb{C}^*\times\mathbb{C}^*$ which is infinitesimally generated by the action of $\mathfrak{h}$.

Lemma 7.19. *$\mathcal{H}(6)^G$ is of type $\mathcal{W}(2^3,3^5,4^7,5^9,6^4,7^2)$. In other words, a minimal strong generating set consists of 3 fields in weight 2, 5 fields in weight 3, 7 fields in weight 4, 9 fields in weight 5, 4 fields in weight 6, and 2 fields in weight 7.*

Proof. By classical invariant theory, $\mathcal{H}(6)^G$ has a strong generating set consisting of the normally ordered monomials

$$\begin{aligned}&q^{12}_{i,j}=\,:\partial^i\alpha^{12}\partial^j\alpha^{21}:,\qquad q^{13}_{i,j}=\,:\partial^i\alpha^{13}\partial^j\alpha^{31}:,\qquad q^{23}_{i,j}=\,:\partial^i\alpha^{23}\partial^j\alpha^{32}:,\qquad i,j\geq 0,\\ &c_{i,j,k}=\,:\partial^i\alpha^{12}\partial^j\alpha^{23}\partial^k\alpha^{31}:,\qquad c'_{i,j,k}=\,:\partial^i\alpha^{21}\partial^j\alpha^{32}\partial^k\alpha^{13}:,\qquad i,j,k\geq 0.\end{aligned}\tag{7.4}$$

Not all of these generators are necessary. In fact, $\{q^{12}_{0,i}|\ 0\leq i\leq 3\}$, $\{q^{23}_{0,i}|\ 0\leq i\leq 3\}$, and $\{q^{13}_{0,i}|\ 0\leq i\leq 3\}$ generate three commuting copies of $\mathcal{W}(2,3,4,5)$, and all the above quadratics lie in one of these copies. Similarly, we need at most $\{c_{i,j,k},c'_{i,j,k}|\ i,j,k\leq 2\}$. This follows from the decoupling relations

$$\begin{aligned}&:q^{12}_{0,0}c_{i,j,k}:-:q^{12}_{i,0}c_{0,j,k}:=-\frac{i}{2i+4}c_{i+2,j,k},\qquad i\geq 1,\qquad j,k\geq 0,\\ &:q^{23}_{0,0}c_{i,j,k}:-:q^{23}_{j,0}c_{i,0,k}:=-\frac{j}{2j+4}c_{i,j+2,k},\qquad j\geq 1,\qquad i,k\geq 0,\\ &:q^{31}_{0,0}c_{i,j,k}:-:q^{31}_{i,0}c_{i,j,0}:=-\frac{k}{2k+4}c_{i,j,k+2},\qquad k\geq 1,\qquad i,j\geq 0,\\ &:q^{21}_{0,0}c'_{i,j,k}:-:q^{21}_{i,0}c'_{0,j,k}:=-\frac{i}{2i+4}c'_{i+2,j,k},\qquad i\geq 1,\qquad j,k\geq 0,\\ &:q^{32}_{0,0}c'_{i,j,k}:-:q^{32}_{j,0}c'_{i,0,k}:=-\frac{j}{2j+4}c'_{i,j+2,k},\qquad j\geq 1,\qquad i,k\geq 0,\\ &:q^{13}_{0,0}c'_{i,j,k}:-:q^{13}_{i,0}c'_{i,j,0}:=-\frac{k}{2k+4}c'_{i,j,k+2},\qquad k\geq 1,\qquad i,j\geq 0.\end{aligned}\tag{7.5}$$

There are some relations among the above cubics and their derivatives, such as $\partial c_{0,0,0}=c_{1,0,0}+c_{0,1,0}+c_{0,0,1}$. It is not difficult to check that a minimal strong generating set for $\mathcal{H}(6)^G$ consists of

$$\{q^{12}_{0,i},q^{23}_{0,i},q^{13}_{0,i}|\ 0\leq i\leq 3\}\cup\{c_{0,j,k},c'_{0,j,k}|\ 0\leq j,k\leq 2\}.$$

In particular, $\mathcal{H}(6)^G$ is of type $\mathcal{W}(2^3,3^5,4^7,5^9,6^4,7^2)$. ◻

Corollary 7.20. *For generic values of k, $\mathcal{C}_k(\mathfrak{sl}_3)$ is also of type $\mathcal{W}(2^3, 3^5, 4^7, 5^9, 6^4, 7^2)$.*

A similar procedure will yield minimal strong generating sets for $\mathcal{C}_k(\mathfrak{g})$ for any simple $\mathfrak{g}$ when k is generic.

8. Cosets of simple affine vertex algebras inside larger structures

Let $\mathcal{A}_k$ be a vertex algebra depending on a parameter k with a weight grading by $\mathbb{Z}_{\geq 0}$, such that all weight spaces are finite-dimensional. Let $\mathfrak{g}$ be simple and assume that $\mathcal{A}_k$ admits an injective map $V_k(\mathfrak{g}) \to \mathcal{A}_k$, and let $\mathcal{C}_k = \text{Com}(V_k(\mathfrak{g}), \mathcal{A}_k)$ as before. Suppose that k is a parameter value for which $\mathcal{A}_k$ is not simple. Let $\mathcal{I}$ be the maximal proper ideal of $\mathcal{A}_k$ graded by conformal weight, so that $\bar{\mathcal{A}}_k = \mathcal{A}_k/\mathcal{I}$ is simple. Let $\mathcal{J}$ denote the kernel of the map $V_k(\mathfrak{g}) \to \bar{\mathcal{A}}_k$, and suppose that $\mathcal{J}$ is maximal so that $V_k(\mathfrak{g})/\mathcal{J} = L_k(\mathfrak{g})$. Let

$$\bar{\mathcal{C}}_k = \text{Com}(L_k(\mathfrak{g}), \bar{\mathcal{A}}_k)$$

denote the corresponding coset. There is always a vertex algebra homomorphism

$$\pi_k : \mathcal{C}_k \to \bar{\mathcal{C}}_k,$$

but in general this map need not be surjective. Of particular interest is the case where k is a positive integer and $\bar{\mathcal{A}}_k$ is C_2-cofinite and rational. It is then expected that $\bar{\mathcal{C}}_k$ will also be C_2-cofinite and rational. In order to apply our results on the generic behavior of $\mathcal{C}_k$ to the structure of $\bar{\mathcal{C}}_k$, the two problems must be solved.

(1) Find conditions for which π_k is surjective, so that a strong generating set for $\mathcal{C}_k$ descends to a strong generating set for $\bar{\mathcal{C}}_k$.
(2) Suppose that $S \subset \mathcal{C}_k$ is a strong generating set for $\mathcal{C}_k$ for generic values of k. We call $k \in \mathbb{C}$ *nongeneric* if $\mathcal{C}_k$ is not strongly generated by S. Find an algorithm for determining which values of k are generic.

Theorem 8.1. *Suppose that $k + h^\vee$ is a positive real number and all zero modes of the currents of the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ in $V_k(\mathfrak{g})$ act semisimply on $\mathcal{A}_k$. Then $\pi_k : \mathcal{C}_k \to \bar{\mathcal{C}}_k$ is surjective.*

Proof. First, $\mathcal{A}_k$ decomposes into a direct sum of indecomposable $V_k(\mathfrak{g}) \otimes \mathcal{C}_k$ modules, and this sum is bigraded by the conformal weights of the two conformal vectors. Each bigraded subspace is finite-dimensional since the two conformal weights add up to the total one, and that weight space is finite-dimensional.

Thus every such indecomposable $V_k(\mathfrak{g})$-module M appearing in $\mathcal{A}_k$ must have finite-dimensional lowest weight subspace. Since the zero modes of the Heisenberg subalgebra corresponding to $\mathfrak{h}$ act semisimply, M must be a subquotient of a finite direct sum of Verma modules, each of which is induced from an irreducible, finite-dimensional $\mathfrak{g}$-module with highest weight Λ. The conformal dimension of such a module is given by

$$\frac{(\Lambda + \rho|\Lambda)}{2(k + h^\vee)}$$

and hence the vacuum module has lowest possible conformal dimension. It therefore cannot appear in any composition factor of the other modules. Of course quotients of the vacuum module can appear in the decomposition of $\mathcal{A}_k$, but their multiplicity spaces will be part of $\mathcal{C}_k$. In other words,

$$\mathcal{A}_k = \Big(\bigoplus_i M_i \otimes C_i\Big) \bigoplus D,$$

where each M_i is a homomorphic image of $V_k(\mathfrak{g})$, C_i is a $\mathcal{C}_k$-module, and D is a sum of indecomposable $V_k(\mathfrak{g}) \otimes \mathcal{C}_k$ modules which do not contain the vacuum module $V_k(\mathfrak{g})$ or any of its quotients. In particular, note that $\mathcal{C}_k = \bigoplus_i C_i$.

Similarly, the indecomposable summands of the ideal $\mathcal{I} \subset \mathcal{A}_k$ will be submodules of the indecomposable summands of $\mathcal{A}_k$. Letting $\mathcal{I}_i = \mathcal{I} \cap (M_i \otimes C_i)$ and $\mathcal{I}_D = \mathcal{I} \cap D$, we have a decomposition

$$\mathcal{A}_k/\mathcal{I} \cong \left(\bigoplus_i (M_i \otimes C_i)/\mathcal{I}_i \right) \bigoplus D/\mathcal{I}_D.$$

Note that each module $(M_i \otimes C_i)/\mathcal{I}_i$ is a module for $L_k(\mathfrak{g})$, and hence must be either zero or $L_k(\mathfrak{g}) \otimes \bar{C}_i$ where $\bar{C}_i$ is a homomorphic image of C_i. Since the remaining summand $D/\mathcal{I}_D$ does not contain any quotient of the vacuum module, it follows that

$$\bar{\mathcal{C}}_k = \bigoplus_i \bar{C}_i$$

is a homomorphic image of $\mathcal{C}_k = \bigoplus_i C_i$. □

This theorem applies in particular to the case where $\mathfrak{g}$ is simple and simply laced, and $\mathcal{A} = V_{k-1}(\mathfrak{g}) \otimes L_1(\mathfrak{g})$. We obtain

Corollary 8.2. *Let $\mathfrak{g}$ be simple and simply laced, and $\mathcal{A}_k = V_{k-1}(\mathfrak{g}) \otimes L_1(\mathfrak{g})$, and let $k > 1$ be a positive integer. Let $\mathcal{C}_k = Com(V_k(\mathfrak{g}), V_{k-1}(\mathfrak{g}) \otimes L_1(\mathfrak{g}))$ and $\bar{\mathcal{C}}_k = Com(L_k(\mathfrak{g}), L_{k-1}(\mathfrak{g}) \otimes L_1(\mathfrak{g}))$. Then the map $\pi_k : \mathcal{C}_k \to \bar{\mathcal{C}}_k$ is surjective.*

Remark 8.3. *If $V_k(\mathfrak{g})$ is replaced by $V_k(\mathfrak{g}, B)$ where $\mathfrak{g}$ is semisimple and B is a sum of positive scalar multiples of the normalized Killing forms of the simple summand, a similar statement to Theorem 8.1 follows by induction on the number of simple summands. Furthermore, this can be generalized to the case where $\mathfrak{g}$ is reductive since the analogous theorem for cosets of Heisenberg algebras is also straightforward.*

Let $\mathcal{C}_k$ be a coset of the form $\text{Com}(V_k(\mathfrak{g}), \mathcal{A}_k)$ as above, and suppose that $S \subset \mathcal{C}_k$ is a strong generating set for $\mathcal{C}_k$ for generic values of k. It is an important problem to determine which values of k are generic, since Theorem 8.1 gives conditions for $\pi_k : \mathcal{C}_k \to \bar{\mathcal{C}}_k$ to be surjective, and this implies that strong generators for $\mathcal{C}_k$ give rise to strong generators for $\bar{\mathcal{C}}_k$. In this rest of this section, we shall describe some examples where the set of nongeneric values of k can be determined either completely or conjecturally.

Parafermion algebras and related structures. Recall that $\mathcal{C}_k(\mathfrak{sl}_2) = \text{Com}(\mathcal{H}, V_k(\mathfrak{sl}_2))$ is of type $\mathcal{W}(2,3,4,5)$ for all $k \neq 0$. To this best of our knowledge, this is the first example in the literature of a coset of the form $\mathcal{C}_k = \text{Com}(V_k(\mathfrak{g}), \mathcal{A}_k)$ where the set of nongeneric points has been completely determined. (Here $\mathfrak{g} = \mathbb{C}$). A similar example involving the coset of the Heisenberg algebra inside the universal Bershadsky-Polyakov algebra $\mathcal{W}^k$ appears in [ACL]. This example differ slightly from the main examples in this paper since $\mathcal{W}^k$ is not built from affine and free field algebras. However, using a similar approach we proved that $\mathcal{C}_k = \text{Com}(\mathcal{H}, \mathcal{W}^k)$ is of type $\mathcal{W}(2,3,4,5,6,7)$ for all $k \neq -1, -\frac{3}{2}$. The strong generating set descends to a strong generating set for the simple quotient of $\mathcal{C}_k$. It is essential for establishing the C_2-cofiniteness and rationality of this quotient when $\mathcal{W}^k$ has a rational quotient.

Based on these examples, one might be tempted to conjecture that the set of nongeneric values of k for cosets of the form $\mathcal{C}_k$ should be finite. However, based on computational evidence, the next example indicates that this is generally not the case.

The case of Com$(V_k(\mathfrak{sp}_2), V_k(\mathfrak{osp}(1|2)))$**.** Fix even generators $H, X^{\pm}$ and odd generators $\phi^{\pm}$ for $V_k(\mathfrak{osp}(1|2)))$, satisfying

$$H(z)X^{\pm}(w) \sim \pm X^{\pm}(w)(z-w)^{-1}, \quad H(z)H(w) \sim \frac{k}{2}(z-w)^{-2},$$

$$X^{+}(z)X^{-}(w) \sim k(z-w)^{-2} + 2H(w)(z-w)^{-1},$$

$$H(x)\phi^{\pm}(w) \sim \pm\frac{1}{2}\phi^{\pm}(z-w)^{-1}, \qquad X^{\pm}(z)\phi^{\mp}(w) \sim -\phi^{\pm}(w)(z-w)^{-1},$$

$$\phi^{\pm}(z)\phi^{\pm}(w) \sim \pm\frac{1}{2}X^{\pm}(w)(z-w)^{-1}, \qquad \phi^{+}(z)\phi^{-}(w) \sim \frac{k}{2}(z-w)^{-2} + \frac{1}{2}H(w)(z-w)^{-1}.$$

Let $\mathcal{C}_k = \text{Com}(V_k(\mathfrak{sp}_2), V_k(\mathfrak{osp}(1|2)))$, where $V_k(\mathfrak{sp}_2)$ is generated by $H, X^{\pm}$. By Corollary 7.4, for generic values of k, $\mathcal{C}_k$ is isomorphic to the Virasoro algebra with $c = -\frac{k(4k+5)}{(k+2)(2k+3)}$. The generator is

$$L = -\frac{4}{2k+3} : \phi^{+}\phi^{-} : + \frac{1}{(k+2)(2k+3)}\big(: X^{+}X^{-} : + : HH : \big) + \frac{1+k}{(k+2)(2k+3)}\partial H,$$

for $k \neq -2, -\frac{3}{2}$. Recall from [CLII] that $\lim_{k\to\infty} \mathcal{C}_k \cong \mathcal{A}(1)^{Sp_2}$, which has the following strong generating set coming from classical invariant theory:

$$w^{2m} = \frac{1}{2}\big(: e\partial^{2m} f : + : (\partial^{2m} e) f : \big), \qquad m \geq 0.$$

Moreover, we have decoupling relations $w^{2m} = P_{2m}(w^2)$ for all $m \geq 1$, where $P_{2m}(w^2)$ is a normally ordered polynomial in w^2 and its derivatives. There exist deformations $W^{2m} \in \mathcal{C}_k$ which strongly generate $\mathcal{C}_k$ for generic values of k, and have the property that $\lim_{k\to\infty} W^{2m} = w^{2m}$. In particular, $W^0 = -\frac{2k+3}{4}L$. There are normally ordered relations $\lambda_{2m}(k)W^{2m} = Q_{2m}(W^2)$, with the property that $\lim_{k\to\infty} Q_{2m}(W^2) = P_{2m}(w^2)$. Here $\lambda_{2m}(k)$ is a rational function of k satisfying $\lim_{k\to\infty} \lambda_{2m}(k) = 1$. We have checked by computer calculation that

$$\lambda_2(k) = -\frac{k+4}{k+3/2}, \qquad \lambda_4(k) = -\frac{(k+4)(k+8/3)}{(k+3/2)^2},$$

$$\lambda_6(k) = -\frac{(k+4)(k+8/3)(k+12/5)}{(k+3/2)^3}, \qquad \lambda_8(k) = -\frac{(k+4)(k+8/3)(k+12/5)(k+16/7)}{(k+3/2)^4}.$$

This suggests that

$$\lambda_{2m}(k) = -\frac{\prod_{i=1}^{m}(k+\frac{4i}{2i-1})}{(k+3/2)^m},$$

and leads to the following conjecture.

Conjecture 8.4. *$\mathcal{C}_k = Com(V_k(\mathfrak{sp}_2), V_k(\mathfrak{osp}(1|2)))$ is isomorphic to the universal Virasoro vertex algebra for all $k \notin K$, where $K = \{-2, -\frac{3}{2}\} \cup \{-\frac{4i}{2i-1} | \, i \geq 1\}$.*

Based on computer calculations, it seems plausible that for $k \in K$, $\mathcal{C}_k$ is not strongly finitely generated.

Remark 8.5. *The set K of conjecturally nongeneric levels splits into two parts. The first one consists of minus the dual Coxeter numbers of $\mathfrak{sp}_2$ and $\mathfrak{osp}(1|2)$ for which the Sugawara construction of the Virasoro field does not work. For the second one, $\mathcal{C}_k$ has central charge*

$$c = 1 - 6\frac{(2i+1)^2}{2(2i+3)}$$

which is the central charge of the non-unitary rational Virasoro minimal model $M(2, 2i+3)$. At these central charges there are interesting logarithmic vertex algebras called singlet and triplet algebras. These are strongly finitely generated vertex algebras which are not simple [TW]*. The triplet algebra is even C_2-cofinite* [TW] *and has also been studied in* [AM]*. The singlet algebra and other extensions of the Virasoro algebra have been studied in* [CMW]*. In view of these relations to logarithmic vertex algebras it is an interesting task to study the nongeneric $\mathcal{C}_k$.*

Although K is not a finite set, it has two good properties: it has compact closure, and it does not contain any positive real values of k, in particular the values for which $V_k(\mathfrak{sp}_2)$ and $V_k(\mathfrak{osp}(1|2)))$ have rational quotients. It is an interesting problem to prove this conjecture, and to establish whether qualitative properties of the nongeneric set K such as compact closure hold for a general class of cosets of the form $\mathcal{C}_k = \text{Com}(V_k(\mathfrak{g}), \mathcal{A}_k)$. In view of Theorem 8.1, this would give a powerful method for studying cosets of simple affine vertex algebras inside larger structures. As an illustration, we shall use this approach to give a new proof of the C_2-cofiniteness and rationality of the simple $N = 2$ superconformal algebra with central charge $c = \frac{3k}{k+2}$ when k is a positive integer. Our argument makes use of a coset realization has been known for many years [DPYZ], and is the case $n = 1$ of Example 7.11. The rationality and regularity of these algebras was first established by Adamovic in [AII].

Rational $N = 2$ superconformal algebras. First, for $k \in \mathbb{C}$, consider the tensor product $V_k(\mathfrak{sl}_2) \otimes \mathcal{E}$, where $V_k(\mathfrak{sl}_2)$ has generators $H, X^{\pm}$ satisfying

$$H(z)X^{\pm}(w) \sim \pm X^{\pm}(w)(z-w)^{-1}, \quad H(z)H(w) \sim \frac{k}{2}(z-w)^{-2},$$
$$X^{+}(z)X^{-}(w) \sim k(z-w)^{-2} + 2H(w)(z-w)^{-1},$$

and $\mathcal{E}$ is the bc-system with odd generators b, c satisfying

$$b(z)b(w) \sim 0, \qquad c(z)c(w) \sim 0, \qquad b(z)c(w) \sim (z-w)^{-1}.$$

Let $\mathcal{H} \subset V_k(\mathfrak{sl}_2) \otimes \mathcal{E}$ denote the Heisenberg algebra with generator $J = H - : bc :$. The zero mode J_0 integrates to a $U(1)$ action on $V_k(\mathfrak{sl}_2) \otimes \mathcal{E}$, and we consider the $U(1)$-invariant algebra $(V_k(\mathfrak{sl}_2) \otimes \mathcal{E})^{U(1)}$, which is easily seen to have the following strong generating set:

$$(8.1) \qquad :\partial^i X^{+} \partial^j X^{-}:, \qquad :\partial^i X^{+} \partial^j b:, \qquad :\partial^i X^{-} \partial^j c:, \qquad :\partial^i b \partial^j c: \qquad i, j \geq 0.$$

Note that

$$(8.2) \qquad :\partial^i X^{+} X^{-}:, \qquad :\partial^i X^{+} b:, \qquad :\partial^i X^{-} c:, \qquad :\partial^i bc: \qquad i \geq 0,$$

is also a strong generating set for $(V_k(\mathfrak{sl}_2) \otimes \mathcal{E})^{U(1)}$, since the span of (8.2) and their derivatives coincides with the span on (8.1).

Lemma 8.6. *For all $k \in \mathbb{C}$, $(V_k(\mathfrak{sl}_2) \otimes \mathcal{E})^{U(1)}$ has a minimal strong generating set*

$$\{H, :X^{+}X^{-}:, :bc:, :X^{+}b:, :X^{-}c:\}.$$

Proof. This is immediate from the following normally ordered relations that exist for all $i \geq 0$.

$$: (: \partial^i X^+ b :)(: X^- c :) : =$$

$$(-1)^i \frac{2}{i+1} : H(\partial^{i+1} b)c : + : (\partial^i X^+) X^- bc : + : (\partial^{i+1} X^+) X^- : - \frac{k}{i+2} : (\partial^{i+2} b)c : .$$

$$: (: \partial^i X^+ b :)(: bc :) = - : \partial^{i+1} X^+ b :,$$

$$: (: \partial^i X^- c :)(: bc :) = : \partial^{i+1} X^- c :,$$

$$: (: \partial^i bc :)(: bc :) : = - \frac{i+2}{i+1} : (\partial^{i+1} b)c : + \partial \omega,$$

where ω is a linear combination of elements of the form $\partial^{i-r} : (\partial^r b)c :$ for $r = 0, 1, \dots, i$. □

Next, we replace H and $: bc :$ with $J = H - : bc :$ and $F = H + \frac{k}{2} : bc :$, respectively, and we replace $: X^+ X^- :$ with

$$L = \frac{1}{k+2} : X^+ X^- : + \frac{2}{k+2} : Hbc : - \frac{k}{2(k+2)} : b\partial c : + \frac{k}{2(k+2)} : (\partial b)c : - \frac{1}{k+2} \partial H.$$

Clearly $F, L, : X^+ b :, : X^- c :$ commute with J, and L is a Virasoro element of central charge $c = \frac{3k}{k+2}$, and F is primary weight one, others are primary weight $\frac{3}{2}$. Moreover, they generate the $N = 2$ superconformal algebra. Since

$$(V_k(\mathfrak{sl}_2) \otimes \mathcal{E})^{U(1)} = \mathcal{H} \otimes \text{Com}(\mathcal{H}, V_k(\mathfrak{sl}_2) \otimes \mathcal{E}),$$

we obtain

Lemma 8.7. *For all $k \neq -2$, $\mathcal{C}_k = \text{Com}(\mathcal{H}, V_k(\mathfrak{sl}_2) \otimes \mathcal{E})$ has a minimal strong generating set*

$$\{F, L, : X^+ b :, : X^- c :\},$$

and is isomorphic to the universal $N = 2$ superconformal vertex algebra with $c = \frac{3k}{k+2}$.

Next, for k a positive integer, we consider the tensor product $L_k(sl_2) \otimes \mathcal{E}$. By abuse of notation, we denote the generators by $H, X^{\pm}, b, c$, as above. Recall that $L_k(sl_2) \otimes \mathcal{E}$ is the simple quotient of $V_k(\mathfrak{sl}_2) \otimes \mathcal{E}$ by the ideal $\mathcal{I}_k$ generated by $(X^+)^{k+1}$. Let $\mathcal{L}_k = \mathcal{C}_k / (\mathcal{I}_k \cap \mathcal{C}_k)$.

Lemma 8.8. *$\mathcal{L}_k = \text{Com}(\mathcal{H}, L_k(\mathfrak{sl}_2) \otimes \mathcal{E})$ where $\mathcal{H}$ is the Heisenberg algebra generated by $J = H - : bc :$. In particular, $\mathcal{L}_k$ is simple.*

Proof. This is immediate from the fact that $V_k(\mathfrak{sl}_2) \otimes \mathcal{E}$ is completely reducible as an $\mathcal{H}$-module, and $\mathcal{I}_k$ is an $\mathcal{H}$-submodule of $V_k(\mathfrak{sl}_2) \otimes \mathcal{E}$. □

It is well known [DPYZ] that the simple $N = 2$ superconformal algebra has a realization inside $\text{Com}(\mathcal{H}, L_k(\mathfrak{sl}_2) \otimes \mathcal{E})$ with generators $\{F, L, : X^+ b :, : X^- c :\}$, which are just the images of the generators of $\mathcal{C}_k$. An immediate consequence of Lemmas 8.7 and 8.8 is that this $N = 2$ algebra is the *full* commutant.

Corollary 8.9. *For $k = 1, 2, 3, \dots$, $\mathcal{L}_k$ is isomorphic to the simple $N = 2$ superconformal algebra with central charge $c = \frac{3k}{k+2}$.*

It is well known that $L_k(\mathfrak{sl}_2)$ contains a copy of the lattice vertex algebra $V_{\sqrt{2k}\mathbb{Z}}$ with generators $\{H, :(X^{\pm})^k:\}$. Moreover,

$$\mathrm{Com}(\mathrm{Com}(\langle H\rangle, L_k(\mathfrak{sl}_2))) = V_{\sqrt{2k}\mathbb{Z}},$$

where $\langle H\rangle$ denotes the Heisenberg algebra generated by H. In particular, $V_{\sqrt{2k}\mathbb{Z}}$ and the parafermion algebra $N_k(\mathfrak{sl}_2)$ form a Howe pair (i.e., a pair of mutual commutants) inside $L_k(\mathfrak{sl}_2)$, and $\mathcal{L}_k$ is an extension of $V_{\sqrt{2k}\mathbb{Z}} \otimes N_k(\mathfrak{sl}_2)$. Both $V_{\sqrt{2k}\mathbb{Z}}$ and $N_k(\mathfrak{sl}_2)$ are rational, and the discriminant $\mathbb{Z}/\sqrt{2k}\mathbb{Z}$ of the lattice $\sqrt{2k}\mathbb{Z}$ acts on $\mathcal{L}_k$ as automorphism subgroup. The orbifold is $V_{\sqrt{2k}\mathbb{Z}} \otimes N_k(\mathfrak{sl}_2)$ and as a module for the orbifold

$$\mathcal{L}_k = \bigoplus_{t=0}^{2k-1} M_t,$$

where each M_t is a simple $V_{\sqrt{2k}\mathbb{Z}} \otimes N_k(\mathfrak{sl}_2)$-module [DM]. Each M_t is in fact also C_1-cofinite as the orbifold is C_2-cofinite, hence Proposition 20 of [MiII] implies it is a simple current. We thus have a simple current extension of a rational, C_2-cofinite vertex algebra of CFT-type and hence by [Y] the extension $\mathcal{L}_k$ is also C_2-cofinite and rational. This provides an alternative proof of Adamovic's theorem that the simple $N = 2$ superconfomal algebra with central charge $c = \frac{3k}{k+2}$ for $k = 1, 2, 3, \dots$, is C_2-cofinite and rational.

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