

SIMPLE CURRENT EXTENSIONS BEYOND SEMI-SIMPLICITY

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ABSTRACT. Let V be a simple VOA and consider a representation category of V that is a vertex tensor category in the sense of Huang-Lepowsky. In particular, this category is a braided tensor category. Let J be an object in this category that is a simple current of order two of either integer or half-integer conformal dimension. We prove that $V \oplus J$ is either a VOA or a super VOA. If the representation category of V is in addition ribbon, then the categorical dimension of J decides this parity question. Combining with Carnahan's work, we extend this result to simple currents of arbitrary order. Our next result is a simple sufficient criterion for lifting indecomposable objects that only depends on conformal dimensions. Several examples of simple current extensions that are C_2 -cofinite and non-rational are then given and induced modules listed.

1. INTRODUCTION AND SUMMARY OF RESULTS

Let V be a simple vertex operator algebra (VOA) a fundamental and unfortunately difficult question is whether V can be extended to a larger VOA by some of its modules. If the representation category \mathcal{C} under consideration is a vertex tensor category in the sense of Huang-Lepowsky (see [HL1]), then this question is equivalent to the existence of a haploid algebra in the category \mathcal{C} due to work by Huang-Kirillov-Lepowsky [HKL] following older work by Kirillov-Ostrik [KO]. This result provides a new direction of constructing extension of VOAs. The nicest possible extensions are those by simple currents, that is by invertible objects in \mathcal{C} . Simple currents appeared first in the context of two-dimensional conformal field theory (CFT) in the physics literature [SY, FG, GW] and have then later been introduced also for VOAs [DLM1], especially Höhn was able to relate the extension problem to the categorical context [Hö]. The famous moonshine module VOA of [FLM] is a simple current extension of a certain VOA. It can also be constructed using vertex tensor categories [H1]. Interesting further results on simple current extensions of VOAs are for example [Y, S, Li1, FRS, LaLaY, La]. Most importantly to us, last year Carnahan elaborated further on the simple current extension problem [C], and he basically solved the problem for integer weight simple currents up to extensions by self-dual ones, i.e., objects in the category that are their own inverses.

Motivation. We are interested in VOAs beyond the semi-simple setting, i.e., VOAs whose representation category has indecomposable but reducible modules. We also do not necessarily restrict to categories with only finitely many simple objects. Such VOAs are sometimes called logarithmic as they are the mathematicians' reformulation of logarithmic CFT. Presently, there is one well understood type of C_2 -cofinite but non-rational simple VOAs, the $\mathcal{W}(p)$ -triplet algebras [FGST, AdM1, TW, CF] and the order two simple current extension of $\mathcal{W}(2)$ called symplectic fermions [AA, Ab] as well as the super triplet [AdM2]. One can now use these to construct new VOAs via orbifolds [AdLM1, AdLM2] and orbifolds of tensor products [Ab]. Also note that both $\mathcal{W}(p)$ and the symplectic fermion super VOA are rigid [TW, DR]. The main objective of this work is to develop a theory of simple current extensions of VOAs beyond semi-simplicity. The main questions one needs to ask are

- Can a given VOA V be extended to a larger VOA or super VOA by simple currents?

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- Which generalized modules¹ of V lift to those of the extension V_e ?

We will use our answers to these questions to construct three new families of C_2 -cofinite VOAs together with all modules that lift at the end of this work. Our main intention is however to find genuinely new C_2 -cofinite VOAs, that is VOAs that are not directly related to the well-known triplet VOA. For example consider the following diagram

$$\begin{array}{ccc}
 \mathcal{A}_k & \xrightarrow{\text{extension}} & \mathcal{E}_k = \bigoplus_{n \in \mathbb{Z}} J^n \\
 \downarrow \text{coset} & & \downarrow \text{coset} \\
 \text{Com}(\mathcal{H}, \mathcal{A}_k) & \xrightarrow{\text{extension}} & \text{Com}(\mathcal{H}, \mathcal{E}_k).
 \end{array}$$

Here \mathcal{A}_k is a family of VOAs with one-dimensional associated variety [A1] containing a Heisenberg sub VOA \mathcal{H} and having a simple current J of infinite order. Our picture is that the Heisenberg coset $\text{Com}(\mathcal{H}, \mathcal{A}_k)$ still has a one-dimensional associated variety while the extension \mathcal{E}_k only has finitely many irreducible objects. Especially $\text{Com}(\mathcal{H}, \mathcal{E}_k)$ is our candidate for new C_2 -cofinite VOAs.

A natural example is $\mathcal{A}_k = L_k(\mathfrak{sl}_2)$ for $k + 2 \in \mathbb{Q}_{>0} \setminus \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$ as it has one-dimensional variety of modules [AdM3]. In [CR1, CR2] it is conjectured that these VOAs allow for infinite order simple current extensions \mathcal{E}_k , which then would only have finitely many simple modules. These VOAs would be somehow unusual as they would not be of CFT-type, and these VOAs are not our final goal, but rather $\text{Com}(\mathcal{H}, \mathcal{E}_k)$. In the example of $k = -1/2$ this construction would just yield $\mathcal{W}(2)$ and in the case of $k = -4/3$ it would just give $\mathcal{W}(3)$ [Ad1, CRW, R2]. In all other cases we expect new C_2 -cofinite VOAs. Another potential candidate is the Bershadsky-Polyakov algebra whose Heisenberg coset is studied in [ACL]. In a subsequent work, we will thus develop general properties of Heisenberg cosets beyond semi-simplicity [CKLR]. This work will rely on our findings here and will then further be used for interesting examples.

The extension problem leads to interesting number theory if restricted to characters. Namely it seems that those types of functions that appear in characters of logarithmic VOAs also appear in current research of modular forms and beyond. For example, the characters of the singlet algebra $\mathcal{M}(p)$ are sometimes composed of partial theta functions [F, CM], while their infinite order simple current extensions, the triplets $\mathcal{W}(p)$, have as characters of modules just ordinary theta functions and their derivatives [F]. Another example is $V_k(\mathfrak{gl}(1|1))$, while its module characters are built out of ordinary Jacobi theta functions it has many simple current extensions whose module characters are sometimes Mock Jacobi forms [AC, CR4].

In the present work, we will translate results into the corresponding statements in a braided tensor category using the theory of [KO, HKL]. The advantage is that the categorical picture is much better suited for proving properties of the representation category. For us it provides a very nice way to understand the problem of the two questions: Does a module lift to a module of the extended VOA? Is the extension a VOA or a super VOA? The work [KO] assumes categories to be semi-simple and focuses on algebras with trivial twist. We believe that many of the results of [KO] can be modified to our non semi-simple setting, the most important one being the question of rigidity in the category of the extension. We shall look at related generalizations in future work.

Results. In order to describe our first result recall that braidings are the commutativity isomorphisms which we denote by

$$c_{A,B} : A \boxtimes B \rightarrow B \boxtimes A$$

¹Generalized modules are those which need not admit a semi-simple action of $L(0)$ and are graded by generalized eigenvalues of $L(0)$.

for objects A, B in our category. Let J be a self-dual simple current, i.e., $J \boxtimes J \cong V$. It can only give rise to a VOA extension of V if its conformal dimension h_J , that is the conformal weight of its lowest weight state, is in $\frac{1}{2}\mathbb{Z}$. The twist of an object in the representation category of a VOA is given by the action of $e^{2\pi i L(0)}$ on the object, so that on a simple module like the simple current J it just acts as $\theta_J = e^{2\pi i h_J} \text{Id}_J$. The balancing axiom of braided tensor categories applied to this case reads

$$1 = \theta_V = \theta_{J \boxtimes J} = c_{J,J} \circ c_{J,J} \circ (\theta_J \boxtimes \theta_J)$$

so that $\theta_J \in \{\pm \text{Id}_J\}$ implies $c_{J,J} \in \{\pm 1\}$. Our first result is Theorem 3.9, which is:

Theorem 1.1. *Assume that V is a VOA satisfying the conditions required to invoke Huang-Lepowsky-Zhang's theory. We also assume that braiding and twist are as given by Huang-Lepowsky. Let J be a simple current such that $J \boxtimes J \cong V$. If $c_{J,J} = 1$, $V \oplus J$ has a structure of $\frac{1}{2}\mathbb{Z}$ -graded vertex operator algebra and if $c_{J,J} = -1$, $V \oplus J$ has a structure of vertex operator superalgebra.*

We remark that simple current extensions by self-dual simple currents generated by a weight one primary vector were understood in the rational, C_2 -cofinite and CFT-type setting 20 years ago [DLM1, Li1]. Also the extensions of the unitary rational Virasoro VOAs are known [LaLaY].

The proof of this theorem is very similar to the proof of Theorem 4.1 of [H6]. For this theorem, we need a vertex tensor category of V in the sense of [HL1]. The current state of the art is that the representation category of C_2 -cofinite VOAs with natural additional requirements as well as subcategories of VOAs such that all modules in this subcategory are C_1 -cofinite with a few more natural additional properties are vertex tensor categories. Especially, braiding and twist are given as developed by Huang-Lepowsky. For the precise requirements, see Theorems 3.22 and 3.23 which are due to [H3, HLZ, Miy]. For the background on the vertex tensor categories, we refer the reader to [HL1]. The construction of vertex tensor category structure in the non semi-simple case is accomplished in [HLZ] and for a quick perspective on [HLZ] and the related results, see [HL2].

In order to understand whether the extension is a super VOA or just a VOA, one needs to determine the braiding $c_{J,J}$. This quantity is often not directly accessible, however there is a useful spin statistics theorem (we adapt the name from [GL] for a similar theorem, but in the unitary conformal net setting). This spin statistics theorem needs the notion of a trace, so it only holds in rigid braided tensor categories, actually only in ribbon categories.

Theorem 1.2. *Assume that the tensor category of V is ribbon, then, for a self-dual simple current J with conformal dimension h_J ,*

$$c_{J,J} \text{qdim}(J) = e^{2\pi i h_J}, \tag{1.1}$$

where $\text{qdim}(J) = \text{tr}_J(\text{Id}_J)$.

This is a small reformulation of Corollary 2.8. The quantity $\text{qdim}(J)$ is the categorical or quantum dimension of J . In a modular tensor category, that is in the tensor category of a regular VOA, the quantum dimension is determined from the modular S -matrix coefficients as

$$\text{qdim}(J) = \frac{S_{J,V}}{S_{V,V}}$$

which coincides with

$$\lim_{\tau \rightarrow 0} \frac{\text{ch}[J](\tau)}{\text{ch}[V](\tau)} \tag{1.2}$$

if the module of lowest conformal weight is V itself (cf. [DJX]), for instance, in the case of unitary VOAs. The quantity (1.2) is clearly non-negative and hence in this case $\text{qdim}(J) = 1$. For modularity of the categories of modules for vertex operator algebras satisfying suitable finiteness and reductivity conditions, see [H3, H4]. This means Carnahan's evenness conjecture [C] is correct for unitary regular VOAs. But beyond that there are counterexamples: Rational C_2 -cofinite

counterexamples are our Theorem 4.9 as well as Theorem 10.3 of [ACL]. The symplectic fermion super VOA is a C_2 -cofinite but non-rational simple current extension of the triplet VOA $\mathcal{W}(2)$ graded by the integers and hence in this case the simple current must have quantum dimension minus one. See [AA] but also [CG] on symplectic fermions. Interestingly, there are strong indications that quantum dimensions are still determined by the modular properties of characters in the C_2 -cofinite setting [CG] and even beyond that [CM, CMW].

Having solved the extension problem for self-dual simple currents, we can combine our findings with those of Carnahan [C] to get Theorem 3.12.

Theorem 1.3. *Assume that V is a VOA satisfying the conditions required to invoke Huang-Lepowsky-Zhang's theory. We also assume that braiding and twist are as given by Huang-Lepowsky. Let J be a simple current. Assume that $\theta_{J^k} = \pm 1$ with $\theta_{J^{k+2}}$ having the same sign as θ_{J^k} for all $k \in \mathbb{Z}$. Then*

$$V_e = \bigoplus_{j \in G} J^j$$

has a natural structure of a strongly graded vertex operator superalgebra, graded by the abelian group G generated by J . (For the definition of strongly graded, we refer the reader to [HLZ].) If G is finite, we get a vertex operator superalgebra.

Next, we would like to elaborate on the representation category of simple current extensions. The works [KO] and [HKL] together bring us into a good position here, since there is also a categorical notation of VOA extension as a haploid algebra in the category as well as many nice results on the representation category of local modules of the haploid algebra. We only need to adapt the main theorem and its proof of [HKL] to extensions that are super VOAs, which leads us to the notion of superalgebras in the category. Then Theorem 3.13 says that extensions that are super VOAs are equivalent to super algebras in the category, and Theorem 3.14 tells us that the module category of the extended VOA is equivalent to the category of local modules of the corresponding superalgebra in the category. This is very useful, as we can use [KO] to define a functor \mathcal{F} (the induction functor) from the category to the representation category of its superalgebra. For this, let $\mathcal{C}' \subset \mathcal{C}$ be the full subcategory of generalized V -modules, such that objects of \mathcal{C}' are subquotients of objects of the tensor ring generated by the simple objects of \mathcal{C} . Theorems 3.18 and 3.20 are the following.

Theorem 1.4. *Let J be simple current such that the extension V_e exists and let P be an indecomposable generalized V -module, then:*

- (1) *If P is an object of \mathcal{C}' then $\mathcal{F}(P)$ is a generalized V_e -module iff $h_{J \boxtimes P} - h_J - h_P \in \mathbb{Z}$.*
- (2) *If J is of finite order and if P is an object of \mathcal{C} such that both $\dim(\text{Hom}(P, P)) < \infty$ and $\dim(\text{Hom}(J \boxtimes P, J \boxtimes P)) < \infty$. Assume also that $L(0)$ has Jordan blocks of bounded size on both P and $J \boxtimes P$. Then, $\mathcal{F}(P)$ is a generalized V_e -module iff $h_{J \boxtimes P} - h_J - h_P \in \mathbb{Z}$.*

In other words, answering whether an indecomposable module P of the VOA V lifts to a generalized module of the simple current extension amounts to the computation of a few conformal dimensions.

The corresponding theorem for simple modules of C_2 -cofinite, rational, CFT-type VOAs has been proven in [Y, La].

In practice, it is expected that \mathcal{C}' is the category of "most interesting V -modules." For example, in the case of $\mathcal{W}(p)$ it contains the category whose indecomposable objects consist of all simple and all projective modules of $\mathcal{W}(p)$ -mod [NT, TW]. Analogous statement is true for modules of $V_k(\mathfrak{gl}(1|1))$ [CR4] and we expect this to be a generic feature of "nice" logarithmic VOAs. In the case of the Heisenberg VOA, \mathcal{C}' is the category of semi-simple modules; in this case the interesting infinite order simple current extensions are lattice VOAs (of positive definite lattices) and thus

they are rational and C_2 -cofinite. The Heisenberg VOA has indecomposable but reducible objects, but they are not objects in \mathcal{C}' and they do not lift to modules of the lattice VOA.

Carnahan titled his work [C] “Building vertex algebras from parts,” and indeed in the last section we construct various new logarithmic VOAs as simple current extensions of tensor products of known VOAs. Our main examples are three series of C_2 -cofinite but non-rational VOAs constructed from the tensor product of $\mathcal{W}(p)$ with a suitable second VOA. We also list interesting modules, both simple and indecomposable, that lift to modules of the extension. Further examples of resulting VOAs are the small $N = 4$ super Virasoro algebra at central charge $c = -3$ as well as super VOAs associated to $\mathfrak{osp}(1|2)$. A non-logarithmic example is then $L_1(\mathfrak{osp}(1|2))$, which is rational (Theorem 4.9) and has only two inequivalent simple modules (Corollary 4.13). Finally, our results are used in proving that the coset vertex algebras of the rational Bershadsky-Polyakov algebra [A2] with its Heisenberg subalgebra are rational W-algebras of type A [ACL].

We organize this paper as follows. In Section 2, we start with some crucial results in braided tensor categories. Especially we derive the “spin statistics theorem,” as well as results that eventually allow us to deduce the criteria for lifting modules to modules of the simple current extension. Section 3 is the heart of this work and contains all the main theorems. We conclude with some examples in Section 4, focusing on VOAs with non semi-simple representation categories.

A remark on notation In Section 2, when we present several general results for braided tensor categories, we shall denote the tensor products by \otimes . In Section 3, we work with vertex tensor categories, where we use the $P(z)$ -tensor products denoted by $\boxtimes_{P(z)}$ as in [HLZ]. For a fixed value of z , taken to be $z = 1$ for convenience, we get a braided tensor category structure and we shall abbreviate $\boxtimes_{P(1)}$ by \boxtimes . In Section 4, we use \otimes yet again to denote tensor products of vertex operator algebras (see [FHL]) and their modules. We hope that no confusion shall arise with various “tensor products” used in the paper.

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2. BRAIDED TENSOR CATEGORIES

In this section, we shall derive a spin statistics theorem for objects in a ribbon category. We will also study a few properties of the monodromy matrix and we will discuss the notion of a superalgebra inside a category. Our main sources of inspiration are [KO, DGNO].

Our notation for the braiding, associativity isomorphisms, the evaluation map and the coevaluation are

$$\begin{aligned} c_{A,B} : A \otimes B &\rightarrow B \otimes A, & \mathcal{A}_{X,Y,Z} : X \otimes (Y \otimes Z) &\rightarrow (X \otimes Y) \otimes Z \\ \text{ev}_X : X^* \otimes X &\rightarrow \mathbf{1}_{\mathcal{C}}, & \text{coev}_X : \mathbf{1}_{\mathcal{C}} &\rightarrow X \otimes X^*. \end{aligned}$$

Spin Statistics. Let \mathcal{C} be a ribbon category, that is a rigid braided tensor category with pivotal structure $\psi : X \rightarrow X^{**}$. Using [DGNO], we define:

Definition 2.1. For $f \in \text{Hom}(X \otimes Y, X \otimes Z)$ and $g \in \text{Hom}(Y \otimes X, Z \otimes X)$, let

$$\begin{aligned} \text{ptr}_X^L(f) &= (\text{ev}_X \otimes \text{Id}_Z) \circ \mathcal{A}_{X^*,X,Z} \circ (\text{Id}_{X^*} \otimes f) \circ (\text{Id}_{X^*} \otimes (\psi_X^{-1} \otimes \text{Id}_Y)) \circ \mathcal{A}_{X^*,X^{**},Y}^{-1} \circ (\text{coev}_{X^*} \otimes \text{Id}_Y) \\ \text{ptr}_X^R(g) &= (\text{Id}_Z \otimes \text{ev}_{X^*}) \circ \mathcal{A}_{Z,X^{**},X^*}^{-1} \circ ((\text{Id}_Z \otimes \psi_X) \otimes \text{Id}_{X^*}) \circ (g \otimes \text{Id}_{X^*}) \circ \mathcal{A}_{Y,X,X^*} \circ (\text{Id}_Y \otimes \text{coev}_X). \end{aligned}$$

If we take $Y = Z = \mathbf{1}_C$ then we recover the ordinary left and right trace of X . These coincide in spherical categories like a ribbon category.

Definition 2.2. *The braided structure yields a natural morphism $u_X : X \rightarrow X^{**}$ given by*

$$\begin{aligned} X &\xrightarrow{\text{Id}_X \otimes \text{coev}_{X^*}} X \otimes (X^* \otimes X^{**}) \xrightarrow{\mathcal{A}_{X, X^*, X^{**}}} (X \otimes X^*) \otimes X^{**} \\ &\xrightarrow{c_{X, X^*} \otimes \text{Id}_{X^{**}}} (X^* \otimes X) \otimes X^{**} \xrightarrow{\text{ev}_X \otimes \text{Id}_{X^{**}}} X^{**}. \end{aligned}$$

It is actually well-known that u_X is an isomorphism.

Definition 2.3. *The twist θ_X is defined by $\psi_X = u_X \theta_X$, or equivalently $\theta_X^{-1} = \psi_X^{-1} u_X$.*

Theorem 2.4. *In a ribbon category*

$$\text{ptr}_X^L(c_{X, X}^{-1}) = \theta_X^{-1} \quad (2.1)$$

Proof. We follow the proof of Proposition 2.32 in [DGNO], except that we incorporate the associativity morphisms.

The left part of the following diagram commutes by naturality of braiding and the fact that $c_{X, \mathbf{1}_C} = \text{Id}_X$. The right part commutes by hexagon identity.

$$\begin{array}{ccc} X = X \otimes \mathbf{1}_C & \xrightarrow{\text{id}_X \otimes \text{coev}_{X^*}} X \otimes (X^* \otimes X^{**}) & \xrightarrow{c_{X, X^*} \otimes \text{Id}_{X^{**}} \circ \mathcal{A}_{X, X^*, X^{**}}} (X^* \otimes X) \otimes X^{**} \\ \downarrow \text{id}_X = c_{X, \mathbf{1}_C} & & \searrow \mathcal{A}_{X^*, X, X^{**}} \circ \text{Id}_{X^*} \otimes c_{X, X^*}^{-1} \circ \mathcal{A}_{X^*, X^{**}, X}^{-1} \\ X = \mathbf{1}_C \otimes X & \xrightarrow{\text{coev}_{X^*} \otimes \text{id}_X} (X^* \otimes X^{**}) \otimes X & \end{array}$$

Thus, we get that:

$$\theta_X^{-1} = \psi_X^{-1} u_X = \psi_X^{-1} \circ (\text{ev}_X \otimes \text{Id}_{X^{**}}) \circ \mathcal{A}_{X^*, X, X^{**}} \circ (\text{Id}_{X^*} \otimes c_{X, X^*}^{-1}) \circ \mathcal{A}_{X^*, X^{**}, X}^{-1} \circ (\text{coev}_{X^*} \otimes \text{Id}_X)$$

Using naturality of associativity and braiding, we also get that

$$\begin{aligned} &\psi_X^{-1} \circ (\text{ev}_X \otimes \text{Id}_{X^{**}}) \circ \mathcal{A}_{X^*, X, X^{**}} \circ (\text{Id}_{X^*} \otimes c_{X, X^*}^{-1}) \\ &= (\text{ev}_X \otimes \text{Id}_X) \circ (\text{Id}_{X^*} \otimes X \otimes \psi_X^{-1}) \circ \mathcal{A}_{X^*, X, X^{**}} \circ (\text{Id}_{X^*} \otimes c_{X, X^*}^{-1}) \\ &= (\text{ev}_X \otimes \text{Id}_X) \circ \mathcal{A}_{X^*, X, X} \circ (\text{Id}_{X^*} \otimes (\text{Id}_X \otimes \psi_X^{-1})) \circ (\text{Id}_{X^*} \otimes c_{X, X^*}^{-1}) \\ &= (\text{ev}_X \otimes \text{Id}_X) \circ \mathcal{A}_{X^*, X, X} \circ (\text{Id}_{X^*} \otimes c_{X, X}^{-1}) \circ (\text{Id}_{X^*} \otimes (\psi_X^{-1} \otimes \text{Id}_X)) \end{aligned}$$

Putting everything together,

$$\begin{aligned} \theta_X^{-1} &= (\text{ev}_X \otimes \text{Id}_X) \circ \mathcal{A}_{X^*, X, X} \circ (\text{Id}_{X^*} \otimes c_{X, X}^{-1}) \circ (\text{Id}_{X^*} \otimes (\psi_X^{-1} \otimes \text{Id}_X)) \circ \mathcal{A}_{X^*, X^{**}, X}^{-1} \circ (\text{coev}_{X^*} \otimes \text{Id}_X) \\ &= \text{ptr}_X^L(c_{X, X}^{-1}) \end{aligned}$$

□

Taking the trace of left and right hand side of equation (2.1) we get the following Corollary.

Corollary 2.5. *In a ribbon category*

$$\text{tr}_{X \otimes X} \left(c_{X, X}^{-1} \right) = \text{tr}_X \left(\theta_X^{-1} \right).$$

Proof. This follows since $\text{tr}_{X \otimes Y}(f) = \text{tr}_X(\text{ptr}_Y^L(f))$ for any endomorphism $f : Y \otimes X \rightarrow Y \otimes X$. □

Remark 2.6. Recall that our ribbon category is not necessarily semi-simple. If it is not, then the trace might vanish on a tensor ideal. Call such an ideal \mathcal{P} for projective. If it is generated by an ambidextrous element it allows for a modified trace on \mathcal{P} that we call t . In that case one gets an analogous result

$$t_{X \otimes X} \left(c_{X,X}^{-1} \right) = t_X \left(\theta_X^{-1} \right)$$

for X in \mathcal{P} as the projective trace also satisfies $t_{X \otimes Y}(f) = t_X \left(\text{ptr}_Y^L(f) \right)$ for any f in $\text{End}(X \otimes Y)$ and any projective module X . It does not matter whether Y is projective or not. The importance of the modified trace in C_2 -cofinite non-rational VOAs is illustrated in [CG], the ideas on the modified trace there follow [GKP1, GKP2].

Definition 2.7. We call an invertible simple object a simple current and a simple current that is its own inverse self-dual.

Let J be a simple current, we define the short-hand notation $J^2 := J \otimes J$, and J^{-1} for the inverse. Its categorical dimension is

$$\text{qdim}(J) = \text{tr}_J(\text{Id}_J).$$

It satisfies

$$\text{qdim}(J) \text{qdim}(J) = \text{qdim}(J^2) \quad \text{and} \quad \text{qdim}(J) \text{qdim}(J^{-1}) = \text{qdim}(\mathbf{1}_C) = 1.$$

So the dimension of J is non-zero. Let J now be self-dual and $\theta_J \in \{\pm \text{id}_J\}$, then it follows that also $c_{J,J} \in \{\pm 1\}$ (our field is $\text{End}(\mathbf{1}_C)$) due to the balancing axiom of the twists. Also the dimension of J can only be either one or minus one, namely

Corollary 2.8. Let J be a self-dual simple current then

$$c_{J,J} \text{qdim}(J) = \tilde{\theta}_J$$

with $\theta_J = \tilde{\theta}_J \text{id}_J$. In particular, if $\theta_J \in \{\pm \text{id}_J\}$, then $c_{J,J} \in \{\pm 1\}$ and $\text{qdim}(J) \in \{\pm 1\}$.

Monodromy. We now only assume that our monoidal category is braided but not necessarily ribbon.

Definition 2.9. For objects $A, B \in \mathcal{C}$, define the monodromy $M_{A,B} : A \otimes B \rightarrow A \otimes B$ to be $c_{B,A} \circ c_{A,B}$.

Lemma 2.10. Monodromy is natural. In particular,

$$M_{(A \otimes B) \otimes C, D} \circ (\mathcal{A}_{A,B,C} \otimes \text{Id}_D) = (\mathcal{A}_{A,B,C} \otimes \text{Id}_D) \circ M_{A \otimes (B \otimes C), D}.$$

Therefore, if $(Y \otimes J^i)_1$ and $(Y \otimes J^i)_2$ are two different ways to parenthesize $Y \otimes J^{\otimes i}$ and $(J^i \otimes Y)_1$ and $(J^i \otimes Y)_2$ are two different ways to parenthesize $J^{\otimes i} \otimes Y$, then $M_{(Y \otimes J^i)_1, X} = \text{Id}$ implies $M_{(Y \otimes J^i)_2, X} = \text{Id}$ and $M_{X, (J^i \otimes Y)_1} = \text{Id}$ implies $M_{X, (J^i \otimes Y)_2} = \text{Id}$.

Proof. Naturality of monodromy is implied by the naturality of braiding. The rest follows. \square

Theorem 2.11. The following hold for monodromy.

(1) For objects $A, B, C \in \mathcal{C}$ such that $M_{A,C} = \text{Id}_{A \otimes C}$, we have

$$M_{A, B \otimes C} = \mathcal{A}_{A,B,C}^{-1} \circ (M_{A,B} \otimes \text{Id}_C) \circ \mathcal{A}_{A,B,C}, \quad (2.2)$$

$$M_{A \otimes B, C} = \mathcal{A}_{A,B,C} \circ (\text{Id}_A \otimes M_{B,C}) \circ \mathcal{A}_{A,B,C}^{-1} \quad (2.3)$$

(2) If $M_{J,X} = \text{Id}_{J \otimes X}$ and $M_{Y,X} = \text{Id}_{Y \otimes X}$ then $M_{Y \otimes J^i, X} = \text{Id}_{(Y \otimes J^i) \otimes X}$, for all positive integers i , regardless of how $Y \otimes J^i$ is parenthesized.

(3) If $M_{X,J} = \text{Id}_{X \otimes J}$ and $M_{X,Y} = \text{Id}_{X \otimes Y}$ then $M_{X, J^i \otimes Y} = \text{Id}_{X \otimes (J^i \otimes Y)}$, for all positive integers i , regardless of how $J^i \otimes Y$ is parenthesized.

(4) If $M_{J,J} = \text{Id}_{J \otimes J}$ then $M_{J^i, J^j} = \text{Id}_{J^i \otimes J^j}$ for all $i, j \in \mathbb{N}$, regardless of how J^i and J^j are parenthesized.

- (5) If $M_{J,J} = \text{Id}_{J \otimes J}$ and $M_{J,X} = \text{Id}_{J \otimes X}$ then $M_{J^i, J^j \otimes X} = \text{Id}_{J^i \otimes (J^j \otimes X)}$, for all $i, j \in \mathbb{N}$, regardless of how J^i and $J^j \otimes X$ are parenthesized.
- (6) If J is an invertible object then $M_{J,X} = \text{Id}_{J \otimes X}$ implies $M_{J^{-1}, X} = \text{Id}_{J^{-1} \otimes X}$ and $M_{X,J} = \text{Id}_{X \otimes J}$ implies $M_{X, J^{-1}} = \text{Id}_{X \otimes J^{-1}}$.
- (7) If J is an invertible object and $M_{J,J} = \text{Id}_{J \otimes J}$ then $M_{J^i, J^j} = \text{Id}_{J^i \otimes J^j}$ for all $i, j \in \mathbb{Z}$, regardless of how J^i and J^j are parenthesized.
- (8) If J is an invertible object with $M_{J,J} = \text{Id}_{J \otimes J}$ and X is such that $M_{J,X} = \text{Id}_{J \otimes X}$ then $M_{J^i, J^j \otimes X} = \text{Id}_{J^i \otimes (J^j \otimes X)}$, for all $i, j \in \mathbb{Z}$, regardless of how J^i and $J^j \otimes X$ are parenthesized.

Proof. For (2.2), we proceed as follows. By the hexagon diagram, we get:

$$\begin{aligned}
M_{A, B \otimes C} &= c_{B \otimes C, A} \circ c_{A, B \otimes C} \\
&= (\mathcal{A}_{A, B, C}^{-1} \circ (c_{B, A} \otimes \text{Id}_C) \circ \mathcal{A}_{B, A, C} \circ (\text{Id}_B \otimes c_{C, A}) \circ \mathcal{A}_{B, C, A}^{-1}) \\
&\quad \circ (\mathcal{A}_{B, C, A} \circ (\text{Id}_B \otimes c_{A, C}) \circ \mathcal{A}_{B, A, C}^{-1} \circ (c_{A, B} \otimes \text{Id}_C) \circ \mathcal{A}_{A, B, C}) \\
&= \mathcal{A}_{A, B, C}^{-1} \circ (c_{B, A} \otimes \text{Id}_C) \circ (c_{A, B} \otimes \text{Id}_C) \circ \mathcal{A}_{A, B, C}.
\end{aligned}$$

For (2.3) we proceed similarly. Again by the hexagon diagram, we get:

$$\begin{aligned}
M_{A \otimes B, C} &= c_{C, A \otimes B} \circ c_{A \otimes B, C} \\
&= (\mathcal{A}_{A, B, C} \circ (\text{Id}_A \otimes c_{C, B}) \circ \mathcal{A}_{A, C, B}^{-1} \circ (c_{C, A} \otimes \text{Id}_B) \circ \mathcal{A}_{C, A, B}) \\
&\quad \circ (\mathcal{A}_{C, A, B}^{-1} \circ (c_{A, C} \otimes \text{Id}_B) \circ \mathcal{A}_{A, C, B} \circ (\text{Id}_A \otimes c_{B, C}) \circ \mathcal{A}_{A, B, C}^{-1}) \\
&= \mathcal{A}_{A, B, C} \circ (\text{Id}_A \otimes c_{C, B}) \circ (\text{Id}_A \otimes c_{B, C}) \circ \mathcal{A}_{A, B, C}^{-1}.
\end{aligned}$$

We first prove (2) when $Y \otimes J^i = (\cdots ((Y \otimes J) \otimes J) \cdots \otimes J)$ for all positive integers i . Note that (2.3) implies $M_{Y \otimes J, X} = \text{Id}_{(Y \otimes J) \otimes X}$. Therefore, by induction on i , we conclude that $M_{Y \otimes J^i, X} = \text{Id}_{(Y \otimes J^i) \otimes X}$ for all $i \in \mathbb{Z}_+$. Using Lemma 2.10 we can now get the result for all different ways of parenthesizing $Y \otimes J^i$.

Using (2.2), (3) follows in complete analogy.

For (4), note that the assertion holds if $i = 0$ or $j = 0$ because $c_{1_C, X} = \text{Id} = c_{X, 1_C}$ and hence $M_{1_C, X} = M_{X, 1_C}$ for any object X . Now let $i, j \geq 1$. If $i = j = 1$, then the assertion follows by assumption that $M_{J,J} = \text{Id}$. If $i = 1$ or $j = 1$, the claim follows by using $Y = 1_C$ and $X = J$ in (2) and (3). With this we have proved (4) for $i \in \{0, 1\}$ or $j \in \{0, 1\}$. Now if $i, j \geq 2$, using (2) with $Y = J, X = J^j$, we obtain that $M_{J \otimes J^{i-1}, J^j} = \text{Id}$ regardless of how $J \otimes J^{i-1}$ and J^j are parenthesized.

For (5), note that the assertion holds if $i = 0$. Indeed, $c_{1_C, X} = c_{X, 1_C} = \text{Id}_X$ results in $M_{1_C, X} = \text{Id}$; which combined with (3) yields $M_{1_C, J^j \otimes X} = \text{Id}$. The assertion also holds if $j = 0$ by taking $Y = 1_C$ in (2). Using this and (3), we obtain that $M_{J^i, (J^j) \otimes X} = \mathcal{A}^{-1} \circ (M_{J^i, J^j} \otimes \text{Id}) \circ \mathcal{A}$ which in turn equals Id because of (4). Now use Lemma 2.10 to get the result for all parenthesizings of $J^j \otimes X$.

For (6):

$$\begin{aligned}
\text{Id}_X &= M_{1_C, X} \\
&= \mathcal{A}_{J, J^{-1}, X} \circ M_{J \otimes J^{-1}, X} \circ \mathcal{A}_{J, J^{-1}, A}^{-1} \\
&= \mathcal{A}_{J, J^{-1}, X} \circ (\text{Id} \otimes M_{J^{-1}, X}) \circ \mathcal{A}_{J, J^{-1}, A}^{-1}.
\end{aligned}$$

Hence, $M_{J^{-1}, X}$ must be Id . We proceed similarly for the rest.

For (7), using (6) we get that $M_{J,J} = \text{Id}$ implies $M_{J^{-1}, J} = \text{Id}$ and $M_{J, J^{-1}} = \text{Id}$, either of which leads to $M_{J^{-1}, J^{-1}} = \text{Id}$. The rest can be easily obtained as in the proof of (2), (3) and (4).

Lastly, (8) is obtained by following the steps in the proof of (5), (6) and (7). \square

We will use the proposition above to give a lifting criterion for simple modules in Corollary 3.16. Next, we provide some useful lemmata which will help us strengthen Corollary 3.16 to some indecomposable modules.

Lemma 2.12. *Let J be a simple current. Then, for any P and X , such that $\dim(\text{Hom}(P, X)) < \infty$ and either $\dim(\text{Hom}(P \otimes J, X \otimes J)) < \infty$ or $\dim(\text{Hom}(J \otimes P, J \otimes X)) < \infty$, we have that*

$$\dim(\text{Hom}(P, X)) = \dim(\text{Hom}(P \otimes J, X \otimes J)) = \dim(\text{Hom}(J \otimes P, J \otimes X)). \quad (2.4)$$

Proof. Since braiding is an isomorphism, $\dim(\text{Hom}(P \otimes J, X \otimes J)) < \infty$ if and only if $\dim(\text{Hom}(J \otimes P, J \otimes X)) < \infty$.

The conclusion holds if $J = \mathbf{1}_C$. That is, $\bullet \otimes \text{Id}_{\mathbf{1}_C}$ is an isomorphism. Fix an isomorphism $g : \mathbf{1}_C \rightarrow J \otimes J^{-1}$. We first prove that $\dim(\text{Hom}(P, X)) \leq \dim(\text{Hom}(P \otimes J, X \otimes J))$. Consider the map $\bullet \otimes \text{Id}_J : \text{Hom}(P, X) \rightarrow \text{Hom}(P \otimes J, X \otimes J)$. If $f \otimes \text{Id}_J = 0$, then,

$$\begin{aligned} 0 &= \mathcal{A}_{X, J, J^{-1}}^{-1} \circ ((f \otimes \text{Id}_J) \otimes \text{Id}_{J^{-1}}) \\ &= f \otimes (\text{Id}_J \otimes \text{Id}_{J^{-1}}) = f \otimes (\text{Id}_{J \otimes J^{-1}}) \\ &= (\text{Id}_X \otimes g) \circ (f \otimes \text{Id}_{\mathbf{1}_C}) \circ (\text{Id}_P \otimes g^{-1}). \end{aligned}$$

Therefore, $f \otimes \text{Id}_{\mathbf{1}_C} = 0$. Since $\bullet \otimes \text{Id}_{\mathbf{1}_C}$ is an isomorphism, $f = 0$.

We turn to the converse direction. Replacing J by J^{-1} in the argument above, we have that

$$\dim(\text{Hom}(P \otimes J, X \otimes J)) \leq \dim(\text{Hom}((P \otimes J) \otimes J^{-1}, (X \otimes J) \otimes J^{-1})).$$

However, since associativity is an isomorphism and since $J \otimes J^{-1} \cong \mathbf{1}_C$

$$\begin{aligned} \text{Hom}((P \otimes J) \otimes J^{-1}, (X \otimes J) \otimes J^{-1}) &\cong \text{Hom}(P \otimes (J \otimes J^{-1}), X \otimes (J \otimes J^{-1})) \\ &\cong \text{Hom}(P \otimes \mathbf{1}_C, X \otimes \mathbf{1}_C) \\ &\cong \text{Hom}(P, X). \end{aligned}$$

For showing $\dim(\text{Hom}(P, X)) = \dim(\text{Hom}(J \otimes P, J \otimes X))$, one proceeds similarly. \square

Lemma 2.13. *Let J be a finite order simple current such that $J^N \cong \mathbf{1}_C$ for some $N \in \mathbb{Z}_+$. Let P be any object such that $\dim(\text{Hom}(P, P)) < \infty$ and $\dim(\text{Hom}(J \otimes P, J \otimes P)) < \infty$. Assume that $M_{J,P} = \lambda \text{Id}_{J \otimes P} + \pi$ where π is a nilpotent endomorphism of $J \otimes P$. Then, $\pi = 0$, equivalently, $M_{J,P}$ is a semi-simple endomorphism. Moreover, $\lambda^N = 1$.*

Proof. Lemma 2.12 in fact shows that $\text{Id}_J \otimes \bullet$ provides an isomorphism $\text{Hom}(P, P) \cong \text{Hom}(J \otimes P, J \otimes P)$ and we conclude that $\pi = \text{Id}_J \otimes \nu$ for some nilpotent endomorphism ν of P .

We claim that for any $n \in \mathbb{Z}_+$, regardless of how J^n is parenthesized,

$$M_{J^n, P} = \sum_{i=0}^n \binom{n}{i} \lambda^i \text{Id}_{J^n} \otimes \nu^{n-i}. \quad (2.5)$$

Equation (2.5) holds for $n = 0$ since $M_{\mathbf{1}_C, X} = \text{Id}_X$ for any object X and for $n = 1$ by assumption. We proceed by induction. Assume that the claim holds for some $n \in \mathbb{Z}_+$. By Lemma 2.10, it is enough to prove the claim when J^n is parenthesized so that $J^n = J \otimes J^{n-1}$. Exactly as in the proof of Theorem 2.11,

$$\begin{aligned} M_{J \otimes J^n, P} &= c_{P, J \otimes J^n} \circ c_{J \otimes J^n, P} \\ &= \mathcal{A}_{J, J^n, P} \circ (\text{Id}_J \otimes c_{P, J^n}) \circ \mathcal{A}_{J, P, J^n}^{-1} \circ (M_{J, P} \otimes \text{Id}_{J^n}) \circ \mathcal{A}_{J, P, J^n} \circ (\text{Id}_J \otimes c_{J^n, P}) \circ \mathcal{A}_{J, J^n, P}^{-1} \\ &= \mathcal{A}_{J, J^n, P} \circ (\text{Id}_J \otimes c_{P, J^n}) \circ \mathcal{A}_{J, P, J^n}^{-1} \circ (\lambda \text{Id}_{J \otimes P} \otimes \text{Id}_{J^n}) \circ \mathcal{A}_{J, P, J^n} \circ (\text{Id}_J \otimes c_{J^n, P}) \circ \mathcal{A}_{J, J^n, P}^{-1} \\ &\quad + \mathcal{A}_{J, J^n, P} \circ (\text{Id}_J \otimes c_{P, J^n}) \circ \mathcal{A}_{J, P, J^n}^{-1} \circ ((\text{Id}_J \otimes \nu) \otimes \text{Id}_{J^n}) \circ \mathcal{A}_{J, P, J^n} \circ (\text{Id}_J \otimes c_{J^n, P}) \circ \mathcal{A}_{J, J^n, P}^{-1}. \end{aligned}$$

However, using naturality of braiding and associativity and using the induction hypothesis, we observe that:

$$\begin{aligned}
& \mathcal{A}_{J,J^n,P} \circ (\text{Id}_J \otimes c_{P,J^n}) \circ \mathcal{A}_{J,P,J^n}^{-1} \circ (\lambda \text{Id}_{J \otimes P} \otimes \text{Id}_{J^n}) \circ \mathcal{A}_{J,P,J^n} \circ (\text{Id}_J \otimes c_{J^n,P}) \circ \mathcal{A}_{J,J^n,P}^{-1} \\
&= \lambda \mathcal{A}_{J,J^n,P} \circ (\text{Id}_J \otimes M_{J^n,P}) \circ \mathcal{A}_{J,J^n,P}^{-1} \\
&= \lambda \mathcal{A}_{J,J^n,P} \circ \left(\text{Id}_J \otimes \left(\sum_{i=0}^n \binom{n}{i} \lambda^i \text{Id}_{J^n} \otimes \nu^{n-i} \right) \right) \circ \mathcal{A}_{J,J^n,P}^{-1} \\
&= \text{Id}_J \otimes \left(\sum_{i=0}^n \binom{n}{i} \lambda^{i+1} \text{Id}_{J^n} \otimes \nu^{n-i} \right) \\
&= \sum_{i=0}^n \binom{n}{i} \lambda^{i+1} \text{Id}_{J \otimes J^n} \otimes \nu^{n-i}.
\end{aligned}$$

and

$$\begin{aligned}
& \mathcal{A}_{J,J^n,P} \circ (\text{Id}_J \otimes c_{P,J^n}) \circ \mathcal{A}_{J,P,J^n}^{-1} \circ ((\text{Id}_J \otimes \nu) \otimes \text{Id}_{J^n}) \circ \mathcal{A}_{J,P,J^n} \circ (\text{Id}_J \otimes c_{J^n,P}) \circ \mathcal{A}_{J,J^n,P}^{-1} \\
&= \mathcal{A}_{J,J^n,P} \circ (\text{Id}_J \otimes c_{P,J^n}) \circ (\text{Id}_J \otimes (\nu \otimes \text{Id}_{J^n})) \circ (\text{Id}_J \otimes c_{J^n,P}) \circ \mathcal{A}_{J,J^n,P}^{-1} \\
&= \mathcal{A}_{J,J^n,P} \circ (\text{Id}_J \otimes (\text{Id}_{J^n} \otimes \nu)) \circ (\text{Id}_J \otimes M_{J^n,P}) \circ \mathcal{A}_{J,J^n,P}^{-1} \\
&= \mathcal{A}_{J,J^n,P} \circ (\text{Id}_J \otimes (\text{Id}_{J^n} \otimes \nu)) \circ \left(\text{Id}_J \otimes \left(\sum_{i=0}^n \binom{n}{i} \lambda^i \text{Id}_{J^n} \otimes \nu^{n-i} \right) \right) \circ \mathcal{A}_{J,J^n,P}^{-1} \\
&= \mathcal{A}_{J,J^n,P} \circ \left(\text{Id}_J \otimes \left(\sum_{i=0}^n \binom{n}{i} \lambda^i \text{Id}_{J^n} \otimes \nu^{n-i+1} \right) \right) \circ \mathcal{A}_{J,J^n,P}^{-1} \\
&= \sum_{i=0}^n \binom{n}{i} \lambda^i \text{Id}_{J \otimes J^n} \otimes \nu^{n+1-i}.
\end{aligned}$$

Combining the two, we immediately get equation (2.5) for $n + 1$.

Now, using equation (2.5) for $n = N$, we get:

$$\text{Id}_P = M_{\mathbf{1}_C,P} = M_{J^N,P} = \sum_{i=0}^N \binom{N}{i} \lambda^i \text{Id}_{J^N} \otimes \nu^{N-i}.$$

However, since ν is nilpotent, we immediately conclude that $\nu = 0$ and $\lambda^N = 1$. \square

\mathcal{C} -superalgebras. In this section, we generalize the notion of \mathcal{C} -algebra of Kirillov-Ostrik to \mathcal{C} -superalgebra. We closely follow their notation and results [KO].

For the rest of the work, we assume the category \mathcal{C} to be abelian, and we assume that \otimes naturally distributes over \oplus . This distributivity will hold for the categories we shall consider, thanks to Proposition 4.24 of [HLZ].

Definition 2.14. A \mathcal{C} -superalgebra is an object $A = A^0 \oplus A^1 \in \mathcal{C}$ (A^0, A^1 are objects in \mathcal{C}) with morphisms

$$\mu : A \otimes A \rightarrow A \tag{2.6}$$

$$\iota : \mathbf{1}_C \hookrightarrow A^0. \tag{2.7}$$

Such that the following conditions hold.

- (1) μ respects the $\frac{1}{2}\mathbb{Z}$ -grading: $\mu(\theta \otimes \theta) = \theta \circ \mu$.
- (2) μ respects the \mathbb{Z}_2 -grading: $\mu(A^i \otimes A^j) \rightarrow A^{i+j} \hookrightarrow A$.

(3) *Associativity:*

$$\mu \circ (\mu \otimes \text{Id}_A) \circ \mathcal{A} = \mu \circ (\text{Id}_A \otimes \mu)$$

(4) *Commutativity:*

$$\mu|_{A^i \otimes A^j} = (-1)^{ij} \cdot \mu \circ c_{A^i, A^j}$$

(5) *Unit:*

$$\mu \circ (\iota_A \otimes \text{Id}_A) \circ \ell_A^{-1} = \text{Id}_A$$

where

$$\ell_A : \mathbf{1}_C \otimes A \rightarrow A$$

is the left unit isomorphism.

Such an algebra is called *haploid* if it has

(6) *Uniqueness of unit:*

$$\dim \text{Hom}(\mathbf{1}_C, A) = 1.$$

Following [KO], we define a natural category for representations of a \mathcal{C} -superalgebra.

Definition 2.15. Let $(A = A^0 \oplus A^1, \mu, \iota)$ be a \mathcal{C} -superalgebra. Define a category $\text{Rep } A$ as follows. The objects are pairs $(W = W^0 \oplus W^1, \mu_W)$, where $W^0, W^1 \in \mathcal{C}$,

$$\mu_W : A \otimes W \cong \bigoplus_{i,j \in \mathbb{Z}/2\mathbb{Z}} A^i \otimes W^j \rightarrow W$$

is a morphism satisfying:

- (1) $\mu_W : A^i \otimes W^j \rightarrow W^{i+j \pmod{2}}$,
- (2) $\mu_W \circ (\mu \otimes \text{Id}_W) \circ \mathcal{A} = \mu_W \circ (\text{Id}_A \otimes \mu_W) : A \otimes (A \otimes W) \rightarrow W$,
- (3) $\mu_W \circ (\iota_A \otimes \text{Id}_W) = \ell_W : \mathbf{1}_C \otimes W \rightarrow W$.

The morphisms are defined as:

$$\begin{aligned} \text{Hom}_{\text{Rep } A}((M, \mu_M), (N, \mu_N)) \\ = \{\varphi \in \text{Hom}_{\mathcal{C}}(M, N) \mid \mu_N \circ (\text{Id}_A \otimes \varphi) = \varphi \circ \mu_M : A \otimes M \rightarrow N\} \end{aligned}$$

Definition 2.16. Define $\text{Rep}^0 A$ to be the full subcategory of $\text{Rep } A$ consisting of objects (W, μ_W) such that

$$\mu_W \circ (c_{W,A} \circ c_{A,W}) = \mu_W : A \otimes W \rightarrow W.$$

Definition 2.17. Given a \mathcal{C} -superalgebra $(A = A^0 \oplus A^1, \mu, \iota)$, define

$$\mathcal{F}(X) = (A \otimes X = A^0 \otimes X \oplus A^1 \otimes X, (\mu \otimes \text{Id}_X) \circ \mathcal{A}_{A,A,X}),$$

$$\mathcal{F}(f) = \text{Id}_A \otimes f$$

for X an object in \mathcal{C} and f a morphism.

Theorem 2.18. \mathcal{F} is a functor from \mathcal{C} to $\text{Rep } A$.

Proof. Let W be an object in \mathcal{C} . We now prove that $\mathcal{F}(W)$ is an object of $\text{Rep } A$. Since $\mu : A^i \otimes A^j \rightarrow A^{i+j \pmod{2}}$, it is clear that $\mu_W : A^i \otimes (A^j \otimes W) \rightarrow A^{i+j \pmod{2}} \otimes W$. Therefore, condition (1) is satisfied. For condition (2), consider the following commuting diagram, where the unlabeled arrows

correspond to associativity isomorphisms, obtained by using the pentagon diagram, naturality of associativity and by the associativity of μ .

$$\begin{array}{ccccc}
& & A \otimes (A \otimes (A \otimes W)) & & \\
& & \downarrow & \searrow & \\
A \otimes (A \otimes W) & \xleftarrow{\mu \otimes (\text{Id} \otimes \text{Id})} & (A \otimes A) \otimes (A \otimes W) & & A \otimes ((A \otimes A) \otimes W) \\
\downarrow & & \downarrow & & \downarrow \\
(A \otimes A) \otimes W & \xleftarrow{(\mu \otimes \text{Id}) \otimes \text{Id}} & ((A \otimes A) \otimes A) \otimes W & \xleftarrow{} & (A \otimes (A \otimes A)) \otimes W \\
\downarrow \mu \otimes \text{Id} & & \downarrow (\text{Id} \otimes \mu) \otimes \text{Id} & & \downarrow \text{Id} \otimes (\mu \otimes \text{Id}) \\
A \otimes W & \xleftarrow{\mu \otimes \text{Id}} & (A \otimes A) \otimes W & & A \otimes (A \otimes W)
\end{array}$$

This commutative diagram immediately establishes (2).

For (3), we have:

$$\begin{aligned}
\ell_{A \otimes W} &= (\ell_A \otimes \text{Id}) \circ \mathcal{A}_{\mathbf{1}_C, A, W} \\
&= ((\mu \circ (\ell_A \otimes \text{Id})) \otimes \text{Id}) \circ \mathcal{A}_{\mathbf{1}_C, A, W} \\
&= (\mu \otimes \text{Id}) \circ ((\ell_A \otimes \text{Id}) \otimes \text{Id}) \circ \mathcal{A}_{\mathbf{1}_C, A, W} \\
&= (\mu \otimes \text{Id}) \circ \mathcal{A}_{A, A, W} \circ (\ell_A \otimes (\text{Id} \otimes \text{Id})),
\end{aligned}$$

where the first equality follows by the properties of left unit, the second property follows by the left unit property of A and the last equality follows by naturality of associativity.

Now let $f : U \rightarrow W$ be a morphism in \mathcal{C} . Let $\varphi = \text{Id}_A \otimes f : \mathcal{F}(U) = A \otimes U \rightarrow A \otimes W = \mathcal{F}(W)$. Then,

$$\begin{aligned}
\mu_{\mathcal{F}(W)} \circ (\text{Id}_A \otimes \varphi) &= (\mu \otimes \text{Id}_W) \circ \mathcal{A}_{A, A, W} \circ (\text{Id}_A \otimes (\text{Id}_A \otimes f)) \\
&= (\mu \otimes \text{Id}_W) \circ ((\text{Id}_A \otimes \text{Id}_A) \otimes f) \circ \mathcal{A}_{A, A, W} \\
&= (\text{Id}_A \otimes f) \circ (\mu \otimes \text{Id}_W) \circ \mathcal{A}_{A, A, W} \\
&= \varphi \circ \mu_{\mathcal{F}(U)},
\end{aligned}$$

where we have used naturality of associativity in the second equality. \square

3. SIMPLE CURRENT EXTENSIONS AND ALGEBRAS

Definition 3.1. A vertex operator superalgebra is a triple $(V, \mathbf{1}, \omega, Y)$, where V has compatible gradings by $\frac{1}{2}\mathbb{Z}$ and \mathbb{Z}_2 , i.e.,

$$V = V^0 \oplus V^1 = \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} V_n^0 \oplus \bigoplus_{n \in \frac{1}{2}\mathbb{Z}} V_n^1, \quad (3.1)$$

Y is a map

$$Y : V \otimes V \rightarrow V[[x, x^{-1}]], \quad (3.2)$$

such that the following axioms are satisfied. We let $Y(\omega, x) = \sum_{n \in \mathbb{Z}} L(n)x^{-n-2}$.

(1) Axioms for grading:

Lower truncation: $V_n = 0$ for all sufficiently negative n .

Finite dimensionality: Each V_n is finite dimensional.

$L(0)$ -grading property: $V_n = \{v \in V \mid L(0)v = nv\}$.

$\mathbf{1} \in V_0^0, \omega \in V_2^0$.

(2) *Axioms for vacuum:*

Left-identity property: $Y(\mathbf{1}, x)v = v$ for all $v \in V$.

Creation property: $\lim_{x \rightarrow 1} Y(v, 1)\mathbf{1}$ exists and equals v for all $v \in V$.

(3) *$L(-1)$ -derivative property:* $[L(-1), Y(v, x)] = Y(L(-1)v, x) = \frac{d}{dx}Y(v, x)$.

(4) *Virasoro relations:* $[L(m), L(n)] = (m - n)L(m + n) + \frac{m^3 - m}{12}\delta_{m, -n}c$

(5) *Jacobi identity:* For $u \in V^i, v \in V^j$,

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y(u, x_1)Y(v, x_2) - (-1)^{ij}x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right)Y(v, x_2)Y(u, x_1) \\ &= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y(Y(u, x_0)v, x_2). \end{aligned}$$

Definition 3.2. A $\frac{1}{2}\mathbb{Z}$ -graded vertex operator algebra is a vertex operator superalgebra V such that $V^1 = 0$.

Definition 3.3. A vertex operator algebra is a $\frac{1}{2}\mathbb{Z}$ -graded vertex operator algebra that is in fact \mathbb{Z} -graded.

Definition 3.4. Consider a vertex operator superalgebra $(V, \mathbf{1}, \omega, Y)$. A V -module is a vector space W with compatible gradings by \mathbb{R} and $\mathbb{Z}/2\mathbb{Z}$ i.e.,

$$W = W^0 \oplus W^1 = \bigoplus_{n \in \mathbb{R}} W_n^0 \oplus \bigoplus_{n \in \mathbb{R}} W_n^1, \quad (3.3)$$

equipped with a vertex operator map Y_W ,

$$Y_W : V \otimes W \rightarrow W[[x, x^{-1}]], \quad (3.4)$$

such that the following axioms are satisfied. We denote the modes of $Y_W(\omega, x)$ by $L(n)$.

(1) *Axioms for grading:*

Lower truncation: $W_n = 0$ for all sufficiently negative n .

Finite dimensionality: Each W_n is finite dimensional.

$L(0)$ -grading property: $W_n = \{w \in W \mid L(0)w = nw\}$.

$\mathbb{Z}/2\mathbb{Z}$ -grading compatibility: $Y_W : V^i \otimes W^j \rightarrow W^{i+j \pmod{2}}[[x, x^{-1}]]$.

(2) *Axioms for vacuum:*

Left-identity property: $Y_W(\mathbf{1}, x)w = w$ for all $w \in W$.

(3) *$L(-1)$ -derivative property:* $[L(-1), Y(v, x)] = Y(L(-1)v, x) = \frac{d}{dx}Y(v, x)$.

(4) *Jacobi identity:* For $u \in V^i, v \in V^j$,

$$\begin{aligned} & x_0^{-1}\delta\left(\frac{x_1 - x_2}{x_0}\right)Y_W(u, x_1)Y_W(v, x_2) - (-1)^{ij}x_0^{-1}\delta\left(\frac{-x_2 + x_1}{x_0}\right)Y_W(v, x_2)Y_W(u, x_1) \\ &= x_2^{-1}\delta\left(\frac{x_1 - x_0}{x_2}\right)Y_W(Y(u, x_0)v, x_2). \end{aligned}$$

Remark 3.5. For definitions involving complex variables instead of the formal variables, refer to [H6].

Definition 3.6. A module W is called a generalized V -module if the \mathbb{R} -grading on W is by generalized eigenvalues of $L(0)$, i.e., W is a direct sum of generalized eigenspaces of $L(0)$. Thus, a generalized module W is in fact a grading-restricted generalized module in the sense of [H6].

Assumption 3.7. We will work with the following assumption in the next few sections. Assume that V is a vertex operator algebra satisfying the conditions required to invoke Huang-Lepowsky-Zhang's theory. We also assume that braiding and twist are as given by Huang-Lepowsky-Zhang. We assume that the tensor bifunctor is chosen to be $\boxtimes_{P(1)}$, which we abbreviate to be \boxtimes .

Definition 3.8. Recall Definition 3.10, 4.2 and 4.13 of logarithmic intertwining operators, $P(z)$ -intertwining maps and $P(z)$ -tensor products, respectively, from [HLZ] and the definitions of rationality of products, rationality of iterates, commutativity and associativity for the vertex operator map and the module map from [H6].

Theorem 3.9. Let J be a simple current such that $J \boxtimes J \cong V$ (which implies that V is simple, see [CKLR]) and $\theta_J = \pm \text{Id}_J$ (which implies that $c_{J,J} \in \{\text{Id}, -\text{Id}\}$ by balancing). If $c_{J,J} = 1$, $V \oplus J$ has a structure of $\frac{1}{2}\mathbb{Z}$ -graded vertex operator algebra and if $c_{J,J} = -1$, $V \oplus J$ has a structure of vertex operator superalgebra.

Proof. Structurally, the proof is similar to the proof of Theorem 4.1 from [H6]. Note the following implications of the assumptions: Since J is simple, $L(0)$ acts semisimply on J , moreover, since $\theta_J = \pm \text{Id}_J$, J is graded either by \mathbb{Z} or by $\frac{1}{2} + \mathbb{Z}$. Since $L(0)$ acts semi-simply on both V and J , any logarithmic intertwining operator of the type $\binom{C}{AB}$ where $A, B, C \in \{V, J\}$ is free of logarithms, cf. Remark 3.23 of [HLZ]. Since J is simple, we have the following fusion rules:

$$\mathcal{N}_{J,V}^J = \mathcal{N}_{V,J}^J = \dim(\text{Hom}(V \boxtimes J, J)) = \dim(\text{Hom}(J, J)) = 1. \quad (3.5)$$

Also, by assumption,

$$\mathcal{N}_{J,J}^{J \boxtimes J} = \dim(\text{Hom}(J \boxtimes J, J \boxtimes J)) = \dim(\text{Hom}(J \boxtimes J, V)) = \mathcal{N}_{J,J}^V = \dim(\text{Hom}(V, V)) = 1. \quad (3.6)$$

Fix an isomorphism $j : J \boxtimes J \rightarrow V$.

Let \mathcal{Y}_{\boxtimes} be the (non-zero) intertwining operator corresponding to the intertwining map \boxtimes of type $\binom{J \boxtimes J}{J J}$. Let

$$\mathcal{Y} = j \circ \mathcal{Y}_{\boxtimes}.$$

It is clear that \mathcal{Y} is the non-zero intertwining operator of type $\binom{V}{J J}$ corresponding to the $P(1)$ -intertwining map $\bar{j} \circ \boxtimes : J \otimes J \rightarrow \bar{V}$. Since J is graded either by \mathbb{Z} or by $1/2 + \mathbb{Z}$, \mathcal{Y} has only integral powers of the formal variable.

Consider the intertwining operator Y_e of type $\binom{V \oplus J}{V \oplus J V \oplus J}$ defined by

$$Y_e(v_1 \oplus j_1, x)(v_2 \oplus j_2) = (Y(v_1, x)v_2 + \mathcal{Y}(j_1, x)j_2) \oplus (Y_J(v_1, x)j_2 + e^{xL(-1)}Y_J(v_2, -x)j_1)$$

Note that Y_e also has only integral powers of x .

First, we analyze the braiding in order to relate $c_{J,J}$ with the skew-symmetry. We will need this information to prove the associativity for Y_e . The braiding is characterized by (cf. equation (3.9) of [HKL]):

$$\bar{\mathcal{R}}_{J \boxtimes J}(j_1 \boxtimes j_2) = e^{L(-1)}\bar{\mathcal{T}}_{\gamma_1^-}(j_2 \boxtimes_{P(-1)} j_1),$$

where γ_1^- is a path in $\overline{\mathbb{H}} \setminus \{0\}$ from -1 to 1 , and correspondingly, $\mathcal{T}_{\gamma_1^-}$ is the parallel transport isomorphism from $J \boxtimes_{P(-1)} J$ to $J \boxtimes J$. Recall that \mathcal{Y}_{\boxtimes} is the intertwining operator corresponding to the intertwining map \boxtimes of type $\binom{J \boxtimes J}{J J}$. Then, (cf. equation (3.11) of [HKL]),

$$\bar{\mathcal{R}}_{J \boxtimes J}(j_1 \boxtimes j_2) = e^{L(-1)}\mathcal{Y}_{\boxtimes}(j_2, e^{i\pi})j_1.$$

Due to our assumption, we know that $\mathcal{R}_{J \boxtimes J} = c_{J,J}\text{Id}_{J \boxtimes J}$ on $J \boxtimes J$ and hence,

$$c_{J,J}(j_1 \boxtimes j_2) = c_{J,J}\mathcal{Y}_{\boxtimes}(j_1, 1)j_2 = e^{L(-1)}\mathcal{Y}_{\boxtimes}(j_2, e^{i\pi})j_1 = e^{L(-1)}\mathcal{Y}_{\boxtimes}(j_2, -1)j_1. \quad (3.7)$$

Composing with \bar{j} , (recall that we have fixed an isomorphism $j : J \boxtimes J \rightarrow V$), we get:

$$c_{J,J}\mathcal{Y}(j_1, 1)j_2 = e^{L(-1)}\mathcal{Y}(j_2, -1)j_1 \quad (3.8)$$

Note that this can also be written as

$$c_{J,J}Y_e(j_1, 1)j_2 = e^{L(-1)}Y_e(j_2, -1)j_1. \quad (3.9)$$

Now we move to the associativity of Y_e . We would like to prove that for all $u, v, w \in X$ and $|z_1| > |z_2| > |z_1 - z_2| > 0$,

$$Y_e(u, z_1)Y_e(v, z_2)w = Y_e(Y_e(u, z_1 - z_2)v, z_2)w. \quad (3.10)$$

There are a few cases: If all u, v, w are in V then all of Y_e are equal to the vertex operator map Y . So the equality of right-hand sides follows from the associativity for Y . Similarly, if exactly one of u, v, w is in J then equality of the right-hand sides follows from the properties of the module map Y_J . If exactly two of u, v, w are in J then equality follows from the properties of the intertwining operator \mathcal{Y} , see [FHL].

The tricky part is when all u, v, w are in J . In this case, we would like to prove that:

$$Y_e(j_1, z_1)\mathcal{Y}(j_2, z_2)j_3 = Y_e(\mathcal{Y}(j_1, z_1 - z_2)j_2, z_2)j_3$$

Using Theorem 9.24 of [HLZ], we know this statement up to a constant: We know that there exist intertwining operators \mathcal{Y}^1 and \mathcal{Y}^2 of types $\binom{J \boxtimes J}{JJ}$ and $\binom{V}{J \boxtimes J}$ respectively such that

$$Y_e(j_1, z_1)\mathcal{Y}(j_2, z_2)j_3 = \mathcal{Y}^1(\mathcal{Y}^2(j_1, z_1 - z_2)j_2, z_2)j_3.$$

But now, since $J \boxtimes J \cong V$, we can get $\tilde{\mathcal{Y}}^1$ and $\tilde{\mathcal{Y}}^2$ of types $\binom{V}{JJ}$ and $\binom{V}{VJ}$ respectively such that

$$Y_e(j_1, z_1)\mathcal{Y}(j_2, z_2)j_3 = \tilde{\mathcal{Y}}^1(\tilde{\mathcal{Y}}^2(j_1, z_1 - z_2)j_2, z_2)j_3.$$

Using the assumed fusion rules, $\tilde{\mathcal{Y}}^2$ is proportional to \mathcal{Y} and by [FHL], $\tilde{\mathcal{Y}}^1$ is proportional to the module map Y_J . Therefore,

$$Y_e(j_1, z_1)\mathcal{Y}(j_2, z_2)j_3 = \lambda Y_e(Y_e(j_1, z_1 - z_2)j_2, z_2)j_3.$$

We must prove that the proportionality constant $\lambda = 1$.

Let us also gather information about this proportionality constant in all the 8 cases corresponding to each u, v, w being in either V or J . Let us temporarily grade the space $V \oplus J$ with \mathbb{Z}_2 such that $V = (V \oplus J)^{0t}$, $J = (V \oplus J)^{1t}$. Here t stands for temporary. This may or may not be the intended \mathbb{Z}_2 grading as in the definition of vertex operator superalgebra. Using simplicity of V and J and Theorem 11.9 of [DL], we know that arbitrary products and iterates of Y_e are non-zero. We thus obtain a well-defined map

$$F(g_1, g_2, g_3) : \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \mathbb{C}^\times,$$

which measures the failure in associativity. Our aim is to prove that $F \equiv 1$. Similarly, let

$$\Omega : \mathbb{Z}_2 \times \mathbb{Z}_2 \rightarrow \mathbb{C}^\times$$

denote the constants regarding skew-symmetry, i.e.,

$$Y_e(u, x)v = \Omega(i, j)e^{xL(-1)}Y_e(v, -x)u,$$

whenever $u \in (V \oplus J)^{it}$ and $v \in (V \oplus J)^{jt}$ with $i, j \in \mathbb{Z}_2$.

At this point, we can proceed in two ways. One is by deriving and using the equations satisfied by F and Ω or the other way is to proceed as in the proof of Theorem 4.1 of [H6].

For the relations satisfied by F and Ω , a general derivation could be found in [H2, Ch]. In our setup, one can also refer to [C]. With our setup, we have got a one-dimensional \mathcal{T} -commutativity datum with a normalized choice of intertwining operators in the sense of [C]. Therefore, using Lemmas 2.1.5, 2.1.7 and 2.2.7 in [C], we know that (F, Ω) is a normalized abelian 3-cocycle on the group \mathbb{Z}_2 with coefficients in \mathbb{C}^\times , i.e., for all $i, j, k, l \in \mathbb{Z}_2$ the following hold:

$$F(0, i, j) = F(i, 0, j) = F(i, j, 0) \quad (3.11)$$

$$\Omega(0, i) = \Omega(i, 0) = 1 \quad (3.12)$$

$$F(i, j, k)F(i, j + k, l)F(j, k, l) = F(i + j, k, l)F(i, j, k + l) \quad (3.13)$$

$$F(i, j, k)^{-1}\Omega(i, j + k)F(j, k, i)^{-1} = \Omega(i, j)F(j, i, k)^{-1}\Omega(i, k) \quad (3.14)$$

$$F(i, j, k)\Omega(i + j, k)F(k, i, j) = \Omega(j, k)F(i, k, j)\Omega(i, k) \quad (3.15)$$

Letting $i, j, k = 1$ in equation 3.15 gives

$$F(1, 1, 1)\Omega(0, 1)F(1, 1, 1) = \Omega(1, 1)F(1, 1, 1)\Omega(1, 1)$$

Hence,

$$F(1, 1, 1) = \Omega(1, 1)^2.$$

In our case, equation 3.8 implies that $\Omega(1, 1)$ is equal to $c_{J,J}^{-1}$, thereby giving $F(1, 1, 1) = 1$ and thereby giving associativity of Y_e . We conclude that the associativity of Y_e holds.

An alternate and a more direct way to prove $F \equiv 1$ is as in the proof of Theorem 4.1 of [H6]. The approach in [H6] amounts precisely to using equation 3.15.

Now, if $c_{J,J} = 1$, we define a final \mathbb{Z}_2 grading on $X = V \oplus J$ by $X^0 = X, X^1 = 0$ and if $c_{J,J} = -1$, $X^0 = V, X^1 = J$.

Recalling equation 3.9, we see that when $c_{J,J} = 1$, we get the skew-symmetry as in the definition of a $\frac{1}{2}\mathbb{Z}$ -graded vertex operator algebra and when $c_{J,J} = -1$, we get the skew-symmetry as in the definition of a vertex operator superalgebra. Now, we proceed exactly as in [H6]. Since Y_e satisfies associativity and skew-symmetry, using results in [H2], one gets that Y_e satisfies commutativity for $\frac{1}{2}\mathbb{Z}$ -graded vertex operator algebras when $c_{J,J} = 1$ and satisfies commutativity for vertex operator superalgebras when $c_{J,J} = -1$. Since Y_e has only integer powers of the formal variable and hence rationality of products and iterates holds. The other axioms in the definition being easy to verify, we conclude that $V \oplus J$ is a $\frac{1}{2}\mathbb{Z}$ -graded vertex operator algebra when $c_{J,J} = 1$ and a vertex operator superalgebra when $c_{J,J} = -1$. □

Remark 3.10. When $\theta_J = -1, c_{J,J} = 1$, we get a vertex operator algebra with “wrong statistics,” that is, a genuinely $\frac{1}{2}\mathbb{Z}$ -graded vertex operator algebra.

When $\theta_J = 1, c_{J,J} = -1$, we get a vertex operator superalgebra with “wrong statistics,” that is, a \mathbb{Z} -graded vertex operator superalgebra.

Remark 3.11. Any two choices j, j' of isomorphisms $J \boxtimes J \cong V$ are non-zero scalar multiples of each other since V is simple. Let $j' = \lambda j$, for some $\lambda \in \mathbb{C}^\times$. If we use j' in place of j in the proof above, we get the map

$$\begin{aligned} & Y_e'(v_1 \oplus j_1, x)(v_2 \oplus j_2) \\ &= (Y(v_1, x)v_2 + \lambda \mathcal{Y}(j_1, x)j_2) \oplus (Y_J(v_1, x)j_2 + e^{xL(-1)}Y_J(v_2, -x)j_1) \end{aligned}$$

Fix an $l \in \mathbb{C}^\times$ such that $l^2 = \lambda$. Define $f : V \oplus J \rightarrow V \oplus J$ by $f(v \oplus w) = v \oplus lw$ for $v \in V, w \in J$. It is clear that f is invertible and fixes the vacuum and the conformal vector as they both belong to V . Moreover,

$$\begin{aligned} & f(Y_e'(v_1 \oplus j_1, x)(v_2 \oplus j_2)) \\ &= (Y(v_1, x)v_2 + \lambda \mathcal{Y}(j_1, x)j_2) \oplus l(Y_J(v_1, x)j_2 + e^{xL(-1)}Y_J(v_2, -x)j_1) \\ &= (Y(v_1, x)v_2 + \mathcal{Y}(lj_1, x)lj_2) \oplus (Y_J(v_1, x)lj_2 + e^{xL(-1)}Y_J(v_2, -x)lj_1) \\ &= Y_e(f(v_1 \oplus j_1), x)(f(v_2 \oplus j_2)) \end{aligned}$$

We see that f furnishes an isomorphism of the two vertex operator algebra structures on $V \oplus J$.

Beyond self-dual simple currents. Combining our findings with those of Carnahan [C] we now get several results on general simple current extensions of VOAs.

Theorem 3.12. *Let J be a simple current. This implies that $J \boxtimes J^{-1} \cong V$ is simple by [CKLR]. Assume that $\theta_{J^k} = \pm \text{Id}_{J^k}$, with $\theta_{J^{k+2}}$ having the same sign as θ_{J^k} all $k \in \mathbb{Z}$. Then*

$$V_e = \bigoplus_{j \in G} J^j$$

has a natural structure of a strongly G -graded vertex operator superalgebra, where G is the cyclic group generated by J . (For the definition of strongly graded, we refer the reader to [HLZ].) If G is finite, we get a vertex operator superalgebra.

Proof. The assumptions on the twist combined with balancing imply that $c_{J^k, J^k} = \pm \text{Id}$. By assumption, $\dim \binom{J^{i+j}}{J^i J^j} = 1$. For each i, j , choose a non-zero logarithmic intertwining operator $\tilde{\mathcal{Y}}_{i,j} \in \binom{J^{i+j}}{J^i J^j}$, such that $\tilde{\mathcal{Y}}_{0,j}$ is the vertex operator map. By simplicity of J^k s, $L(0)$ acts semi-simply on each of the J^k s and hence $\tilde{\mathcal{Y}}_{i,j}$ are free of logarithms by Remark 3.23 of [HLZ]. Moreover, since $\theta_{J^k} = \pm 1$, each J^k is graded either by \mathbb{Z} or by $1/2 + \mathbb{Z}$. Combining with the rest of the assumptions on θ_{J^k} we get that each $\tilde{\mathcal{Y}}_{i,j}$ has only integer powers of the formal variable.

Since each J^i is simple, it is be easy to see, using Theorem 11.9 of [DL] that the products and iterates of $\tilde{\mathcal{Y}}_{i,j}$ are non-zero. This is needed to ensure that “ F ,” defined below, is well-defined and non-zero.

We know, using Proposition 3.44 of [HLZ] that $e^{xL(-1)}\tilde{\mathcal{Y}}_{i,j}(u, e^{\pi i}x)v = e^{xL(-1)}\tilde{\mathcal{Y}}_{i,j}(u, -x)v$ is a (non-zero) intertwining operator of type $\binom{J^{i+j}}{J^i J^j}$. Also, since $\dim \binom{J^{i+j}}{J^i J^j} = 1$, we know that it must be a (non-zero) scalar multiple of $\tilde{\mathcal{Y}}_{j,i}$. So, we get a \mathcal{T} -commutativity datum in the sense of [C] along with a normalized choice of intertwining operators $\tilde{\mathcal{Y}}$. With this, we get a normalized abelian cocycle $(\tilde{F}, \tilde{\Omega})$ on the group $G = \{J^k \mid k \in \mathbb{Z}\}$.

As before, $\tilde{\Omega}(i, j) = c_{J^i, J^j}^{-1}$, and hence, $\tilde{\Omega}(i, i) = \pm 1$. Using Lemmas 1.2.7 and 1.2.8 of [C], we get that $\tilde{\Omega}(2i, 2i) = 1$ for $i \in G$ and that $(\tilde{F}, \tilde{\Omega})$ is cohomologous to a normalized abelian 3-cocycle (F, Ω) on G , pulled back from a normalized abelian cocycle $(\bar{F}, \bar{\Omega})$ for the abelian group $G/2G$. We now modify our choice of $\{\tilde{\mathcal{Y}}_{i,j}\}$ (by multiplying each intertwining operator with an appropriate scalar) so as to match with (F, Ω) . This is possible by Theorem 2.2.13 (i) of [C].

So, we get, using Proposition 2.4.5 of [C] that $\tilde{V} = \bigoplus_{i \in 2G} J^i$ is a \mathbb{Z} -graded vertex operator algebra (if $|G| = \infty$, we would get a strongly G -graded vertex operator algebra), that $\tilde{J} = \bigoplus_{i \notin 2G} J^i$ is a \tilde{V} -module (or a strongly G -graded \tilde{V} -module if $|\{i \notin 2G\}| = \infty$), and a \tilde{V} -intertwining operator $\mathcal{Y} \in \binom{\tilde{V}}{\tilde{J}}$. Therefore, we are done if $G = 2G$.

Now assume that $G \neq 2G$. Note that since G is cyclic, $G/2G \cong \mathbb{Z}_2$. Let $|z_1| > |z_2| > |z_1 - z_2| > 0$ and $w_{i_t} \in J^{i_t}$ for $t \in \{1, 2, 3\}$, $i_t \in G \setminus 2 \cdot G$. Note that by assumption, we have:

$$\begin{aligned} \mathcal{Y}_{i_1, i_2}(w_{i_1}, z_1)w_{i_2} &= \Omega(i_1, i_2)e^{-z_1 L(-1)}\mathcal{Y}_{i_2, i_1}(w_{i_2}, -z_1)w_{i_1} \\ \mathcal{Y}_{i_1, i_2+i_3}(w_{i_1}, z_1)\mathcal{Y}_{i_2, i_3}(w_{i_2}, z_2)w_{i_3} &= F(i_1, i_2, i_3)\mathcal{Y}_{i_1+i_2, i_3}(\mathcal{Y}_{i_1, i_2}(w_{i_1}, z_1 - z_2)w_{i_2}, z_2)w_{i_3}. \end{aligned}$$

However, using that (F, Ω) is a pull-back of $(\bar{F}, \bar{\Omega})$ on $G/2G$ we get:

$$\begin{aligned} \mathcal{Y}(w_{i_1}, z_1)w_{i_2} &= \bar{\Omega}(\bar{i}_1, \bar{i}_2)e^{-z_1 L(-1)}\mathcal{Y}(w_{i_2}, -z_1)w_{i_1} \\ \mathcal{Y}(w_{i_1}, z_1)\mathcal{Y}(w_{i_2}, z_2)w_{i_3} &= \bar{F}(\bar{i}_1, \bar{i}_2, \bar{i}_3)\mathcal{Y}(\mathcal{Y}(w_{i_1}, z_1 - z_2)w_{i_2}, z_2)w_{i_3}, \end{aligned}$$

here \bar{i} denotes the image of i in $\mathbb{Z}/2\mathbb{Z}$. Moreover, recall that $\Omega(i, i) = \pm 1$ for all $i \in G$ and hence, $\bar{\Omega}(i, i) = \pm 1$ for all $i \in G/2G \cong \mathbb{Z}_2$. In other words, $\bar{\Omega}(i, i)^2 = 1$ for all $i \in G/2G \cong \mathbb{Z}_2$. From the proof of Theorem 3.9 it is clear that $\bar{F} \equiv 1$. Now we proceed as in Theorem 3.9 to prove that we

indeed get a vertex operator (super) algebra structure on $\tilde{V} \oplus \tilde{J}$. If $|G| = \infty$ then we actually get a strongly graded vertex operator (super) algebra with abelian group G (cf. [HLZ]). \square

Theorem 3.13. *Let V be a vertex operator algebra such that its module category \mathcal{C} has a natural vertex tensor category structure in the sense of Huang-Lepowsky. Then, the following are equivalent:*

- A vertex operator superalgebra V_e such that V is a subalgebra of V_e^0 .
- A \mathcal{C} -superalgebra V_e with $\theta^2 = \text{Id}_{V_e}$.

Proof. This proof is almost the same as the proof in the [HKL]. Here, we only give those details that are different from the ones in [HKL].

(i) We first prove that a vertex superalgebra yields a \mathcal{C} -superalgebra.

Let V_e be a vertex operator superalgebra such that V is a subalgebra of V_e^0 . We immediately get a morphism $\iota : V \hookrightarrow V_e^0$. Being a V -module, V_e is an object of \mathcal{C} . Even in the case that V_e is a superalgebra, since $V \subset V_e^0$, Y_{V_e} is an intertwining operator for V -modules of the type $\binom{V_e}{V_e V_e}$. By the universal property of $V_e \boxtimes V_e$, there exists a unique module map $\mu : V_e \boxtimes V_e \rightarrow V_e$ such that $\bar{\mu} \circ \boxtimes = Y_e(\cdot, 1)$.

We now prove that (V_e, μ, ι) along with its $\frac{1}{2}\mathbb{Z}$ and \mathbb{Z}_2 -gradings is a \mathcal{C} -superalgebra.

Clearly, since Y_e only has integral powers of the formal variable,

$$\begin{aligned} \bar{\mu}(\theta u \boxtimes \theta v) &= Y_e(\theta u, 1)\theta v \\ &= Y_e(e^{2i\pi L(0)}u, 1)e^{2i\pi L(0)}v \\ &= e^{2i\pi L(0)}Y_e(u, e^{2i\pi}1)v \\ &= e^{2i\pi L(0)}Y_e(u, 1)v \\ &= \bar{\theta}Y_e(u, 1)v \\ &= \bar{\theta}\bar{\mu}(u \boxtimes v) \end{aligned} \tag{3.16}$$

For $i, j \in \{0, 1\}$ and $u \in V_e^i, v \in V_e^j$, $Y_e(u, x)v \in V_e^{i+j}((x))$, and hence, μ respects the \mathbb{Z}_2 -grading as well.

The proof of associativity of μ given in [HKL] goes through line-by-line.

We now turn to skew-symmetry and the relation of μ to braiding \mathcal{R} . The braiding isomorphism \mathcal{R} is determined uniquely by

$$\bar{\mathcal{R}}(u \boxtimes v) = e^{L(-1)}\bar{\mathcal{T}}_{\gamma_1^-}(v \boxtimes_{P(-1)} u)$$

where $u, v \in V_e$ and γ_1^- is a clockwise path from -1 to 1 in $\overline{\mathbb{H}} \setminus \{0\}$, and $\mathcal{T}_{\gamma_1^-}$ is the corresponding parallel transport isomorphism $V_e \boxtimes_{P(-1)} V_e \rightarrow V_e \boxtimes V_e$. Therefore,

$$\bar{\mu}(\bar{\mathcal{R}}(u \boxtimes v)) = \bar{\mu}(e^{L(-1)}\mathcal{T}_{\gamma_1^-}(v \boxtimes_{P(-1)} u)).$$

Let \mathcal{Y} be the intertwining operator of type $\binom{V_e \boxtimes V_e}{V_e V_e}$ corresponding to the intertwining map $\boxtimes_{P(1)}$. Then,

$$e^{L(-1)}\mathcal{T}_{\gamma_1^-}(v \boxtimes_{P(-1)} u) = e^{L(-1)}\mathcal{Y}(v, e^{i\pi})u.$$

For \mathbb{Z}_2 -homogeneous elements $u \in V_e^i, v \in V_e^j$, for $i, j \in \mathbb{Z}/2\mathbb{Z}$, Y_e satisfies the skew-symmetry:

$$Y_e(u, 1)v = (-1)^{ij}e^{L(-1)}Y_e(v, -1)u. \tag{3.17}$$

Therefore, we have that:

$$(-1)^{ij}\bar{\mu}(\bar{\mathcal{R}}(u \boxtimes v)) = (-1)^{ij}\bar{\mu}(e^{L(-1)}\mathcal{T}_{\gamma_1^-}(v \boxtimes_{P(-1)} u))$$

$$\begin{aligned}
&= (-1)^{ij} \bar{\mu}(e^{L(-1)} \mathcal{Y}(v, e^{i\pi}) u) \\
&= (-1)^{ij} e^{L(-1)} \bar{\mu}(\mathcal{Y}(v, e^{i\pi}) u) \\
&= (-1)^{ij} e^{L(-1)} Y_e(v, e^{i\pi}) u \\
&= (-1)^{ij} e^{L(-1)} Y_e(v, -1) u \\
&= Y_e(u, 1) v \\
&= \bar{\mu}(u \boxtimes v)
\end{aligned} \tag{3.18}$$

The left unit isomorphism property also goes through exactly as in [HKL].

(ii) Now we prove that a \mathcal{C} -superalgebra yields a vertex operator superalgebra.

We are given a \mathcal{C} -superalgebra (V_e, μ, ι) . By definition, V_e is a generalized V -module. The $P(1)$ -intertwining map $\bar{\mu} \circ \boxtimes_{P(1)} : V_e \otimes V_e \rightarrow \bar{V}_e$ corresponds to an intertwining operator Y_e of type $\binom{V_e}{V_e V_e}$ such that

$$\bar{\mu}(u \boxtimes v) = Y_e(u, 1)v.$$

Using ι , we can view vacuum vector $\mathbf{1}$ and conformal vector ω as elements of V_e .

The Virasoro relations and the fact that V_e is graded by generalized eigenvalues of $L(0)$ follow from the fact that V_e is a generalized V -module. $L(-1)$ -derivative property and the $L(0)$ -conjugation formula for Y_e follow from the fact that Y_e is an intertwining operator.

Since $\theta^2 = \text{Id}_{V_e}$, $L(0)$ acts semisimply on V_e and V_e is in fact $\frac{1}{2}\mathbb{Z}$ -graded by eigenvalues of $L(0)$. Since $L(0)$ acts semisimply, Y_e does not have logarithms. Using $\mu(\theta \otimes \theta) = \theta \circ \mu$, the definition of θ , the $L(-1)$ -derivative property of Y_e and the $L(0)$ -conjugation formula for Y_e , we get that

$$\begin{aligned}
\bar{\mu}(\theta u \boxtimes \theta v) &= Y_e(e^{2i\pi L(0)} u, 1) e^{2i\pi L(0)} v = e^{2i\pi L(0)} Y_e(u, e^{2i\pi} 1) v, \\
\bar{\theta} \bar{\mu}(u \boxtimes v) &= e^{2i\pi L(0)} Y_e(u, 1) v
\end{aligned}$$

and hence,

$$Y_e(u, e^{2i\pi} 1) v = Y_e(u, 1) v.$$

Letting u and v to be homogeneous with respect to the $L(0)$ -grading, we immediately conclude that $Y_e(u, x)v$ must have only integral powers of x .

Skew-symmetry with the correct factor of -1 , vacuum property, creation property and associativity follow exactly as in [HKL]. Using [H2], skew-symmetry and the associativity imply the right kind of commutativity for a vertex operator superalgebra. Again using results in [H2], commutativity and associativity yield the right kind of rationality of products and iterates, these yield the desired Jacobi identity. □

Theorem 3.14. *Let V, V_e, \mathcal{C} be as in Theorem 3.13. The category of generalized modules for the vertex operator superalgebra V_e is isomorphic to the category $\text{Rep}^0 V_e$, where V_e is considered as a \mathcal{C} -superalgebra.*

Proof. It is clear that a generalized V_e -module corresponds to an object in $\text{Rep}^0 V_e$.

Let $(W, \mu_W) \in \text{Rep}^0 V_e$. Let Y_W be an intertwining operator corresponding to the intertwining map $\overline{\mu_W} \circ \boxtimes : A \otimes W \rightarrow \overline{W}$, and let \mathcal{Y}_{\boxtimes} be the intertwining operator corresponding to the intertwining map $\boxtimes : A \otimes W \rightarrow \overline{A \boxtimes W}$. We have that $Y_W = \mu_W \circ \mathcal{Y}_{\boxtimes}$. Let us analyze the condition $\mu_W \circ (\mathcal{R}_{W,A} \mathcal{R}_{A,W}) = \mu_W$. This says that

$$\overline{\mu_W} \circ (\mathcal{Y}_{\boxtimes}(v, e^{2\pi i}) w) = \overline{\mu_W}(\mathcal{Y}_{\boxtimes}(v, 1) w),$$

for $v \in V_e, w \in W$. Therefore,

$$Y_W(v, e^{2\pi i}) w = Y_W(v, 1) w. \tag{3.19}$$

Using the notation (3.24) of [HLZ] let

$$Y_W(v, x)w = \sum_{n \in \mathbb{C}, k \in \mathbb{N}} (v_{n;k}^{Y_W} w) x^n (\log x)^k,$$

so that for a complex number ζ ,

$$Y_W(v, e^\zeta)w = \sum_{n \in \mathbb{C}, k \in \mathbb{N}} (v_{n;k}^{Y_W} w) e^{n\zeta} \zeta^k.$$

Equation (3.19) now gives that

$$\sum_{n \in \mathbb{C}, k \in \mathbb{N}} (v_{n;k}^{Y_W} w) e^{2\pi i n} (2\pi i)^k = \sum_{n \in \mathbb{C}} v_{n;0}^{Y_W} w,$$

which in turn immediately implies that Y_W has no logarithms and only integral powers of the formal variable. Now, the proof of Theorem 3.4 of [HKL] goes through. \square

Theorem 3.15. *Consider the set-up of Theorem 3.12. Let $V_e = \bigoplus_{j \in G} J^j$. If $X \in \mathcal{C}$ then, $\mathcal{F}(X) \in \text{Rep}^0 V_e$ if and only if $M_{J,X} = \text{Id}_{J \boxtimes X}$.*

Proof. We assume that $X \neq 0$.

Observe that $V_e \boxtimes X \cong \bigoplus_j J^j \boxtimes X$ by Proposition 4.24 of [HLZ].

Let us first do the only if part. Since $\mathcal{F}(X) \in \text{Rep}^0 V_e$,

$$\mu_{\mathcal{F}(X)}|_{J \boxtimes (J^0 \boxtimes X)} \circ M_{J, J^0 \boxtimes X} = \mu_{\mathcal{F}(X)}|_{J \boxtimes (J^0 \boxtimes X)}.$$

However, by definition of $\mu_{\mathcal{F}(X)}$, $\mu_{\mathcal{F}(X)}|_{J \boxtimes (J^0 \boxtimes X)} = (\mu_{J \boxtimes J^0} \otimes \text{Id}_X) \circ \mathcal{A}_{J, J^0, X}$. Now, $J \boxtimes J^0 = J \boxtimes V$ is spanned by homogeneous weight components of $j \boxtimes v$ as j runs over J and v runs over V (cf. Proposition 4.23 of [HLZ]). By definition, $\bar{\mu}(j \boxtimes v) = e^{L(-1)} Y_J(v, -1)j$, where Y_J is the module map corresponding to J . Hence, $\mu|_{J^1 \boxtimes J^0}$ is non-zero. Since J is simple, $\mu|_{J^1 \boxtimes J^0}$ is a morphism of simple modules, and hence, being non-zero, is invertible. Therefore, $\mu_{\mathcal{F}(X)}|_{J \boxtimes (J^0 \boxtimes X)}$ is invertible also. We conclude that $M_{J, J^0 \boxtimes X}$ must be identity.

Conversely, by assumption, $c_{J,J} = \pm \text{Id}$ and hence $M_{J,J} = \text{Id}$. Moreover, if $M_{J,X} = \text{Id}$ then by Theorem 2.11, $M_{J^i, J^j \boxtimes X} = \text{Id}$ and hence $\mathcal{F}(X) \in \text{Rep}^0 V_e$. \square

Corollary 3.16. *If X is a simple V -module, then $\mathcal{F}(X)$ is a V_e -module iff $h_{J \boxtimes X} - h_J - h_X \in \mathbb{Z}$, where h_\bullet denotes the conformal dimension.*

Proof. Recall that $M_{J,X} = \theta_{J \boxtimes X} \circ (\theta_J^{-1} \boxtimes \theta_X^{-1})$ and that $\theta = e^{2\pi i L(0)}$. Since J , X and $J \boxtimes X$ are simple, $L(0)$ acts semisimply on J , X and $J \boxtimes X$. Hence, $M_{J,X} = e^{2\pi i (h_{J \boxtimes X} - h_J - h_X)} \text{Id}_{J \boxtimes X}$. Now use Theorem 3.15. \square

Now we focus on the case when P is an indecomposable object. In what follows, the point is to prove that under certain conditions, the (locally) nilpotent part of the monodromy $M_{J,P}$ vanishes.

Lemma 3.17. *Let J be a finite order simple current as in Theorem 3.12. Let P be an indecomposable object with $\dim(\text{Hom}(P, P)) < \infty$ and $\dim(\text{Hom}(J \boxtimes P, J \boxtimes P)) < \infty$. Assume that $L(0)$ has Jordan blocks of bounded size on both P and $J \boxtimes P$. Then, $M_{J,P} = (\theta_{J \boxtimes P})_{ss} \circ (\theta_J^{-1} \boxtimes (\theta_P^{-1})_{ss})$, where \bullet_{ss} denotes the semi-simple part. In particular, $M_{J,P} = \lambda \text{Id}_{J \boxtimes P}$ for some $\lambda \in \mathbb{C}$. Moreover, $\lambda^N = 1$, where N is the order of J .*

Proof. Note the very important property of simple currents from a forthcoming article [CKLR] that $\bullet \boxtimes J$ and $J \boxtimes \bullet$ are exact for any invertible simple current J . Moreover, it is easy to see that these two functors take non-zero objects to non-zero objects. Using these and the fact that P is

indecomposable, one can prove that $J \boxtimes P$ is indecomposable. Indeed, if we have a split short exact sequence

$$0 \longrightarrow A \longrightarrow J \boxtimes P \longrightarrow B \longrightarrow 0$$

we get a split short exact sequence

$$0 \longrightarrow J^{-1} \boxtimes A \longrightarrow J^{-1} \boxtimes (J \boxtimes P) \longrightarrow J^{-1} \boxtimes B \longrightarrow 0.$$

But, since $J^{-1} \boxtimes (J \boxtimes P)$ and P are isomorphic via associativity and the property of unit, $J^{-1} \boxtimes (J \boxtimes P)$ is indecomposable as well.

Define θ_{ss} to be the semi-simple part of θ and θ_{nil} to be $\theta - \theta_{ss}$. Since twist given by $\theta = e^{2\pi i L(0)}$ and since $L(0)$ has Jordan blocks of bounded size on both P and $J \boxtimes P$, we indeed have that some finite positive power of θ_{nil} is 0. Note that since J is simple, θ acts semi-simply on J . We have that

$$\begin{aligned} M_{J,P} &= \theta_{J \boxtimes P} \circ (\theta_J^{-1} \boxtimes \theta_P^{-1}) \\ &= (\theta_{J \boxtimes P})_{ss} \circ (\theta_J^{-1} \boxtimes (\theta_P^{-1})_{ss}) + (\theta_{J \boxtimes P})_{ss} \circ (\theta_J^{-1} \boxtimes (\theta_P^{-1})_{nil}) \\ &\quad + (\theta_{J \boxtimes P})_{nil} \circ (\theta_J^{-1} \boxtimes (\theta_P^{-1})_{ss}) + (\theta_{J \boxtimes P})_{nil} \circ (\theta_J^{-1} \boxtimes (\theta_P^{-1})_{nil}). \end{aligned} \quad (3.20)$$

Let,

$$(M_{J,P})_{ss} = (\theta_{J \boxtimes P})_{ss} \circ (\theta_J^{-1} \boxtimes (\theta_P^{-1})_{ss}),$$

and observe that $(M_{J,P})_{ss}$ is indeed semi-simple.

Let $(M_{J,P})_{nil} = M_{J,P} - (M_{J,P})_{ss}$. We now prove that $(M_{J,P})_{nil}$ is indeed nilpotent. By definition, morphisms in our category commute with $L(0)$, and hence commute with $L(0)_{ss}$ and $L(0)_{nil}$. Therefore, θ_{ss} and θ_{nil} are natural. Hence,

$$\begin{aligned} &((M_{J,P})_{nil})^K \\ &= \sum_{\substack{a+b+c=K \\ a,b,c \in \mathbb{N}}} C_{a,b,c} ((\theta_{J \boxtimes P})_{ss} (\theta_J^{-1} \boxtimes (\theta_P^{-1})_{nil}))^a ((\theta_{J \boxtimes P})_{nil} (\theta_J^{-1} \boxtimes (\theta_P^{-1})_{ss}))^b ((\theta_{J \boxtimes P})_{nil} (\theta_J^{-1} \boxtimes (\theta_P^{-1})_{nil}))^c \end{aligned}$$

for some constants $C_{a,b,c}$. However, since Jordan blocks of $L(0)$ on P and $J \boxtimes P$ are bounded in size, one can now pick a large enough K for which $((M_{J,P})_{nil})^K = 0$.

It is easily seen that for any indecomposable module X , all the generalized eigenvalues of $L(0)$ on X belong to a single coset $\mu + \mathbb{Z}$ where $\mu \in \mathbb{C}$, and hence, $(\theta_{J \boxtimes P})_{ss}$ and $(\theta_P^{-1})_{ss}$ are scalar multiples of identity. Hence, we deduce that $(M_{J,P})_{ss} = \lambda \text{Id}_{J \boxtimes P}$ for some scalar $\lambda \in \mathbb{C}$.

Combining everything, we get that $M_{J,P} = \lambda \text{Id}_{J \boxtimes P} + \pi$ where π is some nilpotent endomorphism of $J \boxtimes P$ and $\lambda \in \mathbb{C}$. Now use Lemma 2.13. \square

Combining with Theorem 3.15, we arrive at the following criterion for lifting indecomposable objects.

Theorem 3.18. *Let J be a finite order simple current as in Theorem 3.12. Let P be an indecomposable object with finite dimensional endomorphism ring, $\dim(\text{Hom}(P, P)) < \infty$ and also $\dim(\text{Hom}(J \boxtimes P, J \boxtimes P)) < \infty$. Assume also that $L(0)$ has Jordan blocks of bounded size on both P and $J \boxtimes P$. Then, $\mathcal{F}(P)$ is a generalized V_e -module iff $h_{J \boxtimes P} - h_J - h_P \in \mathbb{Z}$.*

Now we give some useful criteria for the cases when J does not necessarily have finite order.

Lemma 3.19. *Let J be a simple current. Let A_i and Q be objects in \mathcal{C} such that $M_{J,A_i} = \lambda_i \text{Id}_{A_i}$ and $M_{J,Q} = \lambda \text{Id}_{J,Q}$ for some scalars λ_i and λ . Then the following hold.*

$$(1) \quad M_{J, \boxtimes_{i=1}^N A_i} = \left(\prod_{i=1}^N \lambda_i \right) \text{Id}_{\boxtimes_{i=1}^N A_i}.$$

- (2) If $0 \rightarrow A \rightarrow Q \rightarrow B \rightarrow 0$ is a short exact sequence of generalized V -modules then $M_{J,A} = \lambda \text{Id}_{J \boxtimes A}$ and $M_{J,B} = \lambda \text{Id}_{J \boxtimes B}$.
- (3) If P is a subquotient of Q then $M_{J,P} = \lambda \text{Id}_{J \boxtimes P}$.

Proof. (1) follows by retracing the proof of equation (2.2) in Theorem 2.11 and then using induction on i . For (2), by exactness of $J \boxtimes \bullet$ from [CKLR], and by naturality of monodromy (Lemma 2.10), we get the following commutative diagram.

$$\begin{array}{ccccccc}
0 & \longrightarrow & J \boxtimes A & \longrightarrow & J \boxtimes Q & \longrightarrow & J \boxtimes B \longrightarrow 0 \\
& & \downarrow M_{J,A} & & \downarrow M_{J,Q} & & \downarrow M_{J,B} \\
0 & \longrightarrow & J \boxtimes A & \longrightarrow & J \boxtimes Q & \longrightarrow & J \boxtimes B \longrightarrow 0.
\end{array}$$

With this, (2) follows. Now, (3) follows from (2). \square

Since $M = \theta \circ (\theta^{-1} \boxtimes \theta^{-1})$, we see that Lemma 3.19 holds if A_i are simple V -modules. We immediately get the following Theorem.

Theorem 3.20. *Let J be simple current as in Theorem 3.12. J need not necessarily have finite order. Let P be a subquotient of $\boxtimes_{i=1}^N A_i$ for some simple V -modules A_i . Then, $\mathcal{F}(P)$ is a generalized V_e -module iff $h_{J \boxtimes P} - h_J - h_P \in \mathbb{Z}$.*

Proof. Using Lemma 3.19, we see that $M_{J,P} = \lambda \text{Id}_{J \boxtimes P}$ for some scalar λ . Using this, using equation (3.20) and observing that θ_{nil} and $(\theta^{-1})_{nil}$ are locally nilpotent, we get that $M_{J,P} = (\theta_{J \boxtimes P})_{ss} \circ ((\theta_J)^{-1} \boxtimes (\theta_P^{-1})_{ss})$. Now the conclusion follows. \square

Remark 3.21. *In Section 4, we will need the fact that the induction functor \mathcal{F} is exact, in order to deduce the Loewy diagrams of induced modules. In our setup, one can use the fact from [CKLR] that $J \boxtimes \bullet$ is an exact functor for a simple current J to deduce that \mathcal{F} is exact. Alternately, one can proceed as in Theorem 1.6 of [KO].*

We now summarize Huang's theorem [H5] on when the Huang-Lepowsky-Zhang theory can be applied.

Theorem 3.22. (Cf. [H5].)

- Let V be such that (1) V is C_1^a -cofinite, i.e., $\text{Span}\{u_n v, L(-1)u \mid u, v \in V_+\}$ has finite codimension, (2) There exists a positive integer N such that the differences between the real parts of the lowest conformal weights of irreducible V -modules are bounded by N and such that the associative algebra $A_N(V)$ (cf. [DLM3]) is finite dimensional and (3) Irreducible V -modules are \mathbb{R} -graded and C_1 -cofinite. Then the category of grading-restricted generalized V -modules (i.e., lower truncated modules with finite dimensional generalized weight spaces) satisfies the conditions required to invoke Huang-Lepowsky-Zhang's theory.
- If V is a C_2 -cofinite such that $V_n = 0$ for $n < 0$ and $V_0 = \mathbb{C}\mathbf{1}$ then all the three conditions mentioned above are satisfied and the category of generalized grading-restricted modules of V has a natural vertex tensor category structure, in particular, it has a braided tensor category structure.

We can also combine the main result of [Miy] with Theorems 12.15 and 12.16 of [HLZ] to obtain the following.

Theorem 3.23. *Let V be a vertex operator algebra and consider a full sub-category \mathcal{C} of generalized modules of V such that:*

- (1) \mathcal{C} is abelian.
- (2) For each object C , the generalized weights are real numbers, there is a $K \in \mathbb{Z}_+$ such that $(L(0) - L(0)_{ss})^K = 0$, where $L(0)_{ss}$ is the semi-simple part of $L(0)$.

- (3) \mathcal{C} is closed under images, contragredients, taking finite direct sums and V is an object of \mathcal{C} .
(4) Every object of \mathcal{C} satisfies C_1 -cofiniteness ($\text{span}\{u_{-1}w \mid u \in V_+, w \in W\}$ has finite codimension in W), has finite dimensional generalized weight spaces with lower truncated weights and is quasi-finite dimensional ($\bigoplus_{n < N} W_{[n]}$ is finite dimensional for any $N \in \mathbb{R}$, where $W_{[n]}$ denotes the generalized eigenspace for $L(0)$ with generalized eigenvalue n .)

Then \mathcal{C} has a natural vertex tensor category structure, in particular, it has a braided tensor category structure.

Proof. The Main Theorem of [Miy] ensures that \mathcal{C} is closed under \boxtimes . The rest follows from Theorems 12.15 and 12.16 of [HLZ]. \square

4. BUILDING LOGARITHMIC VOAs FROM PARTS

As Carnahan suggested, one can now explicitly build various VOAs from parts. Our interest is in the non semi-simple, i.e., logarithmic type and here we will construct a few examples.

Recall the definition and construction of the contragredient modules from Section 5.2 of [FHL] and Theorem 2.34 of [HLZ]. Recall the skew-symmetry and the adjoint operations on intertwining operators, equations (3.77) and (3.87), respectively, and the corresponding Propositions 3.44 and 3.46, respectively, from [HLZ]. We use the same notations for contragredients (\bullet'), skew-symmetry ($\Omega_r(\bullet)$) and the adjoint ($\mathcal{A}_r(\bullet)$) operations as in [HLZ]. Also recall from [HLZ] Definition 4.29 of a finitely reductive vertex operator algebra.

Let V be a finitely reductive simple vertex operator algebra such that V is isomorphic to its contragredient, i.e., $V \cong V'$. For such a V , by Theorem 4.33 of [HLZ], the category of V -modules is closed under $\boxtimes_{P(z)}$ tensor products. Let J be a simple current for V .

Using Propositions 3.44 and 3.46 of [HLZ] and the assumption that $V \cong V'$, we know that the fusion rules $\mathcal{N}_{V,J}^J = \mathcal{N}_{J,V}^J = \mathcal{N}_{J,J'}^{V'} = \mathcal{N}_{J,J'}^V = \mathcal{N}_{J',J}^V$ are non-zero. Therefore, by the universal property of the tensor products, there exists a (non-zero) morphism $J \boxtimes J' \rightarrow V$. This means that V is in fact a direct summand of $J \boxtimes J'$ because V is simple and finitely reductive. Since J is simple, J' is simple. Moreover, since J is a simple current, $J \boxtimes J'$ is simple as well. Hence, $J \boxtimes J' \cong V$. Hence,

$$J' \cong (J^{-1} \boxtimes J) \boxtimes J' \cong J^{-1} \boxtimes (J \boxtimes J') \cong J^{-1} \boxtimes V \cong J^{-1}.$$

In the particular case that J is a self-dual simple current, i.e., $J \boxtimes J \cong V$, we indeed get that $J \cong J^{-1} \cong J'$.

The following proposition will be used later in the case when V is a simple finitely reductive vertex operator algebra such that $V' \cong V$, $V_n = 0$ for $n < 0$ and J is a self-dual simple current. However, we state the proposition in the most general setting.

Proposition 4.1. *Let V be vertex operator algebra and J be a (non-zero) V -module. There exists an intertwining operator \mathcal{Y} of type $\binom{V'}{J}$ and elements $j \in J$ and $j' \in J'$ of lowest conformal weight, say d , such that*

$$\langle \mathcal{Y}(j, x)j', \mathbf{1} \rangle \neq 0.$$

Moreover, if V is such that $V_n = 0$ for $n < 0$, then, there exists a non-zero $v' \in V'$ of conformal weight 0 such that

$$\mathcal{Y}(j, x)j' = x^{-2d}v' + \dots \quad (4.1)$$

Proof. Let $j \in J$ be any (non-zero) vector of lowest conformal dimension d , and let $j' \in J'$ be of the same conformal dimension such that $\langle j', j \rangle \neq 0$. Let Y be the module map corresponding to J . Pick $r, s \in \mathbb{Z}$ and let \mathcal{Y} be the (non-zero) intertwining operator of type $\binom{V'}{J}$ given by $\mathcal{A}_r(\Omega_s(Y))$.

Being a module map, Y has no monodromy and therefore $\Omega_s(Y)$ has no monodromy and \mathcal{Y} is independent of s . We have:

$$\begin{aligned}
\langle \mathcal{Y}(j, x)j', \mathbf{1} \rangle &= \langle \mathcal{A}_r(\Omega_s(Y))(j, x)j', \mathbf{1} \rangle \\
&= \langle j', \Omega_s(Y)(e^{xL(1)} e^{(2r+1)\pi i L(0)} (x^{-L(0)})^2 j, x^{-1}) \mathbf{1} \rangle \\
&= \langle j', \Omega_s(Y)(e^{(2r+1)\pi i L(0)} (x^{-L(0)})^2 j, x^{-1}) \mathbf{1} \rangle \\
&= e^{d(2r+1)\pi i} x^{-2d} \langle j', \Omega_s(Y)(j, x^{-1}) \mathbf{1} \rangle \\
&= e^{d(2r+1)\pi i} x^{-2d} \langle j', e^{x^{-1}L(-1)} Y(\mathbf{1}, -x^{-1})j \rangle \\
&= e^{d(2r+1)\pi i} x^{-2d} \langle e^{x^{-1}L(1)} j', j \rangle \\
&= e^{d(2r+1)\pi i} x^{-2d} \langle j', j \rangle \neq 0.
\end{aligned} \tag{4.2}$$

Moreover, if V is such that $V_n = 0$ for all $n < 0$, then $(V')_n = 0$ for all $n < 0$ which means that $\mathcal{Y}(j, x)j'$ does not have any powers of x lower than x^{-2d} and equation (4.1) follows. \square

The following proposition will be used later to calculate quantum dimensions of certain simple currents.

Proposition 4.2. *Let J be a self-dual simple current. Let $j \in J$ be a non-zero element of lowest conformal weight, say d . Then,*

$$c_{J,J} = (-1)^N e^{-2\pi i d} \tag{4.3}$$

where $N \in \mathbb{Z}$ is such that

$$\mathcal{Y}(j, x)j = vx^{-2d+N} + \sum_{n>N, n \in \mathbb{Z}} v_n x^{-2d+n},$$

for any non-zero intertwining operator \mathcal{Y} of type $\binom{V}{J J}$ and $v, v_n \in V$ with $v \neq 0$. Moreover, if the category is ribbon then,

$$\text{qdim}(J) = (-1)^N e^{-4\pi i d}. \tag{4.4}$$

Proof. First, note that all intertwining operators of type $\binom{V}{J J}$ are scalar multiples of each other as J is a simple current. Fix an isomorphism $i : J \boxtimes J \rightarrow V$. Let \mathcal{Y} be the intertwining operator corresponding to the intertwining map $\bar{i} \circ \boxtimes$. By the definition of braiding, we know that

$$c_{J,J} \mathcal{Y}(j, 1)j = e^{L(-1)} \mathcal{Y}(j, e^{\pi i})j = e^{L(-1)} \left(v e^{(-2d+N)\pi i} + \sum_{n>N, n \in \mathbb{Z}} v_n e^{(-2d+n)\pi i} \right) = v e^{(-2d+N)\pi i} + \dots,$$

where the ellipses denote a sum over elements that have strictly higher weight than v . However,

$$c_{J,J} \mathcal{Y}(j, 1)j = c_{J,J}(v + \dots).$$

Comparing, we arrive at equation (4.3). If the category is ribbon, we can use the spin statistics theorem, i.e., Corollary 2.8 to deduce (4.4). \square

Rational building blocks. Quantum dimensions are only known for rational VOAs via the Verlinde formula, however note that [CG] suggests that this generalizes to the C_2 -cofinite setting. In a unitary VOA, the quantum dimension of a simple current is always one. In a non-unitary VOA this is not guaranteed anymore. Our building blocks are

- (1) The simple rational Virasoro vertex algebra $\text{Vir}(u, v)$ at central charge $c_{u,v} = 1 - 6(u - v)^2/(uv)$ where u, v are coprime positive integers larger than two. This VOA is non-unitary, except if $|u - v| = 1$. It has a self-dual simple current $J_{u,v}$ of conformal dimension $h_{u,v} = (u - 2)(v - 2)/4$ and quantum dimension $\text{qdim}(J_{u,v}) = (-1)^{u+v+1}$, which can be obtained from [IK].

- (2) Let L be an even positive definite lattice, then the associated lattice VOA V_L has the property that it has the least conformal dimension among all its irreducible modules, therefore, by Proposition 4.17 and Example 4.19 of [DJX], the quantum dimension of a simple current is one. The conformal dimension of a simple current is given by the norm squared over two of the corresponding coset representative. Actually, in this case, each irreducible module is a simple current.
- (3) The simple affine VOA $L_k(\mathfrak{sl}_2)$ for a positive integer k is rational and unitary, it has a self-dual simple current K_k of conformal dimension $\frac{k}{4}$.

Logarithmic extensions. Recall the triplet VOAs $\mathcal{W}(p)$ are C_2 -cofinite non-rational VOAs [FGST, AdM1, TW]. They are defined as

$$\mathcal{W}(p) = \ker_Q \left(V_{\sqrt{2p}\mathbb{Z}} \right)$$

where Q is a certain specific screening operator that intertwines lattice VOA modules. $\mathcal{W}(p)$ has an order two simple current, denoted by X_1^- , obtained from the only self-dual simple current J of $V_{\sqrt{2p}}$ in the straight-forward manner $X_1^- = \ker_Q(J)$. The ordinary Virasoro element is actually not in the kernel, but only a shifted version of central charge $c = 1 - 6(p-1)^2/p$ and the conformal dimension of the two lowest-weight states of X_1^- under the Virasoro-zero mode is $(3p-2)/4$. Now we derive that

$$\text{qdim}(X_1^-) = -(-1)^p. \quad (4.5)$$

Indeed, using the lattice realization of the $\mathcal{W}(p)$ algebra and its simple current X_1^- from [AdM1], we know that the N in Proposition 4.2 is such that $-2(3p-2)/4 + N = p/2$. This immediately yields the equation 4.5. We then get two types of new logarithmic VOAs as

$$\mathfrak{A}_p = \mathcal{W}(p) \otimes L_{p-2}(\mathfrak{sl}_2) \oplus X_1^- \otimes K_{p-2} \quad \text{and} \quad \mathfrak{B}_p = \mathcal{W}(p) \otimes \text{Vir}(3, p) \oplus X_1^- \otimes J_{p,3}$$

the simple currents in the first one have now conformal dimension $p-1$ and hence we get a VOA of correct statistics if p is odd and a super VOA of wrong statistics if p is even. In the second case, the simple currents also have dimension $p-1$ so that for each p it is a super VOA of wrong statistics. We also consider

$$\mathfrak{C}_p = \mathcal{W}(p) \otimes \mathcal{W}(p) \oplus X_1^- \otimes X_1^-.$$

Here the conformal dimension of the simple current is $(3p-2)/2$, and hence, we get a VOA of correct statistics if p is even and a super VOA of correct statistics if p is odd.

We would now like to employ Corollary 3.16 and Theorem 3.18 to decide how the modules lift to extensions. We have the following list of inequivalent indecomposable modules for $\mathcal{W}(p)$, $L_k(\mathfrak{sl}_2)$ and $\text{Vir}(3, p)$ and their conformal dimensions:

VOA	Module	Type	Conformal weight
$\mathcal{W}(p)$	$X_s^\pm, s \in N_p$	Simple	$h_s^+ = \frac{(p-s)^2 - (p-1)^2}{4p}$
	$P_s^\pm, s \in N_{p-1}$	Reducible, indecomposable	$h_s^- = \frac{(2p-s)^2 - (p-1)^2}{4p}$
$L_k(\mathfrak{sl}_2)$	$L((k-t)\Lambda_0 + t\Lambda_1), t \in N_k^0$	Simple	$h_{(k-t)\Lambda_0 + t\Lambda_1} = \frac{t(t+2)}{4(k+2)}$
$\text{Vir}(3, p)$	$\phi_{1,s}, s \in N_{p-1}$	Simple	$h_{\phi_{1,s}} = \frac{(p-3s)^2 - (p-3)^2}{12p}$

TABLE 1. Modules and conformal weights, X_s^\pm and P_s^\pm have conformal weight h_s^\pm .

Where $N_p = \{1, 2, \dots, p\}$ and $N_p^0 = \{0, 1, \dots, p\}$. It is also known how the simple currents permute the indecomposables.

$$\begin{aligned} \mathcal{W}(p) : \quad & X_1^- \boxtimes X_s^\pm \cong X_s^\mp, \quad X_1^- \boxtimes P_s^\pm \cong P_s^\mp, \\ L_k(\mathfrak{sl}_2) : \quad & L(k\Lambda_1) \boxtimes L((k-t)\Lambda_0 + t\Lambda_1) \cong L(t\Lambda_0 + (k-t)\Lambda_1), \\ \text{Vir}(3, p) : \quad & \phi_{1, p-1} \boxtimes \phi_{1, s} \cong \phi_{1, p-s}. \end{aligned}$$

We first look at \mathfrak{A}_p . Consider the following classes of modules.

$$\begin{aligned} I_{\mathfrak{A}_p} &= \{X_s^+ \otimes L((p-2-t)\Lambda_0 + t\Lambda_1), X_s^- \otimes L(t\Lambda_0 + (p-2-t)\Lambda_1) \mid \\ &\quad s \in N_p, t \in N_{p-2}^0, s-t \not\equiv p \pmod{2}\} \\ P_{\mathfrak{A}_p} &= \{P_s^+ \otimes L((p-2-t)\Lambda_0 + t\Lambda_1), P_s^- \otimes L(t\Lambda_0 + (p-2-t)\Lambda_1) \mid \\ &\quad s \in N_{p-1}, t \in N_{p-2}^0, s-t \not\equiv p \pmod{2}\} \end{aligned}$$

For \mathfrak{B}_p , consider the following classes of modules.

$$\begin{aligned} I_{\mathfrak{B}_p} &= \{X_s^+ \otimes \phi_{1, t}, X_s^- \otimes \phi_{1, p-t} \mid s \in N_p, t \in N_{p-1}, s-t \equiv p \pmod{2}\} \\ P_{\mathfrak{B}_p} &= \{P_s^+ \otimes \phi_{1, t}, P_s^- \otimes \phi_{1, p-t} \mid s \in N_{p-1}, t \in N_{p-1}, s-t \equiv p \pmod{2}\} \end{aligned}$$

For \mathfrak{C}_p , consider the following classes of modules.

$$\begin{aligned} I_{\mathfrak{C}_p} &= \{X_s^+ \otimes X_t^+, X_s^- \otimes X_t^- \mid s \in N_p, t \in N_p, s+t \equiv p \pmod{2}\} \\ &\quad \cup \{X_s^+ \otimes X_t^-, X_s^- \otimes X_t^+ \mid s \in N_p, t \in N_p, s-t \equiv 0 \pmod{2}\} \\ P_{\mathfrak{C}_p} &= \{P_s^+ \otimes X_t^+, P_s^+ \otimes X_t^-, P_s^+ \otimes P_t^+, X_s^- \otimes P_t^-, P_s^- \otimes X_t^-, P_s^- \otimes P_t^- \mid \\ &\quad s \in N_p, t \in N_p, s+t \equiv p \pmod{2}\} \\ &\quad \cup \{P_s^+ \otimes X_t^-, P_s^+ \otimes X_t^+, P_s^+ \otimes P_t^-, X_s^- \otimes P_t^+, P_s^- \otimes X_t^+, P_s^- \otimes P_t^+ \mid \\ &\quad s \in N_p, t \in N_p, s-t \equiv 0 \pmod{2}\} \end{aligned}$$

It is clear that the indecomposable modules have finite dimensional endomorphism rings, because they have finite length. It is also known that the Jordan blocks for $L(0)$ are bounded in size. Therefore, in each case, I_\bullet lift to simple modules, P_\bullet to reducible indecomposable modules.

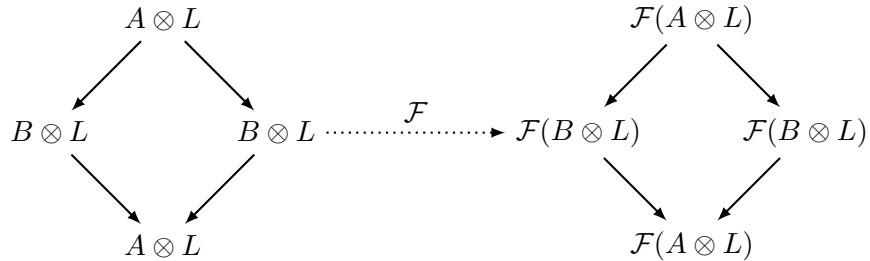


FIGURE 1. An example in the the case \mathfrak{A}_p . Here, $A = X_s^\pm$, $B = X_s^\mp$, $L = L((p-2-t)\Lambda_0 + t\Lambda_1)$.

Using Remark 3.21 and using the known Loewy diagrams of the indecomposables, one can quickly deduce Loewy diagrams of the induced modules. In Figure 1 we show an example in the case \mathfrak{A}_p . For $\mathcal{W}(p) \otimes \mathcal{W}(p)$ one can have indecomposable modules that are not tensor products of indecomposable modules for the individual tensor factors, for example, see Figure 2, (cf. [CR3]). These modules again induce as in Figure 1.

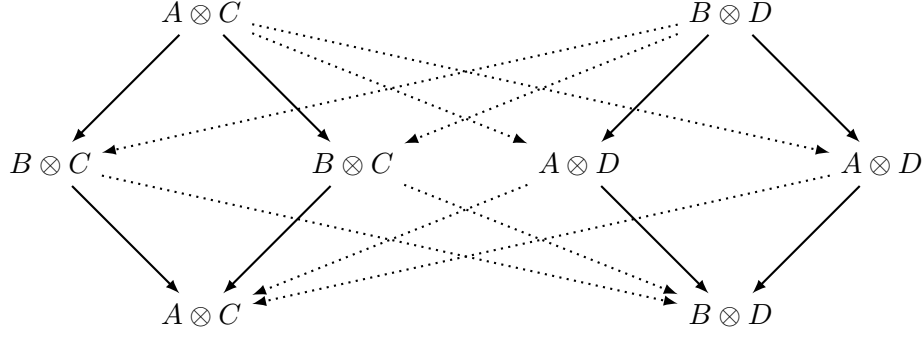


FIGURE 2. An indecomposable $\mathcal{W}(p) \otimes \mathcal{W}(p)$ -module that is not a tensor product of two indecomposable $\mathcal{W}(p)$ -modules. Here, $A = X_s^\pm$, $B = X_s^\mp$, $C = X_t^\pm$, $D = X_t^\mp$. The solid arrows denote the action of the left tensorand and the dashed arrows denote the action of the right tensorand.

On W-algebras. The following conjecture is from the physics literature [B–H].

Conjecture 4.3. Let $\mathcal{W}_r^{(2)}$ be the Feigin-Semikhatov algebra [FS] of level $k = \frac{n-r^2+2r}{r-1}$ then

$$\text{Com} \left(H, \mathcal{W}_r^{(2)} \right) \cong W_{A_{n-1}}(n+1, n+r).$$

Especially $\mathcal{W}_r^{(2)}$ is rational.

Remark 4.4. The Feigin-Semikhatov algebra $\mathcal{W}_r^{(2)}$ in turn is believed to be a quantum Hamiltonian reduction of $V_k(\mathfrak{sl}_r)$ for a certain non-principal nilpotent element. This conjecture is true for $r = 2, 3$ as only in these two cases all involved OPEs are known.

Conjecture 4.3 is true for $r = 3$ [ACL]. The proof uses our results. The following is immediate from the previous subsection:

Proposition 4.5. Let $L = \sqrt{2r}\mathbb{Z}$ and let J be its only self-dual simple current, then

$$\text{Vir}(3, 2+r) \otimes V_L \oplus J_{3,2+r} \otimes J$$

is a W-algebra that is strongly generated by two dimension $\frac{r}{2}$ fields, a Heisenberg field and the Virasoro field and has same central charge as $\mathcal{W}_r^{(2)}$ at level $k = \frac{2-r^2+2r}{r-1}$.

Proof. The conformal dimension of $J_{3,2+r} \otimes J$ is by construction the desired $\frac{r}{2}$ and the quantum dimension is $(-1)^r$ so that the resulting extension is always a VOA. It is strongly generated by the strong generators of $\text{Vir}(3, 2+r) \otimes V_L$ together with the fields of the two lowest-weight vectors of $J_{3,2+r} \otimes J$. However, by Proposition 4.1 together with the well-known lattice VOA operator products the two strong generators of V_L of conformal dimension r must be normal ordered products of the two lowest-weight vectors of $J_{3,2+r} \otimes J$ with themselves. \square

Remark 4.6. Call the dimension $\frac{r}{2}$ fields G^\pm and the dimension one field J , then using the well-known operator products of lattice VOAs it is easy to verify that with appropriate normalization of G^\pm and J , the OPE of J with itself as well as the one of J with G^\pm coincides with the OPE of the corresponding fields of $\mathcal{W}_r^{(2)}$ at level $k = \frac{2-r^2+2r}{r-1}$ as given in [FS]. In the case of the OPE of G^+ with G^- only the first two leading OPE coefficients can be easily computed and they again coincide with those given in [FS].

Remark 4.7. Tweaking the lattice a bit, one gets a W super algebra, that is believed to be an affine W-super algebra associated to $\mathfrak{sl}(r|1)$. This belief is basically due to [FS]. Namely, let

$$N = \sqrt{2(r+2)}\mathbb{Z}$$

and let K be the unique self-dual simple current of V_N , then the lowest-weight states of $J_{3,2+r} \otimes K$ have conformal dimension $\frac{r+1}{2}$ so that

$$\text{Vir}(3, 2+r) \otimes V_N \oplus J_{n+1, n+r} \otimes K$$

is now a super VOA. In the case $r = 2$ this is the $N = 2$ super conformal algebra, which has already known to be rational [Ad3].

More super VOAs. We believe the following:

Conjecture 4.8. $L_k(\mathfrak{sl}_2)$ is a subVOA of $L_k(\mathfrak{osp}(1|2))$ and

$$\text{Com}(L_k(\mathfrak{sl}_2), L_k(\mathfrak{osp}(1|2))) \cong \text{Vir}(k+2, 2k+3) \quad (4.6)$$

for positive integer k , especially $L_k(\mathfrak{osp}(1|2))$ is a rational super VOA.

The conjecture is motivated from [CL1]. Namely, it was shown that the universal coset VOA $\text{Com}(V_k(\mathfrak{sl}_2), V_k(\mathfrak{osp}(1|2)))$ is just the universal Virasoro algebra for generic k . Also computational evidence was given that integral k are generic, and further it was shown that the coset of the universal VOAs surjects on the coset of the corresponding simple quotients. In other words, according to [CL1] the conjecture is true if $L_k(\mathfrak{sl}_2)$ is a subVOA of $L_k(\mathfrak{osp}(1|2))$ and if positive integer k is generic.

Using the singlet and triplet algebras, one can construct interesting new logarithmic VOAs, examples are algebras of Feigin-Semikhatov type [FS] constructed in [Ad1, CRW], but also the small $N = 4$ super conformal algebra at central charge -9 [Ad2]. Here, we will give two further examples and in the same manner prove above conjecture for $k = 1$.

Theorem 4.9. Conjecture 4.8 is true for $k = 1$. Also for $k = -1/2$ the commutant is $\text{Vir}(3, 4)$.

Proof. (1) We first look at $k = 1$. We need to prove that:

$$V = (L_1(\mathfrak{sl}_2) \otimes \text{Vir}(3, 5)) \oplus (J_{\mathfrak{sl}_2} \otimes J_{\text{Vir}}) \cong L_1(\mathfrak{osp}(1|2)).$$

First note that $L_1(\mathfrak{sl}_2) \otimes \text{Vir}(3, 5)$ is a simple VOA, being a tensor product of simple VOAs. Simple current extensions of simple VOAs are simple and hence, V is simple.

It is well known that $\text{Vir}(3, 5)$ is a finitely reductive vertex operator algebra such that $\text{Vir}(3, 5) \cong \text{Vir}(3, 5)'$ and such that $\text{Vir}(3, 5)_n = 0$ for all $n < 0$. Therefore, we can invoke equation (4.1). $L_1(\mathfrak{sl}_2) \cong V_{\sqrt{2}\mathbb{Z}}$ is a lattice VOA. Denote by ϕ_λ the vertex operator associated to the lattice vector λ . Then the three currents are $e = \phi_{\sqrt{2}}$, $f = \phi_{-\sqrt{2}}$ and the Heisenberg field h , the self-dual simple currents has two fields $x = \phi_{1/\sqrt{2}}$, $y = \phi_{-1/\sqrt{2}}$ of conformal dimension $1/4$. The simple current field (associated to the lowest weight vector of J_{Vir}) of $\text{Vir}(3, 5)$ we denote by J . Now, consider the five dimension 1 fields: $(e \otimes \mathbf{1})(z)$, $(f \otimes \mathbf{1})(z)$, $(h \otimes \mathbf{1})(z)$, $(x \otimes J)(z)$, $(y \otimes J)(z)$. We know that

$$\begin{aligned} J(z)J(w) &\sim (z-w)^{-3/2}(\ell + \dots), & x(z)y(w) &\sim (z-w)^{-1/2}(1 + (z-w)h(w) + \dots), \\ x(z)x(w) &\sim (z-w)^{1/2}(e(w) + \dots), & y(z)y(w) &\sim (z-w)^{1/2}(f(w) + \dots) \end{aligned}$$

and ℓ is non-zero by equation (4.1). Now, one can easily verify OPEs to prove that these five fields generate a vertex subalgebra isomorphic to a quotient of $V_1(\mathfrak{osp}(1|2))$.

Finally $\omega_{\mathfrak{osp}} - (\omega_{\mathfrak{sl}_2} \otimes \mathbf{1}) \in \text{Com}(L_1(\mathfrak{sl}_2), V)$ and $\mathbf{1} \otimes \omega_{\text{Vir}}(3, 5) \in \text{Com}(L_1(\mathfrak{sl}_2), V)$ are both conformal vectors and since $\text{Com}(L_1(\mathfrak{sl}_2), V) = \text{Vir}(3, 5)$ they coincide. Therefore, we see that these five dimension 1 fields strongly generate the entire V . Hence, the entire V is a quotient of $V_1(\mathfrak{osp}(1|2))$. Since V is simple, we get that $V \cong L_1(\mathfrak{osp}(1|2))$.

(2) Now we look at $k = -1/2$. As before, let

$$V = (L_{-1/2}(\mathfrak{sl}_2) \otimes \text{Vir}(3, 4)) \oplus (J_{\mathfrak{sl}_2} \otimes J_{\text{Vir}}).$$

We know that $L_{-1/2}(\mathfrak{sl}_2) \oplus J_{\mathfrak{sl}_2} \cong \mathcal{S}(1)$ the rank one $\beta\gamma$ -VOA and hence $L_{-1/2}(\mathfrak{sl}_2) \cong (\mathcal{S}(1))^{\mathbb{Z}/2\mathbb{Z}}$. Also $\text{Vir}(3, 4) \oplus J_{\text{Vir}} \cong \mathcal{F}(1)$, the free fermion super VOA and thus $\text{Vir}(3, 4) \cong (\mathcal{F}(1))^{\mathbb{Z}/2\mathbb{Z}}$. Therefore, the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acts by automorphisms on $\mathcal{S}(1) \otimes \mathcal{F}(1)$ and

$$V = (\mathcal{S}(1) \otimes \mathcal{F}(1))^G$$

where $G = \{(0, 0), (1, 1)\} \subset \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Hence, V is a super VOA. By the same reasoning as in the previous case, V is simple. Just like before, one can check the OPEs of the dimension 1 fields to prove that these five fields generate a vertex subalgebra isomorphic to a quotient of $V_{-1/2}(\mathfrak{osp}(1|2))$, which must be the entire V by analogous arguments. \square

Remark 4.10. *In the $k = -1/2$ case, we haven't really used Theorem 3.9, instead we have given an indirect proof that the "simple current" extension is a super VOA. If one were to use Theorem 3.9, one first has to prove that Huang-Lepowsky-Zhang's theory can be invoked in this case. We expect the results of [Miy] to be useful. For Theorem 4.14 also we shall give an indirect proof.*

We would now like to analyze the irreducible modules of $L_1(\mathfrak{osp}(1|2))$.

Lemma 4.11. *Let $V_e = V \oplus J$ be a simple current extension by a simple current J of finite order. Assume also that $J \boxtimes W \not\cong W$ for a simple V -module W . Let W_e be a simple V_e -module that contains a simple V -submodule W . Then, $W_e \cong \bigoplus_{i \in G} J^i \boxtimes W$, where G is the finite cyclic group generated by J .*

Proof. The proof is the same as the proof of Theorem 3.7 in [La], except that we do not require V to be rational. For $S \subset V_e$ let

$$S \cdot W = \text{Span}\{j_n w \mid j \in S, w \in W, n \in \mathbb{Z}\}. \quad (4.7)$$

It is clear that $J^i \cdot W$ is a V -submodule of W_e for each $i \in G$. Moreover, restriction of the V_e -module map provides an intertwining operator, say \mathcal{Y} , of type $\binom{J^i \cdot W}{J^i \boxtimes W}$. Now, if for some $j \in J^i, w \in W, j_n w = 0$ for all n , then $J^i \cdot W = 0$ by Proposition 11.9 of [DL]. It is easy to see that this implies $V_e \cdot W = 0$. Therefore, we conclude that $J^i \cdot W \neq 0$, and hence $\mathcal{Y} \neq 0$. By universal property of tensor products, there exists a non-zero V -module map $J^i \boxtimes W \rightarrow J^i \cdot W$. This map is clearly surjective. Since $J^i \boxtimes W$ is simple, this map is an isomorphism. We identify $J^i \cdot W$ with $J^i \boxtimes W$. We have that $\sum_{i \in G} J^i \boxtimes W$ is a submodule of W_e and hence $W_e = \sum_{i \in G} J^i \boxtimes W$. The sum is direct because of the assumption that $J \boxtimes W \not\cong W$ for any simple V -module W . \square

Remark 4.12. *The condition on the module W_e of Lemma 4.11 is satisfied if W_e has finite length as a V -module. See [H5] for conditions under which this is guaranteed to happen.*

Corollary 4.13. *$L_1(\mathfrak{osp}(1|2))$ has precisely two inequivalent simple modules.*

Proof. Let $V = L_1(\mathfrak{sl}_2) \otimes \text{Vir}(3, 5)$, $J = L(\Lambda_1) \otimes J_{\text{Vir}}$ and let $V_e = L_1(\mathfrak{osp}(1|2)) \cong V \oplus J$. From [Li1], V is rational. Now, we can invoke Lemma 4.11 to gather that any simple V_e -module is of the form $W \oplus J \boxtimes W$ for some simple V -module W . By Proposition 4.7.4 of [FHL], any simple module for $L_1(\mathfrak{sl}_2) \otimes \text{Vir}(3, 5)$ is a tensor product of simple modules for $L_1(\mathfrak{sl}_2)$ and $\text{Vir}(3, 5)$. Gathering the data from Table 4, it is clear that

$$M_1 = L(\Lambda_0) \otimes \phi_{1,1}, M_2 = L(\Lambda_0) \otimes \phi_{1,3}, M_3 = L(\Lambda_1) \otimes \phi_{1,2}, M_4 = L(\Lambda_1) \otimes \phi_{1,4}$$

are the only $L_1(\mathfrak{sl}_2) \otimes \text{Vir}(3, 5)$ -modules that lift to $L_1(\mathfrak{osp}(1|2))$ -modules by Corollary 3.16. However, M_1 and M_4 lift to isomorphic modules and so do M_2 and M_3 . \square

Theorem 4.14. *Let $\mathcal{W}(2)$ be the C_2 -cofinite $c = -2$ triplet algebra, then $\mathcal{W}(2) \otimes L_{-1/2}(\mathfrak{sl}_2)$ has a simple current extension isomorphic to the small $N = 4$ super Virasoro algebra at $c = -3$.*

Proof. Let

$$V = X_1^+ \otimes L_{-1/2}(\mathfrak{sl}_2) \oplus X_1^- \otimes J_{\mathfrak{sl}_2}.$$

We have as before that $L_{-1/2}(\mathfrak{sl}_2) \oplus J_{\mathfrak{sl}_2} \cong \mathcal{S}(1)$ (the $\beta\gamma$ -VOA) and thus $L_{-1/2}(\mathfrak{sl}_2) \cong (\mathcal{S}(1))^{\mathbb{Z}/2\mathbb{Z}}$, but also $X_1^+ \oplus X_1^- \cong \mathcal{A}(1)$, the rank one symplectic fermion super VOA so that $X_1^+ \cong (\mathcal{A}(1))^{\mathbb{Z}/2\mathbb{Z}}$. Therefore, the group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ acts by automorphisms on $\mathcal{A}(1) \otimes \mathcal{S}(1)$ and $V = (\mathcal{A}(1) \otimes \mathcal{S}(1))^G$ where $G = \{(0, 0), (1, 1)\} \subset \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Hence, V is a vertex operator algebra.

By reasoning as before, V is simple.

The small $N = 4$ super Virasoro algebra is generated by three \mathfrak{sl}_2 fields of weight 1, one Virasoro field of weight 2 and its four superpartners of weight $3/2$. We have got the required number of fields.

We must check that the OPEs match the ones for the small $N = 4$. We denote the lowest weight states of $J_{\mathfrak{sl}_2}$ by β, γ . We denote the lowest weight states of X_1^- by s^+, s^- .

We know that:

$$\begin{aligned} \beta(z)\beta(w) &\sim 2e(w) + (z-w)(\beta(-3/2)\beta(-1/2)\mathbf{1})(w) + \dots \\ \gamma(z)\gamma(w) &\sim 2f(w) + (z-w)(\gamma(-3/2)\gamma(-1/2)\mathbf{1})(w) + \dots \\ \beta(z)\gamma(w) &\sim -(z-w)^{-1} + h(w) + (z-w)(\beta(-3/2)\gamma(-1/2)\mathbf{1})(w) + (z-w)^2(\beta(-5/2)\gamma(-1/2)\mathbf{1})(w) + \dots \\ \gamma(z)\beta(w) &\sim (z-w)^{-1} + h(w) + (z-w)(\gamma(-3/2)\beta(-1/2)\mathbf{1})(w) + (z-w)^2(\gamma(-5/2)\beta(-1/2)\mathbf{1})(w) + \dots \\ s^+(z)s^-(w) &\sim (z-w)^{-2} + (s^+(-1)s^-(-1)\mathbf{1})(w) + (z-w)(s^+(-2)s^-(-1)\mathbf{1})(w) + \dots \\ s^-(z)s^+(w) &\sim -(z-w)^{-2} + (s^-(-1)s^+(-1)\mathbf{1})(w) + (z-w)(s^-(-2)s^+(-1)\mathbf{1})(w) + \dots \\ s^+(z)s^+(w) &\sim (z-w)(\dots) \\ s^-(z)s^-(w) &\sim (z-w)(\dots). \end{aligned}$$

Let $J^+ = -\frac{1}{2} : \beta\beta :$, $J^- = \frac{1}{2} : \gamma\gamma :$ and $h = : \beta\gamma :$, then the OPE of these three is the operator product algebra of $L_{-1/2}(\mathfrak{sl}_2)$ [R1]. Let

$$G^+ = \beta \otimes s^+, \quad G^- = \gamma \otimes s^+, \quad \bar{G}^+ = -\beta \otimes s^-, \quad \bar{G}^- = \gamma \otimes s^-.$$

From the OPEs above, it is easy to calculate the λ -brackets as in [KRW]:

Since the OPE of v_1, v_2 for any $v_1, v_2 \in \{G^+, G^-\}$ or $v_1, v_2 \in \{\bar{G}^+, \bar{G}^-\}$ is regular, their λ -bracket is 0. We also have the following OPEs:

$$\begin{aligned} G^+(z)\bar{G}^+(w) &\sim (z-w)^{-2} \cdot 2e(w) + (z-w)^{-1}(\beta(-3/2)\beta(-1/2)\mathbf{1})(w) + \dots \\ G^-(z)\bar{G}^-(w) &\sim (z-w)^{-2} \cdot 2f(w) + (z-w)^{-1}(\gamma(-3/2)\gamma(-1/2)\mathbf{1})(w) + \dots \\ G^+(z)\bar{G}^-(w) &\sim -(z-w)^{-3} + (z-w)^{-2} \cdot h(w) + (z-w)^{-1}(\beta(-3/2)\gamma(-1/2)\mathbf{1} - s^+(-1)s^-(-1)\mathbf{1})(w) + \dots \\ G^-(z)\bar{G}^+(w) &\sim -(z-w)^{-3} - (z-w)^{-2} \cdot h(w) - (z-w)^{-1}(\gamma(-3/2)\beta(-1/2)\mathbf{1} + s^+(-1)s^-(-1)\mathbf{1})(w) + \dots \end{aligned}$$

We know that $\omega_{\mathcal{W}(2)} = s^-(-1)s^+(-1)\mathbf{1}$ and $\omega_{\mathfrak{sl}_2} = \frac{1}{2}[\beta(-3/2)\gamma(-1/2) - \gamma(-3/2)\beta(-1/2)]\mathbf{1}$. So, the λ -brackets come out to be:

$$\begin{aligned} [J^\pm_\lambda G^\mp] &= G^\pm, \quad [J^\pm_\lambda \bar{G}^\mp] = -\bar{G}^\pm, \quad [G^\pm_\lambda \bar{G}^\pm] = (\partial + 2\lambda)J^\pm, \\ [G^+_\lambda \bar{G}^-] &= -\frac{1}{2}\lambda^2 + \lambda J^0 + L + \frac{1}{2}\partial J^0, \quad [G^-_\lambda \bar{G}^+] = -\frac{1}{2}\lambda^2 - \lambda J^0 + L - \frac{1}{2}\partial J^0. \end{aligned}$$

Therefore, we've got the correct λ -bracket structure for the small $N = 4$ super Virasoro algebra at $c = -3$.

Finally we verify that these fields strongly generate V . For this we have to check that we can obtain the element $s^-(-2)s^+ + s^+(-2)s^- \in \mathcal{W}(2)$ as a normally ordered polynomial in the other generators and their derivatives. We know the following (the subscript denotes the mode):

$$(\beta s^+)_-1(\gamma s^-) = -s^+(-2)s^- + h s^+ s^- + \beta(-5/2)\gamma$$

$$\begin{aligned}
(\gamma s^+)_{-1}(\beta s^-) &= s^+(-2)s^- + h s^+ s^- + \gamma(-5/2)\beta \\
(\beta s^-)_{-1}(\gamma s^+) &= -s^-(-2)s^+ + h s^- s^+ - \beta(-5/2)\gamma \\
(\gamma s^-)_{-1}(\beta s^+) &= s^-(-2)s^+ + h s^- s^+ - \gamma(-5/2)\beta.
\end{aligned}$$

Therefore,

$$(\gamma s^+)_{-1}(\beta s^-) + (\gamma s^-)_{-1}(\beta s^+) - (\beta s^+)_{-1}(\gamma s^-) - (\beta s^-)_{-1}(\gamma s^+) = s^+(-2)s^- + s^-(-2)s^+.$$

□

Some orbifolds with categories of C_1 -cofinite modules. Our results apply to module categories of VOAs that are vertex tensor categories in the sense of Huang-Lepowsky. The main obstacle for the conditions of Theorem 3.23 is in verifying the C_1 -cofiniteness of modules. We thus close this work with a few examples on this question in the context of orbifolds of free field algebras.

Consider first the rank n Heisenberg vertex algebra $\mathcal{H}(n)$, whose full automorphism group is the orthogonal group $O(n)$. By [DLM2], there is a dual reductive pair decomposition

$$\mathcal{H}(n) = \bigoplus_{\nu} L_{\nu} \otimes M^{\nu},$$

where the sum is over all finite-dimensional, irreducible $O(n)$ -modules L_{ν} , and the M^{ν} 's are inequivalent, irreducible $\mathcal{H}(n)^{O(n)}$ -modules.

The C_1 -cofiniteness of the $\mathcal{H}(n)^{O(n)}$ -modules M^{ν} was established in [L3, L4], and we briefly sketch the proof. First, we may view $\mathcal{H}(n)^{O(n)}$ as a deformation of the classical invariant ring $R = (\text{Sym} \bigoplus_{k \geq 0} V_k)^{O(n)}$, where $V_k \cong \mathbb{C}^n$ as $O(n)$ -modules. In particular, $\mathcal{H}(n)$ admits an $O(n)$ -invariant good increasing filtration in the sense of Li [Li2], and $\text{gr}(\mathcal{H}(n)^{O(n)}) \cong R$ as differential graded commutative rings. Using Weyl's first fundamental theorem of invariant theory for $O(n)$ [We], it is not difficult to find an (infinite) strong generating set for $\text{gr}(\mathcal{H}(n)^{O(n)})$ consisting of an element in each weight $2, 4, 6, \dots$. A consequence is that the Zhu algebra of $\mathcal{H}(n)^{O(n)}$ is abelian. This implies that all irreducible, admissible $\mathcal{H}(n)^{O(n)}$ -modules are highest-weight modules, i.e., they are generated by a single vector. In particular, this holds for each M^{ν} above.

It follows from Weyl's second fundamental theorem of invariant theory for $O(n)$ [We] that the relation of minimal weight among the generators of $\mathcal{H}(n)^{O(n)}$ occurs at weight $n^2 + 3n + 2$. In [L3], it was conjectured that this gives rise to a decoupling relation expressing the generator in weight $n^2 + 3n + 2$ as a normally ordered polynomial in the generators of lower weight. Starting with this relation, it is easy to construct decoupling relations for all the higher weight generators, so that $\mathcal{H}(n)^{O(n)}$ is of type $\mathcal{W}(2, 4, \dots, n^2 + 3n)$. In the case $n = 1$, the fact that $\mathcal{H}(1)^{\mathbb{Z}/2\mathbb{Z}}$ is of type $\mathcal{W}(2, 4)$ is a celebrated theorem of Dong and Nagatomo [DN], and this conjecture was verified for $n \leq 6$ in [L4]. Even though it remains open in general, the strong finite generation of $\mathcal{H}(n)^{O(n)}$ was established for all n in [L4].

The proof that each M^{ν} is C_1 -cofinite depends on the strong finite generation of $\mathcal{H}(n)^{O(n)}$, together with the fact that the non-negative Fourier modes of the generators of $\mathcal{H}(n)^{O(n)}$ preserve the filtration on $\mathcal{H}(n)$. Note that Lemma 6.7 of [L3] is precisely the statement that each M^{ν} is C_1 -cofinite according to Miyamoto's definition [Miy]. This was originally proven modulo the above conjecture in [L3], but the proof only requires that $\mathcal{H}(n)^{O(n)}$ is strongly finitely generated. Therefore Lemma 6.7 of [L3] holds unconditionally.

Similar results have been established for several other orbifolds of free field algebras, and the proof is the same. The key ingredients are the strong finite generation of the orbifold and the fact that the non-negative Fourier modes of the generators preserve a filtration on the free field algebra. For the rank n $\beta\gamma$ -system $\mathcal{S}(n)$, $\mathcal{S}(n)^{\text{GL}(n)}$ and $\mathcal{S}(n)^{\text{Sp}(2n)}$ are of types $\mathcal{W}(1, 2, \dots, n^2 + 2n)$ and $\mathcal{W}(2, 4, \dots, 2n^2 + 4n)$, respectively, and every irreducible submodule of $\mathcal{S}(n)$ for either of these

orbifolds is C_1 -cofinite [L1, L2, L4]. The same holds for the rank n bc -system $\mathcal{E}(n)$ and the orbifold $\mathcal{E}(n)^{\text{GL}(n)}$ which is isomorphic to $\mathcal{W}(\mathfrak{gl}_n)$ with central charge n [FKRW]. Similarly, it holds for the free fermion algebra $\mathcal{F}(n)$ and the orbifold $\mathcal{F}(n)^{\text{O}(n)}$, which is of type $\mathcal{W}(2, 4, \dots, 2n)$ [L5]. Finally, it holds for the rank n symplectic fermion algebra $\mathcal{A}(n)$ and the orbifolds $\mathcal{A}(n)^{\text{Sp}(2n)}$ and $\mathcal{A}(n)^{\text{GL}(n)}$, which are of types $\mathcal{W}(2, 4, \dots, 2n)$ and $\mathcal{W}(2, 3, \dots, 2n+1)$, respectively [CL2]. In all these cases, the orbifolds have abelian Zhu algebras. This makes the arguments easier, but it is not essential and we expect the C_1 -cofiniteness to hold for a more general class of orbifolds of free field algebras. There are a few other examples in [CL1, CL3] where an explicit minimal strong generating set has been found using similar methods, including $(\mathcal{E}(n) \otimes \mathcal{S}(n))^{\text{GL}(n)}$, $(\mathcal{A}(n) \otimes \mathcal{S}(n))^{\text{Sp}(2n)}$, $(\mathcal{A}(n) \otimes \mathcal{S}(n))^{\text{GL}(n)}$, and $(\mathcal{H}(n) \otimes \mathcal{F}(n))^{\text{O}(n)}$. In these cases, all irreducible modules for the orbifold inside the ambient free field algebra can be shown to be C_1 -cofinite using similar ideas.

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