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SHORT NOTE

A natural characterization of semilattices of rectangular bands and groups of exponent two

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Abstract In a recent paper, Monzo characterized semilattices of rectangular bands and groups of exponent 2 as the semigroups that satisfy the following conditions: $x = x^3$ and $xyx \in \{xy^2x, y^2xy^2\}$. However, this definition does not seem to point directly to the properties of rectangular bands and groups of exponent 2 (namely, idempotency and commutativity). So, in order to provide a more natural characterization of the class of semigroups under consideration we prove the following theorem:

Main Theorem *In a semigroup S, the following are equivalent:*

- *S is a semilattice of rectangular bands and groups of exponent* 2*;*
- *for all* $x, y \in S$*, we have* $x = x^3$ *and* $xy \in \{yx, (xy)^2\}$.

The paper ends with a list of problems.

Keywords Rectangular bands · Groups of exponent two

1 Completely simple semigroups and rectangular groups

For basics in semigroup theory we refer to [1]. For the reader's convenience we recall here some basic definitions and facts.

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Definition 1.1 Let E be a rectangular band and G a group. The direct product $G \times E$ is called a rectangular group.

Clearly, rectangular groups are completely simple semigroups.

Every rectangular band E is isomorphic to a semigroup $I \times \Lambda$ (for some non-empty sets I and Λ) with the product defined by

$$(i_1, \lambda_1)(i_2, \lambda_2) = (i_1, \lambda_2).$$

Therefore, the direct product of a rectangular band E and a group G can be seen as follows: let $\Lambda \times I$ be the semigroup isomorphic to E; in $S = I \times G \times \Lambda$ define the product

$$(i, a, \mu)(j, b, \lambda) = (i, ab, \lambda).$$

In this setting, rectangular groups are of course Rees matrix semigroups (over groups), in which all elements of the sandwich matrix are the group identity.

Theorem 1.2 *Let S be a completely simple semigroup. If the products of any two idempotents in S is an idempotent, then S is isomorphic to a rectangular group.*

Proof [2, Theorem III.5.2]. □

Definition 1.3 Let Y be a semilattice and S_α $(\alpha \in Y)$ a family of semigroups. Then a semigroup S is said to be a semilattice of semigroups S_α $(\alpha \in Y)$ if

$$(\forall \alpha, \beta \in Y)(\forall a \in S_\alpha)(\forall b \in S_\beta)\ ab \in S_{\alpha \wedge \beta}.$$

Theorem 1.4 *Let S be a semigroup. The following are equivalent:*

(1) For every $a \in S$ we have $a \in aSa^2$;
(2) S is isomorphic to a semilattice of completely simple semigroups.

Proof [2, Theorem II.1.4]. □

The two previous results immediately imply the following theorem.

Theorem 1.5 *Let S be a semigroup such that $x^3 = x$ and the product of two idempotents is an idempotent. Then S is a semilattice of rectangular groups.*

Proof Let S be a semigroup and $x \in S$. Since $x = x^3$, then $x^3 = x^5$. But,

$$x = x^3 \text{ and } x^3 = x^5 \Rightarrow x = x^5 \Rightarrow x = xx^2x^2 \in xSx^2.$$

Then, by Theorem 1.4, S is isomorphic to a semilattice of completely simple semigroups, that is, S is a semilattice Y of completely simple semigroups S_α $(\alpha \in Y)$. In addition, we know that in S the product of two idempotents is an idempotent, and so that also holds in each S_α; by Theorem 1.2 this means that S_α is a rectangular group and hence S is a semilattice of rectangular groups. □

2 The main Theorem

The following is the main result in Monzo's paper. (It is worth observing that Monzo states his theorem in terms of *groups of order* 2, but clearly he has in mind *groups of exponent* 2.)

Theorem 2.1 [3, Theorem 5] *Let S be a semigroup. The following are equivalent:*

(1) S is a semilattice Y of semigroups S_α $(\alpha \in Y)$ with each S_α being a rectangular band or a group of exponent 2;
(2) for all $x, y \in S$ we have $x^3 = x$ and $xyx \in \{xy^2x, y^2x^2y\}$.

Certainly, rectangular bands are semigroups of idempotents ($x^2 = x$) and groups of exponent 2 are commutative ($xy = yx$). A natural characterization of this class of semigroups should in some way include these two identities, something that does not happen with Monzo's. The aim of this note, hence, is to provide a characterization for the class of semigroups under consideration, more natural than the one provided by the previous theorem.

Theorem 2.2 *Let S be a semigroup. Then the following are equivalent:*

(1) S is a semilattice Y of semigroups $S_\alpha(\alpha \in Y)$ with S_α a rectangular band or a group of exponent 2, for all $\alpha \in Y$;
(2) for all $x, y \in S$ we have $x = x^3$ and $xy \in \{yx, (xy)^2\}$.

Proof We follow closely Monzo's proof.

(1) $\Rightarrow$ (2). We can easily see that, for every $x \in S$, we have $x = x^3$ (in the rectangular bands, $x = xx = xxx$; in the groups of exponent 2, $x^3 = xxx = (xx)x = 1x = x$). Now let $x \in S$, $\{\alpha, \beta\} \subseteq Y$, with $x \in S_\alpha$ and $y \in S_\beta$. If $S_{\alpha\beta}$ is a rectangular band, then $xy = (xy)^2$, as required.

If $S_{\alpha\beta}$ is a group of exponent 2, let 1 denote its identity; then

$$
\begin{aligned}
(xy)(xy) &= (xyx)y \\
&= (xyx1)y && \text{(because } xyx \in S_{\alpha\beta}) \\
&= (xyx)(1y) && \text{(by associatvity)} \\
&= x(yx)(1y) \\
&= x(1y)(yx) && \text{(because } 1y, yx \in S_{\alpha\beta}, \text{ an abelian group)} \\
&= x(1yyx) \\
&= x(yyx) && \text{(because } yyx \in S_{\alpha\beta}, \text{ and 1 is the identity in } S_{\alpha\beta}) \\
&= (xy)(yx).
\end{aligned}
$$

Therefore, $(xy)(xy) = (xy)(yx)$. Multiplying both sides by $(xy)^{-1}$, on the left, we get $xy = yx$. Hence, for all $x, y \in S$, we have $x = x^3$ and $xy \in \{yx, (xy)^2\}$, as claimed.

(2) $\Rightarrow$ (1). Since $x = x^3$, we need to prove that the product of two idempotents is an idempotent in order to conclude that S is a semilattice of rectangular groups (by

Theorem 1.5). Let $e \in S$ and $f \in S$ be two idempotents. From our assumptions it follows that $ef \in \{fe, (ef)^2\}$. If $ef = (ef)^2$, then it is proved that ef is idempotent. So we suppose that $ef = fe$. It is well known that if two idempotents commute, then their product is also idempotent.

It is proved that in either case the product of two idempotents is an idempotent and then, by Theorem 1.5, S is a semilattice of rectangular groups. Thus each S_α is isomorphic to $G_\alpha \times E_\alpha$, where G_α is a group and E_α is a rectangular band.

Fix $\alpha \in Y$ and let $x, y \in S_\alpha := G_\alpha \times E_\alpha$, with $x = (1, (i, \lambda))$ and $y = (h, (j, \sigma))$ where h is an arbitrary element of the group G_α and (i, λ) and (j, σ) are arbitrary elements of the rectangular band E_α. Then,

$$\begin{aligned} & xy \in \left\{yx, (xy)^2\right\} \\ \Leftrightarrow\ & (1, (i, \lambda))(h, (j, \sigma)) \in \{(h, (j, \sigma))(1, (i, \lambda)), (1, (i, \lambda))(h, (j, \sigma))(1, (i, \lambda)) \\ & \qquad (h, (j, \sigma))\} \\ \Leftrightarrow\ & (h, (i, \sigma)) \in \left\{(h, (j, \lambda)), (h^2, (i, \sigma))\right\}. \end{aligned}$$

Now, either $h \neq h^2$ and hence $(i, \sigma) = (j, \lambda)$, which means that E_α has only one element, so that $G_\alpha \times E_\alpha$ is isomorphic to G_α; or $h = h^2$ (for every $h \in G_\alpha$) which means that $h = 1$, for all $h \in G_\alpha$. Thus either G_α or E_α have just one element and hence S_α is isomorphic to either G_α or E_α. In the former case, since S satisfies $x = x^3$, each group G_α is a group of exponent 2. □

We finish this note with three questions.

Problem 2.3 Let V be a variety of groups. Characterize the class of semigroups that are semilattices of rectangular bands and groups in V.

Problem 2.4 Let V be a variety of groups. Characterize the class of semigroups that are semilattices of groups in V.

Problem 2.5 Let V_1 and V_2 be two varieties of groups. Characterize the class of semigroups that are semilattices of groups contained in V_1 or V_2.

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