

CONVERGENCE OF QUOTIENTS OF AF ALGEBRAS IN QUANTUM PROPINQUITY BY CONVERGENCE OF IDEALS

KONRAD AGUILAR

ABSTRACT. We provide conditions for when quotients of AF algebras are quasi-Leibniz quantum compact metric spaces building from our previous work with F. Latrémolière. Given a C^* -algebra, the ideal space may be equipped with natural topologies. Next, we impart criteria for when convergence of ideals of an AF algebra can provide convergence of quotients in quantum propinquity, while introducing a metric on the ideal space of a C^* -algebra. We then apply these findings to a certain class of ideals of the Boca-Mundici AF algebra by providing a continuous map from this class of ideals equipped with various topologies including the Jacobson and Fell topologies to the space of quotients with the quantum propinquity topology. Lastly, we introduce new Leibniz Lip-norms on any unital AF algebra motivated by Rieffel's work on Leibniz seminorms and best approximations.

CONTENTS

1. Introduction	1
2. Quantum Metric Geometry and AF algebras	4
3. Convergence of AF algebras with Quantum Propinquity	10
4. Metric on Ideal Space of C^* -Inductive Limits	18
5. Metric on Ideal Space of C^* -Inductive Limits: AF case	27
6. Convergence of Quotients of AF algebras with Quantum Propinquity	39
6.1. The Boca-Mundici AF algebra	43
7. Leibniz Lip-norms for Unital AF algebras	56
Acknowledgements	59
References	59

1. INTRODUCTION

The Gromov-Hausdorff propinquity [29, 26, 24, 28, 27], a family of noncommutative analogues of the Gromov-Hausdorff distance, provides a new framework to study the geometry of classes of C^* -algebras, opening new avenues of research in noncommutative geometry by work of Latrémolière building off notions introduced by Rieffel [37, 45]. In collaboration with Latrémolière, our previous work

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in [1] served to introduce AF algebras into the realm of noncommutative metric geometry. In particular, given a unital AF algebra with a faithful tracial state, we endowed such an AF algebra, viewed as an inductive limit, with quantum metric structure and showed that these AF algebras are indeed limits of the given inductive sequence of finite dimensional algebras in the quantum propinquity topology [1, Theorem 3.5]. From here, we were able to construct a Hölder-continuous map from the Baire space onto the class of UHF algebras of Glimm [16] and a continuous map from the Baire space onto the class Effros-Shen AF algebras introduced in [13] viewed as quantum metric spaces, and therefore both these classes inherit the metric geometry of the Baire Space via continuous maps. We continue this journey in this work by focusing on the ideal structure of AF algebras. In particular, we produce a metric geometry on the space of ideals of an AF algebra and provide criteria for when their quotients are quantum metric spaces and when convergence of ideals provide convergence of quotients in quantum propinquity. Thus, providing a metric geometry for classes of quotients induced from the metric geometry on the ideal space.

This paper begins by providing useful background, definitions, and theorems from quantum metric geometry, in which a core focus is that of finite-dimensional approximations — a motivating factor in our interest of applying the notions of quantum metric geometry to the study of AF algebras. Introduced by Rieffel [37, 38], a quantum metric is provided by a choice of a particular seminorm on a dense subalgebra of a C^* -algebra, called a Lip-norm, which plays an analogue role as the Lipschitz seminorm does in classical metric space theory (see also [22] for the notion of quantum locally compact metric spaces). The key property that such a seminorm must possess is that its dual must induce a metric on the state space of the underlying C^* -algebra which metrizes the weak- $*$ topology. This dual metric is a noncommutative analogue of the Monge-Kantorovich metric, and the idea of this approach to quantum metrics arose in Connes' work [7, 8] and Rieffel's work [37]. A pair of a unital C^* -algebra and a Lip-norm is called a quantum compact metric space, and can be seen as a generalized Lipschitz algebra [46]. However, recent developments in noncommutative metric geometry suggests that some form of relation between the multiplicative structure of C^* -algebras and Lip-norms is beneficial [40, 41, 42, 43, 29, 26, 24, 28]. A general form of such a connection is given by the quasi-Leibniz property [28], which are satisfied by our Lip-norms for unital AF algebras equipped with faithful tracial state constructed in [1].

Various notions of finite dimensional approximations of C^* -algebras are found in C^* -algebra theory, from nuclearity to quasi-diagonality, passing through exactness, to name a few of the more common notions. They are also a core focus and major source of examples for our research in noncommutative metric geometry. Introduced by Latrémolière, the Gromov-Hausdorff propinquity [29, 26, 24, 28, 27, 31], provides a new avenue to study finite-dimensional approximations and continuous families of noncommutative spaces via quantum metric structures. Examples of convergence and finite dimensional approximations in the sense of the propinquity include the approximations of quantum tori by fuzzy tori as well as certain metric perturbations [21, 23, 25] and the full matrix approximations C^* -algebras of continuous functions on coadjoint orbits of semisimple Lie groups

[39, 41, 44]. And, in our previous work [1], we established that finite dimensional subalgebras of an AF algebra provide finite dimensional approximations in the sense of propinquity [1, Theorem 3.5] along with various continuous families.

Before we discuss our analytical relationship between ideals and quotients, we develop some useful generalizations to our work in [1] in Section (3). In particular, we note that the given a unital AF algebra with faithful tracial state, the Lip-norm constructed in [1] is constructed by three structures: the inductive sequence, the faithful tracial state, and a positive sequence vanishing at infinity, which is usually taken to be some form of the reciprocal of the dimensions of the finite dimensional subspaces of the inductive sequence. From this, we provide suitable notions of convergence for all of these three structures, which all together imply convergence of families of AF algebras, in which we introduce the notion of fusing families of inductive limits. This also imparts a generalization that implies the continuity results of [1] pertaining to the UHF and Effros-Shen AF algebras. The results of this section also greatly simplifies the task of providing convergence of AF algebras in the quantum propinquity topology, which proves useful in Section (6) and should prove useful in future projects.

In Section (4), we develop a metric on the ideal space of any C^* -inductive limit. In general, this space is a zero-dimensional ultrametric space. The main application of this metric is to provide a notion of convergence for inductive sequences that determine the quotient spaces as fusing families (Definition (3.4))- a first step towards convergence in quantum propinquity. But, this topology on ideals has close connections to the Fell topology on the ideal space formed by the Jacobson topology on the primitive ideal space. The Fell topology was introduced by Fell in [15] as a topology on closed sets of a given topology. Fell then applied this topology to the closed sets of the Jacobson topology in [14] to provide a compact Hausdorff topology on the set of all ideals on a C^* -algebra. The metric on the ideal space of C^* -inductive limits introduced in this paper is always stronger than the Fell topology. Furthermore, in the AF case, Section (5) produces a formulation of the metric on ideals in terms of Bratteli diagrams that yields the result that the metric space on ideals is compact, and therefore, its topology equals the Fell topology. We make other comparisons including taking into consideration the restriction to primitive ideals and comparison of the Jacobson topology as well as an analysis on unital commutative AF algebras.

Next, Section (6) provides an answer to the question of when convergence of ideals can provide convergence of quotients. This essentially comes as a combination of Section (3) and Section (5). In Section (6.1), we define the Boca-Mundici AF algebra [4, 33], which arises from the Farey tessellation. Next, we prove some basic results pertaining to its Bratteli diagram structure and ideal structure, and then apply our criteria for quotients converging to a subclass of ideals of the Boca-Mundici AF algebra, in which each quotient is $*$ -isomorphic to an Effros-Shen AF algebra. In [4], Boca proved that this subclass of ideals with its relative Jacobson topology is homeomorphic to the irrationals in $(0, 1)$ with its usual topology, which provided our initial interest in our question about convergence of quotients. The main result of this section, Theorem (6.20), provides a continuous function from a subclass of ideals of the Boca-Mundici AF algebra to its quotients as quantum metric spaces

in the quantum propinquity topology, where the topology of the subclass ideals can either be the Jacobson, metric, or Fell topology. Thus, providing an example of when a metric geometry on quotients is inherited from a metric geometry on ideals.

Lastly, in Section (7), we approach the question of Lip-norms for AF algebras via best approximations instead of conditional expectations. An immediately advantageous consequence of this is that by Rieffel's work in [42], we may produce Leibniz Lip-norms for any unital AF algebra with or without faithful tracial state. Furthermore, these Lip-norms still provide convergence of finite-dimensional subspaces to the AF algebra. We finish this section and this paper with a comparison of the best approximation Lip-norms and the conditional expectation Lip-norms of [1] as a consequence of Latrémolière's work in [30].

2. QUANTUM METRIC GEOMETRY AND AF ALGEBRAS

The purpose of this section is to discuss our progress thus far in the realm of quantum metric spaces with regard to AF algebras, and thus places more focus on the AF algebra results, but we also provide a cursory overview of the material on quantum compact metric spaces. We refer the reader to the survey by Latrémolière [27] for a much more detailed and insightful introduction to the study of quantum metric spaces.

Notation 2.1. When E is a normed vector space, then its norm will be denoted by $\|\cdot\|_E$ by default.

Notation 2.2. Let \mathfrak{A} be a unital C^* -algebra. The unit of \mathfrak{A} will be denoted by $1_{\mathfrak{A}}$. The state space of \mathfrak{A} will be denoted by $\mathcal{S}(\mathfrak{A})$ while the self-adjoint part of \mathfrak{A} will be denoted by $\mathfrak{sa}(\mathfrak{A})$.

Definition 2.3 ([37, 29, 28]). A (C, D) -quasi-Leibniz quantum compact metric space (\mathfrak{A}, L) , for some $C \geq 1$ and $D \geq 0$, is an ordered pair where \mathfrak{A} is unital C^* -algebra and L is a seminorm defined on some dense Jordan-Lie subalgebra $\text{dom}(L)$ of $\mathfrak{sa}(\mathfrak{A})$ such that:

- (1) $\{a \in \mathfrak{sa}(\mathfrak{A}) : L(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$,
- (2) the seminorm L is a (C, D) -quasi-Leibniz Lip-norm, i.e. for all $a, b \in \text{dom}(L)$:
$$\max \left\{ L \left(\frac{ab + ba}{2} \right), L \left(\frac{ab - ba}{2i} \right) \right\} \leq C (\|a\|_{\mathfrak{A}} L(b) + \|b\|_{\mathfrak{A}} L(a)) + DL(a)L(b),$$
- (3) the Monge-Kantorovich metric defined, for all two states $\varphi, \psi \in \mathcal{S}(\mathfrak{A})$, by:
$$\text{mk}_L(\varphi, \psi) = \sup \{ |\varphi(a) - \psi(a)| : a \in \text{dom}(L), L(a) \leq 1 \}$$
metrizes the weak* topology of $\mathcal{S}(\mathfrak{A})$,
- (4) the seminorm L is lower semi-continuous with respect to $\|\cdot\|_{\mathfrak{A}}$.

The seminorm L of a quantum compact metric space (\mathfrak{A}, L) is called a *Lip-norm*.

A primary interest in developing a theory of quantum metric spaces is the introduction of various hypertopologies on classes of such spaces, thus allowing us to study the geometry of classes of C^* -algebras and perform analysis on these classes. A classical model for our hypertopologies is given by the Gromov-Hausdorff distance [17, 18]. While several noncommutative analogues of the Gromov-Hausdorff

distance have been proposed — most importantly Rieffel’s original construction of the quantum Gromov-Hausdorff distance [45] — we shall work with a particular metric introduced by Latrémolière, [29], as we did in [1]. This metric, known as the quantum propinquity, is designed to be best suited to quasi-Leibniz quantum compact metric spaces, and in particular, is zero between two such spaces if and only if they are isometrically isomorphic. We now propose a summary of the tools needed to compute upper bounds on this metric.

Convention 2.4. When L is a seminorm defined on some dense subset F of a vector space E , we will implicitly extend L to E by setting $L(e) = \infty$ whenever $e \notin F$.

Definition 2.5 ([29, Definition 3.1]). The 1-level set $\mathcal{S}_1(\mathfrak{D}|\omega)$ of an element ω of a unital C^* -algebra \mathfrak{D} is:

$$\{\varphi \in \mathcal{S}(\mathfrak{D}) : \varphi((1 - \omega^*\omega)) = \varphi((1 - \omega\omega^*)) = 0\}.$$

Next, we define the notion of a Latrémolière bridge, which is not only crucial in the definition of the quantum propinquity but also the convergence results of Latrémolière in [23] and Rieffel in [44]. In particular, the pivot of Definition (2.6) is of utmost importance in the convergence results of [23, 44].

Definition 2.6 ([29, Definition 3.6]). A bridge from \mathfrak{A} to \mathfrak{B} , where \mathfrak{A} and \mathfrak{B} are unital C^* -algebras, is a quadruple $(\mathfrak{D}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, \omega)$ where:

- (1) \mathfrak{D} is a unital C^* -algebra,
- (2) the element ω , called the *pivot* of the bridge, satisfies $\omega \in \mathfrak{D}$ and $\mathcal{S}_1(\mathfrak{D}|\omega) \neq \emptyset$,
- (3) $\pi_{\mathfrak{A}} : \mathfrak{A} \hookrightarrow \mathfrak{D}$ and $\pi_{\mathfrak{B}} : \mathfrak{B} \hookrightarrow \mathfrak{D}$ are unital $*$ -monomorphisms.

In the next few definitions, we denote by Haus_d the *Hausdorff (pseudo)distance* induced by a (pseudo)distance d on the compact subsets of a (pseudo)metric space (X, d) [20].

Definition 2.7 ([29, Definition 3.16]). Let $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. The *height* $\zeta(\gamma|L_{\mathfrak{A}}, L_{\mathfrak{B}})$ of a bridge $\gamma = (\mathfrak{D}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, \omega)$ from \mathfrak{A} to \mathfrak{B} , and with respect to $L_{\mathfrak{A}}$ and $L_{\mathfrak{B}}$, is given by:

$$\max \left\{ \text{Haus}_{\text{mk}_{L_{\mathfrak{A}}}}(\mathcal{S}(\mathfrak{A}), \pi_{\mathfrak{A}}^*(\mathcal{S}_1(\mathfrak{D}|\omega))), \text{Haus}_{\text{mk}_{L_{\mathfrak{B}}}}(\mathcal{S}(\mathfrak{B}), \pi_{\mathfrak{B}}^*(\mathcal{S}_1(\mathfrak{D}|\omega))) \right\},$$

where $\pi_{\mathfrak{A}}^*$ and $\pi_{\mathfrak{B}}^*$ are the dual maps of $\pi_{\mathfrak{A}}$ and $\pi_{\mathfrak{B}}$, respectively.

Definition 2.8 ([29, Definition 3.10]). Let $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ be two unital C^* -algebras. The *bridge seminorm* $\text{bn}_{\gamma}(\cdot)$ of a bridge $\gamma = (\mathfrak{D}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, \omega)$ from \mathfrak{A} to \mathfrak{B} is the seminorm defined on $\mathfrak{A} \oplus \mathfrak{B}$ by:

$$\text{bn}_{\gamma}(a, b) = \|\pi_{\mathfrak{A}}(a)\omega - \omega\pi_{\mathfrak{B}}(b)\|_{\mathfrak{D}}$$

for all $(a, b) \in \mathfrak{A} \oplus \mathfrak{B}$.

We implicitly identify \mathfrak{A} with $\mathfrak{A} \oplus \{0\}$ and \mathfrak{B} with $\{0\} \oplus \mathfrak{B}$ in $\mathfrak{A} \oplus \mathfrak{B}$ in the next definition, for any two spaces \mathfrak{A} and \mathfrak{B} .

Definition 2.9 ([29, Definition 3.14]). Let $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. The *reach* $\varrho(\gamma|_{L_{\mathfrak{A}}, L_{\mathfrak{B}}})$ of a bridge $\gamma = (\mathfrak{D}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, \omega)$ from \mathfrak{A} to \mathfrak{B} , and with respect to $L_{\mathfrak{A}}$ and $L_{\mathfrak{B}}$, is given by:

$$\text{Haus}_{\text{bn}_{\gamma}(\cdot)}(\{a \in \text{sa}(\mathfrak{A}) : L_{\mathfrak{A}}(a) \leq 1\}, \{b \in \text{sa}(\mathfrak{B}) : L_{\mathfrak{B}}(b) \leq 1\}).$$

We thus choose a natural quantity to synthesize the information given by the height and the reach of a bridge:

Definition 2.10 ([29, Definition 3.17]). Let $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. The *length* $\lambda(\gamma|_{L_{\mathfrak{A}}, L_{\mathfrak{B}}})$ of a bridge $\gamma = (\mathfrak{D}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, \omega)$ from \mathfrak{A} to \mathfrak{B} , and with respect to $L_{\mathfrak{A}}$ and $L_{\mathfrak{B}}$, is given by:

$$\max\{\zeta(\gamma|_{L_{\mathfrak{A}}, L_{\mathfrak{B}}}), \varrho(\gamma|_{L_{\mathfrak{A}}, L_{\mathfrak{B}}})\}.$$

Theorem-Definition 2.11 ([29, 28]). Fix $C \geq 1$ and $D \geq 0$. Let $\mathcal{QQCMS}_{C,D}$ be the class of all (C, D) -quasi-Leibniz quantum compact metric spaces. There exists a class function $\Lambda_{C,D}$ from $\mathcal{QQCMS}_{C,D} \times \mathcal{QQCMS}_{C,D}$ to $[0, \infty) \subseteq \mathbb{R}$ such that:

(1) for any $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}) \in \mathcal{QQCMS}_{C,D}$ we have:

$$\Lambda_{C,D}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) \leq \max\{\text{diam}(\mathcal{S}(\mathfrak{A}), \text{mk}_{L_{\mathfrak{A}}}), \text{diam}(\mathcal{S}(\mathfrak{B}), \text{mk}_{L_{\mathfrak{B}}})\},$$

(2) for any $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}) \in \mathcal{QQCMS}_{C,D}$ we have:

$$0 \leq \Lambda_{C,D}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) = \Lambda_{C,D}((\mathfrak{B}, L_{\mathfrak{B}}), (\mathfrak{A}, L_{\mathfrak{A}}))$$

(3) for any $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}), (\mathfrak{C}, L_{\mathfrak{C}}) \in \mathcal{QQCMS}_{C,D}$ we have:

$$\Lambda_{C,D}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{C}, L_{\mathfrak{C}})) \leq \Lambda_{C,D}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) + \Lambda_{C,D}((\mathfrak{B}, L_{\mathfrak{B}}), (\mathfrak{C}, L_{\mathfrak{C}})),$$

(4) for all for any $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}) \in \mathcal{QQCMS}_{C,D}$ and for any bridge γ from \mathfrak{A} to \mathfrak{B} , we have:

$$\Lambda_{C,D}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) \leq \lambda(\gamma|_{L_{\mathfrak{A}}, L_{\mathfrak{B}}}),$$

(5) for any $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}) \in \mathcal{QQCMS}_{C,D}$, we have:

$$\Lambda_{C,D}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) = 0$$

if and only if $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ are isometrically isomorphic, i.e. if and only if there exists a $*$ -isomorphism $\pi : \mathfrak{A} \rightarrow \mathfrak{B}$ with $L_{\mathfrak{B}} \circ \pi = L_{\mathfrak{A}}$, or equivalently there exists a $*$ -isomorphism $\pi : \mathfrak{A} \rightarrow \mathfrak{B}$ whose dual map π^* is an isometry from $(\mathcal{S}(\mathfrak{B}), \text{mk}_{L_{\mathfrak{B}}})$ into $(\mathcal{S}(\mathfrak{A}), \text{mk}_{L_{\mathfrak{A}}})$,

(6) if Ξ is a class function from $\mathcal{QQCMS}_{C,D} \times \mathcal{QQCMS}_{C,D}$ to $[0, \infty)$ which satisfies Properties (2), (3) and (4) above, then:

$$\Xi((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) \leq \Lambda_{C,D}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}}))$$

for all $(\mathfrak{A}, L_{\mathfrak{A}})$ and $(\mathfrak{B}, L_{\mathfrak{B}})$ in $\mathcal{QQCMS}_{C,D}$

The quantum propinquity is, in fact, a special form of the dual Gromov-Hausdorff propinquity [26, 24, 28] also introduced by Latrémolière, which is a complete metric, up to isometric isomorphism, on the class of Leibniz quantum compact metric spaces, and which extends the topology of the Gromov-Hausdorff distance as well. Thus, as the dual propinquity is dominated by the quantum propinquity [26], we conclude that *all the convergence results in this paper are valid for the dual Gromov-Hausdorff propinquity as well.*

In this paper, all our quantum metrics will be $(2,0)$ -quasi-Leibniz quantum compact metric spaces, except for Section (7). Thus, we will simplify our notation as follows:

Convention 2.12. In this paper, Λ will be meant for $\Lambda_{2,0}$ except for Section (7).

Now that the quantum Gromov-Hausdorff propinquity is defined, we provide some results from [1]. For our work in AF algebras, it turns out that our Lip-norms are $(2,0)$ -quasi-Leibniz Lip-norms. The following Theorem (2.15) is [1, Theorem 3.5]. But, first some notation.

Notation 2.13. Let $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n \in \mathbb{N}}$ be an inductive sequence, in which \mathfrak{A}_n is a C^* -algebra and α_n is a $*$ -homomorphism for all $n \in \mathbb{N}$, with limit $\mathfrak{A} = \varinjlim \mathcal{I}$. We denote the canonical $*$ -homomorphisms $\mathfrak{A}_n \rightarrow \mathfrak{A}$ by $\underline{\alpha}_n^*$ for all $n \in \mathbb{N}$, (see [34, Chapter 6.1]).

Definition 2.14. A conditional expectation $\mathbb{E}(\cdot|\mathfrak{B}) : \mathfrak{A} \rightarrow \mathfrak{B}$ onto \mathfrak{B} , where \mathfrak{A} is a C^* -algebra and \mathfrak{B} is a C^* -subalgebra of \mathfrak{A} , is a linear positive map of norm 1 such that for all $b, c \in \mathfrak{B}$ and $a \in \mathfrak{A}$ we have:

$$\mathbb{E}(bac|\mathfrak{B}) = b\mathbb{E}(a|\mathfrak{B})c.$$

Theorem 2.15 ([1, Theorem 3.5]). *Let \mathfrak{A} be a unital AF algebra endowed with a faithful tracial state μ . Let $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n \in \mathbb{N}}$ be an inductive sequence of finite dimensional C^* -algebras with C^* -inductive limit \mathfrak{A} , with $\mathfrak{A}_0 \cong \mathbb{C}$ and where α_n is unital and injective for all $n \in \mathbb{N}$.*

Let π be the GNS representation of \mathfrak{A} constructed from μ on the space $L^2(\mathfrak{A}, \mu)$.

For all $n \in \mathbb{N}$, let:

$$\mathbb{E}\left(\cdot \Big| \underline{\alpha}_n^*(\mathfrak{A}_n)\right) : \mathfrak{A} \rightarrow \mathfrak{A}$$

be the unique conditional expectation of \mathfrak{A} onto the canonical image $\underline{\alpha}_n^(\mathfrak{A}_n)$ of \mathfrak{A}_n in \mathfrak{A} ,*

and such that $\mu \circ \mathbb{E}\left(\cdot \Big| \underline{\alpha}_n^(\mathfrak{A}_n)\right) = \mu$.*

Let $\beta : \mathbb{N} \rightarrow (0, \infty)$ have limit 0 at infinity. If, for all $a \in \mathfrak{sa}\left(\bigcup_{n \in \mathbb{N}} \underline{\alpha}_n^(\mathfrak{A}_n)\right)$, we set:*

$$L_{\mathcal{I}, \mu}^\beta(a) = \sup \left\{ \frac{\|a - \mathbb{E}\left(a \Big| \underline{\alpha}_n^*(\mathfrak{A}_n)\right)\|_{\mathfrak{A}}}{\beta(n)} : n \in \mathbb{N} \right\}$$

and $L_{\mathcal{I}, \mu}^\beta(a) = \infty$ for all $a \in \mathfrak{sa}\left(\mathfrak{A} \setminus \bigcup_{n \in \mathbb{N}} \underline{\alpha}_n^(\mathfrak{A}_n)\right)$, then $(\mathfrak{A}, L_{\mathcal{I}, \mu}^\beta)$ is a 2-quasi-Leibniz quantum compact metric space. Moreover, for all $n \in \mathbb{N}$:*

$$\Lambda\left(\left(\mathfrak{A}_n, L_{\mathcal{I}, \mu}^\beta \circ \underline{\alpha}_n^*\right), \left(\mathfrak{A}, L_{\mathcal{I}, \mu}^\beta\right)\right) \leq \beta(n)$$

and thus:

$$\lim_{n \rightarrow \infty} \Lambda\left(\left(\mathfrak{A}_n, L_{\mathcal{I}, \mu}^\beta \circ \underline{\alpha}_n^*\right), \left(\mathfrak{A}, L_{\mathcal{I}, \mu}^\beta\right)\right) = 0.$$

The fact that the defining finite-dimensional subalgebras provide approximations of the inductive limit with respect to the quantum Gromov-Hausdorff propinquity allowed us to prove that both the UHF algebras and the Effros-Shen AF algebras are continuous images of the Baire space with respect to the quantum propinquity. First, we recall the definition of the Baire space.

Definition 2.16 ([32]). The Baire space \mathcal{N} is the set $(\mathbb{N} \setminus \{0\})^{\mathbb{N}}$ endowed with the metric d defined, for any two $(x(n))_{n \in \mathbb{N}}, (y(n))_{n \in \mathbb{N}}$ in \mathcal{N} , by:

$$d((x(n))_{n \in \mathbb{N}}, (y(n))_{n \in \mathbb{N}}) = \begin{cases} 0 & \text{if } \forall n \in \mathbb{N}, x(n) = y(n), \\ 2^{-\min\{n \in \mathbb{N}: x(n) \neq y(n)\}} & \text{otherwise.} \end{cases}$$

Now, for our UHF algebra result.

Theorem 2.17 ([1, Theorem 4.9]). For any $\beta = (\beta(n))_{n \in \mathbb{N}} \in \mathcal{N}$, we define the sequence $\boxtimes \beta$ by:

$$\boxtimes \beta = n \in \mathbb{N} \mapsto \begin{cases} 1 & \text{if } n = 0, \\ \prod_{j=0}^{n-1} (\beta(j) + 1) & \text{otherwise.} \end{cases}$$

We then define, for all $\beta \in \mathcal{N}$, the unital inductive sequence:

$$\mathcal{I}(\beta) = (\mathfrak{M}(\boxtimes \beta(n)), \alpha_n)_{n \in \mathbb{N}}$$

where $\mathfrak{M}(d)$ is the algebra of $d \times d$ matrices and for all $n \in \mathbb{N}$, the unital $*$ -monomorphism α_n is the canonical map.

The map u from \mathcal{N} to the class of UHF algebras is now defined by:

$$(\beta(n))_{n \in \mathbb{N}} \in \mathcal{N} \mapsto u((\beta(n))_{n \in \mathbb{N}}) = \varinjlim \mathcal{I}(\beta).$$

Let $k \in (0, \infty)$ and $\beta \in \mathcal{N}$. Let $L_{\mathcal{I}(\beta), \mu}^{\vartheta}$ be the Lip-norm on $u(\beta)$ given by Theorem (2.15), the sequence $\vartheta : n \in \mathbb{N} \mapsto \boxtimes \beta(n)^k$ and the unique faithful trace μ on $u(\beta)$.

The $(2, 0)$ -quasi-Leibniz quantum compact metric space $(u(\beta), L_{\mathcal{I}(\beta), \mu}^{\vartheta})$ will be denoted simply by $\text{uhf}(\beta, k)$.

For all $k \in (0, \infty)$, the map:

$$\text{uhf}(\cdot, k) : (\mathcal{N}, d) \longrightarrow (\mathcal{Q}QCM\mathcal{S}_{2,0}, \Lambda)$$

is a $(2, k)$ -Hölder surjection.

Thus, we may provide explicit estimates for distances between UHF algebras, and if $k = 1$, the map $\text{uhf}(\cdot, 1)$ is 2-Lipschitz.

Moving on to the Effros-Shen AF algebras, The original classification of irrational rotation algebras, due to Pimsner and Voiculescu [36], relied on certain embeddings into the AF algebras constructed from continued fraction expansions by Effros and Shen [13]. In [23], Latrémolière proved that the irrational rotational algebras vary continuously in quantum propinquity with respect to their irrational parameter. It is natural to wonder whether the AF algebras constructed by Pimsner and Voiculescu vary continuously with respect to the quantum propinquity if parametrized by the irrational numbers at the root of their construction. We provided a positive answer to this question in [1, Theorem 5.14] stated shortly as Theorem (2.20).

We begin by recalling the construction of the AF C^* -algebras $\mathfrak{A}\mathfrak{F}_{\theta}$ constructed in [13] for any irrational θ in $(0, 1)$. For any $\theta \in (0, 1) \setminus \mathbb{Q}$, let $(a_j)_{j \in \mathbb{N}}$ be the unique

sequence in \mathbb{N} such that:

$$(2.1) \quad \theta = \lim_{n \rightarrow \infty} a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{\ddots + \frac{1}{a_n}}}}} = \lim_{n \rightarrow \infty} [a_0, a_1, \dots, a_n].$$

The sequence $(a_j)_{j \in \mathbb{N}}$ is called the continued fraction expansion of θ , and we will simply denote it by writing $\theta = [a_0, a_1, a_2, \dots] = [a_j]_{j \in \mathbb{N}}$. We note that $a_0 = 0$ (since $\theta \in (0, 1)$) and $a_n \in \mathbb{N} \setminus \{0\}$ for $n \geq 1$.

We fix $\theta \in (0, 1) \setminus \mathbb{Q}$, and let $\theta = [a_j]_{j \in \mathbb{N}}$ be its continued fraction decomposition.

We then obtain a sequence $\left(\frac{p_n^\theta}{q_n^\theta}\right)_{n \in \mathbb{N}}$ with $p_n^\theta \in \mathbb{N}$ and $q_n^\theta \in \mathbb{N} \setminus \{0\}$ by setting:

$$(2.2) \quad \begin{cases} \begin{pmatrix} p_1^\theta & q_1^\theta \\ p_0^\theta & q_0^\theta \end{pmatrix} = \begin{pmatrix} a_0 a_1 + 1 & a_1 \\ a_0 & 1 \end{pmatrix} \\ \begin{pmatrix} p_{n+1}^\theta & q_{n+1}^\theta \\ p_n^\theta & q_n^\theta \end{pmatrix} = \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} p_n^\theta & q_n^\theta \\ p_{n-1}^\theta & q_{n-1}^\theta \end{pmatrix} \end{cases} \text{ for all } n \in \mathbb{N} \setminus \{0\}.$$

We then note that $\frac{p_n^\theta}{q_n^\theta} = [a_0, a_1, \dots, a_n]$ for all $n \in \mathbb{N}$, and therefore $\left(\frac{p_n^\theta}{q_n^\theta}\right)_{n \in \mathbb{N}}$ converges to θ (see [19]).

Expression (2.2) contains the crux for the construction of the Effros-Shen AF algebras.

Notation 2.18. Throughout this paper, we shall employ the notation $x \oplus y \in X \oplus Y$ to mean that $x \in X$ and $y \in Y$ for any two vector spaces X and Y whenever no confusion may arise, as a slight yet convenient abuse of notation.

Notation 2.19. Let $\theta \in (0, 1) \setminus \mathbb{Q}$ and $\theta = [a_j]_{j \in \mathbb{N}}$ be the continued fraction expansion of θ . Let $(p_n^\theta)_{n \in \mathbb{N}}$ and $(q_n^\theta)_{n \in \mathbb{N}}$ be defined by Expression (2.2). We set $\mathfrak{A}_{\mathfrak{F}, \theta, 0} = \mathbb{C}$ and, for all $n \in \mathbb{N} \setminus \{0\}$, we set:

$$\mathfrak{A}_{\mathfrak{F}, \theta, n} = \mathfrak{M}(q_n^\theta) \oplus \mathfrak{M}(q_{n-1}^\theta),$$

and:

$$\alpha_{\theta, n} : a \oplus b \in \mathfrak{A}_{\mathfrak{F}, \theta, n} \mapsto \begin{pmatrix} a & & & \\ & \ddots & & \\ & & a & \\ & & & b \end{pmatrix} \oplus a \in \mathfrak{A}_{\mathfrak{F}, \theta, n+1},$$

where a appears a_{n+1} times on the diagonal of the right hand side matrix above. We also set α_0 to be the unique unital $*$ -morphism from \mathbb{C} to $\mathfrak{A}_{\mathfrak{F}, \theta, 1}$.

We thus define the Effros-Shen C^* -algebra $\mathfrak{A}_{\mathfrak{F}, \theta}$, after [13]:

$$\mathfrak{A}_{\mathfrak{F}, \theta} = \varinjlim (\mathfrak{A}_{\mathfrak{F}, \theta, n}, \alpha_{\theta, n})_{n \in \mathbb{N}} = \varinjlim \mathcal{I}_\theta.$$

We now present our continuity result for Effros-Shen AF Algebras from [1].

Theorem 2.20 ([1, Theorem 5.14]). *Using Notation (2.19), the function:*

$$\theta \in ((0, 1) \setminus \mathbb{Q}, |\cdot|) \longmapsto \left(\mathfrak{A}\mathfrak{F}_\theta, \mathbb{L}_{\mathcal{I}_\theta, \sigma_\theta}^{\beta_\theta} \right) \in (\mathcal{QCM}\mathcal{S}_{2,0}, \Lambda)$$

is continuous from $(0, 1) \setminus \mathbb{Q}$, with its topology as a subset of \mathbb{R} , to the class of $(2, 0)$ -quasi-Leibniz quantum compact metric spaces metrized by the quantum propinquity Λ , where σ_θ is the unique faithful tracial state, and β_θ is the sequence of the reciprocal of dimensions of the inductive sequence, \mathcal{I}_θ .

In [1], the approaches to proving Theorem (2.17) and Theorem (2.20) were somewhat different in nature. The purpose of this was to provide a more powerful continuity result in Theorem (2.17) via a Hölder map. But, if the purpose of Theorem (2.17) and Theorem (2.20) is viewed as to only provide continuity, then we will see in Section (3) that an appropriate generalization (Theorem (3.9)) can provide continuity for both Theorem (2.17) and Theorem (2.20) as a consequence. Now, this generalization will not only serve as a pleasing insight to the quantum metric structure of AF algebras with our conditional expectation Lip-norms but as a tool to provide continuity of many more classes of AF algebras as we will see in Section (6).

3. CONVERGENCE OF AF ALGEBRAS WITH QUANTUM PROPINQUITY

Taking stock of our construction of Lip-norms for unital AF algebras with faithful tracial state in Theorem (2.15), it is apparent that the construction relies on the inductive sequence, faithful tracial state, and some positive sequence converging to 0. Thus, this section provides suitable notions of convergence for all 3 of these structures, which in turn produce convergence of AF algebras in the quantum propinquity. This is motivated by our arguments of continuity in [1] for the UHF and Effros-Shen AF algebras, and therefore, provides these continuity results as a consequence of Theorem (3.9). First, some useful notation.

Notation 3.1. For all $d \in \mathbb{N}$, we denote the full matrix algebra of $d \times d$ matrices over \mathbb{C} by $\mathfrak{M}(d)$.

Let $\mathfrak{B} = \bigoplus_{j=1}^N \mathfrak{M}(n(j))$ for some $N \in \mathbb{N}$ and $n(1), \dots, n(N) \in \mathbb{N} \setminus \{0\}$. For each $k \in \{1, \dots, N\}$ and for each $j, m \in \{1, \dots, n(k)\}$, we denote the matrix $((\delta_{u,v}^{j,m}))_{u,v=1, \dots, n(k)}$ by $e_{k,j,m}$, where we used the Kronecker symbol:

$$\delta_a^b = \begin{cases} 1 & \text{if } a = b, \\ 0 & \text{otherwise,} \end{cases}$$

where $\{e_{k,j,m} \in \mathfrak{B} : k \in \{1, \dots, N\}, j, m \in \{1, \dots, n(k)\}\}$ is called the set of *matrix units* of \mathfrak{B} .

Notation 3.2. Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ denote the Alexandroff compactification of \mathbb{N} with respect to the discrete topology of \mathbb{N} . For $N \in \mathbb{N}$, let $\mathbb{N}_{\geq N} = \{k \in \mathbb{N} : k \geq N\}$, and similarly, for $\overline{\mathbb{N}}_{\geq N}$.

Proposition 3.3. *Let \mathfrak{B} be a unital C^* -algebra. Let \mathfrak{A} be a finite-dimensional unital C^* -subalgebra of \mathfrak{B} such that $\mathfrak{A} \cong \bigoplus_{j=1}^N \mathfrak{M}(n(j))$ for some $N \in \mathbb{N}$ and $n(1), \dots, n(N) \in$*

$\mathbb{N} \setminus \{0\}$ with $*$ -isomorphism $\alpha : \bigoplus_{j=1}^N \mathfrak{M}(n(j)) \rightarrow \mathfrak{A}$. Let E be the set of matrix units for $\bigoplus_{j=1}^N \mathfrak{M}(n(j))$ of Notation (3.1).

If $\{\tau^n : \mathfrak{B} \rightarrow \mathbb{C}\}_{n \in \overline{\mathbb{N}}}$ is a family of faithful tracial states, then for all $n \in \overline{\mathbb{N}}, b \in \mathfrak{B}$:

$$\mathbb{E}^n(b) = \sum_{e \in E} \frac{\tau^n(\alpha(e^*)b)}{\tau^n(\alpha(e^*e))} \alpha(e),$$

where $\mathbb{E}^n : \mathfrak{B} \rightarrow \mathfrak{A}$ is the unique τ^n -preserving conditional expectation.

Furthermore, if $(\tau^n)_{n \in \mathbb{N}}$ converges to τ^∞ in the weak- $*$ topology on $\mathcal{S}(\mathfrak{B})$, then the map:

$$(n, b) \in \overline{\mathbb{N}} \times \mathfrak{B} \mapsto \|b - \mathbb{E}^n(b)\|_{\mathfrak{B}} \in \mathbb{R},$$

is continuous with respect to the product topology on $\overline{\mathbb{N}} \times (\mathfrak{B}, \|\cdot\|_{\mathfrak{B}})$.

Proof. For $n \in \overline{\mathbb{N}}$, by [1, Section 4.1] and [1, Expression 4.1], we have that for each $n \in \overline{\mathbb{N}}$:

$$\mathbb{E}^n(b) = \sum_{e \in E} \frac{\tau^n(\alpha(e)^*b)}{\tau^n(\alpha(e^*e))} \alpha(e)$$

since τ^n is a faithful tracial state on \mathfrak{B} . By faithfulness, for $e \in e_{\mathfrak{A}}$ and $b \in \mathfrak{B}$:

$$\lim_{n \rightarrow \infty} \tau^n(\alpha(e^*e)) = \tau^\infty(\alpha(e^*e)) > 0$$

by weak- $*$ convergence. Since our sum is finite by finite dimensionality, again by weak- $*$ convergence:

$$(3.1) \quad \lim_{n \rightarrow \infty} \sum_{e \in E} \frac{\tau^n(\alpha(e)^*b)}{\tau^n(\alpha(e^*e))} = \sum_{e \in E} \frac{\tau^\infty(\alpha(e)^*b)}{\tau^\infty(\alpha(e^*e))}$$

Furthermore, if we let $C = \max_{e \in E} \{\|\alpha(e)\|_{\mathfrak{B}}\}$, then:

$$\begin{aligned} \|\mathbb{E}^n(b) - \mathbb{E}(b)\|_{\mathfrak{B}} &= \left\| \sum_{e \in E} \frac{\tau^n(\alpha(e)^*b)}{\tau^n(\alpha(e^*e))} \alpha(e) - \sum_{e \in E} \frac{\tau^\infty(\alpha(e)^*b)}{\tau^\infty(\alpha(e^*e))} \alpha(e) \right\|_{\mathfrak{B}} \\ &\leq \left(\sum_{e \in E} \left| \frac{\tau^n(\alpha(e)^*b)}{\tau^n(\alpha(e^*e))} - \frac{\tau^\infty(\alpha(e)^*b)}{\tau^\infty(\alpha(e^*e))} \right| \right) C, \end{aligned}$$

and $\lim_{n \rightarrow \infty} \|\mathbb{E}^n(b) - \mathbb{E}(b)\|_{\mathfrak{B}} = 0$ by Expression (3.1).

Fix $n, m \in \overline{\mathbb{N}}$ and $b, b' \in \mathfrak{B}$. Then, as conditional expectations are contractive:

$$\begin{aligned} \left| \|b - \mathbb{E}^n(b)\|_{\mathfrak{B}} - \|b' - \mathbb{E}^m(b')\|_{\mathfrak{B}} \right| &\leq \|(b - \mathbb{E}^n(b)) - (b' - \mathbb{E}^m(b'))\|_{\mathfrak{B}} \\ &\leq \|\mathbb{E}^n(b) - \mathbb{E}^n(b') + \mathbb{E}^n(b') - \mathbb{E}^m(b')\|_{\mathfrak{B}} \\ &\quad + \|b - b'\|_{\mathfrak{B}} \\ &\leq 2\|b - b'\|_{\mathfrak{B}} + \|\mathbb{E}^n(b') - \mathbb{E}^m(b')\|_{\mathfrak{B}}, \end{aligned}$$

and continuity follows. \square

We now introduce a notion of convergence of inductive sequences in Definition (3.4).

Definition 3.4. We consider 2 cases of inductive sequences in this definition.

Case 1. Closure of union

For each $k \in \overline{\mathbb{N}}$, let \mathfrak{A}^k be a C^* -algebra with $\mathfrak{A}^k = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_{k,n}}^{\|\cdot\|_{\mathfrak{A}^k}}$ such that $\mathcal{U}^k = (\mathfrak{A}_{k,n})_{n \in \mathbb{N}}$ is a non-decreasing sequence of C^* -subalgebras of \mathfrak{A}^k , then we say $\{\mathfrak{A}^k : k \in \overline{\mathbb{N}}\}$ is a *fusing family* if:

- (1) There exists $(c_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}$ non-decreasing such that $\lim_{n \rightarrow \infty} c_n = \infty$, and
- (2) for all $N \in \mathbb{N}$, if $k \in \mathbb{N}_{\geq c_N}$, then $\mathfrak{A}_{k,n} = \mathfrak{A}_{\infty,n}$ for all $n \in \{0, 1, \dots, N\}$.

Case 2. Inductive limit

For each $k \in \overline{\mathbb{N}}$, let $\mathcal{I}(k) = (\mathfrak{A}_{k,n}, \alpha_{k,n})_{n \in \mathbb{N}}$ be an inductive sequence with inductive limit, \mathfrak{A}^k . We say that the family of C^* -algebras $\{\mathfrak{A}^k : k \in \overline{\mathbb{N}}\}$ is an *IL-fusing family* of C^* -algebras if:

- (1) There exists $(c_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}$ non-decreasing such that $\lim_{n \rightarrow \infty} c_n = \infty$, and
- (2) for all $N \in \mathbb{N}$, if $k \in \mathbb{N}_{\geq c_N}$, then $(\mathfrak{A}_{k,n}, \alpha_{k,n}) = (\mathfrak{A}_{\infty,n}, \alpha_{\infty,n})$ for all $n \in \{0, 1, \dots, N\}$.

In either case, we call the sequence $(c_n)_{n \in \mathbb{N}}$ the *fusing sequence*.

Remark 3.5. The results in this section are phrased in terms of *IL-fusing families* since our propinquity convergence results are all in terms of inductive limits. But, we note that all the results of this section are valid for the closure of union case as well with appropriate translations. We will see the closure of union case appear when working with ideals in Sections (4 - 6). Also, note that any IL-fusing family may be viewed as a fusing family via the canonical $*$ -homomorphisms of Notation (2.13), which is why we don't decorate the term *fusing family* in the closure of union case.

Hypotheses (2) and (3) in the following Lemma (3.6) introduce the remaining notions of convergence that together with fusing families will imply convergence of quantum propinquity of AF algebras in Theorem (3.9). Indeed, (2) is simply an appropriate use of weak- $*$ convergence for the faithful tracial states in relation to fusing families, and (3) is an appropriate use of pointwise convergence of the sequences that provide convergence of the finite dimensional subspaces in Theorem (2.15).

Furthermore, Lemma (3.6) provides that the Lip-norms induced on the finite dimensional subspaces form a continuous field of Lip-norms, a notion introduced by Rieffel in [45].

Lemma 3.6. *For each $k \in \overline{\mathbb{N}}$, let $\mathcal{I}(k) = (\mathfrak{A}_{k,n}, \alpha_{k,n})_{n \in \mathbb{N}}$ be an inductive sequence of finite dimensional C^* -algebras with C^* -inductive limit \mathfrak{A}^k , such that $\mathfrak{A}_{k,0} = \mathfrak{A}_{k',0} \cong \mathbb{C}$ and $\alpha_{k,n}$ is unital and injective for all $k, k' \in \overline{\mathbb{N}}, n \in \mathbb{N}$. If:*

- (1) $\{\mathfrak{A}^k : k \in \overline{\mathbb{N}}\}$ is an *IL-fusing family* with fusing sequence $(c_n)_{n \in \mathbb{N}}$,
- (2) $\{\tau^k : \mathfrak{A}^k \rightarrow \mathbb{C}\}_{k \in \overline{\mathbb{N}}}$ is a family of faithful tracial states such that for each $N \in \mathbb{N}$,

we have that $\left(\tau^k \circ \alpha_{k,n}^N \right)_{k \in \mathbb{N}_{\geq c_N}}$ converges to $\tau^\infty \circ \alpha_\infty^N$ in the weak- $*$ topology on

$\mathcal{S}(\mathfrak{A}_{\infty,N})$, and

- (3) $\{\beta^k : \overline{\mathbb{N}} \rightarrow (0, \infty)\}_{k \in \overline{\mathbb{N}}}$ is a family of convergent sequences such that for all $N \in \mathbb{N}$ if $k \in \mathbb{N}_{\geq c_N}$, then $\beta^k(n) = \beta^\infty(n)$ for all $n \in \{0, 1, \dots, N\}$ and there exists $B : \overline{\mathbb{N}} \rightarrow (0, \infty)$ with $B(\infty) = 0$ and $\beta^m(l) \leq B(l)$ for all $m, l \in \overline{\mathbb{N}}$,

then for all $N \in \mathbb{N}$, if $n \in \{0, 1, \dots, N\}$, then the map:

$$l_n^N : (k, a) \in \overline{\mathbb{N}}_{\geq c_N} \times \mathfrak{A}_{\infty, n} \longmapsto \mathbb{L}_{\mathcal{I}(k), \tau^k}^{\beta^k} \circ \underline{\alpha}_k^n(a) \in \mathbb{R}$$

is well-defined and continuous with respect to the product topology on $\overline{\mathbb{N}} \times (\mathfrak{A}_{\infty, n}, \|\cdot\|_{\mathfrak{A}_{\infty, n}})$, where $\mathbb{L}_{\mathcal{I}(k), \tau^k}^{\beta^k}$ is given by Theorem (2.15).

Proof. First, we establish a weak-* convergence result implied by (2). Let $N \in \overline{\mathbb{N}}$.

Claim 3.7. $\left(\tau^k \circ \underline{\alpha}_k^m \right)_{k \in \mathbb{N}_{\geq c_N}}$ converges to $\tau^\infty \circ \underline{\alpha}_\infty^m$ in the weak* topology on $\mathcal{S}(\mathfrak{A}_{\infty, m})$ for each $m \in \{0, 1, \dots, N\}$.

Let $m \in \{0, 1, \dots, N\}$. The case $m = N$ is given by assumption. So, assume that $N \geq 1$ and $m \in \{0, \dots, N-1\}$. Fix $a \in \mathfrak{A}_{\infty, m}$, we have by definition of inductive limit and IL-fusing family:

$$\tau^k \circ \underline{\alpha}_k^m(a) = \tau^k \circ \underline{\alpha}_k^N(\alpha_{k, N-1} \circ \dots \circ \alpha_{k, m}(a)) = \tau^k \circ \underline{\alpha}_k^N(\alpha_{\infty, N-1} \circ \dots \circ \alpha_{\infty, m}(a))$$

for $k \in \overline{\mathbb{N}}_{\geq c_N}$, which proves our claim since $\left(\tau^k \circ \underline{\alpha}_k^N \right)_{k \in \mathbb{N}_{\geq c_N}}$ converges to $\tau^\infty \circ \underline{\alpha}_\infty^N$ in the weak* topology on $\mathcal{S}(\mathfrak{A}_{\infty, N})$.

Next, we establish a more explicit form of our Lip-norms on the finite-dimensional subspaces. Fix $N \in \mathbb{N}$ and $n \in \{0, 1, \dots, N\}$. l_n^N is well-defined by definition of a IL-fusing family. Furthermore, as $\mathbb{E} \left(\cdot \left| \underline{\alpha}_k^j(\mathfrak{A}_{k, j}) \right. \right)$ is a conditional expectation for all $k \in \overline{\mathbb{N}}, j \in \mathbb{N}$, we have that:

$$\mathbb{E} \left(\underline{\alpha}_k^n(a) \left| \underline{\alpha}_k^j(\mathfrak{A}_{k, j}) \right. \right) = \underline{\alpha}_k^n(a)$$

for $j \geq n, a \in \mathfrak{A}_{\infty, n}$.

Therefore:

$$(3.2) \quad l_n^N(k, a) = \max \left\{ \frac{\left\| \underline{\alpha}_k^n(a) - \mathbb{E} \left(\underline{\alpha}_k^n(a) \left| \underline{\alpha}_k^m(\mathfrak{A}_{\infty, m}) \right. \right) \right\|_{\mathfrak{A}^k}}{\beta^\infty(m)} : m \in \{0, \dots, n-1\} \right\},$$

which will allow use to apply Proposition (3.3).

Fix $m \in \{0, \dots, n-1\}, k \geq N, a \in \mathfrak{A}_{\infty, n}$. Since $\mathfrak{A}_{\infty, m}$ is finite dimensional, the C*-algebra $\mathfrak{A}_{\infty, m} \cong \bigoplus_{j=1}^N \mathfrak{M}(n(j))$ for some $N \in \mathbb{N}$ and $n(1), \dots, n(N) \in \mathbb{N} \setminus \{0\}$ with *-isomorphism $\gamma : \bigoplus_{j=1}^N \mathfrak{M}(n(j)) \rightarrow \mathfrak{A}_{\infty, m}$. Let E be the set of matrix units for $\bigoplus_{j=1}^N \mathfrak{M}(n(j))$. Now, define $\alpha_{k, m \rightarrow n-1} = \alpha_{k, n-1} \circ \dots \circ \alpha_{k, m}$, and by definition of IL-fusing family, we have that $\alpha_{k, m \rightarrow n-1} = \alpha_{\infty, m \rightarrow n-1}$. Therefore, by the definition

of inductive limit and Proposition (3.3):

$$\begin{aligned}
(3.3) \quad & \left\| \alpha_{\xrightarrow{k}}^n(a) - \mathbb{E} \left(\alpha_{\xrightarrow{k}}^n(a) \middle| \alpha_{\xrightarrow{k}}^m(\mathfrak{A}_{\infty, m}) \right) \right\|_{\mathfrak{A}^k} \\
&= \left\| \alpha_{\xrightarrow{k}}^n(a) - \sum_{e \in E} \frac{\tau^k(\alpha_{\xrightarrow{k}}^m \circ \gamma(e^*) \alpha_{\xrightarrow{k}}^n(a))}{\tau^k(\alpha_{\xrightarrow{k}}^m \circ \gamma(e^*e))} \alpha_{\xrightarrow{k}}^m \circ \gamma(e) \right\|_{\mathfrak{A}^k} \\
&= \left\| \alpha_{\xrightarrow{k}}^n(a) - \sum_{e \in E} \frac{\tau^k(\alpha_{\xrightarrow{k}}^n(\alpha_{k, m \rightarrow n-1}(\gamma(e^*))a))}{\tau^k(\alpha_{\xrightarrow{k}}^n(\alpha_{k, m \rightarrow n-1}(\gamma(e^*e))))} \alpha_{\xrightarrow{k}}^n(\alpha_{k, m \rightarrow n-1}(\gamma(e))) \right\|_{\mathfrak{A}^k} \\
&= \left\| \alpha_{\xrightarrow{k}}^n \left(a - \sum_{e \in E} \frac{\tau^k(\alpha_{\xrightarrow{k}}^n(\alpha_{\infty, m \rightarrow n-1}(\gamma(e^*))a))}{\tau^k(\alpha_{\xrightarrow{k}}^n(\alpha_{\infty, m \rightarrow n-1}(\gamma(e^*e))))} \alpha_{\infty, m \rightarrow n-1}(\gamma(e)) \right) \right\|_{\mathfrak{A}^k} \\
&= \left\| a - \sum_{e \in E} \frac{\tau^k \circ \alpha_{\xrightarrow{k}}^n(\alpha_{\infty, m \rightarrow n-1} \circ \gamma(e^*)a)}{\tau^k \circ \alpha_{\xrightarrow{k}}^n(\alpha_{\infty, m \rightarrow n-1} \circ \gamma(e^*e))} \alpha_{\infty, m \rightarrow n-1} \circ \gamma(e) \right\|_{\mathfrak{A}_{\infty, n}}
\end{aligned}$$

Hence, by Claim (3.7) and Proposition (3.3), the map:

$$(k, a) \in \overline{\mathbb{N}}_{\geq c_N} \times \mathfrak{A}_{\infty, n} \mapsto \frac{\left\| \alpha_{\xrightarrow{k}}^n(a) - \mathbb{E} \left(\alpha_{\xrightarrow{k}}^n(a) \middle| \alpha_{\xrightarrow{k}}^m(\mathfrak{A}_{\infty, m}) \right) \right\|_{\mathfrak{A}^k}}{\beta^\infty(m)} \in \mathbb{R}$$

is continuous for each $m \in \{0, \dots, n-1\}$. As the maximum of finitely many continuous real-valued functions is continuous, our lemma is proven by Expression (3.2). \square

The proof of the following Theorem (3.8) follows the same process as the proof of [1, Lemma 5.13], but we include the proof for convenience and clarity. This Theorem (3.8) establishes convergence of the finite dimensional subspaces.

Theorem 3.8. *For each $k \in \overline{\mathbb{N}}$, let $\mathcal{I}(k) = (\mathfrak{A}_{k, n}, \alpha_{k, n})_{n \in \mathbb{N}}$ be an inductive sequence of finite dimensional C^* -algebras with C^* -inductive limit \mathfrak{A}^k , such that $\mathfrak{A}_{k, 0} = \mathfrak{A}_{k', 0} \cong \mathbb{C}$ and $\alpha_{k, n}$ is unital and injective for all $k, k' \in \overline{\mathbb{N}}, n \in \mathbb{N}$. If:*

- (1) $\{\mathfrak{A}^k : k \in \overline{\mathbb{N}}\}$ is an IL-fusing family with fusing sequence $(c_n)_{n \in \mathbb{N}}$,
- (2) $\{\tau^k : \mathfrak{A}^k \rightarrow \mathbb{C}\}_{k \in \overline{\mathbb{N}}}$ is a family of faithful tracial states such that for each $N \in \mathbb{N}$, we have that $\left(\tau^k \circ \alpha_{\xrightarrow{k}}^N \right)_{k \in \mathbb{N}_{\geq c_N}}$ converges to $\tau^\infty \circ \alpha_{\xrightarrow{\infty}}^N$ in the weak- $*$ topology on $\mathcal{S}(\mathfrak{A}_{\infty, N})$, and
- (3) $\{\beta^k : \overline{\mathbb{N}} \rightarrow (0, \infty)\}_{k \in \overline{\mathbb{N}}}$ is a family of convergent sequences such that for all $N \in \mathbb{N}$ if $k \in \mathbb{N}_{\geq c_N}$, then $\beta^k(n) = \beta^\infty(n)$ for all $n \in \{0, 1, \dots, N\}$ and there exists $B : \overline{\mathbb{N}} \rightarrow (0, \infty)$ with $B(\infty) = 0$ and $\beta^m(l) \leq B(l)$ for all $m, l \in \overline{\mathbb{N}}$,

then for every $N \in \mathbb{N}$ and $n \in \{0, \dots, N\}$, the map:

$$F_n^N : k \in \overline{\mathbb{N}}_{\geq c_N} \mapsto \left(\mathfrak{A}_{k, n}, \mathbb{L}_{\mathcal{I}(k), \tau^k}^{\beta^k} \circ \alpha_{\xrightarrow{k}}^n \right) \in (\mathcal{QQCMS}_{2,0}, \Lambda)$$

is well-defined and continuous, and therefore:

$$\lim_{k \rightarrow \infty} \Lambda \left(\left(\mathfrak{A}_{k,n}, \mathbb{L}_{\mathcal{I}(k), \tau^k}^{\beta^k} \circ \alpha_k^n \right), \left(\mathfrak{A}_{\infty,n}, \mathbb{L}_{\mathcal{I}(\infty), \tau^\infty}^{\beta^\infty} \circ \alpha_\infty^n \right) \right) = 0,$$

where $\mathbb{L}_{\mathcal{I}(k), \tau^k}^{\beta^k}$ is given by Theorem (2.15).

Proof. Fix $N \in \mathbb{N}$ and $n \in \{0, \dots, N\}$. If $n = 0$, then $\mathfrak{A}_{k,0} = \mathfrak{A}_{\infty,0} \cong \mathbb{C}$ and since Lip-norms vanish only on scalars by Definition (2.3), the map F_0^N is constant up to quantum isometry and therefore continuous.

Assume that $n \in \{1, \dots, N\}$ and $k \geq c_N$. Set $\mathfrak{B}_n = \mathfrak{A}_{k,n} = \mathfrak{A}_{\infty,n}$ by definition of IL-fusing family. Let \mathfrak{W} be any complementary subspace of $\mathbb{R}1_{\mathfrak{A}}$ in $\mathfrak{sa}(\mathfrak{B}_n)$ — which exists since $\mathfrak{sa}(\mathfrak{B}_n)$ is finite dimensional. We shall denote by \mathfrak{S} the unit sphere $\{a \in \mathfrak{W} : \|a\|_{\mathfrak{B}_n} = 1\}$ in \mathfrak{W} . Note that since \mathfrak{W} is finite dimensional, \mathfrak{S} is a compact set. Set $S = \overline{\mathbb{N}}_{\geq c_N} \times \mathfrak{S}$, which is a compact set in the product topology. Therefore, since the function l_n^N is continuous by Lemma (3.6), it reaches a minimum on S . Thus, there exists $(K, c) \in S$ such that: $\min_S l_n^N = l_n^N(K, c)$. In particular, since Lip-norms are zero only on the scalars, we have $l_n^N(K, c) > 0$ as $\|c\|_{\mathfrak{W}} = 1$ yet the only scalar multiple of $1_{\mathfrak{B}_n}$ in \mathfrak{W} is 0. We denote $m_S = l_n^N(K, a) > 0$ in the rest of this proof.

Moreover, the function l_n^N is continuous on the compact set S , and thus, it is uniformly continuous with respect to any metric that metrizes the product topology. In particular, consider the max metric, denoted by m , with respect to the norm on \mathfrak{S} and the metric on $\overline{\mathbb{N}}$ defined by $d_A(n, m) = |1/(n+1) - 1/(m+1)|$ for all $n, m \in \overline{\mathbb{N}}$ with the convention that $1/(\infty+1) = 0$, in which the metric d_A metrizes the topology on $\overline{\mathbb{N}}$.

Let $\varepsilon > 0$. As l_n^N is uniformly continuous on the metric space (S, m) , there exists $\delta > 0$ such that if $m(s, s') < \delta$, then $|l_n^N(s) - l_n^N(s')| \leq m_S^2 \varepsilon$. Now, there exists $M \in \overline{\mathbb{N}}_{\geq c_N}$ such that $1/M < \delta$. Let $m \geq M$ and $a \in \mathfrak{S}$, then by definition of m and d_A :

$$m((m, a), (\infty, a)) = 1/(m+1) < 1/m \leq 1/M < \delta.$$

Thus, for all $m \geq M$ and for all $a \in \mathfrak{S}$ we have:

$$|l_n^N(m, a) - l_n^N(\infty, a)| \leq m_S^2 \varepsilon.$$

We then have, for all $a \in \mathfrak{S}$ and $m \geq M$, since l_n^N is positive on S :

$$(3.4) \quad \left\| a - \frac{l_n^N(m, a)}{l_n^N(\infty, a)} a \right\|_{\mathfrak{B}_n} = \frac{|l_n^N(m, a) - l_n^N(\infty, a)|}{l_n^N(\infty, a)} \|a\|_{\mathfrak{B}_n} \leq \frac{\varepsilon m_S^2}{m_S} \leq m_S \varepsilon.$$

Similarly:

$$(3.5) \quad \left\| a - \frac{l_n^N(\infty, a)}{l_n^N(m, a)} a \right\|_{\mathfrak{B}_n} \leq m_S \varepsilon.$$

We are now ready to provide an estimate for the quantum propinquity. Let $m \geq M$ be fixed. Writing id for the identity of \mathfrak{B}_n , the quadruple:

$$\gamma = (\mathfrak{B}_n, 1_{\mathfrak{B}_n}, \text{id}, \text{id})$$

is a bridge in the sense of Definition (2.6) from $\left(\mathfrak{B}_n, L_{\mathcal{I}(m), \tau^m}^{\beta^m} \circ \alpha_{\frac{m}{\tau}}^n\right)$ to $\left(\mathfrak{B}_n, L_{\mathcal{I}(\infty), \tau^\infty}^{\beta^\infty} \circ \alpha_{\frac{\infty}{\tau}}^n\right)$.

As the pivot of γ is the unit, the height of γ is null. We are left to compute the reach of γ .

Let $a \in \mathfrak{sa}(\mathfrak{B}_n)$. We proceed with three case.

Case 1. Assume that $a \in \mathbb{R}1_{\mathfrak{B}_n}$.

We then have that $l_n^N(\infty, a) = l_n^N(m, a) = 0$, and that $\|a - a\|_{\mathfrak{B}_n} = 0$.

Case 2. Assume that $a \in \mathfrak{S}$.

We note again that $l_n^N(\infty, a) \geq m_S > 0$. Thus, we may define $a' = \frac{l_n^N(\infty, a)}{l_n^N(m, a)} a$. By Inequality (3.5), we have:

$$\|a - a'\|_{\mathfrak{B}_n} = \left\| a - \frac{l_n^N(\infty, a)}{l_n^N(m, a)} a \right\|_{\mathfrak{B}_n} \leq \varepsilon m_S \leq \varepsilon l_n^N(\infty, a),$$

while $l_n^N(m, a') = l_n^N\left(m, \frac{l_n^N(\infty, a)}{l_n^N(m, a)} a\right) = l_n^N(\infty, a)$.

Case 3. Assume that $a \in \mathfrak{sa}(\mathfrak{B}_n)$.

By definition of \mathfrak{S} there exists $r, t \in \mathbb{R}$ such that $a = rb + t1_{\mathfrak{B}_n}$ with $b \in \mathfrak{S}$. We may assume $r \neq 0$ since this case would reduce to Case 1. If $r < 0$, then $-r > 0$, $-b \in \mathfrak{S}$ and $a = -r(-b) + t1_{\mathfrak{B}_n}$. Hence, we may assume that $r > 0$.

Since Lip-norms vanish on scalars, note that $l_n^N(\infty, a) = l_n^N(\infty, rb)$. Let $b' \in \mathfrak{sa}(\mathfrak{B}_n)$ be constructed from $b \in \mathfrak{S}$ as in Case 2. Now, consider $a' = rb' + t1_{\mathfrak{B}_n}$. Thus, by Case 2 and $r > 0$:

$$\begin{aligned} \|a - a'\|_{\mathfrak{B}_n} &= \|rb + t1_{\mathfrak{B}_n} - (rb' + t1_{\mathfrak{B}_n})\|_{\mathfrak{B}_n} \\ &= r \|b - b'\|_{\mathfrak{B}_n} \\ &\leq r \varepsilon l_n^N(\infty, b) \\ &= \varepsilon l_n^N(\infty, rb) = \varepsilon l_n^N(\infty, a), \end{aligned}$$

while $l_n^N(m, a') = l_n^N(m, rb') = r l_n^N(m, b') \leq r l_n^N(\infty, b) = l_n^N(\infty, rb) = l_n^N(\infty, a)$ by Case 2, $r > 0$, and since Lip-norms vanish on scalars.

Thus, from Case 3, we conclude that:

$$(3.6) \quad \forall a \in \mathfrak{sa}(\mathfrak{B}_n), \exists a' \in \mathfrak{sa}(\mathfrak{B}_n) \text{ with } \|a - a'\|_{\mathfrak{B}_n} \leq \varepsilon l_n^N(\infty, a), \quad l_n^N(m, a') \leq l_n^N(\infty, a).$$

By symmetry in the roles of ∞ and m and Inequality (3.4), we can conclude as well that:

$$(3.7) \quad \forall a \in \mathfrak{sa}(\mathfrak{B}_n), \exists a' \in \mathfrak{sa}(\mathfrak{B}_n) \text{ with } \|a - a'\|_{\mathfrak{B}_n} \leq \varepsilon l_n^N(m, a), \quad l_n^N(\infty, a') \leq l_n^N(m, a).$$

Now, Expressions (3.6) and (3.7) together imply that the reach, and hence the length of the bridge γ is no more than ε .

Thus, by definition of length and Theorem-Definition (2.11), we gather:

$$\Lambda \left((\mathfrak{B}_n, l_n^N(\infty, \cdot)), (\mathfrak{B}_n, l_n^N(m, \cdot)) \right) \leq \varepsilon$$

for all $m \geq M$, which concludes our proof. \square

Next, we are now in a position to provide criteria for convergence of AF algebras.

Theorem 3.9. *For each $k \in \overline{\mathbb{N}}$, let $\mathcal{I}(k) = (\mathfrak{A}_{k,n}, \alpha_{k,n})_{n \in \mathbb{N}}$ be an inductive sequence of finite dimensional C^* -algebras with C^* -inductive limit \mathfrak{A}^k , such that $\mathfrak{A}_{k,0} = \mathfrak{A}_{k',0} \cong \mathbb{C}$ and $\alpha_{k,n}$ is unital and injective for all $k, k' \in \overline{\mathbb{N}}, n \in \mathbb{N}$. If:*

- (1) $\{\mathfrak{A}^k : k \in \overline{\mathbb{N}}\}$ is an IL-fusing family with fusing sequence $(c_n)_{n \in \mathbb{N}}$,
- (2) $\{\tau^k : \mathfrak{A}^k \rightarrow \mathbb{C}\}_{k \in \overline{\mathbb{N}}}$ is a family of faithful tracial states such that for each $N \in \mathbb{N}$,

we have that $\left(\tau^k \circ \alpha_{k, \cdot}^N \right)_{k \in \mathbb{N}_{\geq c_N}}$ converges to $\tau^\infty \circ \alpha_\infty^N$ in the weak- $*$ topology on

$\mathcal{S}(\mathfrak{A}_{\infty, N})$, and

- (3) $\{\beta^k : \overline{\mathbb{N}} \rightarrow (0, \infty)\}_{k \in \overline{\mathbb{N}}}$ is a family of convergent sequences such that for all $N \in \mathbb{N}$ if $k \in \mathbb{N}_{\geq c_N}$, then $\beta^k(n) = \beta^\infty(n)$ for all $n \in \{0, 1, \dots, N\}$ and there exists $B : \overline{\mathbb{N}} \rightarrow (0, \infty)$ with $B(\infty) = 0$ and $\beta^m(l) \leq B(l)$ for all $m, l \in \overline{\mathbb{N}}$

then, for each $N \in \mathbb{N}$, we have for all $k \geq c_N$:

$$(3.8) \quad \Lambda \left(\left(\mathfrak{A}^k, \mathbb{L}_{\mathcal{I}(k), \tau^k}^{\beta^k} \right), \left(\mathfrak{A}^\infty, \mathbb{L}_{\mathcal{I}(\infty), \tau^\infty}^{\beta^\infty} \right) \right) \leq \frac{2}{B(N)} + \Lambda \left(F_N^N(k), F_N^N(\infty) \right),$$

where $\mathbb{L}_{\mathcal{I}(k), \tau^k}^{\beta^k}$ is given by Theorem (2.15) and $F_N^N(k)$ is given by Theorem (3.8).

Furthermore:

$$\lim_{k \rightarrow \infty} \Lambda \left(\left(\mathfrak{A}^k, \mathbb{L}_{\mathcal{I}(k), \tau^k}^{\beta^k} \right), \left(\mathfrak{A}^\infty, \mathbb{L}_{\mathcal{I}(\infty), \tau^\infty}^{\beta^\infty} \right) \right) = 0.$$

Proof. Fix $N \in \mathbb{N}$. Then, for all $k \in \overline{\mathbb{N}}$:

$$\Lambda \left(\left(\mathfrak{A}^k, \mathbb{L}_{\mathcal{I}(k), \tau^k}^{\beta^k} \right), \left(\mathfrak{A}_{k,N}, \mathbb{L}_{\mathcal{I}(k), \tau^k}^{\beta^k} \circ \alpha_{k, \cdot}^N \right) \right) \leq \frac{1}{\beta^k(N)} \leq \frac{1}{B(N)}$$

by assumption and Theorem (2.15). And, by the triangle inequality:

$$\begin{aligned} & \Lambda \left(\left(\mathfrak{A}^k, \mathbb{L}_{\mathcal{I}(k), \tau^k}^{\beta^k} \right), \left(\mathfrak{A}^\infty, \mathbb{L}_{\mathcal{I}(\infty), \tau^\infty}^{\beta^\infty} \right) \right) \\ & \leq \frac{2}{B(N)} + \Lambda \left(\left(\mathfrak{A}_{k,N}, \mathbb{L}_{\mathcal{I}(k), \tau^k}^{\beta^k} \circ \alpha_{k, \cdot}^N \right), \left(\mathfrak{A}_{\infty, N}, \mathbb{L}_{\mathcal{I}(\infty), \tau^\infty}^{\beta^\infty} \circ \alpha_{\infty, \cdot}^N \right) \right) \end{aligned}$$

Now, assume $k \geq c_N$. Then, we have:

$$\Lambda \left(\left(\mathfrak{A}^k, \mathbb{L}_{\mathcal{I}(k), \tau^k}^{\beta^k} \right), \left(\mathfrak{A}^\infty, \mathbb{L}_{\mathcal{I}(\infty), \tau^\infty}^{\beta^\infty} \right) \right) \leq \frac{2}{B(N)} + \Lambda \left(F_N^N(k), F_N^N(\infty) \right),$$

and:

$$\limsup_{\substack{k \rightarrow \infty \\ k \in \mathbb{N}_{\geq c_N}}} \Lambda \left(\left(\mathfrak{A}^k, \mathbb{L}_{\mathcal{I}(k), \tau^k}^{\beta^k} \right), \left(\mathfrak{A}^\infty, \mathbb{L}_{\mathcal{I}(\infty), \tau^\infty}^{\beta^\infty} \right) \right) \leq \frac{2}{B(N)},$$

since F_N^N is continuous by Theorem (3.8). And, thus:

$$(3.9) \quad \limsup_{k \rightarrow \infty} \Lambda \left(\left(\mathfrak{A}^k, \mathbb{L}_{\mathcal{I}(k), \tau^k}^{\beta^k} \right), \left(\mathfrak{A}^\infty, \mathbb{L}_{\mathcal{I}(\infty), \tau^\infty}^{\beta^\infty} \right) \right) \leq \frac{2}{B(N)}.$$

Hence, as the left hand side of Inequality (3.9) does not depend on N , we gather:

$$\limsup_{k \rightarrow \infty} \Lambda \left(\left(\mathfrak{A}^k, \mathbb{L}_{\mathcal{I}(k), \tau^k}^{\beta^k} \right), \left(\mathfrak{A}^\infty, \mathbb{L}_{\mathcal{I}(\infty), \tau^\infty}^{\beta^\infty} \right) \right) \leq \lim_{N \rightarrow \infty} \frac{2}{B(N)} = 0,$$

which concludes the proof. \square

Theorem (3.9) provides a satisfying insight to the quantum metric structure of the Lip-norms of Theorem (2.15). Indeed, hypotheses (1), (2), and (3) of Theorem (3.9) are simply appropriate notions of convergence relying on the criteria used to construct the Lip-norms of Theorem (2.15) and nothing more.

Another powerful and immediate consequence of Theorem (3.9) is that, in the Effros-Shen AF algebra case, since the proof of Theorem (2.20) in [1] uses sequential continuity and convergence of irrationals in the Baire Space \mathcal{N} (Definition (2.16)), it is not difficult to see how one may use this Theorem (3.9) to achieve the same result. For the, UHF case (Theorem (2.17)), one could also apply Theorem (3.9) to achieve continuity, but although, Theorem (3.9) does not directly provide the fact that the map in Theorem (2.17) is Hölder, one may use Inequality (3.8) in the statement of Theorem (3.9), to deduce such a result.

4. METRIC ON IDEAL SPACE OF C*-INDUCTIVE LIMITS

For a fixed C*-algebra, the ideal space may be endowed with various natural topologies. We may identify each ideal with a quotient, which is a C*-algebra itself. Now, this defines a function from the ideal space, which has natural topologies, to the class of C*-algebras. But, if each quotient has a quasi-Leibniz Lip-norm, then this function becomes much more intriguing as we may now discuss its continuity or lack thereof since we now have topology on the codomain provided by quantum propinquity. Towards this, we develop a metric topology on ideals of any C*-inductive limit. The purpose of this is to allow fusing families of ideals to provide fusing families of quotients in Proposition (4.15) — a first step in providing convergence of quotients in quantum propinquity. But, our metric is greatly motivated by the Fell topology on the ideal space and is stronger than the Fell topology. Hence, we define the Fell topology on ideals and prove some basic results. But, to discuss the Fell topology, we must first introduce the Jacobson topology. As the definition is quite involved, we do not provide a complete definition of the Jacobson topology, but we provide a reference and a characterization of the closed sets in Definition (4.1).

Definition 4.1. Let \mathfrak{A} be a C*-algebra. Denote the set of norm closed two-sided ideals of \mathfrak{A} by $\text{Ideal}(\mathfrak{A})$, in which we include the trivial ideals \emptyset, \mathfrak{A} . Define:

$$\begin{aligned} \text{Prim}(\mathfrak{A}) \\ = \{J \in \text{Ideal}(\mathfrak{A}) : J = \ker \pi, \pi \text{ is a non-zero irreducible } * \text{-representation of } \mathfrak{A}\}. \end{aligned}$$

The *Jacobson topology* on $\text{Prim}(\mathfrak{A})$, denoted *Jacobson* is defined in [34, Theorem 5.4.2 and Theorem 5.4.6]. Let F be a closed set in the Jacobson topology, then there exists $I_F \in \text{Ideal}(\mathfrak{A})$ such that $F = \{J \in \text{Prim}(\mathfrak{A}) : J \supseteq I_F\}$ [34, Theorem 5.4.7].

Convention 4.2. Given a C^* -algebra, \mathfrak{A} , and $I \in \text{Ideal}(\mathfrak{A})$, an element of the quotient \mathfrak{A}/I will be denoted by $a + I$ for some $a \in \mathfrak{A}$. Furthermore, the quotient norm will be denoted $\|a + I\|_{\mathfrak{A}/I} = \inf \{\|a + b\|_{\mathfrak{A}} : b \in I\}$.

Now, we may define the Fell topology, which is a topology on all ideals of a C^* -algebra. Once again, we do not provide a complete definition, but we will soon be able to characterize net convergence in the Fell topology in Lemma (4.4), which in turn determines the closed sets of the Fell topology.

Definition 4.3 ([14]). Let \mathfrak{A} be a C^* -algebra. Let $\mathcal{C}(\text{Prim}(\mathfrak{A}))$ be the set of closed subsets of $(\text{Prim}(\mathfrak{A}), \text{Jacobson})$ with compact Hausdorff topology, $\tau_{\mathcal{C}(\text{Prim}(\mathfrak{A}))}$, given by [14, Theorem 2.2], where $\check{\mathfrak{A}} = \text{Prim}(\mathfrak{A})$. Let $fell : \text{Ideal}(\mathfrak{A}) \rightarrow \mathcal{C}(\text{Prim}(\mathfrak{A}))$ denote the map:

$$fell(I) = \{J \in \text{Prim}(\mathfrak{A}) : J \supseteq I\},$$

which is a one-to-one correspondence [34, Theorem 5.4.7]. The *Fell topology* on $\text{Ideal}(\mathfrak{A})$, denoted $Fell$, is the initial topology on $\text{Ideal}(\mathfrak{A})$ induced by $fell$, which is the weakest topology for which $fell$ is continuous. Equivalently,

$$Fell = \left\{ U \subseteq \text{Ideal}(\mathfrak{A}) : U = fell^{-1}(V), V \in \tau_{\mathcal{C}(\text{Prim}(\mathfrak{A}))} \right\},$$

and $(\text{Ideal}(\mathfrak{A}), Fell)$ is therefore compact Hausdorff since $fell$ is a bijection.

The following Lemma (4.4) is stated in [2, Section 2], where the Fell topology, $Fell$, is denoted τ_s . We provide a proof.

Lemma 4.4. *Let \mathfrak{A} be a C^* -algebra. Let $(I_\mu)_{\mu \in \Delta} \subseteq \text{Ideal}(\mathfrak{A})$ be a net and $I \in \text{Ideal}(\mathfrak{A})$. The net $(I_\mu)_{\mu \in \Delta}$ converges to I with respect to the Fell topology if and only if the net $\left(\|a + I_\mu\|_{\mathfrak{A}/I_\mu} \right)_{\mu \in \Delta} \subseteq \mathbb{R}$ converges to $\|a + I\|_{\mathfrak{A}/I} \in \mathbb{R}$ with respect to the usual topology on \mathbb{R} for all $a \in \mathfrak{A}$.*

Proof. By [14, Theorem 2.2], let $Y \in \mathcal{C}(\text{Prim}(\mathfrak{A}))$, define:

$$M_Y : a \in \mathfrak{A} \mapsto \sup \{ \|a + I\|_{\mathfrak{A}/I} : I \in Y \} \in \mathbb{R},$$

since in Fell's notation, given an ideal S , we have $S_a = a + S$ according to his definition of *transform* in [14, Section 2.1] in the context of the primitive ideal space $\check{\mathfrak{A}} = \text{Prim}(\mathfrak{A})$. But, by the first line of the proof of [14, Theorem 2.2], we note that $\bigcap_{I \in Y} I \in \text{Ideal}(\mathfrak{A})$ and:

$$(4.1) \quad M_Y(a) = \|a + \bigcap_{I \in Y} I\|_{\mathfrak{A}/(\bigcap_{I \in Y} I)},$$

for all $a \in \mathfrak{A}$.

Let $P \in \text{Ideal}(\mathfrak{A})$, then $fell(P) = \{J \in \text{Prim}(\mathfrak{A}) : J \supseteq P\} \in \mathcal{C}(\text{Prim}(\mathfrak{A}))$ by Definition (4.3). Note that $\bigcap_{H \in fell(P)} H = P$ by [34, Theorem 5.4.3]. Thus, by Expression (4.1):

$$(4.2) \quad M_{fell(P)}(a) = \|a + P\|_{\mathfrak{A}/P}.$$

Now, assume that $(I_\mu)_{\mu \in \Delta} \subset \text{Ideal}(\mathfrak{A})$ converges to $I \in \text{Ideal}(\mathfrak{A})$ with respect to the Fell topology. Since $fell$ is continuous, the net $(fell(I_\mu))_{\mu \in \Delta} \subseteq \mathcal{C}(\text{Prim}(\mathfrak{A}))$ converges to $fell(I) \in \mathcal{C}(\text{Prim}(\mathfrak{A}))$ with respect to the topology on $\mathcal{C}(\text{Prim}(\mathfrak{A}))$.

By [14, Theorem 2.2], the net of functions $\left(M_{f_{\text{ell}}(I_\mu)}\right)_{\mu \in \Delta}$ converges pointwise to $M_{f_{\text{ell}}(I)}$, which completes the forward implication by Equation (4.2).

For the reverse implication, assume that $\left(\|a + I_\mu\|_{\mathfrak{A}/I_\mu}\right)_{\mu \in \Delta} \subseteq \mathbb{R}$ converges to $\|a + I\|_{\mathfrak{A}/I} \in \mathbb{R}$ with respect to the usual topology on \mathbb{R} for all $a \in \mathfrak{A}$ and for some net $(I_\mu)_{\mu \in \Delta} \subseteq \text{Ideal}(\mathfrak{A})$ and $I \in \text{Ideal}(\mathfrak{A})$. But, then by Equation (4.2) and assumption, the net $\left(M_{f_{\text{ell}}(I_\mu)}\right)_{\mu \in \Delta}$ converges pointwise to $M_{f_{\text{ell}}(I)}$. By [14, Theorem 2.2], the net $\left(f_{\text{ell}}(I_\mu)\right)_{\mu \in \Delta} \subseteq \mathcal{C}(\text{Prim}(\mathfrak{A}))$ converges to $f_{\text{ell}}(I) \in \mathcal{C}(\text{Prim}(\mathfrak{A}))$ with respect to the topology on $\mathcal{C}(\text{Prim}(\mathfrak{A}))$. However, as f_{ell} is a continuous bijection between the compact Hausdorff spaces $(\text{Ideal}(\mathfrak{A}), \mathbf{Fell})$ and $(\mathcal{C}(\text{Prim}(\mathfrak{A})), \tau_{\mathcal{C}(\text{Prim}(\mathfrak{A}))})$, the map f_{ell} is a homeomorphism. Thus, we conclude that $(I_\mu)_{\mu \in \Delta}$ converges to I with respect to the Fell topology. \square

Now, the Fell topology induces a topology on $\text{Prim}(\mathfrak{A})$ via its relative topology. But, the set $\text{Prim}(\mathfrak{A})$ can also be equipped with the Jacobson topology. Thus, a comparison of both topologies is in order in Proposition (4.5), which can be proven using Lemma (4.4).

Proposition 4.5. *The relative topology induced by the Fell topology of Definition (4.3) on $\text{Prim}(\mathfrak{A})$ contains the Jacobson topology of Definition (4.1) on $\text{Prim}(\mathfrak{A})$.*

Proof. Let $F \subseteq \text{Prim}(\mathfrak{A})$ be closed in the Jacobson topology. Then, there exists a unique $I_F \in \text{Ideal}(\mathfrak{A})$ such that $F = \{J \in \text{Prim}(\mathfrak{A}) : J \supseteq I_F\}$ by Definition (4.3).

Let $J \in \text{Prim}(\mathfrak{A})$ such that there exists a convergent net $(J^\mu)_{\mu \in \Delta} \subseteq F$ that converges to $J \in \text{Prim}(\mathfrak{A})$ in the Fell topology. Let $x \in I_F$, then $x \in J^\mu$ for all $\mu \in \Delta$. Thus, the net $\left(\|x + J^\mu\|_{\mathfrak{A}/J^\mu}\right)_{\mu \in \Delta} = (0)_{\mu \in \Delta}$, which is a net that converges to $\|x + J\|_{\mathfrak{A}/J}$ by Lemma (4.4). Thus, the limit $\|x + J\|_{\mathfrak{A}/J} = 0$, which implies that $x \in J$. Hence, $J \supseteq I_F$ and since $J \in \text{Prim}(\mathfrak{A})$, we have $J \in F$.

Thus, F is closed in the relative topology on $\text{Prim}(\mathfrak{A})$ induced by the Fell topology, which verifies the containment of the topologies. \square

The next two Lemmas concern the question of how the Jacobson and Fell topologies behave with respect to *-isomorphic C*-algebras. First, we discuss the Jacobson topology.

Lemma 4.6. *If $\mathfrak{A}, \mathfrak{B}$ are C*-algebras that are *-isomorphic, then using notation from Definition (4.1), the topological spaces $(\text{Prim}(\mathfrak{A}), \mathbf{Jacobson})$ and $(\text{Prim}(\mathfrak{B}), \mathbf{Jacobson})$ are homeomorphic.*

*In particular, if $\alpha : \mathfrak{A} \rightarrow \mathfrak{B}$ is a *-isomorphism, then:*

$$\alpha_i : I \in \text{Prim}(\mathfrak{A}) \mapsto \alpha(I) \in \text{Prim}(\mathfrak{B})$$

is well-defined and a homeomorphism from $(\text{Prim}(\mathfrak{A}), \mathbf{Jacobson})$ to $(\text{Prim}(\mathfrak{B}), \mathbf{Jacobson})$.

Proof. Let $\alpha : \mathfrak{A} \rightarrow \mathfrak{B}$ be a *-isomorphism. We begin by establishing that α_i is well-defined. Let $I \in \text{Prim}(\mathfrak{A})$. By Definition (4.1), there exists a non-zero irreducible *-representation $\pi_I : \mathfrak{A} \rightarrow \mathcal{B}(H)$ such that $\ker \pi_I = I$, where $\mathcal{B}(H)$ denotes the C*-algebra of bounded operators on some Hilbert space, H . But, the composition

$\pi_I \circ \alpha^{-1} : \mathfrak{B} \rightarrow \mathcal{B}(H)$ is a non-zero irreducible *-representation on \mathfrak{B} since α^{-1} is a *-isomorphism and π_I is a non-zero irreducible *-representation. We show that the kernel of $\pi_I \circ \alpha^{-1}$ is $\alpha(I)$.

Consider $\alpha(I) \subseteq \mathfrak{A}$. The set $\alpha(I) \in \text{Ideal}(\mathfrak{A})$ since α is a *-isomorphism. However:

$$\begin{aligned} a \in \alpha(I) &\iff \alpha(a)^{-1} \in I \\ &\iff \alpha(a)^{-1} \in \ker \pi_I \\ &\iff a \in \ker \pi_I \circ \alpha^{-1}, \end{aligned}$$

and thus, the ideal $\alpha(I) = \ker \pi_I \circ \alpha^{-1} \in \text{Prim}(\mathfrak{A})$ by Definition (4.1).

Therefore, the following map is well-defined:

$$\alpha_i : I \in \text{Prim}(\mathfrak{A}) \mapsto \alpha(I) \in \text{Prim}(\mathfrak{B}),$$

and is injective since α is a *-isomorphism. For surjectivity, let $I \in \text{Prim}(\mathfrak{B})$. The fact that $\alpha^{-1}(I) \in \text{Prim}(\mathfrak{A})$ follows the same argument for proving that α_i is well-defined. Also, the image $\alpha_i(\alpha^{-1}(I)) = \alpha(\alpha^{-1}(I)) = I$ since α is a bijection. Hence, the map α_i is a well-defined bijection.

Now, we establish continuity. Let $F \subseteq \text{Prim}(\mathfrak{B})$ be closed. By Definition (4.1), there exists an $I_F \in \text{Ideal}(\mathfrak{B})$ such that $F = \{J \in \text{Prim}(\mathfrak{B}) : J \supseteq I_F\}$. Consider $\alpha_i^{-1}(F) = \{I \in \text{Prim}(\mathfrak{A}) : \alpha_i(I) \in F\}$. Assume that $I \in \alpha_i^{-1}(F)$. Then, we have that $\alpha_i(I) \in \text{Prim}(\mathfrak{A})$ by well-defined, and moreover:

$$\alpha(I) \supseteq I_F = \alpha(\alpha^{-1}(I_F)) \implies I \supseteq \alpha^{-1}(I_F) \in \text{Ideal}(\mathfrak{A})$$

since α is a bijection and α^{-1} is a *-isomorphism. Next, assume that $I \in \text{Prim}(\mathfrak{A})$ such that $I \supseteq \alpha^{-1}(I_F)$, then $\alpha(I) \supseteq I_F$ since α is a bijection, which implies that $\alpha_i(I) \in F$ and $I \in \alpha_i^{-1}(F)$ by well-defined. Combing the inclusions, the set $\alpha_i^{-1}(F) = \{I \in \text{Prim}(\mathfrak{A}) : I \supseteq \alpha^{-1}(I_F)\}$, which is closed by Definition (4.1). The continuity argument for α_i^{-1} follows similarly, which completes the proof. \square

Let's continue by proving that the Fell topology also satisfies the conclusions of Lemma (4.6), which will prove useful later in Corollary (5.14) by showing that the metric topology we develop is preserved homeomorphically by *-isomorphisms in the case of AF algebras.

Lemma 4.7. *If $\mathfrak{A}, \mathfrak{B}$ are C^* -algebras that are *-isomorphic, then using notation from Definition (4.3), the topological spaces $(\text{Ideal}(\mathfrak{A}), \text{Fell})$ and $(\text{Ideal}(\mathfrak{B}), \text{Fell})$ are homeomorphic.*

*In particular, if $\alpha : \mathfrak{A} \rightarrow \mathfrak{B}$ is a *-isomorphism, then:*

$$\alpha_i : I \in \text{Ideal}(\mathfrak{A}) \mapsto \alpha(I) \in \text{Ideal}(\mathfrak{B})$$

is well-defined and a homeomorphism from $(\text{Prim}(\mathfrak{A}), \text{Fell})$ to $(\text{Prim}(\mathfrak{B}), \text{Fell})$.

Proof. Let $\alpha : \mathfrak{A} \rightarrow \mathfrak{B}$ be a *-isomorphism, then the map $\alpha_i : I \in \text{Ideal}(\mathfrak{A}) \mapsto \alpha(I) \in \text{Ideal}(\mathfrak{B})$ is a well-defined bijection. Assume that $\left(I_{\mathfrak{A}}^{\mu}\right)_{\mu \in \Delta} \subset \text{Ideal}(\mathfrak{A})$ is a net that converges with respect to the Fell topology to $I_{\mathfrak{A}} \in \text{Id}(\mathfrak{A})$. We show that $\left(\alpha_i\left(I_{\mathfrak{A}}^{\mu}\right)\right)_{\mu \in \Delta} \subset \text{Ideal}(\mathfrak{B})$ converges with respect to the Fell topology to

$\alpha_i(I_{\mathfrak{A}}) \in \text{Ideal}(\mathfrak{B})$. Let $b \in \mathfrak{B}$, then $\alpha^{-1}(b) \in \mathfrak{A}$. Thus, by Lemma (4.4), we have $\left(\left\| \alpha^{-1}(b) + I_{\mathfrak{A}}^\mu \right\|_{\mathfrak{A}/I_{\mathfrak{A}}^\mu} \right)_{\mu \in \Delta}$ converges to $\left\| \alpha^{-1}(b) + I_{\mathfrak{A}} \right\|_{\mathfrak{A}/I_{\mathfrak{A}}}$. But, fix $\mu \in \Delta$, then since α is a $*$ -isomorphism:

$$\begin{aligned} \left\| \alpha^{-1}(b) + I_{\mathfrak{A}}^\mu \right\|_{\mathfrak{A}/I_{\mathfrak{A}}^\mu} &= \inf \left\{ \left\| \alpha^{-1}(b) + a \right\|_{\mathfrak{A}} : a \in I_{\mathfrak{A}}^\mu \right\} \\ &= \inf \left\{ \|b + \alpha(a)\|_{\mathfrak{B}} : a \in I_{\mathfrak{A}}^\mu \right\} \\ &= \inf \left\{ \|b + b'\|_{\mathfrak{B}} : b' \in \alpha(I_{\mathfrak{A}}^\mu) \right\} \\ &= \left\| b + \alpha_i \left(I_{\mathfrak{A}}^\mu \right) \right\|_{\mathfrak{B}/\alpha_i(I_{\mathfrak{A}}^\mu)}, \end{aligned}$$

and similarly, the limit $\left\| \alpha^{-1}(b) + I_{\mathfrak{A}} \right\|_{\mathfrak{A}/I_{\mathfrak{A}}} = \|b + \alpha_i(I_{\mathfrak{A}})\|_{\mathfrak{B}/\alpha_i(I_{\mathfrak{A}})}$.

Hence, the net $\left(\left\| b + \alpha_i \left(I_{\mathfrak{A}}^\mu \right) \right\|_{\mathfrak{B}/\alpha_i(I_{\mathfrak{A}}^\mu)} \right)_{\mu \in \Delta}$ converges to $\|b + \alpha_i(I_{\mathfrak{A}})\|_{\mathfrak{B}/\alpha_i(I_{\mathfrak{A}})}$.

Therefore, since $b \in \mathfrak{B}$ was arbitrary, the net $\left(\alpha_i \left(I_{\mathfrak{A}}^\mu \right) \right)_{\mu \in \Delta} \subset \text{Ideal}(\mathfrak{B})$ converges with respect to the Fell topology to $\alpha_i(I_{\mathfrak{A}}) \in \text{Ideal}(\mathfrak{B})$ by Lemma (4.4). Thus, α_i is continuous, and since both topologies are compact Hausdorff, the proof is complete. \square

As stated earlier, it is with the Fell topology for which we will provide a notion of convergence of quotients from ideals of AF algebras. But, it seems that a metric notion is in order to move from fusing family of ideals to a fusing family of quotients as we will see in Proposition (4.15).

Next, we develop a metric on the ideal space on any inductive limit in the sense of Notation (2.13). But, first, a remark on our change in the language of inductive limits for some of the following results.

Remark 4.8. By [34, Chapter 6.1], if $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n \in \mathbb{N}}$ is an inductive sequence with inductive limit $\mathfrak{A} = \varinjlim \mathcal{I}$ as in Notation (2.13), then $(\alpha_n^n(\mathfrak{A}_n))_{n \in \mathbb{N}}$ is a non-decreasing sequence of C^* -subalgebras of \mathfrak{A} , in which $\mathfrak{A} = \overline{\cup_{n \in \mathbb{N}} \alpha_n^n(\mathfrak{A}_n)}^{\|\cdot\|_{\mathfrak{A}}}$. Thus, in some of the following definitions and results, when we say, "Let \mathfrak{A} be a C^* -algebra with a non-decreasing sequence of C^* -subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ such that $\mathfrak{A} = \overline{\cup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$," we are also including the case of inductive limits. The purpose of this will be to avoid notational confusion later on if we were to work with multiple inductive limits (see for example Proposition (4.15)), and the purpose of this remark is to note that this does not weaken our results.

The following Proposition (4.9) is key for defining our metric.

Proposition 4.9. *Let \mathfrak{A} be a C^* -algebra with a non-decreasing sequence of C^* -subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ such that $\mathfrak{A} = \overline{\cup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. The map:*

$$i(\cdot, \mathcal{U}) : I \in \text{Ideal}(\mathfrak{A}) \mapsto (I \cap \mathfrak{A}_n)_{n \in \mathbb{N}} \in \prod_{n \in \mathbb{N}} \text{Ideal}(\mathfrak{A}_n)$$

is a well-defined injection.

Proof. Since $I \in \text{Ideal}(\mathfrak{A})$ and \mathfrak{A}_n is a C^* -subalgebra for all $n \in \mathbb{N}$, we have that $I \cap \mathfrak{A}_n \in \text{Ideal}(\mathfrak{A}_n)$ for all $n \in \mathbb{N}$. Thus, the map $i(\cdot, \mathcal{U})$ is well-defined.

Next, for injectivity, assume that $I, J \in \text{Ideal}(\mathfrak{A})$ such that $i(I, \mathcal{U}) = i(J, \mathcal{U})$. Hence, the sets $I \cap \mathfrak{A}_n = J \cap \mathfrak{A}_n$ for all $n \in \mathbb{N}$, which implies that $\bigcup_{n \in \mathbb{N}} (I \cap \mathfrak{A}_n) = \bigcup_{n \in \mathbb{N}} (J \cap \mathfrak{A}_n)$. Therefore, the closures $\overline{\bigcup_{n \in \mathbb{N}} (I \cap \mathfrak{A}_n)}^{\|\cdot\|_{\mathfrak{A}}} = \overline{\bigcup_{n \in \mathbb{N}} (J \cap \mathfrak{A}_n)}^{\|\cdot\|_{\mathfrak{A}}}$. But, by [10, Lemma III.4.1], we conclude $I = J$. \square

With this injection, we may define a metric.

Definition 4.10. Let \mathfrak{A} be a C^* -algebra with a non-decreasing sequence of C^* -subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. We define a map from $\text{Ideal}(\mathfrak{A}) \times \text{Ideal}(\mathfrak{A})$ to $[0, 1]$ such that for all $I, J \in \text{Ideal}(\mathfrak{A})$:

$$m_{i(\mathcal{U})}(I, J) = \begin{cases} 0 & \text{if } \forall n \in \mathbb{N}, I \cap \mathfrak{A}_n = J \cap \mathfrak{A}_n \\ 2^{-n} & \text{if } n = \min\{m \in \mathbb{N} : I \cap \mathfrak{A}_m \neq J \cap \mathfrak{A}_m\} \end{cases}$$

Proposition 4.11. *If \mathfrak{A} is a C^* -algebra with a non-decreasing sequence of C^* -subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, then:*

$$\left(\text{Ideal}(\mathfrak{A}), m_{i(\mathcal{U})} \right)$$

is a zero-dimensional ultrametric space, where $m_{i(\mathcal{U})}$ is given by Definition (4.10).

Proof. Consider the metric on $\prod_{n \in \mathbb{N}} \text{Ideal}(\mathfrak{A}_n)$ defined by:

$$m((I_n)_{n \in \mathbb{N}}, (J_n)_{n \in \mathbb{N}}) = \begin{cases} 0 & \text{if } \forall n \in \mathbb{N}, I_n = J_n \\ 2^{-n} & \text{if } n = \min\{m \in \mathbb{N} : I_m \neq J_m\} \end{cases}.$$

Thus, $(\prod_{n \in \mathbb{N}} \text{Ideal}(\mathfrak{A}_n), m)$ is a zero-dimensional metric space since it metrizes the product topology on $\prod_{n \in \mathbb{N}} \text{Ideal}(\mathfrak{A}_n)$, in which $\text{Ideal}(\mathfrak{A}_n)$ is given the discrete topology for all $n \in \mathbb{N}$. But, the identification $m_{i(\mathcal{U})} = m \circ (i(\cdot, \mathcal{U}) \times i(\cdot, \mathcal{U}))$ implies that $(\text{Ideal}(\mathfrak{A}), m_{i(\mathcal{U})})$ is a zero-dimensional metric space since $i(\cdot, \mathcal{U})$ is injective by Proposition (4.9). \square

Remark 4.12. If \mathfrak{A} is any C^* -algebra, then, $\mathfrak{A} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, where $\mathfrak{A}_n = \mathfrak{A}$ for all $n \in \mathbb{N}$. If we set $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$, then, the metric $m_{i(\mathcal{U})}$ of Proposition (4.11) is a metric on the ideal space of any C^* -algebra, but we see in this case that this metric simply metrizes the discrete topology. However, the metric of Proposition (4.11) is not always trivial as we shall see in the case of AF algebras (Theorem (5.12)), in which the metric spaces will always be compact. In particular, if an AF algebra were to contain at least infinitely many ideals (see Section (6.1) for an example of such an AF algebra), then the metric of Proposition (4.11) could not be discrete. Furthermore, this implies that the conclusion of Theorem (4.16) is not trivial.

Remark 4.13. The metric of Proposition (4.11) can be seen as an explicit presentation of a metric on a metrizable topology on ideals presented in [3], where this metrizable topology is presented only in the case of AF algebras and metrizes the Fell topology in the AF case, which we also prove for the metric of Proposition (4.11) via a different approach in Theorem (5.12). But, we note that the metric of Proposition (4.11) is more general as it exists on the ideal space of any C^* -inductive

limit — and on any C^* -algebra by Remark (4.12)—, and in the AF case (Section (5)), we define a metric entirely in the graph setting of a Bratteli diagram on the space of directed and hereditary subsets of the diagram (Theorem (5.12)), which in turn is isometric to the metric of Proposition (4.11). This allows us to explicitly calculate distances between ideals in Remark (6.12), and therefore, make interesting comparisons with certain classical metrics on irrationals. And, in Proposition (4.15), the metric of Proposition (4.11) will explicitly provide fusing families of quotients.

In the context of this paper, the main motivation for the metric of Proposition (4.11) is to provide a fusing family of quotients via convergence of ideals. First, for a fixed ideal of an inductive limit of the form $\mathfrak{A} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, we provide an inductive limit in the sense of Notation (2.13) that is $*$ -isomorphic to the quotient. The reason for this is that given $I \in \text{Ideal}(\mathfrak{A})$, then \mathfrak{A}/I has a canonical closure of union form as $\mathfrak{A}/I = \overline{\bigcup_{n \in \mathbb{N}} ((\mathfrak{A}_n + I)/I)}^{\|\cdot\|_{\mathfrak{A}/I}}$ (see Proposition (4.15)), but if two ideals satisfy $I \cap \mathfrak{A}_n = J \cap \mathfrak{A}_n$ for some $n \in \mathbb{N}$, then even though this provides that $(\mathfrak{A}_n + I)/I$ is $*$ -isomorphic to $(\mathfrak{A}_n + J)/J$ as they are both $*$ -isomorphic to $(\mathfrak{A}_n/(I \cap \mathfrak{A}_n))$ (see Proposition (4.15)), the two algebras $(\mathfrak{A}_n + J)/J$ and $(\mathfrak{A}_n + I)/I$ are not equal in any way if $I \neq J$, yet, equality is a requirement for fusing families (see Definition (3.4)). Thus, Notation (4.14) will allow us to present, up to $*$ -isomorphism, quotients as IL-fusing families as we will see in Proposition (4.15) from convergence of ideals in the metric of Proposition (4.11).

Notation 4.14. Let \mathfrak{A} be a C^* -algebra with a non-decreasing sequence of C^* -subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. Let $I \in \text{Ideal}(\mathfrak{A})$. For $n \in \mathbb{N}$, define:

$$\gamma_{I,n} : a + I \cap \mathfrak{A}_n \in \mathfrak{A}_n / (I \cap \mathfrak{A}_n) \longmapsto a + I \in \mathfrak{A}_{n+1} / (I \cap \mathfrak{A}_{n+1}),$$

which is an injective $*$ -homomorphism by the same argument of Claim (4.17). Let $\mathcal{I}(\mathfrak{A}/I) = (\mathfrak{A}_n / (I \cap \mathfrak{A}_n), \gamma_{I,n})_{n \in \mathbb{N}}$, and denote the C^* -inductive limit by $\varinjlim \mathcal{I}(\mathfrak{A}/I)$.

Proposition 4.15. Let \mathfrak{A} be a C^* -algebra with a non-decreasing sequence of C^* -subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. Using Notation (4.14), if $I \in \text{Ideal}(\mathfrak{A})$, then there exists a $*$ -isomorphism $\phi_I : \varinjlim \mathcal{I}(\mathfrak{A}/I) \rightarrow \mathfrak{A}/I$ such that for all $n \in \mathbb{N}$ the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{A}_n / (I \cap \mathfrak{A}_n) & \xrightarrow{\gamma_I^n} & \varinjlim \mathcal{I}(\mathfrak{A}/I) , \\ & \searrow \phi_I^n & \downarrow \phi_I \\ & & \mathfrak{A}/I \end{array}$$

where for all $n \in \mathbb{N}$, the maps $\phi_I^n : a + \mathfrak{A}_n / (I \cap \mathfrak{A}_n) \mapsto a + I \in (\mathfrak{A}_n + I)/I \subseteq \mathfrak{A}/I$ are injective $*$ -homomorphisms onto $(\mathfrak{A}_n + I)/I$, in which $\mathfrak{A}_n + I = \{a + b \in \mathfrak{A} : a \in \mathfrak{A}_n, b \in I\}$ a C^* -subalgebra of \mathfrak{A} is and $\bigcup_{n \in \mathbb{N}} ((\mathfrak{A}_n + I)/I)$ is a dense $*$ -subalgebra of \mathfrak{A}/I with $((\mathfrak{A}_n + I)/I)_{n \in \mathbb{N}}$ non-decreasing.

Furthermore, if $(I^n)_{n \in \mathbb{N}} \subseteq \text{Ideal}(\mathfrak{A})$ converges to $I^\infty \in \text{Ideal}(\mathfrak{A})$ with respect to $m_{i(\mathcal{U})}$ of Proposition (4.11), then using Definition (3.4), we have

$\{I^n = \overline{\cup_{k \in \mathbb{N}} I^n \cap \mathfrak{A}_k}^{\|\cdot\|_{\mathfrak{A}}} : n \in \overline{\mathbb{N}}\}$ is a fusing family with respect to some fusing sequence $(c_n)_{n \in \mathbb{N}}$ such that $\{\varinjlim \mathcal{I}(\mathfrak{A}/I^n) : n \in \overline{\mathbb{N}}\}$ is an IL-fusing family with fusing sequence $(c_n)_{n \in \mathbb{N}}$.

Proof. Let $I \in \text{Ideal}(\mathfrak{A})$. Fix $n \in \mathbb{N}$, then ϕ_I^n is an injective *-homomorphism by Claim (4.17). Let $a \in \mathfrak{A}_n$. We have that $\phi_I^n(a + \mathfrak{A}_n/(I \cap \mathfrak{A}_n)) = a + \mathfrak{A}/I$. Also, the composition $\phi_I^{n+1}(\gamma_{I,n}(a + \mathfrak{A}_n/(I \cap \mathfrak{A}_n))) = \phi_I^{n+1}(a + \mathfrak{A}_{n+1}/(I \cap \mathfrak{A}_{n+1})) = a + \mathfrak{A}/I$. Therefore, for all $n \in \mathbb{N}$, the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{A}_n/(I \cap \mathfrak{A}_n) & \xrightarrow{\gamma_{I,n}} & \mathfrak{A}_{n+1}/(I \cap \mathfrak{A}_{n+1}) \\ & \searrow \phi_I^n & \downarrow \phi_I^{n+1} \\ & & \mathfrak{A}/I \end{array}$$

Hence, by [34, Theorem 6.1.2], there exists a unique *-homomorphism $\phi_I : \varinjlim \mathcal{I}(\mathfrak{A}/I) \rightarrow \mathfrak{A}/I$ such that for all $n \in \mathbb{N}$ the diagram in the statement of this theorem commutes.

Now, since the maps $\gamma_{I,n}$ are injective for all $n \in \mathbb{N}$, then so are the maps $\varinjlim \gamma_I^n$ for all $n \in \mathbb{N}$ by definition [34, Chapter 6.1]. Hence, we have by the commuting diagram in the statement of the theorem that ϕ_I is an injective *-homomorphism on the dense *-subalgebra $\cup_{n \in \mathbb{N}} \varinjlim \gamma_I^n(\mathfrak{A}_n/(I \cap \mathfrak{A}_n))$ of $\varinjlim \mathcal{I}(\mathfrak{A}/I)$. Thus, ϕ_I is an isometry on $\cup_{n \in \mathbb{N}} \varinjlim \gamma_I^n(\mathfrak{A}_n/(I \cap \mathfrak{A}_n))$, and therefore, is an isometry on $\varinjlim \mathcal{I}(\mathfrak{A}/I)$, and thus an injective *-homomorphism on $\varinjlim \mathcal{I}(\mathfrak{A}/I)$.

Furthermore, fix $n \in \mathbb{N}$. As $I \in \text{Ideal}(\mathfrak{A})$, note that $\mathfrak{A}_n + I = \{a + b \in \mathfrak{A} : a \in \mathfrak{A}_n, b \in I\}$ is a C*-subalgebra of \mathfrak{A} that contains $I \in \text{Ideal}(\mathfrak{A}_n + I)$. Next, let $x \in (\mathfrak{A}_n + I)/I$ so that $x = a + b + I$, where $a \in \mathfrak{A}_n, b \in I$. Thus, we have $a + b - a = b \in I \implies x - (a + I) = 0 + I \implies x = a + I$. But, then, the image $\phi_I^n(a + I) = x$. Hence, the map ϕ_I^n is onto $(\mathfrak{A}_n + I)/I$. We thus have:

$$\phi_I \left(\cup_{n \in \mathbb{N}} \varinjlim \gamma_I^n(\mathfrak{A}_n/(I \cap \mathfrak{A}_n)) \right) = \cup_{n \in \mathbb{N}} ((\mathfrak{A}_n + I)/I),$$

in which the right-hand side is a dense *-subalgebra of \mathfrak{A}/I by continuity of the quotient map and the assumption that $\cup_{n \in \mathbb{N}} \mathfrak{A}_n$ is dense in \mathfrak{A} . Hence, since $\varinjlim \mathcal{I}(\mathfrak{A}/I)$ is complete and ϕ_I is a linear isometry on $\varinjlim \mathcal{I}(\mathfrak{A}/I)$, we have ϕ_I surjects onto \mathfrak{A}/I . Thus, the function $\phi_I : \varinjlim \mathcal{I}(\mathfrak{A}/I) \rightarrow \mathfrak{A}/I$ is a *-isomorphism.

Next, assume that $(I^n)_{n \in \mathbb{N}} \subseteq \text{Ideal}(\mathfrak{A})$ converges to $I^\infty \in \text{Ideal}(\mathfrak{A})$ with respect to $m_{i(\mathcal{U})}$. Thus, we have $\lim_{n \rightarrow \infty} m_{i(\mathcal{U})}(I^n, I^\infty) = 0$. From this, construct an increasing sequence $(b_n)_{n \in \mathbb{N}} \subseteq \mathbb{N} \setminus \{0\}$ such that:

$$m_{i(\mathcal{U})}(I^k, I^\infty) \leq 2^{-(n+1)}$$

for all $k \geq b_n$. In particular, if $N \in \mathbb{N}$, then $I^k \cap \mathfrak{A}_n = I^\infty \cap \mathfrak{A}_n$ if $n \in \{0, \dots, N\}$ and $k \geq b_N$, which implies that $\{I^n = \overline{\cup_{k \in \mathbb{N}} I^n \cap \mathfrak{A}_k}^{\|\cdot\|_{\mathfrak{A}}} : n \in \overline{\mathbb{N}}\}$ is a fusing family with fusing sequence $(b_n)_{n \in \mathbb{N}}$ by Definition (3.4).

Let $c_n = b_{n+1}$ for all $n \in \mathbb{N}$. Then, the sequence $(c_n)_{n \in \mathbb{N}}$ is a fusing sequence for $\left\{ I^n = \overline{\cup_{k \in \mathbb{N}} I^n \cap \mathfrak{A}_k}^{\|\cdot\|_{\mathfrak{A}}} : n \in \overline{\mathbb{N}} \right\}$. Fix $N \in \mathbb{N}$, $n \in \{0, \dots, N\}$, and $k \in \mathbb{N}_{\geq c_N}$. Then, the equality $I^k \cap \mathfrak{A}_n = I^\infty \cap \mathfrak{A}_n$ implies that $\mathfrak{A}_n / (I^k \cap \mathfrak{A}_n) = \mathfrak{A}_n / (I^\infty \cap \mathfrak{A}_n)$. But, also, we gather $\gamma_{I^k, n} = \gamma_{I^\infty, n}$ since $\mathfrak{A}_{n+1} / (I^k \cap \mathfrak{A}_{n+1}) = \mathfrak{A}_{n+1} / (I^\infty \cap \mathfrak{A}_{n+1})$ as $c_n = b_{n+1}$. Hence, the family of inductive limits $\left\{ \varinjlim \mathcal{I}(\mathfrak{A} / I^n) : n \in \overline{\mathbb{N}} \right\}$ is an IL-fusing family with fusing sequence $(c_n)_{n \in \mathbb{N}}$. \square

For the ideal space, Proposition (4.11) provides a zero-dimensional Hausdorff space metrized by an ultrametric. We will see that if the sequence of C^* -subalgebras $(\mathfrak{A}_n)_{n \in \mathbb{N}}$ are all assumed to be finite dimensional (or if \mathfrak{A} is AF), then the metric space of Proposition (4.11) will be compact in Theorem (5.12). But, we will approach this by first providing a compact metric on the directed hereditary subsets of a Bratteli diagram in Proposition (5.9), and then translating this metric back to the setting of Proposition (4.11), which will provide compactness with ease. Thus, providing another in the line of many applications of the novel Bratteli diagram. But, before we continue in this path, we see that in the very least, the Proposition (4.11) metric can be utilized as a tool to provide convergence in the Fell topology as the metric topology is stronger. This is the content of following Theorem (4.16). Later on, this will show in the AF algebra case that the Fell and metric topologies agree by maximal compactness in Theorem (5.12).

Theorem 4.16. *If $\mathfrak{A} = \overline{\cup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ is a C^* -algebra in which $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ is a non-decreasing sequence of C^* -subalgebras of \mathfrak{A} , then on $\text{Ideal}(\mathfrak{A})$, the Fell topology is contained in the metric topology of $\mathfrak{m}_i(\mathcal{U})$.*

Proof. First, we prove the following claim to provide norm calculations.

Claim 4.17. *Let $J \in \text{Ideal}(\mathfrak{A})$. For each $k \in \mathbb{N}$, the map:*

$$(4.3) \quad \phi_j^k : a + J \cap \mathfrak{A}_k \in \mathfrak{A}_k / (J \cap \mathfrak{A}_k) \longmapsto a + J \in \mathfrak{A} / J.$$

is an injective $$ -homomorphism.*

Assume that $a, b \in \mathfrak{A}_k$ such that $a + J \cap \mathfrak{A}_k = b + J \cap \mathfrak{A}_k$, which implies that $a - b \in J \cap \mathfrak{A}_k \subseteq J \implies a + J = b + J$, and thus, ϕ_j^k is well-defined. Next, assume that $a, b \in \mathfrak{A}_k$ such that $a + J = b + J$, which implies that $a - b \in J$. But, we have $a - b \in \mathfrak{A}_k \implies a - b \in J \cap \mathfrak{A}_k$ and $a + J \cap \mathfrak{A}_k = b + J \cap \mathfrak{A}_k$, which provides injectivity. Thus, for each $k \in \mathbb{N}$, we have ϕ_j^k is a well-defined injective $*$ -homomorphism since J is an ideal. Hence, the map ϕ_j^k is an isometry for each $k \in \mathbb{N}$ and any $J \in \text{Ideal}(\mathfrak{A})$, which proves the claim.

Let $F \subseteq \text{Ideal}(\mathfrak{A})$ be closed with respect to *Fell*. We show that F is closed with respect to the metric topology of $\mathfrak{m}_i(\mathcal{U})$. Since the topology is metric, we may use sequences. Thus, let $\left(I^l \right)_{l \in \mathbb{N}} \subseteq F$ and $I \in \text{Ideal}(\mathfrak{A})$ such that $\lim_{l \rightarrow \infty} \mathfrak{m}_i(\mathcal{U}) \left(I^l, I \right) = 0$. Now, we claim that this sequence converges with respect to the Fell topology, and thus, we will approach by Lemma (4.4).

Let $\varepsilon > 0$, $a \in \mathfrak{A}$. By density of $\cup_{n \in \mathbb{N}} \mathfrak{A}_n$ in \mathfrak{A} , there exists $N \in \mathbb{N}$ such that $a_N \in \mathfrak{A}_N$ and $\|a - a_N\|_{\mathfrak{A}} < \varepsilon/2$. By convergence in $\mathfrak{m}_i(\mathcal{U})$, there exists $k_N \in \mathbb{N}$ such that $I^l \cap \mathfrak{A}_N = I \cap \mathfrak{A}_N$ for all $l \geq k_N$. Furthermore, since $\phi_{I^l}^{k_N}$ is an isometry by Claim

(4.17), we have that $\|a_N + I^l \cap \mathfrak{A}_N\|_{\mathfrak{A}_N/(I^l \cap \mathfrak{A}_N)} = \|a_N + I^l\|_{\mathfrak{A}/I^l}$ for all $l \geq k_N$. But, we have $\|a_N + I^l \cap \mathfrak{A}_N\|_{\mathfrak{A}_N/(I^l \cap \mathfrak{A}_N)} = \|a_N + I \cap \mathfrak{A}_N\|_{\mathfrak{A}_N/(I \cap \mathfrak{A}_N)} = \|a_N + I\|_{\mathfrak{A}/I}$ for all $l \geq k_N$ since $I^l \cap \mathfrak{A}_N = I \cap \mathfrak{A}_N$. Therefore, for $l \geq k_N$, we conclude:

$$(4.4) \quad \|a_N + I^l\|_{\mathfrak{A}/I^l} = \|a_N + I\|_{\mathfrak{A}/I}.$$

Now, let $l \geq k_N$, then by Expression (4.4) and the fact that any quotient norm of \mathfrak{A} with respect to $\|\cdot\|_{\mathfrak{A}}$ is bounded above by $\|\cdot\|_{\mathfrak{A}}$, we gather:

$$\begin{aligned} \left| \|a + I^l\|_{\mathfrak{A}/I^l} - \|a + I\|_{\mathfrak{A}/I} \right| &\leq \left| \|a + I^l\|_{\mathfrak{A}/I^l} - \|a_N + I^l\|_{\mathfrak{A}/I^l} \right| \\ &\quad + \left| \|a_N + I^l\|_{\mathfrak{A}/I^l} - \|a_N + I\|_{\mathfrak{A}/I} \right| \\ &\quad + \left| \|a_N + I\|_{\mathfrak{A}/I} - \|a + I\|_{\mathfrak{A}/I} \right| \\ &\leq \left| \|a_N + I^l\|_{\mathfrak{A}/I^l} - \|a_N + I\|_{\mathfrak{A}/I} \right| \\ &\quad + \left| \|a - a_N + I^l\|_{\mathfrak{A}/I^l} + \|a - a_N + I\|_{\mathfrak{A}/I} \right| \\ &\leq 2 \|a - a_N\|_{\mathfrak{A}} + \left| \|a_N + I^l\|_{\mathfrak{A}/I^l} - \|a_N + I\|_{\mathfrak{A}/I} \right| \\ &< \varepsilon + 0 \end{aligned}$$

Therefore, we may conclude $\lim_{l \rightarrow \infty} \|a + I^l\|_{\mathfrak{A}/I^l} = \|a + I\|_{\mathfrak{A}/I}$, and by Lemma (4.4), since $a \in \mathfrak{A}$ was arbitrary, the net $(I^l)_{l \in \mathbb{N}}$ converges with respect to the Fell topology to I . But, as F is closed in **Fell**, we have that $I \in F$. Thus, F is closed with respect to $m_i(\mathcal{U})$. This completes the containment argument. \square

5. METRIC ON IDEAL SPACE OF C*-INDUCTIVE LIMITS: AF CASE

In this section, the ultrametric of Proposition (4.11) is greatly strengthened in the AF case. For instance, its induced topology will be compact. The notion of a Bratteli diagram will prove quite useful in providing these advantages. Thus, for the moment, we introduce a new metric based entirely on the diagram structure. And, we will see in Theorem (5.12) that, when AF algebras are reintroduced, the inductive limit metric and diagram metrics are isometric and form a topology that equals the Fell topology on ideals. First, we recall the definition of a Bratteli diagram.

Definition 5.1 ([5]). Let $\mathcal{D} = (V^{\mathcal{D}}, E^{\mathcal{D}})$ be a directed graph with labelled vertices and multiple edges between two vertices is allowed. The set $V^{\mathcal{D}} \subset \mathbb{N}^2$ is the set of labeled vertices and $E^{\mathcal{D}} \subset \mathbb{N}^2 \times \mathbb{N}^2$ is the set of edges, which consist of ordered pairs from $V^{\mathcal{D}}$. For each $n \in \mathbb{N}$, let $v_n^{\mathcal{D}} \in \mathbb{N}$.

Define $V^{\mathcal{D}} = \cup_{n \in \mathbb{N}} V_n^{\mathcal{D}}$, where for $n \in \mathbb{N}$, we let:

$$V_n^{\mathcal{D}} = \left\{ (n, k) \in \mathbb{N} \times \mathbb{N} : k \in \{0, \dots, v_n^{\mathcal{D}}\} \right\},$$

and we denote the label of the vertices $(n, k) \in V^{\mathcal{D}}$ by $[n, k]_{\mathcal{D}} \in \mathbb{N} \setminus \{0\}$.

Next, let $E^{\mathcal{D}} \subset V^{\mathcal{D}} \times V^{\mathcal{D}}$. Now, we list some axioms for $V^{\mathcal{D}}$ and $E^{\mathcal{D}}$.

- (i) For all $n \in \mathbb{N}$, if $m \in \mathbb{N} \setminus \{n+1\}$, then $((n, k), (m, q)) \notin E^{\mathcal{D}}$ for all $k \in \{0, \dots, v_n^{\mathcal{D}}\}, q \in \{0, \dots, v_m^{\mathcal{D}}\}$.
- (ii) If $(n, k) \in V^{\mathcal{D}}$, then there exists $q \in \{0, \dots, v_{n+1}^{\mathcal{D}}\}$ such that $((n, k), (n+1, q)) \in E^{\mathcal{D}}$.
- (iii) If $n \in \mathbb{N} \setminus \{0\}$ and $(n, k) \in V^{\mathcal{D}}$, then there exists $q \in \{0, \dots, v_{n-1}^{\mathcal{D}}\}$ such that $((n-1, q), (n, k)) \in E^{\mathcal{D}}$.

If \mathcal{D} satisfies the all of the above properties, then we call \mathcal{D} a *Bratteli diagram*, and we denote the set of all Bratteli diagrams by \mathcal{BD} .

We also introduce the following notation. For each $n \in \mathbb{N}$, let:

$$E_n^{\mathcal{D}} = (V_n^{\mathcal{D}} \times V_{n+1}^{\mathcal{D}}) \cap E^{\mathcal{D}},$$

which by axiom (i), we have that $E^{\mathcal{D}} = \cup_{n \in \mathbb{N}} E_n^{\mathcal{D}}$. Also, for $((n, k), (n+1, q)) \in E_n^{\mathcal{D}}$, we denote $[(n, k), (n+1, q)]_{\mathcal{D}} \in \mathbb{N} \setminus \{0\}$ as the number of edges from (n, k) to $(n+1, q)$. Let $(n, k) \in V^{\mathcal{D}}$, define:

$$R_{(n,k)}^{\mathcal{D}} = \left\{ (n+1, q) \in V_{n+1}^{\mathcal{D}} : ((n, k), (n+1, q)) \in E^{\mathcal{D}} \right\},$$

which is non-empty by axiom (ii). Also, for $n \in \mathbb{N}$, we refer to $V_n^{\mathcal{D}}, E_n^{\mathcal{D}}$, and $(V_n^{\mathcal{D}}, E_n^{\mathcal{D}})$ as the vertices at level n , edges at level n , and diagram at level n , respectively.

Remark 5.2. It is easy to see that this definition coincides with Bratteli's of [5, Section 1.8] in that we simply trade his arrow notation with that of edges and number of edges. That is, given a Bratteli diagram \mathcal{D} , the correspondence is: $(n, k) \searrow^p (n+1, q)$ if and only if $((n, k), (n+1, q)) \in E^{\mathcal{D}}$ and $[(n, k), (n+1, q)]_{\mathcal{D}} = p$.

One of the first of many useful properties of Bratteli diagram is that given a Bratteli diagram there exists a unique AF algebra up to *-isomorphism associated to the diagram [5, Section 2.8], [10, Proposition III.2.7]. How we associate a Bratteli diagram to an AF algebra is described in the following Definition (5.3) following [5, Section 1.8].

Definition 5.3 ([5]). Let $\mathcal{I} = (\mathfrak{A}_n, \alpha_n)_{n \in \mathbb{N}}$ be an inductive sequence of finite dimensional C*-algebras with C*-inductive limit \mathfrak{A} , where α_n is injective for all $n \in \mathbb{N}$. Thus, \mathfrak{A} is an AF algebra by [34, Chapter 6.1]. Let $\mathcal{D}_b(\mathfrak{A})$ be a diagram associated to \mathfrak{A} constructed as follows.

Fix $n \in \mathbb{N}$. Since \mathfrak{A}_n is finite dimensional, $\mathfrak{A}_n \cong \oplus_{k=0}^{a_n} \mathfrak{M}(n(k))$ such that $a_n \in \mathbb{N}$ and $n(k) \in \mathbb{N} \setminus \{0\}$ for $k \in \{0, \dots, a_n\}$. Define:

$$v_n^{\mathcal{D}_b(\mathfrak{A})} = a_n, \quad V_n^{\mathcal{D}_b(\mathfrak{A})} = \left\{ (n, k) \in \mathbb{N}^2 : k \in \{0, \dots, v_n^{\mathcal{D}_b(\mathfrak{A})}\} \right\},$$

and label $[n, k]_{\mathcal{D}_b(\mathfrak{A})} = \sqrt{\dim(\mathfrak{M}(n(k)))}$ for $k \in \{0, \dots, v_n^{\mathcal{D}_b(\mathfrak{A})}\}$.

Let A_n be the $a_{n+1} + 1 \times a_n + 1$ -partial multiplicity matrix associated to $\alpha_n : \mathfrak{A}_n \rightarrow \mathfrak{A}_{n+1}$ with entries $(A_n)_{i,j} \in \mathbb{N}, i \in \{1, \dots, a_{n+1} + 1\}, j \in \{1, \dots, a_n + 1\}$ given by [10, Lemma III.2.2]. Define:

$$E_n^{\mathcal{D}_b(\mathfrak{A})} = \left\{ ((n, k), (n+1, q)) \in \mathbb{N}^2 \times \mathbb{N}^2 : (A_n)_{q+1, k+1} \neq 0 \right\},$$

and if $((n, k), (n + 1, q)) \in E_n^{\mathcal{D}_b(\mathfrak{A})}$, then let the number of edges be $[(n, k), (n + 1, q)]_{\mathcal{D}_b(\mathfrak{A})} = (A_n)_{q+1, k+1}$.

Let $V^{\mathcal{D}_b(\mathfrak{A})} = \cup_{n \in \mathbb{N}} V_n^{\mathcal{D}_b(\mathfrak{A})}$, $E^{\mathcal{D}_b(\mathfrak{A})} = \cup_{n \in \mathbb{N}} E_n^{\mathcal{D}_b(\mathfrak{A})}$, and $\mathcal{D}_b(\mathfrak{A}) = (V^{\mathcal{D}_b(\mathfrak{A})}, E^{\mathcal{D}_b(\mathfrak{A})})$. By [5, Section 1.8], we conclude $\mathcal{D}_b(\mathfrak{A}) \in \mathcal{BD}$ is a Bratteli diagram as in Definition (5.1).

If \mathfrak{A} is an AF algebra of the form $\mathfrak{A} = \overline{\cup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ where $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ is a non-decreasing sequence of finite dimensional C^* -subalgebras of \mathfrak{A} , then the diagram $\mathcal{D}_b(\mathfrak{A})$ has the same vertices, and the edges are formed by the partial multiplicity matrix built from the partial multiplicities of the inclusion mappings $\iota_n : \mathfrak{A}_n \rightarrow \mathfrak{A}_{n+1}$ for all $n \in \mathbb{N}$.

Remark 5.4. We note that the converse of the Definition (5.3) is true in the sense that given a Bratteli diagram, one may construct an AF algebra associated to it. The process is described in [5, Section 1.8], and in particular, one may construct partial multiplicity matrices from the edge set, which then provide injective *-homomorphisms to build an inductive limit.

As an example, which will be used in Section (6.1), we display the Bratteli diagram for the Effros-Shen AF algebras of Notation (2.19).

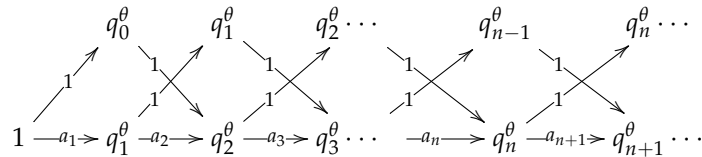
Example 5.5. Fix $\theta \in (0, 1) \setminus \mathbb{Q}$ with continued fraction expansion $\theta = [a_j]_{j \in \mathbb{N}}$ using Expression (2.1) with rational approximations $\left(\frac{p_n^\theta}{q_n^\theta}\right)_{n \in \mathbb{N}}$ given by Expression (2.2). Let $\mathfrak{A}_{\mathfrak{F}_\theta}$ be the Effros-Shen AF algebra from Notation (2.19). Thus, $v_0^{\mathcal{D}_b(\mathfrak{A}_{\mathfrak{F}_\theta})} = 0$ and $V_0^{\mathcal{D}_b(\mathfrak{A}_{\mathfrak{F}_\theta})} = \{(0, 0)\}$ with $[0, 0]_{\mathcal{D}_b(\mathfrak{A}_{\mathfrak{F}_\theta})} = 1$. For $n \in \mathbb{N} \setminus \{0\}$, we have $v_n^{\mathcal{D}_b(\mathfrak{A}_{\mathfrak{F}_\theta})} = 1$ and $V_n^{\mathcal{D}_b(\mathfrak{A}_{\mathfrak{F}_\theta})} = \{(n, 0), (n, 1)\}$ with $[n, 0]_{\mathcal{D}_b(\mathfrak{A}_{\mathfrak{F}_\theta})} = q_n^\theta$, $[n, 1]_{\mathcal{D}_b(\mathfrak{A}_{\mathfrak{F}_\theta})} = q_{n-1}^\theta$. The partial multiplicity matrix for $n = 0$ is:

$$A_0 = \begin{pmatrix} a_1 \\ 1 \end{pmatrix},$$

and let $n \in \mathbb{N} \setminus \{0\}$, then the partial multiplicity matrix is:

$$A_n = \begin{pmatrix} a_{n+1} & 1 \\ 1 & 0 \end{pmatrix},$$

by Notation (2.19) and [10, Lemma III.2.1], which determines the edges. We now provide the diagram as a graph, where the label in the edges denotes number of edges and the top row contains the vertices $(n, 1)$ with their labels with n increasing from left to right with the bottom row having vertices $(n, 0)$ with their labels with n increasing from left to right. Let $n \geq 4$:



Returning to the diagram setting, we define what an ideal of a diagram is.

Definition 5.6. Let $\mathcal{D} = (V^{\mathcal{D}}, E^{\mathcal{D}})$ be a Bratteli diagram defined in Definition (5.1). We call $\mathcal{D}(I) = (V^I, E^I)$ an *ideal diagram* of \mathcal{D} if $V^I \subseteq V^{\mathcal{D}}, E^I \subseteq E^{\mathcal{D}}$ and:

- (i) (directed) if $(n, k) \in V^I$ and $((n, k), (n+1, q)) \in E^{\mathcal{D}}$, then $(n+1, q) \in V^I$.
- (ii) (hereditary) if $(n, k) \in V^{\mathcal{D}}$ and $R_{(n,k)}^{\mathcal{D}} \subseteq V^I$, then $(n, k) \in V^I$.
- (iii) (edges) If $(n, k), (n+1, q) \in V^I$ such that $((n, k), (n+1, q)) \in E^{\mathcal{D}}$, then $((n, k), (n+1, q)) \in E^I$.

Furthermore, if $(n, k) \in V^{\mathcal{D}} \cap V^I$, then $[n, k]_{\mathcal{D}} = [n, k]_{\mathcal{D}(I)}$. And, if $((n, k), (n+1, q)) \in E^{\mathcal{D}} \cap E^I$, then $[(n, k), (n+1, q)]_{\mathcal{D}} = [(n, k), (n+1, q)]_{\mathcal{D}(I)}$.

Also, for $n \in \mathbb{N}$, denote $V_n^I = V_n^{\mathcal{D}} \cap V^I$ and $E_n^I = E_n^{\mathcal{D}} \cap E^I$ with $I_n = (V_n^I, E_n^I)$ to also include all associated labels and number of edges, and we will refer to V_n^I as the vertices at level n of the diagram. Let $\text{Ideal}(\mathcal{D})$ denote the set of ideals of \mathcal{D} .

Directly, from this definition, we may prove a Lemma that will serve us later.

Lemma 5.7. Using Definition (5.1), let $\mathcal{D} = (V^{\mathcal{D}}, E^{\mathcal{D}})$ be a Bratteli diagram. Using Definition (5.6), if $I, J \in \text{Ideal}(\mathcal{D})$ such that there exists $n \in \mathbb{N}$ with $V_n^I \neq \emptyset$ and $V_n^I = V_n^J$, then $I_m = J_m$ for all $m \leq n$.

Proof. Assume that $n \in \mathbb{N} \setminus \{0\}$ and $V_n^I = V_n^J \neq \emptyset$. Let $(n-1, k) \in V_{n-1}^I$. By directed, for all $(n, q) \in R_{(n-1,k)}^{\mathcal{D}}$, we have that $(n, q) \in V_n^I = V_n^J$. Therefore, the set $R_{(n-1,k)}^{\mathcal{D}} \subset V_n^J$. Hence, by hereditary, we have that $(n-1, k) \in V_{n-1}^J$. Thus, the set $V_{n-1}^I \subseteq V_{n-1}^J$ and the fact that the argument is symmetric in the other direction implies that $V_{n-1}^I = V_{n-1}^J$. We may continue in this fashion to show that vertices of the ideals agree up to n . By the edge axiom in Definition (5.6), we thus have that $E_m^I = E_m^J$ for all $m \leq n-1$, but by the directed property, we also have that $E_n^I = E_n^J$. As the labels of vertices and number of edges for both I and J are both inherited from \mathcal{D} , our proof is finished. \square

We now define a metric on ideals of a Bratteli diagram.

Definition 5.8. Using Definition (5.1), let $\mathcal{D} \in \mathcal{BD}$ be a Bratteli diagram and for each $n \in \mathbb{N}$, let $Z_{v_n^{\mathcal{D}}} = \prod_{k=0}^{v_n^{\mathcal{D}}} \mathbb{Z}_2$.

Let $\mathcal{C}_{\mathcal{D}} = \prod_{n \in \mathbb{N}} Z_{v_n^{\mathcal{D}}}$. Denote an element in $x \in \mathcal{C}_{\mathcal{D}}$ by $x = (x(0), x(1), \dots)$, where $x(n) = (x(n)_0, x(n)_1, \dots, x(n)_{v_n^{\mathcal{D}}}) \in Z_{v_n^{\mathcal{D}}}$ for all $n \in \mathbb{N}$. Define a metric on $\mathcal{C}_{\mathcal{D}}$ by:

$$m_{\mathcal{C}}(x, y) = \begin{cases} 0 & \text{if } x(n) = y(n), \forall n \in \mathbb{N} \\ 2^{-n} & \text{if } n = \min\{m \in \mathbb{N} : x(m) \neq y(m)\}. \end{cases}$$

We note that it is a routine argument that $m_{\mathcal{C}}$ is a metric. Furthermore, if each $Z_{v_n^{\mathcal{D}}}$ is given the discrete topology and $\mathcal{C}_{\mathcal{D}}$ is given the product topology, then $m_{\mathcal{C}}$ metrizes this topology. As each $Z_{v_n^{\mathcal{D}}}$ is finite and nonempty, $(\mathcal{C}_{\mathcal{D}}, m_{\mathcal{C}})$ is a Cantor space, a nonempty perfect zero-dimensional compact ultrametric space.

Proposition 5.9. Using Definition (5.1), let \mathcal{D} be a Bratteli diagram. Using Definitions (5.6, 5.8), if we define:

$$i_m(\cdot, \mathcal{D}) : \text{Ideal}(\mathcal{D}) \rightarrow \mathcal{C}_{\mathcal{D}}$$

coordinate-wise in the following way:

$$i_m(I, \mathcal{D})(n)_k = \begin{cases} 1 & \text{if } (n, k) \in V^I \\ 0 & \text{if } (n, k) \in V^{\mathcal{D}} \setminus V^I \end{cases}$$

then $i_m(\cdot, \mathcal{D})$ is a well-defined injection such that $(i_m(\text{Ideal}(\mathcal{D}), \mathcal{D}), m_{\mathcal{C}})$ is a zero-dimensional compact ultrametric space.

Furthermore, let $m_{i_m(\mathcal{D})} = m_{\mathcal{C}} \circ (i_m(\cdot, \mathcal{D}) \times i_m(\cdot, \mathcal{D}))$. Then, the metric space $(\text{Ideal}(\mathcal{D}), m_{i_m(\mathcal{D})})$ is a zero-dimensional compact ultrametric space.

Proof. The map $i_m(\cdot, \mathcal{D})$ is well-defined by construction. For injectivity, assume that there exist $I, J \in \text{Ideal}(\mathcal{D})$ such that $i_m(I, \mathcal{D}) = i_m(J, \mathcal{D})$. By definition, this implies that $i_m(I, \mathcal{D})(n) = i_m(J, \mathcal{D})(n)$ for each $n \in \mathbb{N}$, and therefore, the vertices $V_n^I = V_n^J$ for each $n \in \mathbb{N}$. Thus, applying Lemma (5.7), we have that $I = J$.

For compactness of $(i_m(\text{Ideal}(\mathcal{D}), \mathcal{D}), m_{\mathcal{C}})$, we need only to show that $i_m(\text{Ideal}(\mathcal{D}), \mathcal{D})$ is closed as $(\mathcal{C}_{\mathcal{D}}, m_{\mathcal{C}})$ is compact. Thus, assume $j \in \mathcal{C}_{\mathcal{D}}$ such that there exists $(J^n)_{n \in \mathbb{N}} \subseteq \text{Ideal}(\mathcal{D})$ with $\lim_{n \rightarrow \infty} m_{\mathcal{C}}(i_m(J^n, \mathcal{D}), j) = 0$. With this, construct an increasing sequence $(c_n)_{n \in \mathbb{N}} \subset \mathbb{N}$ such that for fixed $n \in \mathbb{N}$, we have:

$$m_{\mathcal{C}}(i_m(J^l, \mathcal{D}), j) \leq 2^{-(n+1)}$$

for all $l \geq c_n$.

By definition of $m_{\mathcal{C}}$, for each $n \in \mathbb{N}, p \in \{0, \dots, n\}$, we gather that $j(p) = i_m(J^l, \mathcal{D})(p)$ for all $l \geq c_n$. In particular, we also have that:

$$(5.1) \quad j(p) = i_m(J^{c_n}, \mathcal{D})(p) = i_m(J^{c_n+q}, \mathcal{D})(p)$$

for all $n \in \mathbb{N}, q \in \mathbb{N}, p \in \{0, \dots, n\}$ since $(c_n)_{n \in \mathbb{N}}$ is increasing. Thus:

$$(5.2) \quad V_p^{J^{c_n}} = V_p^{J^{c_n+q}}$$

for all $q \in \mathbb{N}, n \in \mathbb{N}, p \in \{0, \dots, n\}$ by definition of $i_m(\cdot, \mathcal{D})$. Thus, for each $n \in \mathbb{N}$, define $V_n^J = V_n^{J^{c_n}}$. Now, let:

$$V^J = \bigcup_{n \in \mathbb{N}} V_n^{J^{c_n}}.$$

Form E^J by imposing the edge axiom (iii) from Definition (5.6). For $J = (V^J, E^J)$, inherit the vertex labels and number of edges from \mathcal{D} as done in Definition (5.6). We claim that $J \in \text{Ideal}(\mathcal{D})$ and that $i_m(J, \mathcal{D}) = j$.

First, let $(n, k) \in V^J$ such that $((n, k), (n+1, q)) \in E^{\mathcal{D}}$. But, $(n, k) \in V_n^{J^{c_n}} = V_n^{J^{c_{n+1}}}$ by Equation (5.2). Since $V^{J^{c_{n+1}}}$ is an ideal, by the directed axiom (i), we have that $(n+1, q) \in V^{J^{c_{n+1}}} \cap V_{n+1}^{\mathcal{D}} = V_{n+1}^{J^{c_{n+1}}} \subseteq V^J$, which provides directed axiom (i) for J .

Next, for the hereditary axiom (ii), let $(n, k) \in V^{\mathcal{D}}$ and $R_{(n,k)}^{\mathcal{D}} \subseteq V^J$. Now, the set $R_{(n,k)}^{\mathcal{D}} \subseteq V_{n+1}^{J^{c_{n+1}}}$. Thus, since $V^{J^{c_{n+1}}}$ is an ideal, then $(n, k) \in V_n^{J^{c_{n+1}}} = V_n^{J^{c_n}} \subseteq V^J$ by Equation (5.2), which proves the hereditary axiom (ii) for J . Axiom (iii) for edges is given by construction. Furthermore, as the labels of vertices and number of edges are inherited from \mathcal{D} , we have that $J \in \text{Ideal}(\mathcal{D})$ by Definition (5.6).

Next, fix $n \in \mathbb{N}, k \in \{0, \dots, v_n^{\mathcal{D}}\}$, then by Equation (5.1), we have $j(n)_k = 1 \iff i_m(J^{c_n}, \mathcal{D})(n)_k = 1 \iff (n, k) \in V^{J^{c_n}} \iff (n, k) \in V_n^{J^{c_n}} = V_n^J \subseteq V^J \iff i_m(J, \mathcal{D})(n)_k = 1$.

Now, assume that $j(n)_k = 0$. Then, by Equation (5.1), we have $0 = j(n)_k = i_m(J^{c_n}, \mathcal{D})(n)_k$ implies that $(n, k) \in V^{\mathcal{D}} \setminus V^{J^{c_n}} = \cap_{l \in \mathbb{N}} (V^{\mathcal{D}} \setminus V_l^{J^{c_n}})$. Thus, the vertex $(n, k) \in V^{\mathcal{D}} \setminus V_n^{J^{c_n}} = V^{\mathcal{D}} \setminus V_n^J$. However, for all $m \in \mathbb{N} \setminus \{n\}$, the set V_m^J does not contain a vertex of the form (n, l) for any l , and thus the vertex $(n, k) \notin V_m^J$ for all $m \in \mathbb{N} \setminus \{n\}$ as well. Hence, the vertex $(n, k) \in V^{\mathcal{D}} \setminus V^J \iff i_m(J, \mathcal{D})(n)_k = 0$.

For the reverse implication, assume that $i_m(J, \mathcal{D})(n)_k = 0$, then $(n, k) \in V^{\mathcal{D}} \setminus V^J = \cap_{l \in \mathbb{N}} (V^{\mathcal{D}} \setminus V_l^J)$. Hence, it must be the case that $(n, k) \notin V_n^{J^{c_n}} = V_n^J$. Again, the vertex $(n, k) \notin V_m^{J^{c_n}}$ for all $m \in \mathbb{N} \setminus \{n\}$ as well. Thus, the vertex $(n, k) \in V^{\mathcal{D}} \setminus V^{J^{c_n}}$, which implies that $j(n)_k = i_m(J^{c_n}, \mathcal{D})(n)_k = 0$ by Equation (5.1). Hence, we conclude $j(n)_k = 0 \iff i_m(J, \mathcal{D})(n)_k = 0$ for $n \in \mathbb{N}, k \in \{0, \dots, v_n^{\mathcal{D}}\}$.

Therefore, we have $i_m(J, \mathcal{D})(n) = j(n)$ for all $n \in \mathbb{N}$. Hence, the space $(i_m(\text{Ideal}(\mathcal{D}), \mathcal{D}), m_{\mathcal{C}})$ is a compact metric space. Zero-dimensional is inherited from $(\mathcal{C}_{\mathcal{D}}, m_{\mathcal{C}})$. The fact that the metric space $(\text{Ideal}(\mathcal{D}), m_{i_m(\mathcal{D})})$ is a zero-dimensional compact metric space follows from the fact that $i_m(\cdot, \mathcal{D})$ is injective and that $i_m(\text{Ideal}(\mathcal{D}), \mathcal{D})$ is compact in $(\mathcal{C}_{\mathcal{D}}, m_{\mathcal{C}})$. \square

The metric of Proposition (5.9) is stated entirely in the setting of Bratteli diagram without acknowledging an AF algebra. But, we would like utilize Proposition (5.9) to provide compactness of the metric of Proposition (4.11) in the case of AF algebras. Thus, we now begin this transition.

Notation 5.10. Let $\mathfrak{A} = \overline{\cup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ be an AF algebra where $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ is a non-decreasing sequence of finite dimensional C^* -subalgebras of \mathfrak{A} . Let $\mathcal{D}_b(\mathfrak{A})$ be the diagram given by Definition (5.3).

Let $I \in \text{Ideal}(\mathfrak{A})$ be a norm closed two-sided ideal of \mathfrak{A} , then by [5, Lemma 3.2], the subset Λ of $\mathcal{D}_b(\mathfrak{A})$ formed by I is an ideal in the sense of Definition (5.6), and denote this by $\mathcal{D}_b(\mathfrak{A})(I) \in \text{Ideal}(\mathcal{D}_b(\mathfrak{A}))$, where $\text{Ideal}(\mathcal{D}_b(\mathfrak{A}))$ is the set of ideals of $\mathcal{D}_b(\mathfrak{A})$ from Definition (5.6).

We state the following classic result for notation and convenience.

Proposition 5.11. [5, Lemma 3.2] *Let $\mathfrak{A} = \overline{\cup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ be an AF algebra where $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ is a non-decreasing sequence of finite dimensional C^* -subalgebras of \mathfrak{A} and Bratteli diagram $\mathcal{D}_b(\mathfrak{A})$ from Definition (5.3). Using Notation (5.10) and Definition (5.6), the map:*

$$i(\cdot, \mathcal{D}_b(\mathfrak{A})) : I \in \text{Ideal}(\mathfrak{A}) \mapsto \mathcal{D}_b(\mathfrak{A})(I) \in \text{Ideal}(\mathcal{D}_b(\mathfrak{A}))$$

given by [5, Lemma 3.2] is a well-defined bijection, where the vertices of $V_n^{\mathcal{D}_b(\mathfrak{A})(I)}$ are determined by $I \cap \mathfrak{A}_n$ for each $n \in \mathbb{N}$.

We are now prepared to strengthen Proposition (4.11) in the case of AF algebras.

Theorem 5.12. *If $\mathfrak{A} = \overline{\cup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ is a C^* -algebra in which where $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ is a non-decreasing sequence of finite dimensional C^* -subalgebras of \mathfrak{A} , and is thus AF, then*

using Definition (4.10), we have that the map $i(\cdot, \mathcal{D}_b(\mathfrak{A}))$ of Proposition (5.11) is an isometry from the metric space $(\text{Ideal}(\mathfrak{A}), m_{i(\mathcal{U})})$ of Proposition (4.11) onto the metric space $(\text{Ideal}(\mathcal{D}_b(\mathfrak{A})), m_{i_m(\mathcal{D}_b(\mathfrak{A}))})$ using notation of Definition (5.3) and Proposition (5.9).

Therefore, the space $(\text{Ideal}(\mathfrak{A}), m_{i(\mathcal{U})})$ is a zero-dimensional compact ultrametric space, and moreover, the topology induced by $m_{i(\mathcal{U})}$ on $\text{Ideal}(\mathfrak{A})$ is the Fell topology, Definition (4.3).

Proof. The isometry is given by Proposition (5.11). Indeed, since the vertices of $V_n^{\mathcal{D}_b(\mathfrak{A})}(I)$ are determined by $I \cap \mathfrak{A}_n$ for each $n \in \mathbb{N}$ for any $I \in \text{Ideal}(\mathfrak{A})$, if $I, J \in \text{Ideal}(\mathfrak{A})$, then $i(I, \mathcal{D}_b(\mathfrak{A}))(n) = i(J, \mathcal{D}_b(\mathfrak{A}))(n)$ if and only if $I \cap \mathfrak{A}_n = J \cap \mathfrak{A}_n$ by Lemma (5.7). Thus:

$$i(\cdot, \mathcal{D}_b(\mathfrak{A})) : I \in \text{Ideal}(\mathfrak{A}) \longmapsto \mathcal{D}_b(\mathfrak{A})(I) \in \text{Ideal}(\mathcal{D}_b(\mathfrak{A}))$$

is an isometry from $(\text{Ideal}(\mathfrak{A}), m_{i(\mathcal{U})})$ onto the metric space $(\text{Ideal}(\mathcal{D}_b(\mathfrak{A})), m_{i_m(\mathcal{D}_b(\mathfrak{A}))})$. Therefore, $(\text{Ideal}(\mathfrak{A}), m_{i(\mathcal{U})})$ is a zero-dimensional compact ultrametric space. But, by Theorem (4.16), the metric topology of $(\text{Ideal}(\mathfrak{A}), m_{i(\mathcal{U})})$ is a compact Hausdorff topology that contains the compact Hausdorff topology, *Fell*. Therefore, by maximal compactness, the two topologies equal. \square

We now begin a sequence of Corollaries that highlight the consequences of Theorem (5.12). All of these Corollaries are phrased in terms of $m_{i(\mathcal{U})}$, but can be translated in terms of the diagram metric $m_{i_m(\mathcal{D})}$ by Theorem (5.12), and we note that $m_{i_m(\mathcal{D})}$ will prove useful in its own right in the proof of Theorem (5.21), Proposition (6.11), and the main result of Section (6.1), Theorem (6.20), as many results and constructions with regard to AF algebras are phrased diagrammatically. First, Theorem (5.12) provides the same comparison with the Jacobson topology as the Fell topology.

Corollary 5.13. *If \mathfrak{A} be a C^* -algebra with a non-decreasing sequence of finite dimensional C^* -subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ such that $\mathfrak{A} = \overline{\cup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, then using notation from Proposition (4.11), the space $(\text{Prim}(\mathfrak{A}), m_{i(\mathcal{U})})$ is a totally bounded zero-dimensional ultrametric space in which the relative topology on $\text{Prim}(\mathfrak{A})$ induced by the metric topology $m_{i(\mathcal{U})}$ or the Fell topology contains the Jacobson topology on $\text{Prim}(\mathfrak{A})$.*

Proof. Apply Theorem (5.12) to Proposition (4.5). And, since total boundedness and zero-dimensionality are hereditary properties, the proof is complete. \square

Another immediate consequence of Theorem (5.12) is that, although the metric is built using a fixed inductive sequence, the metric topology with respect to an inductive sequence is homeomorphic to the metric topology on the same AF algebra with respect to any other inductive sequence. In particular, concerning continuity or convergence results, Corollary (5.14) provides that one need not worry about the possibility of choosing the wrong inductive sequence, and therefore, one may choose any inductive sequence without worry to suit the needs of the problem at hand.

Corollary 5.14. *Let $\mathfrak{A}, \mathfrak{B}$ be C^* -algebras with non-decreasing sequences of finite dimensional C^* -subalgebras $\mathcal{U}_{\mathfrak{A}} = (\mathfrak{A}_n)_{n \in \mathbb{N}}, \mathcal{U}_{\mathfrak{B}} = (\mathfrak{B}_n)_{n \in \mathbb{N}}$, respectively, such that $\mathfrak{A} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ and $\mathfrak{B} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{B}_n}$.*

If \mathfrak{A} and \mathfrak{B} are $$ -isomorphic, then the metric spaces $(\text{Ideal}(\mathfrak{A}), m_{i(\mathcal{U}_{\mathfrak{A}})})$ and $(\text{Ideal}(\mathfrak{B}), m_{i(\mathcal{U}_{\mathfrak{B}})})$ are homeomorphic.*

In particular, if $\mathfrak{A} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_{1,n}}^{\|\cdot\|_{\mathfrak{A}}} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_{2,n}}^{\|\cdot\|_{\mathfrak{A}}}$, where $\mathcal{U}_1 = (\mathfrak{A}_{1,n})_{n \in \mathbb{N}}, \mathcal{U}_2 = (\mathfrak{A}_{2,n})_{n \in \mathbb{N}}$ are non-decreasing sequences of finite dimensional C^ -subalgebras of \mathfrak{A} , then the metric spaces $(\text{Ideal}(\mathfrak{A}), m_{i(\mathcal{U}_1)})$ and $(\text{Ideal}(\mathfrak{A}), m_{i(\mathcal{U}_2)})$ are homeomorphic.*

Proof. Apply Lemma (4.7) to Theorem (5.12). \square

Furthermore, as another consequence of Theorem (5.12), we may strengthen Proposition (4.14) with the Fell topology in the case of AF algebras.

Corollary 5.15. *Let \mathfrak{A} be a C^* -algebra with a non-decreasing sequence of finite dimensional C^* -subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$.*

If $(I^n)_{n \in \mathbb{N}} \subseteq \text{Ideal}(\mathfrak{A})$ converges to $I^\infty \in \text{Ideal}(\mathfrak{A})$ with respect to $m_{i(\mathcal{U})}$ or the Fell topology, then using Definition (3.4), the family $\{I^n = \overline{\bigcup_{k \in \mathbb{N}} I^n \cap \mathfrak{A}_k}^{\|\cdot\|_{\mathfrak{A}}} : n \in \overline{\mathbb{N}}\}$ is a fusing family with fusing sequence $(c_n)_{n \in \mathbb{N}}$ such that $\{\varinjlim \mathcal{I}(\mathfrak{A}/I^n) : n \in \overline{\mathbb{N}}\}$ is an IL-fusing family with fusing sequence $(c_n)_{n \in \mathbb{N}}$.

Proof. Apply Theorem (5.12) Proposition (4.14). \square

Now, that we have this identification with the Fell topology, we finish our discussion of the metric topology by considering it in the commutative case. The reason for this is because if \mathfrak{A} is a commutative C^* -algebra, then the Jacobson topology on the primitive ideals of \mathfrak{A} is homeomorphic to the maximal ideal space with its weak- $*$ topology, which is a classic result for which we provide a proof of in the unital case, Theorem (5.18). Furthermore, in the unital case, we will show that the relative topology on the primitive ideals induced by the metric topology will be compact, which will provide that this metric topology agrees with the Jacobson topology since it is compact in the unital case. This result rests on a characterization of Bratteli diagrams associated to unital commutative AF algebras provided by Bratteli as [6, Expression 3.1] along with his diagrammatic characterization of primitive ideals found as [5, Theorem 3.8], [6, Expression 2.7]. We begin with some classical results about the ideal space of unital commutative C^* -algebras.

Lemma 5.16. *Let (X, τ) be a compact Hausdorff space. If $U \subset X$, then $I_U = \{f \in C(X) : \forall u \in U, f(u) = 0\} \in \text{Ideal}(C(X))$ and $I_{\overline{U}^\tau} = I_U$, where \overline{U}^τ denotes the closure of U with respect to τ .*

Proof. Fix $x \in X$. Consider $I_{\{x\}}$. It is routine to check that $I_{\{x\}} \in \text{Ideal}(C(X))$. Now, let $U \subseteq X$. It is clear that I_U is a two-sided ideal. But, note that $I_U = \bigcap_{x \in U} I_{\{x\}}$ is the intersection of closed sets and is therefore closed. Hence, the ideal $I_U \in \text{Ideal}(C(X))$.

The ideal $I_{\overline{U}^\tau} \subseteq I_U$ since $U \subseteq \overline{U}^\tau$. Let $f \in I_U$. Let $v \in \overline{U}^\tau$, then there exists a net $(u_\lambda)_{\lambda \in \Lambda} \subseteq U$ converging to v . Hence, we have $f(u_\lambda) = 0$ for all $\lambda \in \Lambda$, and since

f is continuous, we conclude $f(v) = 0$. Therefore, the function $f \in \bigcap_{x \in \bar{U}^\tau} I_{\{x\}} = I_{\bar{U}^\tau}$. Thus, the ideal $I_{\bar{U}^\tau} = I_U$. \square

Proposition 5.17. *Let (X, τ) be a compact Hausdorff space. If $I \in \text{Ideal}(C(X))$, then $F = \{x \in X : \forall f \in I, f(x) = 0\}$ is closed and $I = I_F$ of Lemma (5.16). Moreover, $F \mapsto I_F$ establishes a one-to-one correspondence between closed subsets of (X, τ) and $\text{Ideal}(C(X))$.*

Proof. Let $F \subseteq X$ be closed. By Lemma (5.16), we have that $I_F \in \text{Ideal}(C(X))$.

Next, assume $I \in \text{Ideal}(C(X))$. If $I = \{0\}$ or $C(X)$, then $F = X$ or $F = \emptyset$, respectively. Also, if I were maximal, then by [9, Theorem VII.8.7], the ideal $I = I_{\{x\}}$ for some $x \in X$. Assume that $I \in \text{Ideal}(C(X))$ and not maximal with $\{0\} \subsetneq I \subsetneq C(X)$. Define:

$$(5.3) \quad F = \{x \in X : \forall f \in I, f(x) = 0\} = \bigcap_{f \in I} f^{-1}(\{0\}),$$

which shows that F is closed since each $f \in I$ is continuous. As in the statement of the proposition, define $I_F = \{f \in C(X) : \forall x \in F, f(x) = 0\} \in \text{Ideal}(C(X))$ by Lemma (5.16).

First, we show that $I \subseteq I_F$. Let $f \in I$. Let $x \in F$, then by definition of F , we have $f(x) = 0$. Since $x \in F$ was arbitrary, the function $f \in I_F$ by definition and by Lemma (5.16) since F is closed.

For the reverse containment, assume $f \notin I$. Let $\Gamma : C(X)/I \rightarrow C(M)$ denote the Gelfand transform, which is a *-isomorphism map given by evaluation, where M the space of multiplicative linear functionals associated to maximal ideals of $C(X)/I$ as kernels [9, Theorem VIII.2.1]. Since $f \notin I$, we have $f + I \neq 0 + I$. Thus, $\Gamma(f + I) \neq 0 \implies \widehat{f + I} \neq 0$. So, there exists $\varphi_m \in M$, where $\ker \varphi_m = m$, a maximal ideal of $C(X)/I$, such that:

$$(5.4) \quad 0 \neq \widehat{f + I}(\varphi_m) = \varphi_m(f + I).$$

In particular, we have $f + I \notin \ker \varphi_m = m$.

From this, define $m' = \{g \in C(X) : (g + I) \in m\}$. Clearly, the ideal $I \subseteq m'$. But, the set m' is closed since quotient map is continuous and m is closed. Also, we have m' is a two-sided ideal since m and I are two-sided ideals. Thus, the set $m' \in \text{Ideal}(C(X))$. Therefore, the quotient m'/I is well-defined and it is routine to check that $m'/I = m$. Furthermore, by [11, Proposition 2.11.5] along with commutativity and [34, Theorem 5.4.4], the ideal m' is maximal since m is maximal. Therefore, by [9, Theorem VII.8.7], there exists $y \in X$ such that $m' = I_{\{y\}} = \{g \in C(X) : g(y) = 0\}$. But, the containment $I \subseteq m' = I_{\{y\}}$ implies that $g(y) = 0$ for all $g \in I$. Thus, we gather that $y \in F$ by definition of F in Expression (5.3). Now, Expression (5.4) implies that $f + I \notin m$, but then, the function $f \notin m' = I_{\{y\}}$ by definition of m' . Hence, we have $f(y) \neq 0$, but $y \in F$. Hence, since F is closed, we have $f \notin I_F$ by Lemma (5.16), and thus, the ideal $I_F \subseteq I$, which completes the argument for $I = I_F$.

Lastly, we have already established that the map $I \mapsto I_F$ is well-defined and onto. What remains is injectivity. Assume $F \neq E$ are closed subsets of X , then choose $e \in E$ such that $e \notin F$. By Urysohn's lemma, there exists $f \in C(X)$ such

that $f(v) = 0$ for all $v \in F$, but $f(e) \neq 0$. Since F is closed, we have $f \in I_F$ by Lemma (5.16). But, also, we have that $f \notin I_E$. Thus, the ideal $I_E \neq I_F$. \square

We can now present a classical result for convenience.

Theorem 5.18. *If (X, τ) is compact Hausdorff space, then the map:*

$$\mathfrak{c}_{\text{Prim}, X} : x \in X \longmapsto I_{\{x\}} \in \text{Prim}(C(X))$$

is a homeomorphism from (X, τ) onto $\text{Prim}(C(X))$ with its Jacobson topology (Definition (4.1)), which is therefore compact Hausdorff.

Moreover, let \mathfrak{A} be a unital commutative C^ -algebra, if $M_{\mathfrak{A}}$ denotes its maximal ideal space with its weak- $*$ topology, then the map:*

$$\varphi \in M_{\mathfrak{A}} \longmapsto \ker \varphi \in \text{Prim}(\mathfrak{A}).$$

is a homeomorphism onto $\text{Prim}(\mathfrak{A})$ with its Jacobson topology, and therefore $\text{Prim}(\mathfrak{A})$ with its Jacobson topology is a compact Hausdorff space.

Proof. By [34, Theorem 5.4.4], the set $\text{Prim}(C(X))$ is the set of maximal ideals. Furthermore, by [9, Theorem VII.8.7], every maximal ideal is of the form $I_{\{x\}}$ for some $x \in X$, and $I_{\{x\}} = I_{\{y\}}$ if and only if $x = y$ by Proposition (5.17). Therefore, the map $\mathfrak{c}_{\text{Prim}, X}$ is a well-defined bijection.

We continue with continuity of $\mathfrak{c}_{\text{Prim}, X}$. Let P be a closed set in the Jacobson topology of $\text{Prim}(C(X))$. Thus, there exists an ideal $I \subseteq C(X)$ such that $P = \{J \in \text{Prim}(C(X)) : J \supseteq I\}$ by Definition (4.1). But, by Proposition (5.17), we have that there exists a closed set $F \subseteq X$ such that $I = I_F$. Next, let $(x_\mu)_{\mu \in \Lambda} \subset \mathfrak{c}_{\text{Prim}}^{-1}(P)$ be a net that converges to some $y \in X$. Hence, for each $\mu \in \Lambda$, we have $I_{\{x_\mu\}} \supseteq I_F$. Now, fix $f \in I_F$, this implies $f(x_\mu) = 0$ for all $\mu \in \Lambda$. But, as f is continuous and the net converges to y , we conclude $f(y) = 0$. Therefore, the ideal $I_{\{y\}} \supseteq I_F$ and $y \in \mathfrak{c}_{\text{Prim}, X}^{-1}(P)$, which implies that $\mathfrak{c}_{\text{Prim}, X}^{-1}(P)$ is closed, which establishes continuity of $\mathfrak{c}_{\text{Prim}, X}$.

Next, we establish continuity of $\mathfrak{c}_{\text{Prim}, X}^{-1}$. Let $F \subseteq X$ be closed. Then, the set $(\mathfrak{c}_{\text{Prim}, X}^{-1})^{-1}(F) = \mathfrak{c}_{\text{Prim}, X}(F) = \{I_{\{x\}} \in \text{Prim}(C(X)) : x \in F\}$ as $\mathfrak{c}_{\text{Prim}, X}$ is a bijection. Consider the ideal $I_F \in \text{Ideal}(C(X))$. By Definition (4.1), we have $P = \{J \in \text{Prim}(C(X)) : J \supseteq I_F\}$ is closed. We claim that $P = \mathfrak{c}_{\text{Prim}, X}(F)$. Clearly, the image $\mathfrak{c}_{\text{Prim}, X}(F) \subseteq P$. Assume that $J \in P$. Thus, by [9, Theorem VII.8.7], there exists $y \in X$ such that $J = I_{\{y\}}$, which implies that $I_F \subseteq I_{\{y\}}$. But, then $I_F \subseteq I_{F \cup \{y\}} \subseteq I_F$ and $I_F = I_{F \cup \{y\}}$. However, as $F \cup \{y\}$ is closed, the set $F = F \cup \{y\}$ by Proposition (5.17), which implies $y \in F$. And thus, the ideal $J = I_{\{y\}} \in \mathfrak{c}_{\text{Prim}, X}(F)$, which establishes $P = \mathfrak{c}_{\text{Prim}, X}(F)$ is closed, which completes the continuity argument and homeomorphism.

Lastly, let \mathfrak{A} be a unital commutative C^* -algebra. By [9, Theorem VIII.2.1], the Gelfand transform: $\Gamma : a \in \mathfrak{A} \mapsto \hat{a} \in C(M_{\mathfrak{A}})$ is a $*$ -isomorphism, where \hat{a} denotes evaluation and where $M_{\mathfrak{A}}$ is the maximal ideal space of \mathfrak{A} with its weak- $*$ topology, which is compact Hausdorff. By Lemma (4.6), the map $\Gamma_i^{-1} : I \in \text{Prim}(C(M_{\mathfrak{A}})) \mapsto \Gamma^{-1}(I) \in \text{Prim}(\mathfrak{A})$ is a homeomorphism with respect to the associated Jacobson topologies.

But, since we have already verified that $\mathfrak{c}_{\text{Prim}, X}$ is a homeomorphism given any compact Hausdroff space X , we gather that:

$$\Gamma_i^{-1} \circ \mathfrak{c}_{\text{Prim}, M_{\mathfrak{A}}} : \varphi \in M_{\mathfrak{A}} \longmapsto \Gamma^{-1} \left(I_{\{\varphi\}} \right) \in \text{Prim}(\mathfrak{A})$$

is a homeomorphism onto $\text{Prim}(\mathfrak{A})$ with its Jacobson topology. However, fix $\varphi \in M_{\mathfrak{A}}$, then:

$$\begin{aligned} \Gamma^{-1} \left(I_{\{\varphi\}} \right) &= \left\{ a \in \mathfrak{A} : \hat{a} \in I_{\{\varphi\}} \right\} \\ &= \left\{ a \in \mathfrak{A} : \hat{a}(\varphi) = 0 \right\} \\ &= \left\{ a \in \mathfrak{A} : \varphi(a) = 0 \right\} \\ &= \ker \varphi, \end{aligned}$$

which completes the proof. \square

Now, that sufficient classical results are established, we return to diagrams and some notation common with respect to Bratteli diagrams.

Notation 5.19. Let $\mathcal{D} \in \mathcal{BD}$ be a Bratteli diagram of Definition (5.1). For $(n, k), (m, r) \in V^{\mathcal{D}}, m \geq n$, we write $(n, k) \downarrow (m, r)$ if there exists a sequence $((n, k_p))_{p=n}^m \subset V^{\mathcal{D}}$ such that $(n, k_n) = (n, k)$ and $(m, r) = (m, k_m)$ and $((n, k_p), (p+1, k_{p+1})) \in E^{\mathcal{D}}$ for all $p \in \{n, \dots, m-1\}$.

We require more information for the diagram associated to the AF algebra as a quotient of an ideal of an AF algebra. This is Remark (5.20).

Remark 5.20. Let \mathfrak{A} be a unital C^* -algebra with a non-decreasing sequence of finite dimensional unital C^* -subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$. Let $I \in \text{Ideal}(\mathfrak{A})$. Recall the map, $i(\cdot, \mathcal{D}_b(\mathfrak{A}))$ defined in Proposition (5.11). We define the graph $\mathcal{D}(\mathfrak{A}/I) = \left(V^{\mathcal{D}_b(\mathfrak{A})} \setminus V^{i(I, \mathcal{D}_b(\mathfrak{A}))}, E^{\mathfrak{A}/I} \right)$, where $E^{\mathfrak{A}/I}$ is all edges from $E^{\mathcal{D}_b(\mathfrak{A})}$ between vertices in $V^{\mathcal{D}_b(\mathfrak{A})} \setminus V^{i(I, \mathcal{D}_b(\mathfrak{A}))}$ along with the induced labels and number of edges from $\mathcal{D}_b(\mathfrak{A})$. By [5, Proposition 3.7], the diagram $\mathcal{D}(\mathfrak{A}/I)$ satisfies axioms (i),(ii),(iii) of Definition (5.1). Furthermore, the diagram $\mathcal{D}(\mathfrak{A}/I)$ forms the diagram associated to the Bratteli diagram $\mathcal{D}_b(\mathfrak{A}/I)$ from Definition (5.3) up to shifting the placement of vertices as done in [5, Proposition 3.7].

Thus, we are now in a position to compare the relative metric topology with the Jacobson topology on the primitive ideals of a unital commutative C^* -algebra.

Theorem 5.21. *Let \mathfrak{A} be a unital C^* -algebra with a non-decreasing sequence of unital finite dimensional C^* -subalgebras $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ such that $\mathfrak{A} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$.*

If \mathfrak{A} is commutative, then the metric space $(\text{Prim}(\mathfrak{A}), \mathfrak{m}_{i(\mathcal{U})})$ with relative topology induced by the metric topology of $\mathfrak{m}_{i(\mathcal{U})}$ (Proposition (4.11)):

- (1) *is a zero-dimensional compact ultrametric space,*
- (2) *has the same topology as the Jacobson topology or the relative Fell topology on $\text{Prim}(\mathfrak{A})$, and*
- (3) *is homeomorphic to the maximal ideal space of \mathfrak{A} , with its weak- $*$ topology, denoted $M_{\mathfrak{A}}$, in which the homeomorphism is given by:*

$$\varphi \in M_{\mathfrak{A}} \longmapsto \ker \varphi \in \text{Prim}(\mathfrak{A}).$$

Proof. We start by verifying conclusion (1). For this, we show that $\text{Ideal}(\mathfrak{A}) \setminus \text{Prim}(\mathfrak{A})$ is open in $(\text{Ideal}(\mathfrak{A}), m_{i(\mathcal{U})})$. Note that $\mathfrak{A} \in \text{Ideal}(\mathfrak{A})$ is not primitive by Definition (4.1). Thus, we approach the proof in two cases.

Case 1. Assume $I = \mathfrak{A}$.

We show that \mathfrak{A} is isolated. Note that $\{P \in \text{Ideal}(\mathfrak{A}) : m_{i(\mathcal{U})}(P, \mathfrak{A}) < 2^{-1}\}$ is a basic open set such that $\{\mathfrak{A}\} \subseteq \{P \in \text{Ideal}(\mathfrak{A}) : m_{i(\mathcal{U})}(P, \mathfrak{A}) < 2^{-1}\}$. However, let $K \in \{P \in \text{Ideal}(\mathfrak{A}) : m_{i(\mathcal{U})}(P, \mathfrak{A}) < 2^{-1}\}$, then by definition of $m_{i(\mathcal{U})}$, we have $\mathfrak{A}_0 = \mathfrak{A} \cap \mathfrak{A}_0 = K \cap \mathfrak{A}_0$ and thus K would be unital, which implies that $K = \mathfrak{A}$. Hence, the ideal $I \in \{\mathfrak{A}\} = \{P \in \text{Ideal}(\mathfrak{A}) : m_{i(\mathcal{U})}(P, \mathfrak{A}) < 2^{-1}\} \subseteq \text{Ideal}(\mathfrak{A}) \setminus \text{Prim}(\mathfrak{A})$.

Case 2. Assume $I \in \text{Ideal}(\mathfrak{A}) \setminus \text{Prim}(\mathfrak{A})$ such that $I \neq \mathfrak{A}$

Recall the map $i(\cdot, \mathcal{D}_b(\mathfrak{A}))$ defined in Proposition (5.11). By [6, Expression 2.7], since I is not primitive:

$$(5.5) \quad \begin{aligned} & \text{there exists } N_I \in \mathbb{N} \text{ such that for all } m \geq N_I, (m, r) \in V^{\mathcal{D}_b(\mathfrak{A})} \setminus V^{i(I, \mathcal{D}_b(\mathfrak{A}))} \\ & \text{there exists } (N_I, k) \in V^{\mathcal{D}} \setminus V^{i(I, \mathcal{D}_b(\mathfrak{A}))} \text{ such that } (N_I, k) \not\Downarrow (m, r) \end{aligned}$$

using Notation (5.19).

Next, we consider the vertices of I , where $I \neq \mathfrak{A}$. Assume by way of contradiction that there exists $k \in \mathbb{N}$ such that $V_k^{i(I, \mathcal{D}_b(\mathfrak{A}))} = V_k^{\mathcal{D}_b(\mathfrak{A})}$, then by definition of $i(I, \mathcal{D}_b(\mathfrak{A}))$, this would imply that $I \cap \mathfrak{A}_k = \mathfrak{A}_k$. Since \mathfrak{A}_k is unital, then I would contain the unit, and thus, the ideal $I = \mathfrak{A}$, a contradiction to our assumption that $I \neq \mathfrak{A}$ of Case 2. Therefore, we have that $\emptyset \subsetneq V_M^{\mathcal{D}_b(\mathfrak{A})} \setminus V_M^{i(I, \mathcal{D}_b(\mathfrak{A}))} \subsetneq V_M^{\mathcal{D}_b(\mathfrak{A})}$ for all $M \in \mathbb{N}$. Therefore, at $N_I + 1$, there exists $(N_I + 1, r) \in V^{\mathcal{D}_b(\mathfrak{A})} \setminus V^{i(I, \mathcal{D}_b(\mathfrak{A}))}$. By Expression (5.5), since I is not primitive, there exists $(N_I, k) \in V^{\mathcal{D}} \setminus V^{i(I, \mathcal{D}_b(\mathfrak{A}))}$ such that $(N_I, k) \not\Downarrow (N_I + 1, r)$. Let $\mathcal{D}(\mathfrak{A}/I)$ denote the diagram associated to \mathfrak{A}/I defined in Remark (5.20). Thus, since $\mathcal{D}(\mathfrak{A}/I)$ satisfies axiom (iii) of Definition (5.1) and $(N_I, k) \not\Downarrow (N_I + 1, r)$, we have that there must exist $(N_I, l) \in V_{N_I}^{\mathcal{D}_b(\mathfrak{A})} \setminus V_{N_I}^{i(I, \mathcal{D}_b(\mathfrak{A}))}$ such that $(N_I, l) \Downarrow (N_I + 1, r)$, and so the cardinality of $V_{N_I}^{\mathcal{D}_b(\mathfrak{A})} \setminus V_{N_I}^{i(I, \mathcal{D}_b(\mathfrak{A}))}$ is greater than or equal to 2.

Thus, consider the basic open set

$$B_{m_{i(\mathcal{U})}}(I, 2^{-(N_I+2)}) = \{J \in \text{Ideal}(\mathfrak{A}) : m_{i(\mathcal{U})}(I, J) < 2^{-(N_I+2)}\}. \text{ Let}$$

$J \in B_{m_{i(\mathcal{U})}}(I, 2^{-(N_I+2)})$. Therefore, since $i(\cdot, \mathcal{D}_b(\mathfrak{A}))$ is an isometry by Theorem (5.12), we have $V_{N_I}^{i(I, \mathcal{D}_b(\mathfrak{A}))} = V_{N_I}^{i(J, \mathcal{D}_b(\mathfrak{A}))}$ and $V_{N_I}^{\mathcal{D}_b(\mathfrak{A})} \setminus V_{N_I}^{i(I, \mathcal{D}_b(\mathfrak{A}))} = V_{N_I}^{\mathcal{D}_b(\mathfrak{A})} \setminus V_{N_I}^{i(J, \mathcal{D}_b(\mathfrak{A}))}$, which thus has cardinality greater than or equal to 2, and so there exists $(N_I, k), (N_I, l) \in V_{N_I}^{\mathcal{D}_b(\mathfrak{A})} \setminus V_{N_I}^{i(J, \mathcal{D}_b(\mathfrak{A}))}$ such that $k \neq l$.

We claim that $J \in \text{Ideal}(\mathfrak{A}) \setminus \text{Prim}(\mathfrak{A})$. Assume by way of contradiction that $J \in \text{Prim}(\mathfrak{A})$. Thus by [6, Expression 2.7], there exist $m > N_I$ and $(m, r) \in V^{\mathcal{D}_b(\mathfrak{A})} \setminus V^{i(J, \mathcal{D}_b(\mathfrak{A}))}$ such that $(N_I, k) \Downarrow (m, r)$ and $(N_I, l) \Downarrow (m, r)$.

Let $((N_I, k_p))_{p=N_I}^m \subseteq V^{\mathcal{D}_b(\mathfrak{A})}$ and $((N_I, l_j))_{p=N_I}^m \subseteq V^{\mathcal{D}_b(\mathfrak{A})}$ be the sequences defined by $(N_I, k) \Downarrow (m, r)$ and $(N_I, l) \Downarrow (m, r)$, respectively, and Notation (5.19). Thus, the vertices $(m, k_m) = (m, r)$ and $(m, l_m) = (m, r)$. Hence, since $(N_I, k) \neq (N_I, l)$, there exists $p \in \{N_I + 1, \dots, m\}$ such that $(p-1, k_{p-1}) \neq (p-1, l_{p-1})$ and $(p, k_p) = (p, l_p)$ lest the condition \Downarrow not be satisfied. But, then, the edges $((p-1, k_{p-1}), (p, k_p)), ((p-1, l_{p-1}), (p, k_p)) \in E^{\mathcal{D}_b(\mathfrak{A})}$. Since the diagram $\mathcal{D}_b(\mathfrak{A})$ is a Bratteli diagram of a unital commutative AF algebra, by Bratteli's characterization of Bratteli diagrams of unital commutative AF algebras as [6, Expression 3.1], we have reached a contradiction since $(p-1, k_{p-1}) \neq (p-1, l_{p-1})$. Therefore, the ideal $J \in \text{Ideal}(\mathfrak{A}) \setminus \text{Prim}(\mathfrak{A})$ and $I \in B_{m_i(\mathcal{U})}(I, 2^{-(N_I+2)}) \subseteq \text{Ideal}(\mathfrak{A}) \setminus \text{Prim}(\mathfrak{A})$.

Combining Case 1 and Case 2, the set $\text{Ideal}(\mathfrak{A}) \setminus \text{Prim}(\mathfrak{A})$ is open and $\text{Prim}(\mathfrak{A})$ is closed in the zero-dimensional compact metric space $(\text{Ideal}(\mathfrak{A}), m_{i(\mathcal{U})})$, which is compact by Theorem (5.12). Thus, the space $(\text{Prim}(\mathfrak{A}), m_{i(\mathcal{U})})$ is a zero-dimensional compact metric space with its relative topology.

For conclusion (2), the comment about the relative Fell topology is already established by Theorem (5.12). By Theorem (5.18) and Corollary (5.13), we have that the Jacobson topology on $\text{Prim}(\mathfrak{A})$ is a compact Hausdorff topology contained in the compact Hausdorff topology given by $(\text{Prim}(\mathfrak{A}), m_{i(\mathcal{U})})$, which is compact Hausdorff by part (1). By maximal compactness, the topologies equal.

For conclusion (3), by Theorem (5.18), the set $\text{Prim}(\mathfrak{A})$ with its Jacobson topology is homeomorphic to the maximal ideal space on \mathfrak{A} with its weak-* topology. Thus, by part (2), we conclude that $(\text{Prim}(\mathfrak{A}), m_{i(\mathcal{U})})$ is homeomorphic to the maximal ideal space on \mathfrak{A} with its weak-* topology by the described homeomorphism. \square

6. CONVERGENCE OF QUOTIENTS OF AF ALGEBRAS WITH QUANTUM PROPINQUITY

In the case of unital AF algebras, we provide criteria for when convergence of ideals in the Fell topology provides convergence of quotients in the quantum propinquity topology, when the quotients are equipped with faithful tracial states.

But, first, as we saw in Corollary (5.15) and Proposition (4.15), it seems that an inductive limit is suitable for describing fusing families with regard to convergence of ideals. Thus, in order to avoid the notational trouble of too many inductive limits, we will phrase many results in this section in terms of closure of union. Hence, we now display a version of Theorem (2.15) in the setting of closure of union.

Theorem 6.1. *Let \mathfrak{A} be a unital AF algebra with unit $1_{\mathfrak{A}}$ endowed with a faithful tracial state μ . Let $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ be an increasing sequence of unital finite dimensional C^* -subalgebras such that $\mathfrak{A} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ with $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$.*

Let π be the GNS representation of \mathfrak{A} constructed from μ on the space $L^2(\mathfrak{A}, \mu)$.

For all $n \in \mathbb{N}$, let:

$$\mathbb{E}(\cdot | \mathfrak{A}_n) : \mathfrak{A} \rightarrow \mathfrak{A}$$

be the unique conditional expectation of \mathfrak{A} onto \mathfrak{A}_n , and such that $\mu \circ \mathbb{E}(\cdot | \mathfrak{A}_n) = \mu$.

Let $\beta : \mathbb{N} \rightarrow (0, \infty)$ have limit 0 at infinity. If, for all $a \in \mathfrak{sa}(\cup_{n \in \mathbb{N}} \mathfrak{A}_n)$, we set:

$$L_{\mathcal{U}, \mu}^\beta(a) = \sup \left\{ \frac{\|a - \mathbb{E}(a|\mathfrak{A}_n)\|_{\mathfrak{A}}}{\beta(n)} : n \in \mathbb{N} \right\}$$

and $L_{\mathcal{U}, \mu}^\beta(a) = \infty$ for all $a \in \mathfrak{sa}(\mathfrak{A} \setminus \cup_{n \in \mathbb{N}} \mathfrak{A}_n)$, then $(\mathfrak{A}, L_{\mathcal{U}, \mu}^\beta)$ is a 2-quasi-Leibniz quantum compact metric space. Moreover, for all $n \in \mathbb{N}$:

$$\Lambda \left((\mathfrak{A}_n, L_{\mathcal{U}, \mu}^\beta), (\mathfrak{A}, L_{\mathcal{U}, \mu}^\beta) \right) \leq \beta(n)$$

and thus:

$$\lim_{n \rightarrow \infty} \Lambda \left((\mathfrak{A}_n, L_{\mathcal{U}, \mu}^\beta), (\mathfrak{A}, L_{\mathcal{U}, \mu}^\beta) \right) = 0.$$

Proof. The proof follows the same process of the proof of Theorem (2.15) found in [1, Theorem 3.5]. \square

Now, when a quotient has a faithful tracial state, it turns out that the *-isomorphism provided in Proposition (4.15) is an isometric isomorphism (Theorem-Definition (2.11)) between the induced quantum compact metric spaces of Theorem (2.15) and Theorem (6.1), which preserves the finite-dimensional structure as well in Theorem (6.2). The purpose of this is to apply the results of Section (3) pertaining to fusing families directly to the quotient spaces.

Theorem 6.2. Let \mathfrak{A} be a unital AF algebra with unit $1_{\mathfrak{A}}$ such that $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ is an increasing sequence of unital finite dimensional C^* -subalgebras such that $\mathfrak{A} = \overline{\cup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ with $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$. Let $I \in \text{Ideal}(\mathfrak{A}) \setminus \{\mathfrak{A}\}$. By Proposition (4.15), the C^* -algebra $\mathfrak{A}/I = \overline{\cup_{n \in \mathbb{N}} ((\mathfrak{A}_n + I)/I)}^{\|\cdot\|_{\mathfrak{A}/I}}$ and denote $\mathcal{U}/I = ((\mathfrak{A}_n + I)/I)_{n \in \mathbb{N}}$, and note that $(\mathfrak{A}_0 + I)/I = \mathbb{C}1_{\mathfrak{A}/I}$.

If \mathfrak{A}/I is equipped with a faithful tracial state, μ , then using notation from Proposition (4.15), the map $\mu \circ \phi_I$ is a faithful tracial state on $\varinjlim \mathcal{I}(\mathfrak{A}/I)$.

Furthermore, let $\beta : \mathbb{N} \rightarrow (0, \infty)$ have limit 0 at infinity. If $L_{\mathcal{I}(\mathfrak{A}/I), \mu \circ \phi_I}^\beta$ is the $(2, 0)$ -quasi-Leibniz Lip norm on $\varinjlim \mathcal{I}(\mathfrak{A}/I)$ given by Theorem (2.15) and $L_{\mathcal{U}/I, \mu}^\beta$ is the $(2, 0)$ -quasi-Leibniz Lip norm on \mathfrak{A}/I given by Theorem (6.1), then:

$$\phi_I^{-1} : (\mathfrak{A}/I, L_{\mathcal{U}/I, \mu}^\beta) \rightarrow (\varinjlim \mathcal{I}(\mathfrak{A}/I), L_{\mathcal{I}(\mathfrak{A}/I), \mu \circ \phi_I}^\beta)$$

is an isometric isomorphism of Theorem-Definition (2.11) and:

$$\Lambda \left((\varinjlim \mathcal{I}(\mathfrak{A}/I), L_{\mathcal{I}(\mathfrak{A}/I), \mu \circ \phi_I}^\beta), (\mathfrak{A}/I, L_{\mathcal{U}/I, \mu}^\beta) \right) = 0$$

Moreover, for all $n \in \mathbb{N}$, we have:

$$\Lambda \left((\mathfrak{A}_n/(I \cap \mathfrak{A}_n), L_{\mathcal{I}(\mathfrak{A}/I), \mu \circ \phi_I}^\beta \circ \gamma_I^n), ((\mathfrak{A}_n + I)/I, L_{\mathcal{U}/I, \mu}^\beta) \right) = 0$$

Proof. Since $I \neq \mathfrak{A}$, the AF algebra \mathfrak{A}/I is unital and $(\mathfrak{A}_0 + I)/I = \mathbb{C}1_{\mathfrak{A}/I}$ as $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$. Since μ is faithful on \mathfrak{A}/I , we have $\mu \circ \phi_I$ is faithful since ϕ_I is a *-isomorphism by Proposition (4.15). Thus, we may define $L_{\mathcal{I}(\mathfrak{A}/I), \mu \circ \phi_I}^\beta$ on $\varinjlim \mathcal{I}(\mathfrak{A}/I)$ and $L_{\mathcal{U}/I, \mu}^\beta$ on \mathfrak{A}/I .

Fix $m \in \mathbb{N}, a \in (\mathfrak{A}_m + I)/I$. Let $n \in \mathbb{N}$. Since $\mathfrak{A}_n/(I \cap \mathfrak{A}_n)$ is finite dimensional, the C^* -algebra $\mathfrak{A}_n/(I \cap \mathfrak{A}_n) \cong \bigoplus_{j=1}^N \mathfrak{M}(n(j))$ for some $N \in \mathbb{N}$ and $n(1), \dots, n(N) \in \mathbb{N} \setminus \{0\}$ with $*$ -isomorphism $\pi : \bigoplus_{j=1}^N \mathfrak{M}(n(j)) \rightarrow \mathfrak{A}_n/(I \cap \mathfrak{A}_n)$. Let E be the set of matrix units for $\bigoplus_{j=1}^N \mathfrak{M}(n(j))$ given in Notation (3.1). Define $E_\pi = \{\pi(b) \in \mathfrak{A}_n/(I \cap \mathfrak{A}_n) : b \in E\}$.

By Proposition (3.3) and the commuting diagram of Proposition (4.15), we gather that $\phi_I \circ \underline{\gamma}_I^n = \phi_I^n$ and:

$$\begin{aligned}
& \left\| \phi_I^{-1}(a) - \mathbb{E} \left(\phi_I^{-1}(a) \Big| \underline{\gamma}_I^n(\mathfrak{A}_n/(I \cap \mathfrak{A}_n)) \right) \right\|_{\lim_{\rightarrow} \mathcal{I}(\mathfrak{A}/I)} \\
&= \left\| \phi_I^{-1}(a) - \sum_{e \in E_\pi} \frac{\mu \circ \phi_I \left(\underline{\gamma}_I^n(e) * \phi_I^{-1}(a) \right)}{\mu \circ \phi_I \left(\underline{\gamma}_I^n(e * e) \right)} \underline{\gamma}_I^n(e) \right\|_{\lim_{\rightarrow} \mathcal{I}(\mathfrak{A}/I)} \\
&= \left\| \phi_I^{-1}(a) - \sum_{e \in E_\pi} \frac{\mu(\phi_I^n(e) * a)}{\mu(\phi_I^n(e * e))} \underline{\gamma}_I^n(e) \right\|_{\lim_{\rightarrow} \mathcal{I}(\mathfrak{A}/I)} \\
&= \left\| \phi_I \left(\phi_I^{-1}(a) - \sum_{e \in E_\pi} \frac{\mu(\phi_I^n(e) * a)}{\mu(\phi_I^n(e * e))} \underline{\gamma}_I^n(e) \right) \right\|_{\mathfrak{A}/I} \\
&= \left\| a - \sum_{e \in E_\pi} \frac{\mu(\phi_I^n(e) * a)}{\mu(\phi_I^n(e * e))} \phi_I(\underline{\gamma}_I^n(e)) \right\|_{\mathfrak{A}/I} \\
&= \left\| a - \sum_{e \in E_\pi} \frac{\mu(\phi_I^n(e) * a)}{\mu(\phi_I^n(e * e))} \phi_I^n(e) \right\|_{\mathfrak{A}/I} \\
&= \|a - \mathbb{E}(a | (\mathfrak{A}_n + I)/I)\|_{\mathfrak{A}/I}.
\end{aligned}$$

Thus, since $n \in \mathbb{N}$ was arbitrary, we have:

$$\mathbb{L}_{\mathcal{I}(\mathfrak{A}/I), \mu \circ \phi_I}^\beta \circ \phi_I^{-1}(a) = \mathbb{L}_{\mathcal{U}/I, \mu}^\beta(a)$$

for all $a \in (\mathfrak{A}_m + I)/I$. However, since $\phi_I \circ \underline{\gamma}_I^m = \phi_I^m$ by Proposition (4.15), we thus have:

$$\underline{\gamma}_I^m(\mathfrak{A}_m/(I \cap \mathfrak{A}_m)) = \phi_I^{-1} \circ \phi_I^m(\mathfrak{A}_m/(I \cap \mathfrak{A}_m)) = \phi_I^{-1}((\mathfrak{A}_m + I)/I).$$

Therefore, we have $\left(\underline{\gamma}_I^m(\mathfrak{A}_m/(I \cap \mathfrak{A}_m)), \mathbb{L}_{\mathcal{I}(\mathfrak{A}/I), \mu \circ \phi_I}^\beta \right)$ is isometrically isomorphic to $\left((\mathfrak{A}_m + I)/I, \mathbb{L}_{\mathcal{U}/I, \mu}^\beta \right)$ by the map ϕ_I^{-1} restricted to $(\mathfrak{A}_m + I)/I$. However, the space $\left(\underline{\gamma}_I^m(\mathfrak{A}_m/(I \cap \mathfrak{A}_m)), \mathbb{L}_{\mathcal{I}(\mathfrak{A}/I), \mu \circ \phi_I}^\beta \right)$ is isometrically isomorphic to $\left((\mathfrak{A}_m/(I \cap \mathfrak{A}_m)), \mathbb{L}_{\mathcal{I}(\mathfrak{A}/I), \mu \circ \phi_I}^\beta \circ \underline{\gamma}_I^m \right)$ by the map $\underline{\gamma}_I^m$. Since isometric isomorphism

is an equivlance relation, we conclude that:

$$\wedge \left(\left(\mathfrak{A}_m / (I \cap \mathfrak{A}_m), \mathcal{L}_{\mathcal{I}(\mathfrak{A}/I), \mu \circ \phi_I}^\beta \circ \gamma_I^m \right), \left((\mathfrak{A}_m + I) / I, \mathcal{L}_{\mathcal{U}/I, \mu}^\beta \right) \right) = 0$$

by Theorem-Definition (2.11).

But, as $m \in \mathbb{N}$ was arbitrary, we have that $\mathcal{L}_{\mathcal{I}(\mathfrak{A}/I), \mu \circ \phi_I}^\beta \circ \phi_I^{-1}(a) = \mathcal{L}_{\mathcal{U}/I, \mu}^\beta(a)$ for all $a \in \cup_{m \in \mathbb{N}}((\mathfrak{A}_m + I) / I)$. Hence:

$$\phi_I^{-1} : \left(\mathfrak{A} / I, \mathcal{L}_{\mathcal{U}/I, \mu}^\beta \right) \rightarrow \left(\varinjlim \mathcal{I}(\mathfrak{A}/I), \mathcal{L}_{\mathcal{I}(\mathfrak{A}/I), \mu \circ \phi_I}^\beta \right)$$

is an isometric isomorphism by Theorem-Definition (2.11). \square

Thus, the isometric isomorphism, ϕ_I , of Theorem (6.2) is in some sense the best one could hope for since it preserves the finite-dimensional approximations in the quantum propinquity. Next, we give criteria for when a family of quotients converge in the quantum propinquity with respect to ideal convergence.

Theorem 6.3. *Let \mathfrak{A} be a unital AF algebra with unit $1_{\mathfrak{A}}$ such that $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ is an increasing sequence of unital finite dimensional C^* -subalgebras such that $\mathfrak{A} = \overline{\cup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, with $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$. Let $(I^n)_{n \in \overline{\mathbb{N}}} \subseteq \text{Ideal}(\mathfrak{A}) \setminus \{\mathfrak{A}\}$ such that $\{\mu_k : \mathfrak{A}/I^k \rightarrow \mathbb{C} : k \in \overline{\mathbb{N}}\}$ is a family of faithful tracial states. Let $Q^k : \mathfrak{A} \rightarrow \mathfrak{A}/I^k$ denote the quotient map for all $k \in \overline{\mathbb{N}}$. If:*

- (1) $(I^n)_{n \in \mathbb{N}} \subseteq \text{Ideal}(\mathfrak{A})$ converges to $I^\infty \in \text{Ideal}(\mathfrak{A})$ with respect to $m_{i(\mathcal{U})}$ of Definition (4.10) or the Fell topology (Definition (4.3)) with fusing sequence $(c_n)_{n \in \mathbb{N}}$ for the fusing family $\{I^n = \overline{\cup_{k \in \mathbb{N}} I^n \cap \mathfrak{A}_k}^{\|\cdot\|_{\mathfrak{A}}} : n \in \overline{\mathbb{N}}\}$,
- (2) for each $N \in \mathbb{N}$, we have that $(\mu_k \circ Q^k)_{k \in \mathbb{N}_{\geq c_N}}$ converges to $\mu_\infty \circ Q^\infty$ in the weak-* topology on $\mathcal{S}(\mathfrak{A}_N)$, and
- (3) $\{\beta^k : \overline{\mathbb{N}} \rightarrow (0, \infty)\}_{k \in \overline{\mathbb{N}}}$ is a family of convergent sequences such that for all $N \in \mathbb{N}$ if $k \in \mathbb{N}_{\geq c_N}$, then $\beta^k(n) = \beta^\infty(n)$ for all $n \in \{0, 1, \dots, N\}$ and there exists $B : \overline{\mathbb{N}} \rightarrow (0, \infty)$ with $B(\infty) = 0$ and $\beta^m(l) \leq B(l)$ for all $m, l \in \overline{\mathbb{N}}$,

then using notation from Theorem (6.2):

$$\lim_{n \rightarrow \infty} \wedge \left(\left(\mathfrak{A} / I^n, \mathcal{L}_{\mathcal{U}/I^n, \mu_n}^{\beta^n} \right), \left(\mathfrak{A} / I^\infty, \mathcal{L}_{\mathcal{U}/I^\infty, \mu_\infty}^{\beta^\infty} \right) \right) = 0$$

Proof. By Corollary (5.15), the assumption that $(I^n)_{n \in \mathbb{N}} \subseteq \text{Ideal}(\mathfrak{A})$ converges to $I^\infty \in \text{Ideal}(\mathfrak{A})$ with respect to $m_{i(\mathcal{U})}$ of Definition (4.10) or the Fell topology implies that:

$$\{I^n = \overline{\cup_{k \in \mathbb{N}} I^n \cap \mathfrak{A}_k}^{\|\cdot\|_{\mathfrak{A}}} : n \in \overline{\mathbb{N}}\}$$

is a fusing family with some fusing sequence $(c_n)_{n \in \mathbb{N}}$ such that

$\{\varinjlim \mathcal{I}(\mathfrak{A}/I^n) : n \in \overline{\mathbb{N}}\}$ is an IL-fusing family with fusing sequence $(c_n)_{n \in \mathbb{N}}$.

Fix $N \in \mathbb{N}$ and $k \in \mathbb{N}_{\geq c_N}$. Let $x \in \mathfrak{A}_N$, and let $Q_N^k : \mathfrak{A}_N \rightarrow \mathfrak{A}_N / (I^k \cap \mathfrak{A}_N)$ and $Q_N^\infty : \mathfrak{A}_N \rightarrow \mathfrak{A}_N / (I^\infty \cap \mathfrak{A}_N)$ denote the quotient maps, and let $\phi_{I^k} : \varinjlim \mathcal{I}(\mathfrak{A}/I^k) \rightarrow \mathfrak{A}/I^k$ denote the *-isomorphism given in Proposition (4.15) and

recall that $\mathcal{I}(\mathfrak{A}/I^k) = \left(\mathfrak{A}_n / (I^k \cap \mathfrak{A}_n), \gamma_{I^k, n} \right)_{n \in \mathbb{N}}$ from Notation (4.14). Now, by Proposition (4.15) and its commuting diagram, we gather:

$$\begin{aligned} \mu_k \circ \phi_{I^k} \circ \gamma_{I^k}^N \circ Q_N^k(x) &= \mu_k \circ \phi_{I^k}^N \circ Q_N^k(x) \\ &= \mu_k \circ \phi_{I^k}^N(x + I^k \cap \mathfrak{A}_N) \\ &= \mu_k(x + I^k) \\ &= \mu_k \circ Q^k(x). \end{aligned}$$

Therefore, by hypothesis (2), the sequence $\left(\mu_k \circ \phi_{I^k} \circ \gamma_{I^k}^N \circ Q_N^k \right)_{k \in \mathbb{N}_{\geq c_N}}$ converges to $\mu_\infty \circ \phi_{I^\infty} \circ \gamma_{I^\infty}^N \circ Q_N^\infty$ in the weak-* topology on \mathfrak{A}_N . Hence, the sequence $\left(\mu_k \circ \phi_{I^k} \circ \gamma_{I^k}^N \right)_{k \in \mathbb{N}_{\geq c_N}}$ converges to $\mu_\infty \circ \phi_{I^\infty} \circ \gamma_{I^\infty}^N$ in the weak-* topology on $\mathcal{S}(\mathfrak{A}_N / (I^\infty \cap \mathfrak{A}_N))$ by [9, Theorem V.2.2]. Thus, by hypothesis (3) and by Theorem (3.9), we have that:

$$\lim_{n \rightarrow \infty} \Lambda \left(\left(\varinjlim \mathcal{I}(\mathfrak{A}/I^n), L_{\mathcal{I}(\mathfrak{A}/I^n), \mu_n \circ \phi_{I^n}}^{\beta^n} \right), \left(\varinjlim \mathcal{I}(\mathfrak{A}/I^\infty), L_{\mathcal{I}(\mathfrak{A}/I^\infty), \mu_\infty \circ \phi_{I^\infty}}^{\beta^\infty} \right) \right) = 0.$$

But, as $\phi_{I^n}^{-1}$ is an isometric isomorphism for all $n \in \overline{\mathbb{N}}$ by Theorem (6.2), we conclude:

$$\lim_{n \rightarrow \infty} \Lambda \left(\left(\mathfrak{A}/I^n, L_{\mathfrak{A}/I^n, \mu_n}^{\beta^n} \right), \left(\mathfrak{A}/I^\infty, L_{\mathfrak{A}/I^\infty, \mu_\infty}^{\beta^\infty} \right) \right) = 0,$$

which completes the proof. \square

6.1. The Boca-Mundici AF algebra. The Boca-Mundici AF algebra arose in [4] and [33] independently and is constructed from the Farey tessellation. In both [4], [33], it was shown that the all Effros-Shen AF algebras (Notation (2.19)) arise as quotients up to *-isomorphism of certain primitive ideals of the Boca-Mundici AF algebra, which is the main motivation for our convergence result. In both [4], [33], it was also shown that the center of the Boca-Mundici AF algebra is *-isomorphic to $C([0,1])$, which provided the framework for Eckhardt to introduce a noncommutative analogue to the Gauss map in [12].

We present the construction of this algebra as presented in the paper by F. Boca [4]. We refer mostly to Boca's work as his unique results pertaining to the Jacobson topology (for example [4, Corollary 12], which is the result that led us to begin this paper) are more applicable to our work (see Proposition (6.11)).

As in [4], we define the Boca-Mundici AF algebra recursively by the following Relations (6.1). We note that the relations presented here are the same as in [4, Section 1], but instead of starting at $n = 0$, these relations begin at $n = 1$, so that

this formulation of the Boca-Mundici AF algebra as an inductive limit begins at \mathbb{C} .

(6.1)

$$\left\{ \begin{array}{ll} q(n,0) = q(n,2^{n-1}) = 1, \quad p(n,0) = 0, \quad p(n,2^{n-1}) = 1 & n \in \mathbb{N} \setminus \{0\}; \\ q(n+1,2k) = q(n,k), \quad p(n+1,2k) = p(n,k), & n \in \mathbb{N} \setminus \{0\}, \\ & k \in \{0, \dots, 2^{n-1}\}; \\ q(n+1,2k+1) = q(n,k) + q(n,k+1), & n \in \mathbb{N} \setminus \{0\}, \\ & k \in \{0, \dots, 2^{n-1} - 1\}; \\ p(n+1,2k+1) = p(n,k) + p(n,k+1), & n \in \mathbb{N} \setminus \{0\}, \\ & k \in \{0, \dots, 2^{n-1} - 1\}; \\ r(n,k) = \frac{p(n,k)}{q(n,k)}, & n \in \mathbb{N} \setminus \{0\}, \\ & k \in \{0, \dots, 2^{n-1} - 1\}. \end{array} \right.$$

We now define the finite dimensional algebras which determine the inductive limit \mathfrak{F} .

Definition 6.4. For $n \in \mathbb{N} \setminus \{0\}$, define the finite dimensional C^* -algebras,

$$\mathfrak{F}_n = \bigoplus_{k=0}^{2^n-1} \mathfrak{M}(q(n,k)) \text{ and } \mathfrak{F}_0 = \mathbb{C}.$$

Next, we define $*$ -homomorphisms to complete the inductive limit recipe. We utilize partial multiplicity matrices.

Definition 6.5. For $n \in \mathbb{N} \setminus \{0\}$, let F_n be the $(2^n + 1) \times (2^{n-1} + 1)$ matrix with entries in $\{0, 1\}$ determined entry-wise by:

$$(F_n)_{h,j} = \begin{cases} 1 & \text{if } (h = 2k + 1, k \in \{0, \dots, 2^{n-1}\}, \wedge j = k + 1) \\ & \vee (h = 2k, k \in \{1, \dots, 2^{n-1}\} \wedge (j = k \vee j = k + 1)); \\ 0 & \text{otherwise.} \end{cases}$$

For example,

$$F_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}, F_2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

We would like these matrices to determine unital $*$ -monomorphisms, so that our inductive limit is a unital C^* -algebra, which motivates the following Lemma (6.6).

Lemma 6.6. Using Definition (6.5), if $n \in \mathbb{N} \setminus \{0\}$, then:

$$F_n \begin{pmatrix} q(n,0) \\ q(n,1) \\ \vdots \\ q(n,2^{n-1}) \end{pmatrix} = \begin{pmatrix} q(n+1,0) \\ q(n+1,1) \\ \vdots \\ q(n+1,2^n) \end{pmatrix}$$

Proof. Let $n \in \mathbb{N} \setminus \{0\}$. Let $k \in \{1, \dots, 2^{n-1}\}$ and consider $q(n+1, 2k-1)$. Now, by Definition (6.5), row $2k-1+1 = 2k$ of F_n has 1 in entry k and $k+1$, and 0 elsewhere. Thus:

$$\begin{aligned} ((F_n)_{2k,1}, \dots, (F_n)_{2k,2^{n-1}+1}) \cdot \begin{pmatrix} q(n,0) \\ q(n,1) \\ \vdots \\ q(n,2^{n-1}) \end{pmatrix} &= q(n, k-1) + q(n, k-1+1) \\ &= q(n+1, 2k-1) \end{aligned}$$

by Relations (6.1). Next, let $k \in \{0, \dots, 2^{n-1}\}$ and consider $q(n+1, 2k)$. By Definition (6.5), row $2k+1$ of F_n has 1 in entry $k+1$ and 0 elsewhere. Thus:

$$((F_n)_{2k+1,1}, \dots, (F_n)_{2k+1,2^{n-1}+1}) \cdot \begin{pmatrix} q(n,0) \\ q(n,1) \\ \vdots \\ q(n,2^{n-1}) \end{pmatrix} = q(n, 2k) = q(n+1, 2k)$$

by Relations (6.1). Hence, by matrix multiplication, the proof is complete. \square

Definition 6.7 ([4, 33]). Define $\varphi_0 : \mathfrak{F}_0 \rightarrow \mathfrak{F}_1$ by $\varphi_0(a) = a \oplus a$. For $n \in \mathbb{N} \setminus \{0\}$, by [10, Lemma III.2.1] and Lemma (6.6), we let $\varphi_n : \mathfrak{F}_n \rightarrow \mathfrak{F}_{n+1}$ be a unital *-monomorphism determined by F_n of Definition (6.5). Using Definition (6.4), we let the unital C*-inductive limit (Notation (2.13)):

$$\mathfrak{F} = \varinjlim (\mathfrak{F}_n, \varphi_n)_{n \in \mathbb{N}}$$

denote the *Boca-Mundici AF algebra*.

Let $\mathfrak{F}^n = \varphi^n(\mathfrak{F}_n)$ for all $n \in \mathbb{N}$ and $\mathcal{U}_{\mathfrak{F}} = (\mathfrak{F}^n)_{n \in \mathbb{N}}$, which is a non-decreasing sequence of C*-subalgebras of \mathfrak{F} such that $\mathfrak{F} = \overline{\cup_{n \in \mathbb{N}} \mathfrak{F}^n}^{\|\cdot\|_{\mathfrak{F}}}$, where $\mathfrak{F}^0 = \mathbb{C}1_{\mathfrak{F}}$ (see [34, Chapter 6.1]).

We note that in [4], the AF algebra \mathfrak{F} is constructed by a diagram displayed as [4, Figure 2], so in order to utilize the results of [4], we verify that we have the same diagram up to adding one vertex of label 1 at level $n = 0$ satisfying the conditions at the beginning of [4, Section 1].

Proposition 6.8. *The Bratteli diagram of \mathfrak{F} , denoted $\mathcal{D}_b(\mathfrak{F}) = (V^{\mathcal{D}_b(\mathfrak{F})}, E^{\mathcal{D}_b(\mathfrak{F})})$ of Definition (5.3) satisfies for all $n \in \mathbb{N} \setminus \{0\}$:*

- (i) $V_n^{\mathcal{D}_b(\mathfrak{F})} = \{(n, k) : k \in \{0, \dots, 2^{n-1}\}\}$
- (ii) $((n, k), (n+1, l)) \in E_n^{\mathcal{D}_b(\mathfrak{F})}$ if and only if $|2k - l| \leq 1$. And, there exists only one edge between any two vertices for which there is an edge.

Proof. Property (i) is clear by Definition (6.4). By [10, Section III.2 Definition **Bratteli diagram**], an edge exists from (n, s) to $(n+1, t)$ if and only if its associated entry in the partial multiplicity matrix $(F_n)_{t+1, s+1}$ is non-zero.

Now, assume that $|2s - t| \leq 1$. Assume $t = 2k + 1$ for some $k \in \{0, \dots, 2^{n-1} - 1\}$. We thus have $|2s - t| \leq 1 \iff k \leq s \leq k+1 \iff s \in \{k, k+1\}$, since $s \in \mathbb{N}$.

Next, assume that $t = 2k$ for some $k \in \{0, \dots, 2^{n-1}\}$. We thus have $|2s - t| \leq 1 \iff -1/2 + k \leq s \leq 1/2 + k \iff |s - k| \leq 1/2 \iff s = k$ since $s \in \mathbb{N}$. But, considering both t odd and even, these equivalences are equivalent to the conditions for $(F_n)_{t+1, s+1}$ to be non-zero by Definition (6.5), which determine the edges of $\mathcal{D}_b(\mathfrak{F})$. Furthermore, since the non-zero entries of F_n are all 1, only one edge exists between vertices for which there is an edge. \square

Next, we describe the ideals of \mathfrak{F} , whose quotients are *-isomorphic to the Effros-Shen AF algebras.

Definition 6.9 ([4]). Let $\theta \in (0, 1) \setminus \mathbb{Q}$. We define the ideal $I_\theta \in \text{Ideal}(\mathfrak{F})$ diagrammatically.

By [4, Proposition 4.i], for each $n \in \mathbb{N} \setminus \{0\}$, there exists a unique $j_n(\theta) \in \{0, \dots, 2^{n-1} - 1\}$ such that $r(n, j_n(\theta)) < \theta < r(n, j_n(\theta) + 1)$ of Relations (6.1). The set of vertices is defined by:

$$V^{\mathcal{D}_b(\mathfrak{F})} \setminus \{(n, j_n(\theta)), (n, j_n(\theta) + 1) : n \in \mathbb{N} \setminus \{0\} \cup \{(0, 0)\}\}$$

and we denote this set by $V^{\mathcal{D}(I_\theta)}$. Let $E^{\mathcal{D}(I_\theta)}$ be the set of edges of $\mathcal{D}_b(\mathfrak{F})$, which are between the vertices in $V^{\mathcal{D}(I_\theta)}$ and let $\mathcal{D}(I_\theta) = (V^{\mathcal{D}(I_\theta)}, E^{\mathcal{D}(I_\theta)})$. By [4, Proposition 4.i], the diagram $\mathcal{D}(I_\theta) \in \text{Ideal}(\mathcal{D}_b(\mathfrak{F}))$ is an ideal diagram of Definition (5.6).

Using Proposition (5.11), define:

$$I_\theta = i(\cdot, \mathcal{D}_b(\mathfrak{A}))^{-1}(\mathcal{D}(I_\theta)) \in \text{Ideal}(\mathfrak{A}).$$

By [4, Proposition 4.i], if $n \in \mathbb{N} \setminus \{0, 1\}$ and $1 \leq j_n(\theta) \leq 2^{n-1} - 2$, then:

$$I_\theta \cap \mathfrak{F}^n = \varphi^n \left(\left(\bigoplus_{k=0}^{j_n(\theta)-1} \mathfrak{M}(q(n, k)) \right) \oplus \{0\} \oplus \{0\} \oplus \left(\bigoplus_{k=j_n(\theta)+2}^{2^{n-1}} \mathfrak{M}(q(n, k)) \right) \right).$$

If $j_n(\theta) = 0$, then:

$$I_\theta \cap \mathfrak{F}^n = \varphi^n \left(\{0\} \oplus \{0\} \oplus \left(\bigoplus_{k=j_n(\theta)+2}^{2^{n-1}} \mathfrak{M}(q(n, k)) \right) \right).$$

If $j_n(\theta) = 2^{n-1} - 1$, then:

$$I_\theta \cap \mathfrak{F}^n = \varphi^n \left(\left(\bigoplus_{k=0}^{j_n(\theta)-1} \mathfrak{M}(q(n, k)) \right) \oplus \{0\} \oplus \{0\} \right),$$

and if $n \in \{0, 1\}$, then $I_\theta \cap \mathfrak{F}^n = \{0\}$. We note that $I_\theta \in \text{Prim}(\mathfrak{A})$ by [4, Proposition 4.i].

Before we move on to describing the quantum metric structure of quotients of the ideals of Definition (6.9), let's first capture more properties of the structure of the ideals introduced in Definition (6.9), which are sufficient for later results.

Lemma 6.10. *Using notation from Definition (6.9), if $n \in \mathbb{N} \setminus \{0\}$, $\theta \in (0, 1) \setminus \mathbb{Q}$, then $j_{n+1}(\theta) \in \{2j_n(\theta), 2j_n(\theta) + 1\}$.*

Proof. We first note that the vertices $V^{\mathcal{D}_b(\mathfrak{A})} \setminus V^{\mathcal{D}(I_\theta)}$ determine a Bratteli diagram associated to the AF algebra \mathfrak{F}/I_θ , which we will denote $\mathcal{D}_b(\mathfrak{A}/I_\theta)$, as in Definition (5.3) by [5, Proposition 3.7] up to shifting vertices, in which the edges for $\mathcal{D}_b(\mathfrak{A}/I_\theta)$

are given by all the edges from $E^{\mathcal{D}_b(\mathfrak{A})}$ between vertices all vertices in $V^{\mathcal{D}_b(\mathfrak{A})} \setminus V^{\mathcal{D}(I_\theta)}$. Thus, by Definition (6.9), the vertex set for $\mathcal{D}_b(\mathfrak{A}/I_\theta)$ is:

$$(6.2) \quad V^{\mathcal{D}_b(\mathfrak{A})} \setminus V^{\mathcal{D}(I_\theta)} = \left\{ (n, j_n(\theta)), (n, j_n(\theta) + 1) \in \mathbb{N}^2 : n \in \mathbb{N} \setminus \{0\} \right\} \cup \{(0, 0)\},$$

and in particular, this vertex set along with the edges between the vertices satisfy axioms (i),(ii), (iii) of Definition (5.1).

Consider $n = 1$. Since there are only 3 vertices at level $n = 2$, the conclusion is satisfied since $j_2(\theta), j_2(\theta) + 1 \in \{0, 1, 2\}$ and $j_1(\theta) = 0$ since there are only 2 vertices at level $n = 1$.

Furthermore, note by definition, we have $j_n(\theta) \leq 2^{n-1} - 1$ since $j_n(\theta) + 1 \in \{0, \dots, 2^{n-1}\}$.

Case 1. For $n \geq 2$, we show that $j_{n+1}(\theta) \geq 2j_n(\theta)$.

We note that if $j_n(\theta) = 0$, then clearly $j_{n+1}(\theta) \geq 0 = 2j_n(\theta)$. Thus, we may assume that $j_n(\theta) \geq 1$. Hence, we may assume by way of contradiction that $j_{n+1}(\theta) \leq 2j_n(\theta) - 1$. Consider $j_n(\theta) + 1$. By Expression (6.2), the only vertices at level $n + 1$ of the diagram of \mathfrak{F}/I_θ are $(n + 1, j_{n+1}(\theta))$ and $(n + 1, j_{n+1}(\theta) + 1)$. Consider $j_{n+1}(\theta) + 1$. Now:

$$|2(j_n(\theta) + 1) - (j_{n+1}(\theta) + 1)| = |2j_n(\theta) - j_{n+1}(\theta) + 1|.$$

But, our contradiction assumption, we have $2j_n(\theta) - j_{n+1}(\theta) + 1 \geq 2j_n(\theta) + 1 - 2j_n(\theta) + 1 = 2$. Thus, by Proposition (6.8), there is no edge from $(n, j_n(\theta) + 1)$ to $(n + 1, j_{n+1}(\theta) + 1)$. Next, consider $j_{n+1}(\theta)$. Similarly, we have $|2(j_n(\theta) + 1) - j_{n+1}(\theta)| = |2j_n(\theta) - j_{n+1}(\theta) + 2|$. However, the indices $2j_n(\theta) - j_{n+1}(\theta) + 2 \geq 2j_n(\theta) + 1 - 2j_n(\theta) + 2 = 3$. And, again by Proposition (6.8), there is no edge from $(n, j_n(\theta) + 1)$ to $(n + 1, j_{n+1}(\theta))$. But, by Expression (6.2), this implies that $(n, j_{n+1}(\theta) + 1)$ is a vertex in the quotient diagram \mathfrak{F}/I_θ in which there does not exist a vertex $(n + 1, l)$ in the diagram of \mathfrak{F}/I_θ such that $((n, j_{n+1}(\theta) + 1), (n + 1, l))$ is an edge in the diagram of \mathfrak{F}/I_θ , which is a contradiction since the quotient diagram is a Bratteli diagram that would not satisfy axiom (ii) of Definition (5.1). Therefore, we conclude $j_{n+1}(\theta) \geq 2j_n(\theta)$.

Case 2. For $n \geq 2$, we show that $j_{n+1}(\theta) \leq 2j_n(\theta) + 1$.

Now, if $j_n(\theta) = 2^{n-1} - 1$, then $j_{n+1}(\theta) + 1 \leq 2^n = 2(2^{n-1} - 1) + 2$ and thus $j_{n+1}(\theta) \leq 2(2^{n-1} - 1) + 1 = 2j_n(\theta) + 1$ and we would be done. Thus, we may assume that $j_n(\theta) \leq 2^{n-1} - 2$ and we note that this can only occur in the case that $n \geq 3$, which implies that the case of $n = 2$ is complete. Thus, we may assume by way of contradiction that $j_{n+1}(\theta) \geq 2j_n(\theta) + 2$. Consider $j_n(\theta)$. As in Case 1, we provide a contradiction via a diagram approach. Consider $j_{n+1}(\theta) + 1$. Now, we have $|2j_n(\theta) - (j_{n+1}(\theta) + 1)| = |2j_n(\theta) - j_{n+1}(\theta) - 1|$. But, by our contradiction assumption, we gather that $2j_n(\theta) - j_{n+1}(\theta) - 1 \leq 2j_n(\theta) - 2j_n(\theta) - 2 - 1 = -3$ and $|2j_n(\theta) - (j_{n+1}(\theta) + 1)| \geq 3$. Thus, by Proposition (6.8), there is no edge from $(n, j_n(\theta))$ to $(n + 1, j_{n+1}(\theta) + 1)$. Next, consider $j_{n+1}(\theta)$. Similarly, we have $2j_n(\theta) - j_{n+1}(\theta) \leq 2j_n(\theta) - 2j_n(\theta) - 2 = -2$ and $|2j_n(\theta) - j_{n+1}(\theta)| \geq 2$. Thus, by Proposition (6.8), there is no edge from $(n, j_n(\theta))$ to $(n + 1, j_{n+1}(\theta))$. Thus, by Expression (6.2) and the same diagram argument of Case 1, we have reached a contradiction. Hence, $j_{n+1}(\theta) \leq 2j_n(\theta) + 1$.

Hence, combining Case 1 and Case 2, the proof is complete. \square

Next, on the subspace of ideals of Definition (6.9), we provide a useful topological result about the metric on ideals of Proposition (4.11), in which the equivalence of (1) and (3) is a consequence of [4, Corollary 12], which is unique to Boca's work on the AF algebra, \mathfrak{F} .

Proposition 6.11. *If $(\theta_n)_{n \in \overline{\mathbb{N}}} \subseteq (0, 1) \setminus \mathbb{Q}$, then using notation from Definition (6.7) and Definition (6.9), the following are equivalent:*

- (1) $(\theta_n)_{n \in \mathbb{N}}$ converges to θ_∞ with respect to the usual topology on \mathbb{R} ;
- (2) $(\text{cf}(\theta_n))_{n \in \mathbb{N}}$ converges to $\text{cf}(\theta_\infty)$ with respect to the Baire space, \mathcal{N} and its metric from Definition (2.16), where cf denotes the unique continued fraction expansion of an irrational;
- (3) $(I_{\theta_n})_{n \in \mathbb{N}}$ converges to I_{θ_∞} with respect to the Jacobson topology (Definition (4.1)) on $\text{Prim}(\mathfrak{F})$;
- (4) $(I_{\theta_n})_{n \in \mathbb{N}}$ converges to I_{θ_∞} with respect to the metric topology of $\mathfrak{m}_i(\mathcal{U}_{\mathfrak{F}})$ of Proposition (4.11) or the Fell topology of Definition (4.3).

Proof. The equivalence between (1) and (2) is a classic result, in which a proof can be found in [1, Proposition 5.10]. The equivalence between (1) and (3) is immediate from [4, Corollary 12]. And, therefore, (2) is equivalent to (3). Thus, it remains to prove that (3) is equivalent to (4).

(4) implies (3) is an immediate consequence of Corollary (5.13) as the Fell topology is stronger. Hence, assume (3), then since we have already established (3) implies (2), we may assume (2). For each $n \in \overline{\mathbb{N}}$, let $\text{cf}(\theta_n) = [a_j^n]_{j \in \mathbb{N}}$. By assumption, the coordinates $a_0^n = 0$ for all $n \in \overline{\mathbb{N}}$. Now, assume that there exists $N \in \mathbb{N} \setminus \{0\}$ such that $a_j^n = a_j^\infty$ for all $n \in \mathbb{N}$ and $j \in \{0, \dots, N\}$. Assume without loss of generality, assume that N is odd. Thus, using [4, Figure 5], we have that:

$$(6.3) \quad L_{a_1^n - 1} \circ R_{a_2^n} \circ \cdots \circ L_{a_N^n} = L_{a_1^\infty - 1} \circ R_{a_2^\infty} \circ \cdots \circ L_{a_N^\infty}$$

for all $n \in \mathbb{N}$. But, Equation (6.3) determines the vertices for the diagram of the quotient $\mathfrak{F}/I_{\theta_n}$ for all $n \in \overline{\mathbb{N}}$ by [4, Proposition 4.i] (specifically, the 2nd line of paragraph 2 after [4, Figure 5] in arXiv v6). But, the vertices of the diagram of the quotient $\mathfrak{F}/I_{\theta_n}$ are simply the complement of the vertices of the diagram of I_{θ_n} by [10, Theorem III.4.4]. Now, primitive ideals must have the same vertices at level 0 of the diagram since they cannot equal \mathfrak{A} by Definition (4.1) and are thus non-unital. But, for any $\eta \in (0, 1) \setminus \mathbb{Q}$, the ideals I_η must always have the same vertices at level 1 of the diagram as well since the only two vertices are $(1, 0)$, $(1, 1)$ and $r(1, 0) = 0 < \theta < 1 = r(1, 1)$ by Relations (6.1) for all $\theta \in (0, 1) \setminus \mathbb{Q}$. Thus, Equation (6.3) and the isometry of Theorem (5.12), we gather that $I_{\theta_n} \cap \mathfrak{F}^j = I_{\theta_\infty} \cap \mathfrak{F}^j$ for all $n \in \mathbb{N}$ and:

$$j \in \left\{ 0, \dots, \max \left\{ 1, a_1^\infty - 1 + \left(\sum_{k=2}^N a_k^N \right) \right\} \right\},$$

where $\max \left\{ 1, a_1^\infty - 1 + \left(\sum_{k=2}^N a_k^N \right) \right\} \geq N$ as the terms of the continued fraction expansion are all positive integers for coordinates greater than 0. Thus, by the definition of the metric on the Baire Space and the metric $\mathfrak{m}_i(\mathcal{U}_{\mathfrak{F}})$, we conclude that convergence in the the Baire space metric of $(\text{cf}(\theta_n))_{n \in \mathbb{N}}$ to $\text{cf}(\theta_\infty)$ implies

convergence of $(I_{\theta_n})_{n \in \mathbb{N}}$ to I_{θ_∞} with respect to the metric $m_{i(\mathcal{U}_{\mathfrak{F}})}$ or the Fell topology by Theorem (5.12). \square

Remark 6.12. By [4, Proposition 4.i] and the construction of the ideals of Definition (6.9), the function $\theta \in (0, 1) \setminus \mathbb{Q} \mapsto I_\theta \in \text{Prim}(\mathfrak{F})$ is a bijection onto its image. Thus, a consequence of this and Lemma (6.11) is that if: $(0, 1) \setminus \mathbb{Q}$ is equipped with its relative topology from the usual topology on \mathbb{R} , the set $\{I_\theta \in \text{Prim}(\mathfrak{A}) : \theta \in (0, 1) \setminus \mathbb{Q}\}$ is equipped with its relative topology induced by the Jacobson topology, and the set $\{I_\theta \in \text{Prim}(\mathfrak{A}) : \theta \in (0, 1) \setminus \mathbb{Q}\}$ is equipped with its relative topology induced by the metric topology of $m_{i(\mathcal{U}_{\mathfrak{F}})}$ of Definition (4.11) or the Fell topology of Definition (4.3), then all these spaces are homeomorphic to the Baire space \mathcal{N} with its metric topology from Definition (2.16). In particular, from Corollary (5.13), the totally bounded metric $m_{i(\mathcal{U}_{\mathfrak{F}})}$ topology on the set of ideals

$\{I_\theta \in \text{Prim}(\mathfrak{A}) : \theta \in (0, 1) \setminus \mathbb{Q}\}$ is homeomorphic to $(0, 1) \setminus \mathbb{Q}$ with its totally bounded metric topology inherited from the usual topology on \mathbb{R} . Hence, in some sense, the metric $m_{i(\mathcal{U}_{\mathfrak{F}})}$ topology shares more metric information with $(0, 1) \setminus \mathbb{Q}$ and its metric than the Baire space metric topology as the Baire space is not totally bounded [1, Theorem 6.5]. This can also be displayed in metric calculations as well.

Indeed, consider $\theta, \mu \in (0, 1) \setminus \mathbb{Q}$ with continued fraction expansions $\theta = [a_j]_{j \in \mathbb{N}}$ and $\mu = [b_j]_{j \in \mathbb{N}}$, in which $a_0 = 0, a_1 = 1000, a_j = 1 \forall j \geq 2$ and $b_0, b_1 = 1, b_j = 1 \forall j \geq 2$, and thus $\theta \approx 0.001, \mu \approx 0.618, |\theta - \mu| \approx 0.617$. In the Baire metric $d(\text{cf}(\theta), \text{cf}(\mu)) = 0.5$, and, in the ideal metric $m_{i(\mathcal{U}_{\mathfrak{F}})}(I_\theta, I_\mu) = 0.25$ by Theorem (5.12) since at level $n = 1$ the diagram for \mathfrak{F}/I_θ begins with L_{999} and for \mathfrak{F}/I_μ begins with R_{b_2} by [4, Proposition 4.i], so the ideal diagrams differ first at $n = 2$. Now, assume that for μ we have instead $b_1 = 999, b_j = 1 \forall j \geq 2$, and thus $|\theta - \mu| \approx 0.000000998$, but in the Baire metric, we still have that $d(\text{cf}(\theta), \text{cf}(\mu)) = 0.5$, while $m_{i(\mathcal{U}_{\mathfrak{F}})}(I_\theta, I_\mu) = 2^{-1000}$ by Theorem (5.12) since at level $n = 1$ the diagram for \mathfrak{F}/I_θ begins with L_{999} and for \mathfrak{F}/I_μ begins with L_{998} and then transitions to R_{b_2} by [4, Proposition 4.i], so the ideal diagrams differ first at $n = 1000$. In conclusion, in this example, the absolute value metric $|\cdot|$ behaves much more like the metric $m_{i(\mathcal{U}_{\mathfrak{F}})}$ than the Baire metric.

Fix $\theta \in (0, 1) \setminus \mathbb{Q}$, we present a $*$ -isomorphism from \mathfrak{F}/I_θ to $\mathfrak{A}\mathfrak{F}_\theta$ (Notation (2.19)) as a proposition to highlight a useful property for our purposes. Of course, [4, Proposition 4.i] already established that \mathfrak{F}/I_θ and $\mathfrak{A}\mathfrak{F}_\theta$ are $*$ -isomorphic, but here we simply provide an explicit detail of such a $*$ -isomorphism, which will serve us in the results pertaining to tracial states in Lemma (6.19).

Proposition 6.13. *If $\theta \in (0, 1) \setminus \mathbb{Q}$ with continued fraction expansion $\theta = [a_j]_{j \in \mathbb{N}}$ as in Expression (2.1), then using Notation (2.19) and Definition (6.9), there exists a $*$ -isomorphism $\text{af}_\theta : \mathfrak{F}/I_\theta \rightarrow \mathfrak{A}\mathfrak{F}_\theta$ such that if $x = x_0 \oplus \cdots \oplus x_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$, then:*

$$\text{af}_\theta \left(\underset{\rightarrow}{\varphi}^{a_1}(x) + I_\theta \right) = \underset{\rightarrow}{\alpha}_\theta^1 \left(x_{j_{a_1}(\theta)+1} \oplus x_{j_{a_1}(\theta)} \right) \in \underset{\rightarrow}{\alpha}_\theta^1(\mathfrak{A}\mathfrak{F}_{\theta,1}).$$

Proof. By [4, Proposition 4.i] (specifically, the 2nd line of paragraph 2 after [4, Figure 5] in arXiv v6), the Bratteli diagram of \mathfrak{F}/I_θ begins with the diagram L_{a_1-1} of

[4, Figure 5] at level $n = 1$. Now, the diagram $C_a \circ C_b$ of [4, Figure 6] is a section of the diagram of Example (5.5), in which the left column of $C_{a_1-1} \circ C_{a_2}$ is the bottom row of the first two levels from left to right after level $n = 0$ of Example (5.5). Therefore, by the placement of \otimes at level a_1 in [4, Figure 6], define a map $f : (\mathfrak{F}^{a_1} + I_\theta) / I_\theta \rightarrow \underline{\alpha}_\theta^1(\mathfrak{A}\mathfrak{F}_{\theta,1})$ by:

$$f : \left(\underline{\varphi}^{a_1}(x) + I_\theta \right) \mapsto \underline{\alpha}_\theta^1 \left(x_{j_{a_1}(\theta)+1} \oplus x_{j_{a_1}(\theta)} \right),$$

where $x = x_0 \oplus \cdots \oplus x_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$. We show that f is a *-isomorphism from $(\mathfrak{F}^{a_1} + I_\theta) / I_\theta$ onto $\underline{\alpha}_\theta^1(\mathfrak{A}\mathfrak{F}_{\theta,1})$.

We first show that f is well-defined. Let $c, e \in (\mathfrak{F}^{a_1} + I_\theta) / I_\theta$ such that $c = e$. Now, we have $c = \underline{\varphi}^{a_1}(c') + I_\theta, e = \underline{\varphi}^{a_1}(e') + I_\theta$ where $c' = c'_0 \oplus \cdots \oplus c'_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$ and $e' = e'_0 \oplus \cdots \oplus e'_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$. But, the assumption $c = e$ implies that $\underline{\varphi}^{a_1}(c' - e') \in I_\theta \cap \mathfrak{F}^{a_1}$. Thus, by Definition (6.9) of I_θ , we have that $c'_{j_{a_1}(\theta)+1} \oplus c'_{j_{a_1}(\theta)} = e'_{j_{a_1}(\theta)+1} \oplus e'_{j_{a_1}(\theta)}$, and since $j_{a_1}(\theta) = q_0^\theta$ and $j_{a_1}(\theta) + 1 = q_1^\theta$ by [4, Proposition 4.i] and the discussion at the start of the proof, we gather that f is a well-defined *-homomorphism since the canonical maps $\underline{\alpha}_\theta^1$ and $\underline{\varphi}^{a_1}$ are *-homomorphisms.

For surjectivity of f , let $x = \underline{\alpha}_\theta^1(x_{q_1^\theta} \oplus x_{q_0^\theta})$, where $x_{q_1^\theta} \oplus x_{q_0^\theta} \in \mathfrak{A}\mathfrak{F}_{\theta,1}$. Define $y = y_0 \oplus \cdots \oplus y_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$ such that $y_{j_{a_1}(\theta)} = x_{q_0^\theta}$ and $y_{j_{a_1}(\theta)+1} = x_{q_1^\theta}$ with $y_k = 0$ for all $k \in \{0, \dots, 2^{a_1-1}\} \setminus \{j_{a_1}(\theta), j_{a_1}(\theta) + 1\}$. Hence, the image $f\left(\underline{\varphi}^{a_1}(y) + I_\theta\right) = x$.

For injectivity of f , let $x = x_0 \oplus \cdots \oplus x_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$ and $y = y_0 \oplus \cdots \oplus y_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$ such that $f\left(\underline{\varphi}^{a_1}(x) + I_\theta\right) = f\left(\underline{\varphi}^{a_1}(y) + I_\theta\right)$. Thus, since $\underline{\alpha}_\theta^1$ is injective, we have that $x_{j_{a_1}(\theta)+1} \oplus x_{j_{a_1}(\theta)} = y_{j_{a_1}(\theta)+1} \oplus y_{j_{a_1}(\theta)}$. But, this then implies that $\underline{\varphi}^{a_1}(x - y) \in I_\theta \cap \mathfrak{F}^{a_1} \subseteq I_\theta$ by Definition (6.9), and therefore, the terms $\underline{\varphi}^{a_1}(x) + I_\theta = \underline{\varphi}^{a_1}(y) + I_\theta$, which completes the argument that f is a *-isomorphism from $(\mathfrak{F}^{a_1} + I_\theta) / I_\theta$ onto $\underline{\alpha}_\theta^1(\mathfrak{A}\mathfrak{F}_{\theta,1})$.

Lastly, the proof of [4, Proposition 4.i] provides a Bratteli diagram for \mathfrak{F} / I_θ equal to the Bratteli diagram of $\mathfrak{A}\mathfrak{F}_\theta$ given by Example (5.5) up to the identifications of [4, Figure 6], in which the Bratteli diagram for \mathfrak{F} / I_θ begins at $(\mathfrak{F}^{a_1} + I_\theta) / I_\theta$ and the Bratteli diagram for $\mathfrak{A}\mathfrak{F}_\theta$ begins at $\mathfrak{A}\mathfrak{F}_{\theta,1}$. Therefore, by the construction of the *-isomorphism in [10, Proposition III.2.7] between two AF algebras with the same diagram, we conclude that there exists a *-isomorphism $\mathfrak{af}_\theta : \mathfrak{F} / I_\theta \rightarrow \mathfrak{A}\mathfrak{F}_\theta$ such that $\mathfrak{af}_\theta(z) = f(z)$ for all $z \in (\mathfrak{F}^{a_1} + I_\theta) / I_\theta$, which completes the proof. \square

From the *-isomorphism of Proposition (6.13), we may provide a faithful tracial state for the quotient \mathfrak{F} / I_θ from the unique faithful tracial state of $\mathfrak{A}\mathfrak{F}_\theta$. Indeed:

Notation 6.14. Fix $\theta \in (0, 1) \setminus \mathbb{Q}$. There is a unique faithful tracial state on $\mathfrak{A}\mathfrak{F}_\theta$ denoted σ_θ (see [1, Lemma 5.3 and Lemma 5.5]). Thus,

$$\tau_\theta = \sigma_\theta \circ \mathfrak{af}_\theta$$

is a unique faithful tracial state on \mathfrak{F}/I_θ with af_θ from Proposition (6.13).

Let $Q_\theta : \mathfrak{F} \rightarrow \mathfrak{F}/I_\theta$ denote the quotient map. Thus, by [9, Theorem V.2.2], there exists a unique linear functional on \mathfrak{F} denoted, ρ_θ , such that $\tau_\theta \circ Q_\theta(x) = \rho_\theta(x)$ for all $x \in \mathfrak{F}$. Since τ_θ is a tracial state and:

$$\tau_\theta \circ Q_\theta(x) = \rho_\theta(x)$$

for all $x \in \mathfrak{F}$, we conclude that ρ_θ is also a tracial state that vanishes on I_θ . Furthermore, ρ_θ is faithful on $\mathfrak{F} \setminus I_\theta$ since τ_θ is faithful on \mathfrak{F}/I_θ .

One more ingredient remains before we define the quantum metric structure for the quotient spaces \mathfrak{F}/I_θ .

Lemma 6.15. *Let $\theta \in (0, 1) \setminus \mathbb{Q}$. Using notation from Definition (6.7) and Definition (6.9), if we define:*

$$\beta^\theta : n \in \mathbb{N} \mapsto \frac{1}{\dim((\mathfrak{F}^n + I_\theta)/I_\theta)} \in (0, \infty),$$

then $\beta^\theta(n) = \frac{1}{q(n, j_n(\theta))^2 + q(n, j_n(\theta) + 1)^2} \leq \frac{1}{n^2}$ for all $n \in \mathbb{N} \setminus \{0\}$ and $\beta^\theta(0) = 1$.

Proof. First, the quotient $(\mathfrak{F}^0 + I_\theta)/I_\theta = \mathbb{C}1_{\mathfrak{F}/I_\theta}$. Hence, the term $\beta^\theta(0) = 1$.

Fix $n \in \mathbb{N} \setminus \{0\}$. Since $(\mathfrak{F}^n + I_\theta)/I_\theta$ is *-isomorphic to $\mathfrak{F}^n/(I_\theta \cap \mathfrak{F}^n)$ (see Proposition (4.15)), we have that

$$\dim((\mathfrak{F}^n + I_\theta)/I_\theta) = \dim(\mathfrak{F}^n/(I_\theta \cap \mathfrak{F}^n)) = q(n, j_n(\theta))^2 + q(n, j_n(\theta) + 1)^2$$

by Definition (6.9) and the dimension of the quotient is the difference in dimensions of \mathfrak{F}^n and $I_\theta \cap \mathfrak{F}^n$. Therefore, the term $\beta^\theta(n) = \frac{1}{q(n, j_n(\theta))^2 + q(n, j_n(\theta) + 1)^2}$.

To prove the inequality of the Lemma, we claim that for all $n \in \mathbb{N} \setminus \{0\}$, we have $q(n, j_n(\theta)) \geq n$ or $q(n, j_n(\theta) + 1) \geq n$. We proceed by induction. If $n = 1$, then $q(1, j_1(\theta)) = 1$ and $q(1, j_1(\theta) + 1) = 1$ by Relations (6.1). Next assume the statement of the claim is true for $n = m$. Thus, we have that $q(m, j_m(\theta)) \geq m$ or $q(m, j_m(\theta) + 1) \geq m$. First, assume that $q(m, j_m(\theta)) \geq m$. By Lemma (6.10), assume that $j_{m+1}(\theta) = 2j_m(\theta)$. Thus, we gather $q(m+1, j_{m+1}(\theta) + 1) = q(m+1, 2j_m(\theta) + 1) = q(m, j_m(\theta)) + q(m, j_m(\theta) + 1) \geq m+1$ by Relations (6.1) and since $q(m, j_m(\theta) + 1) \in \mathbb{N} \setminus \{0\}$. The case when $j_{m+1}(\theta) = 2j_m(\theta) + 1$ follows similarly as well as the case when $q(m, j_m(\theta) + 1) \geq m$, which completes the induction argument.

In particular, for all $n \in \mathbb{N} \setminus \{0\}$, we have $q(n, j_n(\theta)) \geq n$ or $q(n, j_n(\theta) + 1) \geq n$, which implies that $q(n, j_n(\theta))^2 \geq n^2$ or $q(n, j_n(\theta) + 1)^2 \geq n^2$. And thus, the term:

$$\frac{1}{q(n, j_n(\theta))^2 + q(n, j_n(\theta) + 1)^2} \leq \frac{1}{n^2}$$

for all $n \in \mathbb{N} \setminus \{0\}$. □

Hence, we have all the ingredients to define the quotient quantum metric spaces of the ideals of Definition (6.9).

Notation 6.16. Fix $\theta \in (0, 1) \setminus \mathbb{Q}$. Using Definition (6.7), Definition (6.9), Notation (6.14), and Lemma (6.15), let:

$$\left(\mathfrak{F}/I_\theta, \mathcal{L}_{\mathfrak{F}/I_\theta, \tau_\theta}^{\beta^\theta} \right)$$

denote the $(2, 0)$ -quasi-Leibniz quantum compact metric space given by Theorem (6.2) associated to the ideal I_θ , faithful tracial state τ_θ , and $\beta^\theta : \mathbb{N} \rightarrow (0, \infty)$ having limit 0 at infinity by Lemma (6.15).

Remark 6.17. Fix $\theta \in (0, 1) \setminus \mathbb{Q}$. Although \mathfrak{F}/I_θ and $\mathfrak{A}\mathfrak{F}_\theta$ are $*$ -isomorphic, it is unlikely that $\left(\mathfrak{F}/I_\theta, \mathcal{L}_{\mathcal{U}_{\mathfrak{F}/I_\theta, \tau_\theta}}^{\beta^\theta}\right)$ is isometrically isomorphic to $\left(\mathfrak{A}\mathfrak{F}_\theta, \mathcal{L}_{\mathcal{I}_\theta, \sigma_\theta}^{\beta^\theta}\right)$ of Theorem (2.20) based on the Lip-norm constructions. Thus, one could not simply apply Proposition (6.11) to Theorem (2.20) to achieve Theorem (6.20).

In order to provide our continuity results, we describe the faithful tracial states on the quotients in sufficient detail through Lemma (6.18) and Lemma (6.19).

Lemma 6.18. Fix $\theta \in (0, 1) \setminus \mathbb{Q}$. Let tr_d be the unique tracial state of $\mathfrak{M}(d)$. Using notation from Definitions (6.7, 6.9), if $n \in \mathbb{N} \setminus \{0\}$ and $a = a_0 \oplus \cdots \oplus a_{2^n-1} \in \mathfrak{F}_n$, then using Notation (6.14):

$$\rho_\theta \circ \varphi_{\rightarrow}^n(a) = c(n, \theta) \text{tr}_{q(n, j_n(\theta))} \left(a_{j_n(\theta)} \right) + (1 - c(n, \theta)) \text{tr}_{q(n, j_n(\theta)+1)} \left(a_{j_n(\theta)+1} \right),$$

where $c(n, \theta) \in (0, 1)$ and $\rho_\theta \circ \varphi_{\rightarrow}^0(a) = a$ for all $a \in \mathfrak{F}_0$.

Furthermore, let $n \in \mathbb{N} \setminus \{0\}$, then:

$$c(n+1, \theta) = \begin{cases} \frac{(q(n, j_n(\theta)) + q(n, j_n(\theta)+1))c(n, \theta) - q(n, j_n(\theta))}{q(n, j_n(\theta)+1)} & \text{if } j_{n+1}(\theta) = 2j_n(\theta) \\ \left(1 + \frac{q(n, j_n(\theta)+1)}{q(n, j_n(\theta))}\right) c(n, \theta) & \text{if } j_{n+1}(\theta) = 2j_n(\theta) + 1 \end{cases}.$$

Proof. Fix $\theta \in (0, 1) \setminus \mathbb{Q}$. If $n = 0$, then $\rho_\theta \circ \varphi_{\rightarrow}^0(a) = a$ for all $a \in \mathfrak{F}_0$ since $\mathfrak{F}_0 = \mathbb{C}$. Let $n \in \mathbb{N} \setminus \{0\}$ and $a = a_0 \oplus \cdots \oplus a_{2^n-1} \in \mathfrak{F}_n$. Now, ρ_θ is a tracial state on \mathfrak{F} , and thus, the composition $\rho_\theta \circ \varphi_{\rightarrow}^n$ is a tracial state on \mathfrak{F}_n . Hence, by [10, Example IV.5.4]:

$$\rho_\theta \circ \varphi_{\rightarrow}^n(a) = \sum_{k=0}^{2^n-1} c_k \text{tr}_{q(n, k)}(a_k),$$

where $\sum_{k=0}^{2^n-1} c_k = 1$ and $c_k \in [0, 1]$ for all $k \in \{0, \dots, 2^n-1\}$. But, since ρ_θ vanishes on I_θ , by Definition (6.9), we conclude that $c_k = 0$ for all $k \in \{0, \dots, 2^n-1\} \setminus \{j_n(\theta), j_n(\theta) + 1\}$. Also, the fact that ρ_θ is faithful on $\mathfrak{F} \setminus I_\theta$ implies that $c_{j_n(\theta)}, c_{j_n(\theta)+1} \in (0, 1)$ and $c_{j_n(\theta)} + c_{j_n(\theta)+1} = 1$. Define $c(n, \theta) = c_{j_n(\theta)}$ and clearly $c_{j_n(\theta)+1} = 1 - c(n, \theta)$.

Next, let $n \in \mathbb{N} \setminus \{0\}$ and let $j_{n+1}(\theta) = 2j_n(\theta)$. Combining Lemma (6.10) and Proposition (6.8), there is one edge from $(n, j_n(\theta))$ to $(n+1, j_{n+1}(\theta))$ and one edge from $(n, j_n(\theta))$ to $(n+1, j_{n+1}(\theta) + 1)$ with no other edges from $(n, j_n(\theta))$ to either $(n, j_n(\theta))$ or $(n+1, j_{n+1}(\theta) + 1)$. Also, there is one edge from $(n, j_n(\theta) + 1)$ to $(n+1, j_{n+1}(\theta) + 1)$ with no other edges from $(n, j_n(\theta) + 1)$ to either $(n, j_n(\theta))$ or $(n+1, j_{n+1}(\theta) + 1)$.

Hence, consider an element $a = a_0 \oplus \cdots \oplus a_{2^n-1} \in \mathfrak{F}_n$ such that $a_k = 0$ for all $k \in \{0, \dots, 2^n-1\} \setminus \{j_n(\theta), j_n(\theta) + 1\}$. Since the edges determine the partial multiplicities of φ_n , we have that $\varphi_n(a) = b_0 \oplus \cdots \oplus b_{2^n}$ such that

$$(6.4) \quad b_{j_{n+1}(\theta)} = Ua_{j_n(\theta)}U^* \text{ and } b_{j_{n+1}(\theta)+1} = V \begin{bmatrix} a_{j_n(\theta)} & 0 \\ 0 & a_{j_n(\theta)+1} \end{bmatrix} V^*,$$

where $U \in \mathfrak{M}(q(n+1, j_{n+1}(\theta)))$, $V \in \mathfrak{M}(q(n+1, j_{n+1}(\theta) + 1))$ are unitary by [10, Lemma III.2.1]. Also, the terms $b_k = 0$ for all $k \in \{0, \dots, 2^n-1\} \setminus \{j_{n+1}(\theta), j_{n+1}(\theta) + 1\}$. But, by definition of the canonical *-homomorphisms φ_n, φ_{n+1} , we have that

$$\varphi_n(a) = \varphi_{n+1}(\varphi_n(a)) \quad [34, \text{Chapter 6.1}].$$

Now, assume that $a_{j_n(\theta)} = 1_{\mathfrak{M}(q(n, j_n(\theta)))}$ and $a_{j_n(\theta)+1} = 0$. Therefore, by Expression (6.4):

$$(6.5) \quad \begin{aligned} c(n, \theta) &= \rho_\theta \circ \varphi_n(a) \\ &= \rho_\theta \circ \varphi_{n+1}(\varphi_n(a)) \\ &= c(n+1, \theta) \text{tr}_{q(n+1, j_{n+1}(\theta))} (Ua_{j_n(\theta)}U^*) \\ &\quad + (1 - c(n+1, \theta)) \text{tr}_{q(n+1, j_{n+1}(\theta)+1)} \left(V \begin{bmatrix} a_{j_n(\theta)} & 0 \\ 0 & 0 \end{bmatrix} V^* \right) \\ &= c(n+1, \theta) \cdot 1 + (1 - c(n+1, \theta)) \text{tr}_{q(n+1, j_{n+1}(\theta)+1)} \left(\begin{bmatrix} 1_{\mathfrak{M}(q(n, j_n(\theta)))} & 0 \\ 0 & 0 \end{bmatrix} \right) \\ &= c(n+1, \theta) + (1 - c(n+1, \theta)) \frac{1}{q(n+1, j_{n+1}(\theta) + 1)} q(n, j_n(\theta)). \end{aligned}$$

Thus, since $q(n+1, 2j_n(\theta) + 1) = q(n, j_n(\theta)) + q(n, j_n(\theta) + 1)$ from Relations (6.1) and $j_{n+1}(\theta) + 1 = 2j_n(\theta) + 1$, we conclude that:

$$c(n+1, \theta) = \frac{(q(n, j_n(\theta)) + q(n, j_n(\theta) + 1))c(n, \theta) - q(n, j_n(\theta))}{q(n, j_n(\theta) + 1)}.$$

Lastly, assume that $j_{n+1}(\theta) = 2j_n(\theta) + 1$. Let $a = a_0 \oplus \cdots \oplus a_{2^n-1} \in \mathfrak{F}_n$ such that $a_k = 0$ for all $k \in \{0, \dots, 2^n-1\} \setminus \{j_n(\theta), j_n(\theta) + 1\}$. A similiar argument shows that $\varphi_n(a) = b_0 \oplus \cdots \oplus b_{2^n}$ such that:

$$b_{j_{n+1}(\theta)} = Y \begin{bmatrix} a_{j_n(\theta)} & 0 \\ 0 & a_{j_n(\theta)+1} \end{bmatrix} Y^* \text{ and } b_{j_{n+1}(\theta)+1} = Za_{j_n(\theta)+1}Z^*,$$

where $Y \in \mathfrak{M}(q(n+1, j_{n+1}(\theta)))$, $Z \in \mathfrak{M}(q(n+1, j_{n+1}(\theta) + 1))$ are unitary. Now, assume that $a_{j_n(\theta)} = 1_{\mathfrak{M}(q(n, j_n(\theta)))}$ and $a_{j_n(\theta)+1} = 0$. Therefore, similarly to Expression (6.5):

$$c(n, \theta) = c(n+1, \theta) \frac{1}{q(n+1, j_{n+1}(\theta))} q(n, j_n(\theta)),$$

and therefore:

$$c(n+1, \theta) = \left(1 + \frac{q(n, j_n(\theta) + 1)}{q(n, j_n(\theta))} \right) c(n, \theta)$$

by Relations (6.1). And, by Lemma (6.10), this exhausts all possibilities for $c(n + 1, \theta)$, and the proof is complete. \square

Lemma 6.19. *Using notation from Lemma (6.18), if $\theta \in (0, 1) \setminus \mathbb{Q}$, then:*

$$c(1, \theta) = 1 - \theta.$$

Moreover, using notation from Definition (6.9), if $\theta, \mu \in (0, 1) \setminus \mathbb{Q}$ such that there exists $N \in \mathbb{N} \setminus \{0\}$ with $I_\theta \cap \mathfrak{F}^N = I_\mu \cap \mathfrak{F}^N$, then there exists $a, b \in \mathbb{R}, a \neq 0$ such that:

$$c(N, \theta) = a\theta + b, \quad c(N, \mu) = a\mu + b.$$

Proof. Let $\theta \in (0, 1) \setminus \mathbb{Q}$, and denote its continued fraction expansion by $\theta = [a_j]_{j \in \mathbb{N}}$. Recall, by Proposition (6.13), we have for all $x = x_0 \oplus \cdots \oplus x_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$:

$$(6.6) \quad \text{af}_\theta \left(\underset{\rightarrow}{\varphi}^{a_1}(x) + I_\theta \right) = \underset{\rightarrow}{\alpha}_\theta^1 \left(x_{j_{a_1}(\theta)+1} \oplus x_{j_{a_1}(\theta)} \right).$$

Next, by Notation (6.14), we note that:

$$(6.7) \quad \rho_\sigma \circ \underset{\rightarrow}{\varphi}^{a_1} = \tau_\theta \circ Q_\theta \circ \underset{\rightarrow}{\varphi}^{a_1} = \sigma_\theta \circ \text{af}_\theta \circ Q_\theta \circ \underset{\rightarrow}{\varphi}^{a_1}$$

Now, consider $x = x_0 \oplus \cdots \oplus x_{2^{a_1-1}} \in \mathfrak{F}_{a_1}$ such that $x_{j_{a_1}(\theta)+1} = 1_{q_1^\theta}$ and $x_k = 0$ for all $k \in \{0, \dots, 2^{a_1-1}\} \setminus \{j_{a_1}(\theta)\}$. Then, by Lemma (6.18) and Expressions (6.6, 6.7), we have that $(1 - c(a_1, \theta)) = \rho_\theta \circ \underset{\rightarrow}{\varphi}^{a_1}(x) = \sigma_\theta \circ \underset{\rightarrow}{\alpha}_\theta^1 \left(1_{q_1^\theta} \oplus 0 \right) = a_1\theta$ by [1, Lemma 5.5]. And, thus:

$$(6.8) \quad c(a_1, \theta) = 1 - a_1\theta.$$

Thus, if $a_1 = 1$, then we would be done.

Assume that $a_1 \geq 2$. By [4, Proposition 4.i] (specifically, the 2nd line of paragraph 2 after [4, Figure 5] in arXiv v6), the Bratteli diagram of \mathfrak{F}/I_θ begins with the diagram L_{a_1-1} of [4, Figure 5] at level $n = 1$. Thus, the term $j_m(\theta) = 0$ for all $m \in \{1, \dots, a_1\}$. Hence, if $m \in \{1, \dots, a_1 - 1\}$, then $j_{m+1}(\theta) = 2j_m(\theta)$.

We claim that for all $m \in \{1, \dots, a_1\}$ we have that:

$$(6.9) \quad c(m, \theta) = mc(1, \theta) - (m - 1).$$

We proceed by induction. The case $m = 1$ is clear. Assume true for $m \in \{1, \dots, a_1 - 1\}$. Consider $m + 1$. Since $j_{m+1}(\theta) = 2j_m(\theta)$, by Lemma (6.18), we have that:

$$(6.10) \quad \begin{aligned} c(m+1, \theta) &= \frac{(q(m, 0) + q(m, 1))c(m, \theta) - q(m, 0)}{q(m, 1)} \\ &= \frac{c(m, \theta) + q(m, 1)c(m, \theta) - 1}{q(m, 1)}. \end{aligned}$$

By Relations (6.1), we gather that $q(m, 1) = m$. Hence, by induction hypothesis and Expression (6.10), we have:

$$\begin{aligned} c(m+1, \theta) &= \frac{mc(1, \theta) - (m - 1) + m(mc(1, \theta) - (m - 1)) - 1}{m} \\ &= c(1, \theta) - 1 + 1/m + mc(1, \theta) - (m - 1) - 1/m \\ &= (m + 1)c(1, \theta) - ((m + 1) - 1), \end{aligned}$$

which completes the induction argument. Hence, by Expression (6.9), we conclude $c(a_1, \theta) = a_1 c(1, \theta) - (a_1 - 1)$, which implies that:

$$(6.11) \quad c(1, \theta) = 1 - \theta$$

by Equation (6.8).

Lastly, let $\theta, \mu \in (0, 1) \setminus \mathbb{Q}$. We prove the remaining claim in the Lemma by induction. Assume $N = 1$. Then, by Equation (6.11), the coefficients $c(1, \mu) = 1 - \mu$ and $c(1, \theta) = 1 - \theta$, which completes the base case.

Assume true for $N \in \mathbb{N} \setminus \{0, 1\}$. Assume that $I_\mu \cap \mathfrak{F}^{N+1} = I_\theta \cap \mathfrak{F}^{N+1}$. Now, since $\mathfrak{F}^N \subseteq \mathfrak{F}^{N+1}$, we thus have $I_\mu \cap \mathfrak{F}^N = I_\theta \cap \mathfrak{F}^N$. Hence, by induction hypothesis, there exists $a, b \in \mathbb{R}, a \neq 0$ such that $c(N, \mu) = a\mu + b$ and $c(N, \theta) = a\theta + b$. But, as $I_\mu \cap \mathfrak{F}^{N+1} = I_\theta \cap \mathfrak{F}^{N+1}$, the vertices a level $N + 1$ agree in the ideal diagrams by Proposition (5.11). In particular, by Definition (6.9), we have $j_{N+1}(\theta) = j_{N+1}(\mu)$, and similarly, the term $j_N(\theta) = j_N(\mu)$ by $I_\mu \cap \mathfrak{F}^N = I_\theta \cap \mathfrak{F}^N$. Therefore, by Lemma (6.18), the desired conclusion follows by basic arithmetic along with the induction hypothesis and irrationality of θ, μ . \square

We can now prove the main result of this section.

Theorem 6.20. *Using Definition (6.9) and Notation (6.16), the map:*

$$I_\theta \in (\text{Prim}(\mathfrak{F}), \tau) \longmapsto \left(\mathfrak{F}/I_\theta, \mathbb{L}_{\mathcal{U}_{\mathfrak{F}}/I_\theta, \tau_\theta}^{\beta^\theta} \right) \in (\mathcal{QQCMS}_{2,0}, \Lambda)$$

is continuous to the class of $(2, 0)$ -quasi-Leibniz quantum compact metric spaces metrized by the quantum propinquity Λ , where τ is either the Jacobson topology, the relative metric topology of $\mathfrak{m}_i(\mathcal{U}_{\mathfrak{F}})$ (Proposition (4.11)), or the relative Fell topology (Definition (4.3)).

Proof. By Proposition (6.11), we only need to show continuity with respect to the metric $\mathfrak{m}_i(\mathcal{U}_{\mathfrak{F}})$. Thus, let $(I_{\theta_n})_{n \in \overline{\mathbb{N}}} \subset \text{Prim}(\mathfrak{F})$ be a sequence such that $(I_{\theta_n})_{n \in \mathbb{N}}$ converges to I_{θ_∞} with respect to $\mathfrak{m}_i(\mathcal{U}_{\mathfrak{F}})$. Therefore, by Corollary (5.15), this implies that:

$$\left\{ I_{\theta_n} = \overline{\bigcup_{k \in \mathbb{N}} I_{\theta_n} \cap \mathfrak{F}^k}^{\|\cdot\|_{\mathfrak{F}}} : n \in \overline{\mathbb{N}}} \right\}$$

is a fusing family with some fusing sequence $(c_n)_{n \in \mathbb{N}}$. Thus, condition (1) of Theorem (6.3) is satisfied.

For condition (2) of Theorem (6.3), let $N \in \mathbb{N}$, then by definition of fusing sequence, if $k \in \mathbb{N}_{\geq c_N}$, then $I_{\theta_k} \cap \mathfrak{F}^N = I_{\theta_\infty} \cap \mathfrak{F}^N$. Now, let $k \in \overline{\mathbb{N}}_{\geq c_N}$. Consider ρ_{θ_k} on \mathfrak{F}^N . By Lemma (6.19), there exists $a, b \in \mathbb{R}, a \neq 0$, such that $c(N, \theta_k) = a\theta_k + b$ for all $k \in \overline{\mathbb{N}}_{\geq c_N}$. But, by Proposition (6.11), we obtain $(\theta_n)_{n \in \mathbb{N}}$ converges to θ_∞ with respect to the usual topology on \mathbb{R} . Hence, the sequence $(c(N, \theta_k))_{k \in \mathbb{N}_{\geq c_N}}$ converges to $c(N, \theta_\infty)$ with respect to the usual topology on \mathbb{R} and the same applies to $(1 - c(N, \theta_k))_{k \in \mathbb{N}_{\geq c_N}}$. However, by Lemma (6.18), the coefficient $c(N, \theta_k)$ determines ρ_k for all $k \in \overline{\mathbb{N}}_{\geq c_N}$. Hence, a similar argument given in [1, Lemma 5.11], provides that $(\rho_{\theta_k})_{k \in \mathbb{N}_{\geq c_N}}$ converges to ρ_{θ_∞} in the weak-* topology on $\mathcal{S}(\mathfrak{F}^N)$.

Condition (3) of Theorem (6.3) follows a similar argument as in the proof of condition (2) since the sequences β^θ of Lemma (6.15) are determined by the terms $j_n(\theta)$. Also, all β^θ are uniformly bounded by the sequence $(1/n^2)_{n \in \mathbb{N}}$ which converges to 0. Therefore, the proof is complete. \square

As an aside to Remark (6.17), we obtain the following analogue to Theorem (2.20) in terms of quotients.

Corollary 6.21. *Using Notation (6.16), the map:*

$$\theta \in ((0, 1) \setminus \mathbb{Q}, |\cdot|) \mapsto \left(\mathfrak{F}/I_\theta, L_{\mathcal{U}_{\mathfrak{F}}/I_\theta, \tau_\theta}^{\beta^\theta} \right) \in (\mathcal{QQCMS}_{2,0}, \Lambda)$$

is continuous from $(0, 1) \setminus \mathbb{Q}$, with its topology as a subset of \mathbb{R} to the class of $(2, 0)$ -quasi-Leibniz quantum compact metric spaces metrized by the quantum propinquity Λ .

Proof. Apply Proposition (6.11) to Theorem (6.20). \square

7. LEIBNIZ LIP-NORMS FOR UNITAL AF ALGEBRAS

Our work in [1] and the rest of this current paper aside from this section rely on the hypothesis of the existence of faithful tracial state for a Unital AF algebra. Of course, every simple unital AF algebra has a faithful tracial state, but in the non-simple case, there exists unital AF algebras without faithful tracial states. For example, consider the unitlization of the compact operators on a separable Hilbert space.

Thus, in this section, we introduce Leibniz Lip-norms that exist on any unital AF algebra, in which Leibniz Lip-norms are defined to be $(1, 0)$ -quasi-Leibniz Lip-norms, built from best approximations and the work of Rieffel in [42], in which he established the Leibniz property for certain quotient norms. Then, in Proposition (7.5), in the case of a unital AF algebra with faithful tracial state, we make a comparison between the best approximation Leibniz Lip-norm and our conditional expectation quasi-Leibniz Lip-norm, which utilizes the recent work of Latrémolière in [30].

Convention 7.1. A Leibniz Lip-norm is a $(1, 0)$ -quasi-Leibniz Lip-norm of Definition (2.3), and in this Section (7), by Λ , we mean $\Lambda_{1,0}$.

Notation 7.2. Let $(\mathfrak{A}, L_{\mathfrak{A}})$ be a quasi-Leibniz quantum compact metric space. Let $\mu \in \mathcal{S}(\mathfrak{A})$. Denote:

$$\begin{aligned} \mathfrak{Lip}_1(\mathfrak{A}, L_{\mathfrak{A}}) &= \{a \in \mathfrak{sa}(\mathfrak{A}) : L_{\mathfrak{A}}(a) \leq 1\} \\ \mathfrak{Lip}_1(\mathfrak{A}, L_{\mathfrak{A}}, \mu) &= \{a \in \mathfrak{sa}(\mathfrak{A}) : L_{\mathfrak{A}}(a) \leq 1, \mu(a) = 0\}. \end{aligned}$$

Lemma 7.3. *Let $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. If $\gamma = (\mathfrak{D}, \omega, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is a bridge of Definition (2.6), then for any two states $\varphi_{\mathfrak{A}} \in \mathcal{S}(\mathfrak{A}), \varphi_{\mathfrak{B}} \in \mathcal{S}(\mathfrak{B})$, we have that:*

$$\begin{aligned} \text{Haus}_{\mathfrak{D}}(\pi_{\mathfrak{A}}(\mathfrak{Lip}_1(\mathfrak{A}, L_{\mathfrak{A}}))\omega, \omega\pi_{\mathfrak{B}}(\mathfrak{Lip}_1(\mathfrak{B}, L_{\mathfrak{B}}))) &\leq \\ \text{Haus}_{\mathfrak{D}}(\pi_{\mathfrak{A}}(\mathfrak{Lip}_1(\mathfrak{A}, L_{\mathfrak{A}}, \varphi_{\mathfrak{A}}))\omega, \omega\pi_{\mathfrak{B}}(\mathfrak{Lip}_1(\mathfrak{B}, L_{\mathfrak{B}}, \varphi_{\mathfrak{B}}))). & \end{aligned}$$

Proof. The proof is the argument in between [29, Notation 3.13] and [29, Definition 3.14]. \square

Theorem 7.4. *Let \mathfrak{A} be a unital AF algebra with unit $1_{\mathfrak{A}}$ such that $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ is an increasing sequence of unital finite dimensional C^* -subalgebras such that $\mathfrak{A} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$, with $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$. For each $n \in \mathbb{N}$, we denote the quotient norm of $\mathfrak{A}/\mathfrak{A}_n$ with respect to $\|\cdot\|_{\mathfrak{A}}$ by S_n . Let $\beta : \mathbb{N} \rightarrow (0, \infty)$ have limit 0 at infinity.*

If, for all $a \in \cup_{n \in \mathbb{N}} \mathfrak{A}_n$, we set:

$$L_{\mathcal{U}}^{\beta}(a) = \sup \left\{ \frac{S_n(a)}{\beta(n)} : n \in \mathbb{N} \right\}$$

and $L_{\mathcal{U}}^{\beta}(a) = \infty$ for all $a \in \mathfrak{A} \setminus \cup_{n \in \mathbb{N}} \mathfrak{A}_n$, then,

- (1) $(\mathfrak{A}, L_{\mathcal{U}}^{\beta})$ and $(\mathfrak{A}_n, L_{\mathcal{U}}^{\beta})$ for all $n \in \mathbb{N}$ are Leibniz quantum compact metric spaces,
and
- (2) $\lim_{n \rightarrow \infty} \wedge \left((\mathfrak{A}_n, L_{\mathcal{U}}^{\beta}), (\mathfrak{A}, L_{\mathcal{U}}^{\beta}) \right) = 0$.

Proof. We begin by proving (1). By [42, Theorem 3.1], for all $n \in \mathbb{N}$, we have that since \mathfrak{A}_n is unital and finite dimensional, the quotient norm S_n satisfies condition (2) of Definition (2.3) for $C = 1$, $D = 0$, and therefore, so does $L_{\mathcal{U}}^{\beta}$. Or, in other words, $L_{\mathcal{U}}^{\beta}$ is a Leibniz seminorm.

By finite dimensionality, the subspace \mathfrak{A}_n is closed in \mathfrak{A} for all $n \in \mathbb{N}$. Also, as $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}} \subsetneq \mathfrak{A}_n$ for each $n \in \mathbb{N} \setminus \{0\}$, we can conclude $L_{\mathcal{U}}^{\beta^{-1}}(\{0\}) = \mathbb{C}1_{\mathfrak{A}}$. Next, let $a \in \cup_{n \in \mathbb{N}} \mathfrak{A}_n$, then there exists $N \in \mathbb{N}$ such that $a \in \mathfrak{A}_k$ for all $k \geq N$. Therefore, the seminorm $S_k(a) = 0$ for all $k \geq N$, and hence, the seminorm $L_{\mathcal{U}}^{\beta}$ evaluated at a is a supremum over finitely many terms, and is thus finite. Thus, $(\mathfrak{A}, L_{\mathcal{U}}^{\beta})$ and $(\mathfrak{A}_n, L_{\mathcal{U}}^{\beta})$ for each $n \in \mathbb{N}$ satisfy up to and including condition (2) of Definition (2.3). But, as a supremum of continuous maps, we have $L_{\mathcal{U}}^{\beta}$ is lower semicontinuous.

Thus, all that remains for (1) is for $L_{\mathcal{U}}^{\beta}$ to satisfy condition (3) of Definition (2.3). First, note that since β has limit 0 at infinity, we have $K = \sup\{\beta(n) : n \in \mathbb{N}\} < \infty$. Next, let $q_0 : \mathfrak{A} \rightarrow \mathfrak{A}/\mathfrak{A}_0 = \mathfrak{A}/\mathbb{C}1_{\mathfrak{A}}$ denote the quotient map. Define:

$$L_1 = \left\{ a \in \cup_{n \in \mathbb{N}} \mathfrak{A}_n : L_{\mathcal{U}}^{\beta}(a) \leq 1 \right\}.$$

By [37, Theorem 1.8], to satisfy condition (3) of Definition (2.3), it is enough to show that $q_0(L_1)$ totally bounded with respect to the quotient norm on $\mathfrak{A}/\mathbb{C}1_{\mathfrak{A}}$, in which the quotient norm is simply S_0 since $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$. Let $\varepsilon > 0$. By definition of $L_{\mathcal{U}}^{\beta}$, there exists $N \in \mathbb{N}$ such that $\beta(N) < \varepsilon/3$, so that $S_N(a) \leq \beta(N) < \varepsilon/3$ for all $a \in L_1$. Since \mathfrak{A}_N is a finite dimensional subspace, there exists a best approximation to a in \mathfrak{A}_N . Thus, for all $a \in L_1$, by axiom of choice, set $b_N(a) \in \mathfrak{A}_N$ to be one best approximation of a . Define:

$$B_N = \{b_N(a) \in \mathfrak{A}_N : a \in L_1\}.$$

If $a \in L_1$, then since $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$:

$$\begin{aligned} S_0(b_N(a)) &= \inf\{\|b_N(a) - \lambda 1_{\mathfrak{A}}\|_{\mathfrak{A}} : \lambda \in \mathbb{C}\} \\ &= \inf\{\|b_N(a) - a + a - \lambda 1_{\mathfrak{A}}\| : \lambda \in \mathbb{C}\} \\ &\leq \|b_N(a) - a\| + \inf\{\|a - \lambda 1_{\mathfrak{A}}\| : \lambda \in \mathbb{C}\} \\ &= S_N(a) + S_0(a) \\ &\leq \beta(N) + \beta(0) \leq 2K. \end{aligned}$$

Hence, the set $q_0(B_N)$ is bounded with respect S_0 on $\mathfrak{A}/\mathbb{C}1_{\mathfrak{A}}$, and therefore totally bounded with respect to S_0 on $\mathfrak{A}/\mathbb{C}1_{\mathfrak{A}}$ since \mathfrak{A}_N is finite dimensional. Let F_N be a

finite $\varepsilon/3$ -net of $q_0(B_N)$, so let $f_N = \{b_N(a_1), \dots, b_N(a_n) : a_j \in L_1 \cap \mathfrak{A}_N, 1 \leq j \leq n < \infty\}$ such that $F_N = q_0(f_N)$. Let $a \in L_1$, then $b_N(a) \in B_N$, so there exists $b_N(a_j) \in f_N$ such that $S_0(b_N(a) - b_N(a_j)) < \varepsilon/3$. Therefore,

$$\begin{aligned} S_0(a - a_j) &\leq S_0(a - b_N(a)) + S_0(b_N(a) - b_N(a_j)) + S_0(b_N(a_j) - a_j) \\ &\leq \|a - b_N(a)\| + \varepsilon/3 + \|b_N(a_j) - a_j\| \\ &= S_N(a) + \varepsilon/3 + S_N(a_j) < \varepsilon. \end{aligned}$$

Hence, F_N serves as a finite ε -net for $q_0(L_1)$. Therefore, by [37, Theorem 1.8], the pair $(\mathfrak{A}, L_{\mathcal{U}}^{\beta})$ is a Leibniz quantum compact metric space, and similarly, so are $(\mathfrak{A}_n, L_{\mathcal{U}}^{\beta})$ for all $n \in \mathbb{N}$. Thus, the proof of (1) is complete.

Next, we prove conclusion (2) of this theorem.

Let $\varepsilon > 0$. Fix $\mu \in \mathcal{S}(\mathfrak{A})$. By part (1) and [35, Proposition 1.3], the set $\mathfrak{Lip}_1(\mathfrak{A}, L_{\mathcal{U}}^{\beta}, \mu)$ of Notation (7.2) is compact. Hence, there exist $a_1, \dots, a_k \in \mathfrak{Lip}_1(\mathfrak{A}, L_{\mathcal{U}}^{\beta}, \mu)$ such that:

$$(7.1) \quad \mathfrak{Lip}_1(\mathfrak{A}, L_{\mathcal{U}}^{\beta}, \mu) \subseteq \bigcup_{j=1}^k B_{\|\cdot\|_{\mathfrak{A}}}(a_j, \varepsilon),$$

where $B_{\|\cdot\|_{\mathfrak{A}}}(a_j, \varepsilon) = \{a \in \mathfrak{A} : \|a - a_j\|_{\mathfrak{A}} < \varepsilon\}$ for each $j \in \{1, \dots, k\}$. Since $a \in \bigcup_{n \in \mathbb{N}} \mathfrak{A}_n$ for all $a \in \mathfrak{A}$ such that $L_{\mathcal{U}}^{\beta}(a) \leq 1 < \infty$, choose $N = \min\{m \in \mathbb{N} : \{a_1, \dots, a_k\} \subseteq \mathfrak{A}_m\}$.

Fix $n \geq N$. Let $a \in \mathfrak{Lip}_1(\mathfrak{A}, L_{\mathcal{U}}^{\beta}, \mu)$. By Expression (7.1), there exists $b \in \mathfrak{A}_N \subseteq \mathfrak{A}_n$ such that $b \in \mathfrak{Lip}_1(\mathfrak{A}_n, L_{\mathcal{U}}^{\beta}, \mu)$, where μ is seen as a state of \mathfrak{A}_n , and:

$$(7.2) \quad \|a - b\|_{\mathfrak{A}} < \varepsilon.$$

Consider the bridge $\gamma = (\mathfrak{A}, 1_{\mathfrak{A}}, \text{id}_{\mathfrak{A}}, \iota_n)$ in the sense of Definition (2.6), where $\text{id}_{\mathfrak{A}} : \mathfrak{A} \rightarrow \mathfrak{A}$ is identity and $\iota_n : \mathfrak{A}_n \rightarrow \mathfrak{A}$ is inclusion. But, since the pivot is $1_{\mathfrak{A}}$, the height is 0. Now, combining Lemma (7.3) and Inequality (7.2), we gather that the reach of the bridge is bounded by ε . Thus, by definition of length and Theorem-Definition (2.11), we conclude:

$$\Lambda\left(\left(\mathfrak{A}_n, L_{\mathcal{U}}^{\beta}\right), \left(\mathfrak{A}, L_{\mathcal{U}}^{\beta}\right)\right) \leq \varepsilon,$$

which completes the proof. \square

By [30], we can make the following comparison.

Proposition 7.5. *Let \mathfrak{A} be a unital AF algebra with unit $1_{\mathfrak{A}}$ endowed with a faithful tracial state μ . Let $\mathcal{U} = (\mathfrak{A}_n)_{n \in \mathbb{N}}$ be an increasing sequence of unital finite dimensional C^* -subalgebras such that $\mathfrak{A} = \overline{\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ with $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$. Let $\beta : \mathbb{N} \rightarrow (0, \infty)$ have limit 0 at infinity.*

If we define $L_{\mathcal{U}, \mu}^{\beta}$ by Theorem (6.1) and $L_{\mathcal{U}}^{\beta}$ by Theorem (7.4), then there exists $K \in (0, 1]$ such that:

$$K \cdot L_{\mathcal{U}, \mu}^{\beta}(a) \leq L_{\mathcal{U}}^{\beta}(a) \leq L_{\mathcal{U}, \mu}^{\beta}(a)$$

for all $a \in \mathfrak{sa}(\bigcup_{n \in \mathbb{N}} \mathfrak{A}_n)$.

Proof. Let $a \in \mathfrak{sa}(\cup_{n \in \mathbb{N}} \mathfrak{A}_n)$ and $n \in \mathbb{N}$, then $\mathbb{E}(a|\mathfrak{A}_n) \in \mathfrak{A}_n$. Therefore:

$$S_n(a) = \inf \{ \|a - b\|_{\mathfrak{A}} : b \in \mathfrak{A}_n \} \leq \|a - \mathbb{E}(a|\mathfrak{A}_n)\|_{\mathfrak{A}}$$

for all $n \in \mathbb{N}$. Hence, the seminorm $L_{\mathcal{U}}^{\beta}(a) \leq L_{\mathcal{U}, \mu}^{\beta}(a)$ for all $a \in \mathfrak{sa}(\cup_{n \in \mathbb{N}} \mathfrak{A}_n)$. The existence of K follows from [30, Corollary 2.5]. \square

Remark 7.6. Appropriate analogues to Theorem (7.4) and Proposition (7.5) can be easily obtained in the inductive limit case of AF algebras.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DENVER, 2199 S. UNIVERSITY BLVD, DENVER
CO 80208

E-mail address: konrad.aguilar@du.edu

URL: <http://www.math.du.edu/~kaguilar/>