

ORBIFOLDS AND COSETS OF MINIMAL \mathcal{W} -ALGEBRAS

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ABSTRACT. Let \mathfrak{g} be a simple, finite-dimensional Lie (super)algebra equipped with an embedding of \mathfrak{sl}_2 inducing the minimal gradation on \mathfrak{g} . The corresponding minimal \mathcal{W} -algebra $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ introduced by Kac and Wakimoto has strong generators in weights $1, 2, 3/2$, and all operator product expansions are known explicitly. The weight one subspace generates an affine vertex (super)algebra $V^{k'}(\mathfrak{g}^{\natural})$ where $\mathfrak{g}^{\natural} \subset \mathfrak{g}$ denotes the centralizer of \mathfrak{sl}_2 . Therefore $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ has an action of a connected Lie group G_0^{\natural} with Lie algebra $\mathfrak{g}_0^{\natural}$, where $\mathfrak{g}_0^{\natural}$ denotes the even part of \mathfrak{g}^{\natural} . We show that for any reductive subgroup $G \subset G_0^{\natural}$, and for any reductive Lie algebra $\mathfrak{g}' \subset \mathfrak{g}^{\natural}$, the orbifold $\mathcal{O}^k = \mathcal{W}^k(\mathfrak{g}, e_{-\theta})^G$ and the coset $\mathcal{C}^k = \text{Com}(V^{\ell}(\mathfrak{g}'), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$ are strongly finitely generated for generic values of k . Here $V^{\ell}(\mathfrak{g}')$ denotes the affine vertex algebra associated to \mathfrak{g}' . We find explicit minimal strong generating sets for \mathcal{C}^k when \mathfrak{g} is either $\mathfrak{sl}_n, \mathfrak{sp}_{2n}, \mathfrak{sl}(2|n)$ for $n \neq 2$, $\mathfrak{psl}(2|2)$, or $\mathfrak{osp}(1|4)$, and $\mathfrak{g}' = \mathfrak{g}^{\natural}$. Finally, we conjecture some surprising coincidences among families of cosets \mathcal{C}_k which are the simple quotients of \mathcal{C}^k , and we prove several cases of our conjecture.

1. INTRODUCTION

Vertex algebras arose out of conformal field theory in the 1980s and have been developed mathematically from various points of view in the literature [B, FLM, K]. In addition to several well-known families such as free field algebras, affine vertex algebras, lattice vertex algebras and \mathcal{W} -algebras, there are some standard ways to construct new vertex algebras from old ones. The *orbifold construction* begins with a vertex algebra \mathcal{V} and a group of automorphisms G , and considers the invariant subalgebra \mathcal{V}^G and its extensions. Similarly, the *coset construction* associates to a vertex algebra \mathcal{V} and a subalgebra \mathcal{A} , the subalgebra $\text{Com}(\mathcal{A}, \mathcal{V}) \subset \mathcal{V}$ which commutes with \mathcal{A} [FZ]. It is believed that nice properties of \mathcal{V} such as strong finite generation, C_2 -cofiniteness, and rationality, will be inherited by \mathcal{V}^G and $\text{Com}(\mathcal{A}, \mathcal{V})$ if G and \mathcal{A} are also nice, but few general results of this kind are known.

The structure of orbifolds of free field algebras has been studied in a series of paper using ideas from classical invariant theory [LI, LII, LIII, LIV, LV, CLIII]. The main result is that for a free field algebra \mathcal{F} (either a Heisenberg algebra, free fermion algebra, symplectic fermion algebra, or $\beta\gamma$ -system), and a reductive group G of automorphisms of \mathcal{F} , the orbifold \mathcal{F}^G is strongly finitely generated. The proof is essentially constructive and proceeds by first finding an explicit strong generating set for $\mathcal{F}^{\text{Aut}(\mathcal{F})}$. Here $\text{Aut}(\mathcal{F})$ denotes the full automorphism group, which is either an orthogonal or symplectic group. The description of $\text{Aut}(\mathcal{F})$ is a formal consequence of Weyl's first and second fundamental theorems of invariant theory for these groups [W]. The strong finite generation of

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$\mathcal{F}^{\text{Aut}(\mathcal{F})}$ implies that all $\mathcal{F}^{\text{Aut}(\mathcal{F})}$ -modules appearing in the decomposition of \mathcal{F} have the C_1 -cofiniteness property according to Miyamoto's definition [Mil]. This fact, together with a classical result (Theorem 2.5A of [W]), allows us to deduce the strong finite generation of \mathcal{F}^G for a general reductive group $G \subset \text{Aut}(\mathcal{F})$.

It was observed in [LIV] that similar results for orbifolds of affine vertex algebras at generic level can be obtained by a deformation process. A large family of cosets of affine vertex algebras inside larger structures can also be studied using this process. Let \mathcal{A}^k be a family of vertex algebras whose structure constants depend continuously on a complex parameter k . If \mathcal{A}^k contains a universal affine vertex algebra $V^k(\mathfrak{g})$ of some reductive Lie algebra \mathfrak{g} at level k , let

$$\mathcal{C}^k = \text{Com}(V^k(\mathfrak{g}), \mathcal{A}^k).$$

The problem of describing \mathcal{C}^k can be formulated and answered using the language of *deformable families* of vertex algebras. A deformable family \mathcal{A} associated to \mathcal{A}^k is roughly speaking a vertex algebra defined over the ring F_K . Here K is a subset of \mathbb{C} which is at most countable, and F_K is the ring of rational functions in a variable κ of degree at most zero, with possible poles along K . One requires that $\mathcal{A}/(\kappa - k) \cong \mathcal{A}^k$ for $k \notin K$. This notion is due to [CLI, CLII] and it allows us to find strong generators of \mathcal{C}^k for generic k by passing to the limit $\lim_{k \rightarrow \infty} \mathcal{C}^k$. This limit can then be identified with an orbifold of a free field algebra, which can be studied using the above methods.

Minimal \mathcal{W} -algebras. We shall adapt these methods to study orbifolds and cosets of the minimal \mathcal{W} -algebra $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ introduced by Kac and Wakimoto in [KWI]. Here \mathfrak{g} is a simple, finite-dimensional Lie (super)algebra equipped with an \mathfrak{sl}_2 -triple $\{e, f, h\}$ inducing the minimal $\frac{1}{2}\mathbb{Z}$ -gradation

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1.$$

In this notation, $\mathfrak{g}_{-1} = \mathbb{C}f$ and $\mathfrak{g}_1 = \mathbb{C}e$, and the above decomposition is the eigenvalue decomposition with respect to $\text{ad}(h)$ where $h = [e, f]$. Define \mathfrak{g}^{\natural} to be the centralizer of $\{e, f, h\}$ in \mathfrak{g} . By Theorem 5.1 of [KWI], $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ is strongly generated by a Virasoro field T , primary weight one fields J^a , $a \in \mathfrak{g}^{\natural}$, which generate an affine vertex (super)algebra $V^{k'}(\mathfrak{g}^{\natural})$, and primary weight $3/2$ fields G^u , $u \in \mathfrak{g}_{-1/2}$. Moreover, all OPE relations are given explicitly in this theorem.

In Section 4, we show that $\lim_{k \rightarrow \infty} \mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ is a *generalized free field algebra* in an appropriate sense. The orbifold theory of generalized free field algebras can be studied using classical invariant theory, and orbifolds and cosets of minimal \mathcal{W} -algebras for generic values of k can be studied by passing to the limit. The even part $\mathfrak{g}_0^{\natural} \subset \mathfrak{g}^{\natural}$ is reductive and $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ decomposes into a sum of finite-dimensional $\mathfrak{g}_0^{\natural}$ -modules. Therefore the action of $\mathfrak{g}_0^{\natural}$ lifts to an action of a connected Lie group G_0^{\natural} on $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ by automorphisms.

Theorem 1.1. *Let $G \subset G_0^{\natural}$ be a reductive group of automorphisms of $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$. Then the orbifold $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})^G$ is strongly finitely generated for generic values of k .*

Theorem 1.2. *Let $\mathfrak{g}' \subset \mathfrak{g}^{\natural}$ be a reductive Lie subalgebra and let $V^{\ell}(\mathfrak{g}') \subset V^{k'}(\mathfrak{g}^{\natural})$ be the corresponding affine vertex subalgebra. Then the coset $\mathcal{C}^k = \text{Com}(V^{\ell}(\mathfrak{g}'), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$ is strongly finitely generated for generic values of k .*

We are mostly interested in the case $\mathfrak{g}' = \mathfrak{g}^{\natural}$, which we consider in Section 5. We give explicit minimal strong generating sets for \mathcal{C}^k when \mathfrak{g} is either \mathfrak{sl}_n , \mathfrak{sp}_{2n} , $\mathfrak{sl}(2|n)$ for $n \neq 2$, $\mathfrak{psl}(2|2)$, or $\mathfrak{osp}(1|4)$.

Theorem 1.3. *For generic values of k , we have the following.*

- (a) For $n \geq 3$, $\text{Com}(V^{k+1}(\mathfrak{gl}_{n-2}), \mathcal{W}^k(\mathfrak{sl}_n, e_{-\theta}))$ is of type $\mathcal{W}(2, 3, \dots, n^2 - 2)$.
- (b) For $n \geq 2$, $\text{Com}(V^{k+1/2}(\mathfrak{sp}_{2n-2}), \mathcal{W}^k(\mathfrak{sp}_{2n}, e_{-\theta}))$ is of type $\mathcal{W}(2, 4, \dots, 2n^2 + 2n - 2)$.
- (c) For $n \neq 2$, $\text{Com}(V^{k-1}(\mathfrak{gl}_n), \mathcal{W}^k(\mathfrak{sl}(2|n), e_{-\theta}))$ is even and of type $\mathcal{W}(2, 3, \dots, 3n + 2)$.
- (d) $\text{Com}(V^{k-1}(\mathfrak{sl}_2), \mathcal{W}^k(\mathfrak{psl}(2|2), e_{-\theta}))$ is even and of type $\mathcal{W}(2, 3^3, 4, 5^3, 6, 7^3, 8)$.
- (e) $\text{Com}(V^{k+1/2}(\mathfrak{osp}(1|2)), \mathcal{W}^k(\mathfrak{osp}(1|4), e_{-\theta}))$ is even and of type $\mathcal{W}(2, 4, \dots, 18)$.

The nongeneric set. In general, it is a difficult problem to determine the structure of the nongeneric set, where the above strong generating set does not work. In the case $\mathfrak{g} = \mathfrak{sl}_3$, $\mathfrak{g}^{\natural} = \mathfrak{gl}_1$ so the affine subalgebra is just the Heisenberg algebra \mathcal{H} . It was shown in [ACL] that $\mathcal{C}^k = \text{Com}(\mathcal{H}, \mathcal{W}^k(\mathfrak{sl}_3, e_{-\theta}))$ is of type $\mathcal{W}(2, 3, 4, 5, 6, 7)$ for all k except for $\{-1, -3/2\}$. However, this example is somewhat degenerate because cosets of Heisenberg algebras are easier to understand than the general case. In Section 6, we analyze the structure of the nongeneric set for \mathcal{C}^k in the cases $\mathfrak{g} = \mathfrak{sl}_4$ and $\mathfrak{g} = \mathfrak{sp}_4$.

- Theorem 1.4.** (a) For all real numbers $k > -5/2$, $\text{Com}(V^{k+1/2}(\mathfrak{sp}_2), \mathcal{W}^k(\mathfrak{sp}_4, e_{-\theta}))$ is of type $\mathcal{W}(2, 4, 6, 8, 10)$.
- (b) For all but finitely many real numbers $k > -3$, $\text{Com}(V^{k+1}(\mathfrak{gl}_2), \mathcal{W}^k(\mathfrak{sl}_4, e_{-\theta}))$ is of type $\mathcal{W}(2, 3, 4, \dots, 14)$.

The proof of this result depends on computer calculations performed using the Mathematica package of Thielemans [T]. In general, we expect that the most interesting values of k , where $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ is highly reducible, are generic. This holds in the above examples $\mathfrak{g} = \mathfrak{sl}_4$ and $\mathfrak{g} = \mathfrak{sp}_4$, but a conceptual rather than computational approach to this problem is currently out of reach.

Simple cosets. The main reason to study $\mathcal{C}^k = \text{Com}(V^{k'}(\mathfrak{g}^{\natural}), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$ is for the purpose of studying the coset

$$\mathcal{C}_k = \text{Com}(L_{k'}(\mathfrak{g}^{\natural}), \mathcal{W}_k(\mathfrak{g}, e_{-\theta}))$$

at special values of k where the maximal proper ideal $\mathcal{I}_k \subset \mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ is large. Here $L_{k'}(\mathfrak{g}^{\natural})$ denotes the simple affine vertex algebra and $\mathcal{W}_k(\mathfrak{g}, e_{-\theta})$ is the simple quotient $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})/\mathcal{I}_k$. For example, when $k + 1/2$ is a non-negative integer, $\mathcal{W}_k(\mathfrak{sp}_{2n}, e_{-\theta})$ is C_2 -cofinite [ArIV], and conjecturally rational. It contains a copy of $L_{k+1/2}(\mathfrak{sp}_{2n-2})$, so

$$\mathcal{C}_k = \text{Com}(L_{k+1/2}(\mathfrak{sp}_{2n-2}), \mathcal{W}_k(\mathfrak{sp}_{2n}, e_{-\theta}))$$

is also expected to be rational and C_2 -cofinite. One would like to understand the structure of \mathcal{C}_k as well as the structure of $\mathcal{W}_k(\mathfrak{g}, e_{-\theta})$ as an extension of $L_{k+1/2}(\mathfrak{sp}_{2n-2}) \otimes \mathcal{C}_k$.

Conjecturally [AM1], for $n \geq 4$, $\mathcal{W}_k(\mathfrak{sl}_n, e_{-\theta})$ is not C_2 -cofinite for any $k \in \mathbb{N}$. However, $\mathcal{W}_k(\mathfrak{sl}_n, e_{-\theta})$ contains a copy of $L_{k+1}(\mathfrak{gl}_{n-2}) := \mathcal{H} \otimes L_{k+1}(\mathfrak{sl}_{n-2})$, and is expected to be “small” in the sense that it has a one-dimensional associated variety [ArII]. One expects that

$$\text{Com}(L_{k+1}(\mathfrak{gl}_{n-2}), \mathcal{W}_k(\mathfrak{sl}_n, e_{-\theta})), \quad k \in \mathbb{N},$$

also has a one-dimensional associated variety. As above, we would like to describe \mathcal{C}_k as well as $\mathcal{W}_k(\mathfrak{g}, e_{-\theta})$ as an extension of $L_{k+1}(\mathfrak{gl}_{n-2}) \otimes \mathcal{C}_k$.

In all these cases, the natural map $\pi_k : \mathcal{C}^k \rightarrow \mathcal{C}_k$ is surjective and \mathcal{C}_k coincides with the simple quotient of \mathcal{C}^k . Therefore a strong generating set for \mathcal{C}^k descends to a strong generating set for \mathcal{C}_k . This illustrates the importance of both finding a strong generating set for \mathcal{C}^k for generic k , and determining the nongeneric set.

Some remarkable coincidences. If \mathcal{A}^k is a vertex algebra containing $V^k(\mathfrak{g})$ and k is a special value for which the simple quotient \mathcal{A}_k contains $L_k(\mathfrak{g})$ and has low-dimensional associated variety, then $\mathcal{C}_k = \text{Com}(L_k(\mathfrak{g}), \mathcal{A}_k)$ also has low-dimensional associated variety. Moreover, vertex algebras with a given strong generating set with low-dimensional associated variety are rare. If there is another family of vertex algebras \mathcal{D}_k with the same central charges and strong generating sets, we take this as enough information to conjecture an isomorphism $\mathcal{C}_k \cong \mathcal{D}_k$. In general, such conjectures can then either be proven by direct OPE computation or by proving a uniqueness theorem for OPE-algebras with a given strong generating set. In the case where \mathcal{A}^k is a minimal \mathcal{W} -algebra, we are indeed able to state such a uniqueness theorem; see Theorem 3.1. Hence if we can prove that $L_k(\mathfrak{g}) \otimes \mathcal{C}_k$ allows for a vertex algebra extension that is of minimal \mathcal{W} -algebra type that is simple, this provides a method to prove coincidences. This is one of the many purposes to establish an effective theory of VOA extensions [CKL, CKM]. Furthermore, once having \mathcal{A}_k as an extension of $L_k(\mathfrak{g}) \otimes \mathcal{C}_k$, one can use the representation theory of VOA extensions to understand the one of \mathcal{A}_k provided the one of $L_k(\mathfrak{g}) \otimes \mathcal{C}_k$ is known.

Conjecture 1.5. *For all $n \geq 2$ and k such that $k + 1/2$ is a positive integer,*

$$\mathcal{C}_k(n) = \text{Com}(L_{k+1/2}(\mathfrak{sp}_{2n-2}), \mathcal{W}_k(\mathfrak{sp}_{2n}, e_{-\theta}))$$

is isomorphic to the C_2 -cofinite, rational principal \mathcal{W} -algebra $\mathcal{W}_s(\mathfrak{sp}_{2m}, f_{\text{prin}})$ where

$$m = k + 1/2, \quad s + (k + 3/2) = \frac{n + k + 1/2}{2n + 2k + 2}.$$

In the case $n = 2$ and $k = 1/2$, this was proven by Kawasetsu using the fact that $\mathcal{W}_{1/2}(\mathfrak{sp}_4, e_{-\theta})$ is a simple current extension of $L_1(\mathfrak{sl}_2) \otimes L(-25/7, 0)$ [Ka]. Here $L(-25/7, 0)$ denotes the simple Virasoro vertex algebra with $c = -25/7$. Since $\mathcal{W}_k(\mathfrak{sp}_{2n}, e_{-\theta})$ is an extension of $L_{k+1/2}(\mathfrak{sp}_{2n-2}) \otimes \mathcal{C}_k(n)$, Conjecture 1.5 implies the rationality of $\mathcal{W}_k(\mathfrak{sp}_{2n}, e_{-\theta})$ for all n and k as above. Using Theorem 1.4, we give an alternative proof of Kawasetsu's result, and we also prove our conjecture for $n = 2$ and $k = 3/2$. This implies the rationality of $\mathcal{W}_{3/2}(\mathfrak{sp}_4, e_{-\theta})$.

For $n \geq 3$, let $\mathcal{W}^\ell(\mathfrak{sl}_n, f_{\text{subreg}})$ be the \mathcal{W} -algebra associated with a subregular nilpotent element f_{subreg} at level ℓ [KRW], and let $\mathcal{W}_\ell(\mathfrak{sl}_n, f_{\text{subreg}})$ be the unique simple quotient of $\mathcal{W}^\ell(\mathfrak{sl}_n, f_{\text{subreg}})$. It was recently conjectured in [A-R], and proven by Genra [G], that $\mathcal{W}^\ell(\mathfrak{sl}_n, f_{\text{subreg}})$ is isomorphic to the vertex algebra $\mathcal{W}_{n,\ell}^{(2)}$ introduced by Feigin and Semikhatov [FS]. Note that $\mathcal{W}^\ell(\mathfrak{sl}_n, f_{\text{subreg}})$ is of type $\mathcal{W}(1, 2, \dots, n-1, n/2, n/2)$ and the weight one field generates a Heisenberg algebra \mathcal{H} . Also, note that for $n = 1$ and $n = 2$, $\mathcal{W}_{n,\ell}^{(2)}$ is well-defined and coincides with the rank one $\beta\gamma$ -system and the affine vertex algebra $V^\ell(\mathfrak{sl}_2)$, respectively.

Conjecture 1.6. *For all integers $n \geq 4$ and $k \geq 0$ such that $n^2 - k^2 - 1 \geq n$,*

$$\mathcal{C}_k(n) = \text{Com}(L_{k+1}(\mathfrak{gl}_{n-2}), \mathcal{W}_k(\mathfrak{sl}_n, e_{-\theta})) \cong \text{Com}(\mathcal{H}, \mathcal{W}_\ell(\mathfrak{sl}_{k+1}, f_{\text{subreg}})), \quad \ell = -\frac{1 + k^2 + kn}{k + n}.$$

To interpret this conjecture for $k = 0$ and $k = 1$, we replace $\mathcal{W}_\ell(\mathfrak{sl}_{k+1}, f_{\text{subreg}})$ with the $\beta\gamma$ -system and the simple affine vertex algebra $L_\ell(\mathfrak{sl}_2)$, respectively. It is worth mentioning that the subregular nilpotent orbit, which is the Lusztig-Spaltenstein dual of the minimal nilpotent orbit, appears in the above correspondence.

We will prove Conjecture 1.6 for all $n \geq 4$ and $k = 0$ using our uniqueness result for minimal \mathcal{W} -algebras, and for $n = 4$ and $k = 1$ using Theorem 1.4. Since the simple algebra $\mathcal{W}_{2,-6/5}^{(2)}$ is isomorphic to $L_{-6/5}(\mathfrak{sl}_2)$, $\mathcal{C}_1(4)$ is isomorphic to the simple \mathfrak{sl}_2 -parafermion algebra at level $-6/5$. In the case $n = 4$ and $k = 2$ there is also considerable computational evidence that the conjecture holds. We remark that conformal embeddings of affine vertex (super)algebras in minimal \mathcal{W} -algebras have been studied recently in [A-PI, A-PII]. Their findings can be viewed as cases of the coincidences where the simple coset $\mathcal{C}_k \cong \mathbb{C}$.

Constructing new C_2 -cofinite but non-rational VOAs. The representation category of grading restricted modules of a simple, CFT-type and C_2 -cofinite VOA that is its own contragredient dual is expected to be a log-modular tensor category [CGI, CGII]. Similar to its rational cousin this comes with a close and powerful relation of the modularity of (pseudo)-trace functions [MiII] and (logarithmic)-Hopf links [CGI, CGII]. In order to understand this relation better, one needs new examples of interesting C_2 -cofinite but non-rational VOAs.

We will now outline how minimal \mathcal{W} -algebra cosets provide suitable candidates for the construction of new C_2 -cofinite VOAs. Consider $\mathcal{W}_k(\mathfrak{g})$ with $\mathfrak{g}^\natural = \mathfrak{gl}_1 \oplus \mathfrak{a}$ for some Lie algebra \mathfrak{a} . Let $L_{k'}(\mathfrak{g}^\natural)$ be the affine subVOA of $\mathcal{W}_k(\mathfrak{g})$. Then the coset satisfies

$$\mathcal{C}_k = \text{Com}(L_{k'}(\mathfrak{g}^\natural), \mathcal{W}_k(\mathfrak{g})) = \text{Com}(\mathcal{H}(1), \text{Com}(L_{k'}(\mathfrak{a}), \mathcal{W}_k(\mathfrak{g}))).$$

Assume that the tensor category of grading restricted modules of \mathcal{C}_k is a vertex tensor category in the sense of [HKL]. Then the following statements of [CKLR, CKL, CKM] hold:

(1)

$$\text{Com}(L_{k'}(\mathfrak{a}), \mathcal{W}_k(\mathfrak{g})) = \bigoplus_{\lambda \in L} \mathcal{F}_\lambda \otimes \mathcal{C}_\lambda$$

where L is a rank one lattice with corresponding generalized lattice VOA

$$V_L = \bigoplus_{\lambda \in L} \mathcal{F}_\lambda$$

the \mathcal{F}_λ are Fock modules of the rank one Heisenberg VOA $\mathcal{H}(1)$. the \mathcal{C}_λ are simple currents and above decomposition is as $\mathcal{H}(1) \otimes \mathcal{C}_k$ -modules.

(2) Let $N \subset L$ be an even sublattice so that $V_N = \bigoplus_{\lambda \in N} \mathcal{F}_\lambda$ is a lattice VOA, then

$$\mathcal{D}_k = \bigoplus_{\lambda \in N} \mathcal{C}_\lambda$$

is a VOA.

(3) Every module of \mathcal{D}_k is an induced module and an indecomposable module X of \mathcal{C}_k lifts to a \mathcal{D}_k -module if and only if there exists $\lambda \in N'$ such that $\mathcal{F}_\lambda \otimes X$ is contained in a $\text{Com}(L_{k'}(\mathfrak{a}), \mathcal{W}_k(\mathfrak{g}))$ -module.

Sloppily phrased, only very few modules of \mathcal{C}_k lift to \mathcal{D}_k -modules. We conjecture

Conjecture 1.7. *Let $\mathcal{W}_k(\mathfrak{g})$ be a minimal \mathcal{W} -super algebra with one-dimensional associated variety and let \mathcal{C}_k be the coset with the weight one affine subVOA. Then there is an infinite order simple current extension \mathcal{D}_k of \mathcal{C}_k with zero-dimensional associated variety.*

The conjecture is true for the cases $k = 0$ and $n \geq 4$ of Conjecture 1.6. Here, we always obtain for \mathcal{D}_k the triplet \mathcal{W} -algebra of central charge $c = -2$. The conjecture is also true for $\mathfrak{g} = \mathfrak{sl}_3$ and $k = -9/4$. In this case \mathcal{D}_k is the triplet \mathcal{W} -algebra of central charge $c = -7$ [CRW]. The triplet VOAs are known to be C_2 -cofinite [AdM]. They are infinite order simple current extensions of the singlet algebras and in these cases the cosets turn out to be the singlet algebras of central charge $c = -2$ respectively $c = -7$.

It is difficult to show that the complete category of grading restricted modules of a VOA is a vertex tensor category, and an alternative route for this conjecture might be to even prove C_2 -cofiniteness. For some progress on this idea in the context of admissible level $L_k(\mathfrak{sl}_2)$ parafermions see [ACR].

2. VERTEX ALGEBRAS

We assume that the reader is familiar with the basics of vertex algebra theory, which has been discussed from several different points of view in the literature (see for example [B, FLM, K, FBZ]). Given an element a in a vertex algebra \mathcal{V} , the corresponding field is denoted by

$$a(z) = \sum_{n \in \mathbb{Z}} a(n)z^{-n-1}.$$

Given $a, b \in \mathcal{V}$, the *operators product expansion* (OPE) formula is given by

$$a(z)b(w) \sim \sum_{n \geq 0} (a_{(n)}b)(w) (z-w)^{-n-1}.$$

Here $(a_{(n)}b)(w) = \text{Res}_z[a(z), b(w)](z-w)^n$ where $[a(z), b(w)] = a(z)b(w) - (-1)^{|a||b|}b(w)a(z)$, and \sim means equal modulo terms which are regular at $z = w$. The *normally ordered product* $: a(z)b(z) :$ is defined to be

$$a(z)_-b(z) + (-1)^{|a||b|}b(z)a(z)_+,$$

where

$$a(z)_- = \sum_{n < 0} a(n)z^{-n-1}, \quad a(z)_+ = \sum_{n \geq 0} a(n)z^{-n-1}.$$

For fields $a_1(z), \dots, a_k(z)$, the k -fold iterated Wick product is defined inductively to be

$$(2.1) \quad : a_1(z)a_2(z) \cdots a_k(z) : = : a_1(z) (: a_2(z) \cdots a_k(z) :) .$$

A subset $S = \{a_i \mid i \in I\}$ of \mathcal{A} *strongly generates* \mathcal{A} if \mathcal{A} is spanned by

$$\{ : \partial^{k_1} a_{i_1}(z) \cdots \partial^{k_m} a_{i_m}(z) : \mid i_1, \dots, i_m \in I, k_1, \dots, k_m \geq 0 \}.$$

We say that S *freely generates* \mathcal{A} if there are no nontrivial normally ordered polynomial relations among the generators and their derivatives. We say that \mathcal{A} is of type

$$\mathcal{W}((d_1)^{n_1}, \dots, (d_r)^{n_r})$$

if it has a minimal strong generating set consisting of n_i fields in each weight d_i for $i = 1, \dots, r$.

A vertex algebra \mathcal{A} is called *simple* if it has no nontrivial two-sided ideals. Here an ideal is a subspace closed under $a_{(k)}(-)$ and $(-)_{(k)}a$ for all $a \in \mathcal{A}$ and $k \in \mathbb{Z}$. The following result will be needed.

Lemma 2.1. *Let \mathcal{V} be a simple vertex algebra and $\mathcal{W} \subset \mathcal{V}$ a simple vertex subalgebra. Suppose that \mathcal{V} decomposes into a direct sum of simple \mathcal{W} -modules W_i , where $W_0 = \mathcal{W}$, and write*

$$\mathcal{V} = \bigoplus_i U_i \otimes W_i,$$

where U_i is the multiplicity space of W_i . Then U_0 is a simple vertex algebra.

Proof. Let $u \in U_i$. Because \mathcal{V} is simple there exists $v \in \mathcal{V}$ and $n \in \mathbb{Z}$ such that $v_{(n)}(u \otimes 1) = 1 \otimes 1$ by Corollary 4.2 of [DM1]. Write

$$v = \sum_i a_i \otimes b_i,$$

with $a_i \in U_i$ and $b_i \in W_i$. Then

$$\sum_i \sum_k (a_i)_{(k)} u \otimes (b_i)_{(n-k)} 1 = 1 \otimes 1.$$

Since $(b_i)_{(n-k)} 1 \in W_i$, by Proposition 5.4.7 of [FHL], this means that

$$\sum_k (a_0)_{(k)} u \otimes (b_0)_{(n-k)} 1 = 1 \otimes 1.$$

So there exists m such that $(a_0)_{(m)} u = 1$, that is, U_0 is simple. \square

This situation we wish to apply this to is the following. Suppose that \mathcal{W} and \mathcal{V} are as above, and that \mathcal{W} is rational. Then \mathcal{V} is a direct sum of irreducible \mathcal{W} -modules, and $U_0 = \text{Com}(\mathcal{W}, \mathcal{V})$. We obtain

Corollary 2.2. *If \mathcal{V} is simple, and $\mathcal{W} \subset \mathcal{V}$ is simple and rational, then $\text{Com}(\mathcal{W}, \mathcal{V})$ is simple.*

3. MINIMAL \mathcal{W} -ALGEBRAS

Let \mathfrak{g} be a simple, finite-dimensional Lie (super)algebra. We recall some basic properties of the minimal \mathcal{W} -(super)algebra $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ which was introduced by Kac and Wakimoto in [KWI]. First, suppose that $\{e, f, h\}$ is an \mathfrak{sl}_2 -triple in \mathfrak{g} inducing the minimal $\frac{1}{2}\mathbb{Z}$ -gradation

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1.$$

Here $\mathfrak{g}_{-1} = \mathbb{C}f$ and $\mathfrak{g}_1 = \mathbb{C}e$, and the above decomposition is the eigenvalue decomposition with respect to $\text{ad}(h)$ where $h = [e, f]$. Define \mathfrak{g}^{\natural} to be the centralizer of $\{e, f, h\}$ in \mathfrak{g} . Recall that $\mathfrak{g}_{-1/2}$ and $\mathfrak{g}_{1/2}$ are isomorphic as \mathfrak{g}^{\natural} -modules, and that $\mathfrak{g}_{1/2}$ possesses a nondegenerate, skew-supersymmetric bilinear form $\langle \cdot, \cdot \rangle_{\text{ne}}$ given by

$$\langle a, b \rangle_{\text{ne}} = (f|[a, b]),$$

where $(\cdot|\cdot)$ denotes the bilinear form on \mathfrak{g} .

By Theorem 5.1 of [KWI], $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ is strongly generated by a Virasoro field T , primary weight one fields J^a , $a \in \mathfrak{g}^{\natural}$, which generate an affine vertex (super)algebra $V^{k'}(\mathfrak{g}^{\natural})$, and primary weight 3/2 fields G^u , $u \in \mathfrak{g}_{-1/2}$. Moreover, all OPE relations are given explicitly in this theorem. A number of well-known vertex algebras can be realized in this way,

such as the Virasoro algebra, $N = 2$ superconformal algebra, small $N = 4$ superconformal algebra, Bershadsky-Polyakov algebra, etc.

The even part $\mathfrak{g}_0^{\natural} \subset \mathfrak{g}^{\natural}$ is reductive and $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ decomposes into a sum of finite-dimensional $\mathfrak{g}_0^{\natural}$ -modules. Therefore the action of $\mathfrak{g}_0^{\natural}$ lifts to an action of a connected Lie group G_0^{\natural} on $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ by automorphisms. In this paper, we are interested in the structure of orbifolds $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})^G$ where $G \subset G_0^{\natural}$ is a reductive subgroup, and cosets $\text{Com}(V^{\ell}(\mathfrak{g}'), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$ where $V^{\ell}(\mathfrak{g}') \subset V^{k'}(\mathfrak{g}^{\natural})$ is the affine vertex algebra corresponding to a reductive Lie algebra $\mathfrak{g}' \subset \mathfrak{g}^{\natural}$.

Uniqueness of minimal \mathcal{W} -algebras. Let $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ and $V^{k'}(\mathfrak{g}^{\natural})$ be as above, and let $\mathcal{W}_k(\mathfrak{g}, e_{-\theta})$ denote the unique simple quotient of $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$. Let \mathcal{A}_k be another vertex (super)algebra with the following properties:

- (1) The central charges of $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ and \mathcal{A}_k coincide.
- (2) The full affine sub(super)algebra of \mathcal{A}_k is a homomorphic image of $V^{k'}(\mathfrak{g}^{\natural})$.
- (3) \mathcal{A}_k is strongly generated by the Virasoro field T , the weight one and the weight $3/2$ fields and the number of weight $3/2$ strong generators of \mathcal{A}_k and $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ are the same, and of the same parity.
- (4) The weight $3/2$ fields in both \mathcal{A}_k and $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ have the same OPE with the Virasoro field and the weight one fields. In particular, they carry the same representation of \mathfrak{g}^{\natural} .

Theorem 3.1. *Let $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ and \mathcal{A}_k be as above. Then the OPE algebra of the strong generators of $\mathcal{W}_k(\mathfrak{g}, e_{-\theta})$ and \mathcal{A}_k coincide up to null fields. In other words \mathcal{A}_k is a homomorphic image of $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$. In particular if \mathcal{A}_k is simple then $\mathcal{A}_k \cong \mathcal{W}_k(\mathfrak{g}, e_{-\theta})$.*

Proof. First, the weight $3/2$ subspace of \mathcal{A}_k has a \mathfrak{g}^{\natural} -invariant supersymmetric bilinear form given by $\langle X, Y \rangle = X_{(2)}Y$. This form is nondegenerate: \mathcal{A}_k has an order two automorphism that acts as (-1) on elements of conformal weight in $\mathbb{Z} + \frac{1}{2}$ and as 1 on those of integer conformal weight. Then \mathcal{A}_k decomposes as

$$\mathcal{A}_k = \mathcal{A}_k^+ \oplus \mathcal{A}_k^-$$

where \mathcal{A}_k^+ is the orbifold subVOA. Proposition 4.1 of [CKL] applied to \mathcal{A}_k^- implies that this form is non-zero. Since $\mathfrak{g}_{1/2}$ is either irreducible or is a direct sum of two contragredient modules of \mathfrak{g}^{\natural} [KWI] this form is nondegenerate.

So we can rescale the weight $3/2$ fields of \mathcal{A}_k so that the 3rd order poles coincide with the corresponding poles in $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$. It remains to compare the first and second order poles between the weight $3/2$ fields of \mathcal{A}_k . For this, we need the following vertex algebra identities. For any elements U, V, W in a vertex superalgebra,

$$(3.1) \quad U_{(1)}(V_{(1)}W) = (U_{(1)}V)_{(1)}W + (-1)^{|U||V|}V_{(1)}(U_{(1)}W) + (U_{(0)}V)_{(2)}W,$$

$$(3.2) \quad \begin{aligned} U_{(3)}(V_{(0)}W) &= (U_{(3)}V)_{(0)}W + (-1)^{|U||V|}V_{(0)}(U_{(3)}W) + \\ &+ 3(U_{(2)}V)_{(1)}W + 3(U_{(1)}V)_{(2)}W + (U_{(0)}V)_{(3)}W. \end{aligned}$$

Take V and W to be weight $3/2$ fields of \mathcal{A}_k and take U to be a weight one field. Since $V_{(1)}W$ has weight one and the bilinear form on the weight one subspace $\mathcal{A}_k[1]$ is nondegenerate, it follows from (3.1) that $V_{(1)}W$ is uniquely determined by the known OPEs.

Next, consider the weight 2 subspace $\mathcal{A}_k[2]$. It splits as the direct sum of the affine part and the span of the Virasoro element T . Suppose first that $V^{k'}(\mathfrak{g}^{\natural})$ has no singular vectors in weight 2 and that $L = T - L_{\mathfrak{g}}$ has nonzero central charge. Then $\mathcal{A}_k[2]$ has a nondegenerate bilinear form $\langle X, Y \rangle = X_{(3)}Y$. Taking V, W to be weight 3/2 fields and taking U to be a weight 2 field, it then follows from (3.2) that $V_{(0)}W$ is uniquely determined by the known OPEs.

Finally, suppose that the form on $\mathcal{A}_k[2]$ is degenerate. Write $\mathcal{A}_k[2] = A_1 \oplus A_2$ where the form is nondegenerate on A_1 and vanishes on A_2 . Then A_2 lies in the maximal proper ideal of \mathcal{A}_k . Applying the above argument only to vectors $U \in A_1$, we see that $V_{(0)}W$ differs from the corresponding OPE in $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ by elements in A_2 . \square

Weak filtrations. By a *compatible filtration* on a vertex algebra \mathcal{A} , we mean an increasing $\mathbb{Z}_{\geq 0}$ -filtration

$$(3.3) \quad \mathcal{A}_{(0)} \subset \mathcal{A}_{(1)} \subset \mathcal{A}_{(2)} \subset \cdots, \quad \mathcal{A} = \bigcup_{d \geq 0} \mathcal{A}_{(d)}$$

such that $\mathcal{A}_{(0)} = \mathbb{C}$, and for all $a \in \mathcal{A}_{(r)}, b \in \mathcal{A}_{(s)}$, we have

$$(3.4) \quad a_{(n)}b \in \begin{cases} \mathcal{A}_{(r+s)} & n < 0 \\ \mathcal{A}_{(r+s-1)} & n \geq 0 \end{cases}.$$

We set $\mathcal{A}_{(-1)} = \{0\}$, and we say that $a(z) \in \mathcal{A}_{(d)} \setminus \mathcal{A}_{(d-1)}$ has degree d . Such filtrations were introduced in [Li], and the key property is that the associated graded algebra

$$\mathrm{gr}(\mathcal{A}) = \bigoplus_{d \geq 0} \mathcal{A}_{(d)} / \mathcal{A}_{(d-1)}$$

is a $\mathbb{Z}_{\geq 0}$ -graded associative, (super)commutative algebra with a unit 1 under a product induced by the Wick product. For $r \geq 1$ we have the projection

$$(3.5) \quad \varphi_r : \mathcal{A}_{(r)} \rightarrow \mathcal{A}_{(r)} / \mathcal{A}_{(r-1)} \subset \mathrm{gr}(\mathcal{A}).$$

Given a vertex algebra \mathcal{A} with such a filtration, we have the following reconstruction property. Let $\{a_i \mid i \in I\}$ be a set of generators for $\mathrm{gr}(\mathcal{A})$ as a differential algebra, so that $\{\partial^k a_i \mid i \in I, k \geq 0\}$ generates $\mathrm{gr}(\mathcal{A})$ as a ring. If a_i is homogeneous of degree d_i and $a_i(z) \in \mathcal{A}_{(d_i)}$ satisfies $\varphi_{d_i}(a_i(z)) = a_i$, then \mathcal{A} is strongly generated as a vertex algebra by $\{a_i(z) \mid i \in I\}$.

In [ACL], a *weak filtration* on a vertex algebra \mathcal{A} was defined to be an increasing $\mathbb{Z}_{\geq 0}$ -filtration

$$(3.6) \quad \mathcal{A}_{(0)} \subset \mathcal{A}_{(1)} \subset \mathcal{A}_{(2)} \subset \cdots, \quad \mathcal{A} = \bigcup_{d \geq 0} \mathcal{A}_{(d)}$$

such that for $a \in \mathcal{A}_{(r)}, b \in \mathcal{A}_{(s)}$, we have

$$(3.7) \quad a_{(n)}b \in \mathcal{A}_{r+s}, \quad n \in \mathbb{Z}.$$

This condition guarantees that $\mathrm{gr}(\mathcal{A}) = \bigoplus_{d \geq 0} \mathcal{A}_{(d)} / \mathcal{A}_{(d-1)}$ is a vertex algebra, but it is no longer abelian in general. As above, a strong generating set for $\mathrm{gr}(\mathcal{A})$ consisting of homogeneous elements always gives rise to a strong generating set for \mathcal{A} .

We define an increasing filtration

$$\mathcal{W}_{(0)}^k \subset \mathcal{W}_{(1)}^k \subset \cdots$$

on $\mathcal{W}^k = \mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ as follows: $\mathcal{W}_{(-1)}^k = \{0\}$, and $\mathcal{W}_{(r)}^k$ is spanned by iterated Wick products of T, J^a, G^u and their derivatives, such that at most r of the fields G^u and their derivatives appear. It is clear from the defining OPE relations [KWI] that this is a weak filtration. Note that $\mathcal{W}_{(0)}^k \cong \text{Vir} \times V^{k'}(\mathfrak{g}^\natural)$ where Vir denotes the Virasoro vertex algebra with generator T . Note also that the associated graded algebra

$$\text{gr}(\mathcal{W}^k) = \bigoplus_{d \geq 0} \mathcal{W}_{(d)}^k / \mathcal{W}_{(d-1)}^k$$

is not abelian since it contains $\mathcal{W}_{(0)}^k$ as a subalgebra, but $G^u(z)G^v(w) \sim 0$ in this algebra for all u, v .

4. DEFORMATIONS AND LIMITS OF MINIMAL \mathcal{W} -ALGEBRAS

In this section, we shall adapt the methods of our previous papers on orbifolds and cosets of free field and affine vertex algebras to the setting of minimal \mathcal{W} -algebras. First, recall the notion of a *deformable family* of vertex algebras [CLI]. Let $K \subset \mathbb{C}$ be a subset which is at most countable, and let F_K denote the \mathbb{C} -algebra of rational functions in a formal variable κ of the form $\frac{p(\kappa)}{q(\kappa)}$ where $\deg(p) \leq \deg(q)$ and the roots of q lie in K . A *deformable family* will be a free F_K -module \mathcal{B} with the structure of a vertex algebra with coefficients in F_K . Vertex algebras over F_K are defined the same way as vertex algebras over \mathbb{C} . We assume that \mathcal{B} possesses a $\mathbb{Z}_{\geq 0}$ -grading $\mathcal{B} = \bigoplus_{m \geq 0} \mathcal{B}[m]$ by conformal weight where each $\mathcal{B}[m]$ is free F_K -module of finite rank. For $k \notin K$, we have a vertex algebra

$$\mathcal{B}_k = \mathcal{B}/(\kappa - k),$$

where $(\kappa - k)$ is the ideal generated by $\kappa - k$. Clearly $\dim_{\mathbb{C}}(\mathcal{B}_k[m]) = \text{rank}_{F_K}(\mathcal{B}[m])$ for all $k \notin K$ and $m \geq 0$. We have a vertex algebra $\mathcal{B}_{\infty} = \lim_{\kappa \rightarrow \infty} \mathcal{B}$ with basis $\{\alpha_i \mid i \in I\}$, where $\{\alpha_i \mid i \in I\}$ is any basis of \mathcal{B} over F_K , and $\alpha_i = \lim_{\kappa \rightarrow \infty} a_i$. By construction, $\dim_{\mathbb{C}}(\mathcal{B}_{\infty}[m]) = \text{rank}_{F_K}(\mathcal{B}[m])$ for all $m \geq 0$. The vertex algebra structure on \mathcal{B}_{∞} is defined by

$$(4.1) \quad (\alpha_i)_{(n)}(\alpha_j) = \lim_{\kappa \rightarrow \infty} (a_i)_{(n)}(a_j), \quad i, j \in I, \quad n \in \mathbb{Z}.$$

The F_K -linear map $\varphi : \mathcal{B} \rightarrow \mathcal{B}_{\infty}$ sending $a_i \mapsto \alpha_i$ satisfies

$$(4.2) \quad \varphi(\omega_{(n)}\nu) = \varphi(\omega)_{(n)}\varphi(\nu), \quad \omega, \nu \in \mathcal{B}, \quad n \in \mathbb{Z}.$$

Lemma 4.1 ([CLI], Lemma 8.1). *Let $K \subset \mathbb{C}$ be at most countable, and let \mathcal{B} be a vertex algebra over F_K as above. Let $U = \{\alpha_i \mid i \in I\}$ be a strong generating set for \mathcal{B}_{∞} , and let $T = \{a_i \mid i \in I\}$ be the corresponding subset of \mathcal{B} , so that $\varphi(a_i) = \alpha_i$. There exists a subset $S \subset \mathbb{C}$ containing K which is at most countable, such that $F_S \otimes_{F_K} \mathcal{B}$ is strongly generated by T . Here we have identified T with the set $\{1 \otimes a_i \mid i \in I\} \subset F_S \otimes_{F_K} \mathcal{B}$.*

Corollary 4.2. *For $k \notin S$, the vertex algebra $\mathcal{B}_k = \mathcal{B}/(\kappa - k)$ is strongly generated by the image of T under the map $\mathcal{B} \rightarrow \mathcal{B}_k$.*

If U is a *minimal* strong generating set for \mathcal{B}_{∞} , T need not be a minimal strong generating set for \mathcal{B} , since there may exist relations of the form $\lambda(k)\alpha_j = P$, where P is a normally ordered polynomial in $\{\alpha_i \mid i \in I, i \neq k\}$ and $\lim_{k \rightarrow \infty} \lambda(k) = 0$, although $\lim_{k \rightarrow \infty} P$ is a nontrivial. However, there is one condition which holds in many examples, under which T is a minimal strong generating set for \mathcal{B} .

Proposition 4.3 ([CLII], Proposition 3.4). *Suppose that $U = \{\alpha_i \mid i \in I\}$ is a minimal strong generating set for \mathcal{B}_∞ such that $\text{wt}(\alpha_i) < N$ for all $i \in I$. If there are no normally ordered polynomial relations among $\{\alpha_i \mid i \in I\}$ and their derivatives of weight less than N , the corresponding set $T = \{a_i \mid i \in I\}$ is a minimal strong generating set for \mathcal{B} .*

In our main example $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$, suppose that

$$\dim(\mathfrak{g}_0^{\natural}) = n, \quad \dim(\mathfrak{g}_1^{\natural}) = 2m, \quad \dim(\mathfrak{g}_{-1/2}^{\text{ev}}) = 2r, \quad \dim(\mathfrak{g}_{-1/2}^{\text{odd}}) = s.$$

If we replace the generators J^a , L , and G^u with $\tilde{J}^a = \frac{1}{\sqrt{k}}J^a$, $\tilde{L} = \frac{1}{\sqrt{k}}L$, and $\tilde{G}^u = \frac{1}{k}G^u$, respectively, then it follows from Theorem 5.1 of [KWI] that all structure constants are rational functions of \sqrt{k} of degree at most zero. Moreover, the only structure constants which have degree zero as rational functions of \sqrt{k} are the following.

- (1) The second-order pole of $\tilde{J}^a(z)\tilde{J}^b(w)$ when $(a|b) \neq 0$.
- (2) The third-order pole of $\tilde{G}^u(z)\tilde{G}^v(w)$ when $\langle u, v \rangle_{\text{ne}} \neq 0$.
- (3) The fourth-order pole of $\tilde{L}(z)\tilde{L}(w)$.

Let κ be a formal variable satisfying $\kappa^2 = k$, and let $F = F_K$ for $K = \{0\}$. Let \mathcal{W} be the vertex algebra with coefficients in F which is freely generated by $\bar{J}^a, \bar{L}, \bar{G}^u$ satisfying the rescaled OPE relations. It follows that for $k \neq 0$, we have a surjective vertex algebra homomorphism

$$\mathcal{W} \rightarrow \mathcal{W}^k(\mathfrak{g}, e_{-\theta}), \quad \bar{J}^a \mapsto \frac{1}{\sqrt{k}}J^a, \quad \bar{L} \mapsto \frac{1}{\sqrt{k}}L, \quad \bar{G}^u \mapsto \frac{1}{k}G^u,$$

whose kernel is the ideal $(\kappa - \sqrt{k})$. Then $\mathcal{W}^k(\mathfrak{g}, e_{-\theta}) \cong \mathcal{W}/(\kappa - \sqrt{k})$, and the limit is given by

$$\mathcal{W}^\infty = \lim_{\kappa \rightarrow \infty} \mathcal{W} \cong \mathcal{H}(n) \otimes \mathcal{A}(m) \otimes \mathcal{T} \otimes \mathcal{G}_{\text{ev}}(r) \otimes \mathcal{G}_{\text{odd}}(s).$$

In this notation,

- (1) $\mathcal{H}(n)$ is the rank n Heisenberg algebra, which has even generators $\alpha^i, i = 1, \dots, n$, and OPE relations

$$\alpha^i(z)\alpha^j(w) \sim \delta_{i,j}(z-w)^{-2}.$$

- (2) $\mathcal{A}(m)$ is the rank m symplectic fermion algebra, which has odd generators $e^i, f^i, i = 1, \dots, m$, and operator products

$$e^i(z)f^j(w) \sim \delta_{i,j}(z-w)^{-2}.$$

- (3) \mathcal{T} is a generalized free field algebra with even generator $t(z)$ in weight two satisfying

$$t(z)t(w) \sim (z-w)^{-4}.$$

- (4) $\mathcal{G}_{\text{ev}}(r)$ is a generalized free field algebra with even generators $a^i, b^i, i = 1, \dots, r$, and operator products

$$a^i(z)b^j(w) \sim \delta_{i,j}(z-w)^{-3}.$$

- (5) $\mathcal{G}_{\text{odd}}(s)$ is a generalized free field algebra with odd generators $\varphi^i, i = 1, \dots, s$, and operator products

$$\varphi^i(z)\varphi^j(w) \sim \delta_{i,j}(z-w)^{-3}.$$

Note that \mathcal{T} , $\mathcal{G}_{\text{ev}}(r)$, and $\mathcal{G}_{\text{odd}}(s)$ do not possess conformal vectors. However, they have quasi-conformal structures [FBZ] such that t has weight 2, and a^i, b^i, φ^i have weight $3/2$. To see this, note that \mathcal{T} can be regarded as the subalgebra of the rank one Heisenberg algebra generated by the derivative of the generator. Similarly, $\mathcal{G}_{\text{ev}}(r)$ can be regarded as the subalgebra of the rank r $\beta\gamma$ -system generated by the derivatives of the generators. Finally, $\mathcal{G}_{\text{odd}}(s)$ can be regarded as the subalgebra of the rank s free fermion algebra generated by the derivatives of the generators. The quasi-conformal structures on \mathcal{T} , $\mathcal{G}_{\text{ev}}(r)$, and $\mathcal{G}_{\text{odd}}(s)$ are then inherited from the conformal structures on the Heisenberg algebra, $\beta\gamma$ -system, and free fermion algebra, respectively.

The full automorphism groups of $\mathcal{G}_{\text{ev}}(r)$ and $\mathcal{G}_{\text{odd}}(s)$ which preserves the weight gradation are the symplectic group $\text{Sp}(2r)$ and the orthogonal group $\text{O}(s)$, respectively. It is necessary to understand the structure of orbifolds of $\mathcal{G}_{\text{ev}}(r)$ and $\mathcal{G}_{\text{odd}}(s)$ under arbitrary reductive groups. As in our previous studies of orbifolds of free field algebras, we first need to describe $\mathcal{G}_{\text{ev}}(r)^{\text{Sp}(2r)}$ and $\mathcal{G}_{\text{odd}}(s)^{\text{O}(s)}$.

Theorem 4.4. $\mathcal{G}_{\text{ev}}(r)^{\text{Sp}(2r)}$ has a minimal strong generating set

$$\omega^{2j+1} = \frac{1}{2} \sum_{i=1}^r (: a^i \partial^{2j+1} b^i : - : (\partial^{2j+1} a^i) b^i :), \quad 0 \leq j \leq r^2 + 3r - 1,$$

and is therefore of type $\mathcal{W}(4, 6, \dots, 2r^2 + 6r + 2)$. Moreover, $\mathcal{G}_{\text{ev}}(r)$ is completely reducible as a $\mathcal{G}_{\text{ev}}(r)^{\text{Sp}(2r)}$ -module, and all irreducible modules in this decomposition are highest-weight and C_1 -cofinite according to Miyamoto's definition [MiI].

Proof. The first statement can be reduced to showing that, in the notation of Equation 9.1 of [LV], $R_r(I) \neq 0$ for $I = (1, 2, \dots, 2r + 2)$. The explicit formula for $R_r(I)$ is given by Theorem 4 of [LV], and it is clear that it is nonzero. The Zhu algebra [Z] of $\mathcal{G}_{\text{ev}}(r)^{\text{Sp}(2r)}$ is abelian, which implies that the admissible irreducible modules are highest-weight. The proof of C_1 -cofiniteness is the same as the proof of Lemma 8 of [LII]. \square

Corollary 4.5. For any reductive group $G \subset \text{Sp}(2r)$, $\mathcal{G}_{\text{ev}}(r)^G$ is strongly finitely generated.

Proof. This is the same as the proof of Theorem 15 of [LV]. First, $\mathcal{G}_{\text{ev}}(r)^G$ is completely reducible as a $\mathcal{G}_{\text{ev}}(r)^{\text{Sp}(2r)}$ -module. By a classical theorem of Weyl (Theorem 2.5A of [W]), it has an (infinite) strong generating set that lies in the sum of finitely many irreducible $\mathcal{G}_{\text{ev}}(r)^{\text{Sp}(2r)}$ -modules. The result then follows from the strong finite generation of $\mathcal{G}_{\text{ev}}(r)^{\text{Sp}(2r)}$ and the C_1 -cofiniteness of these modules. \square

There is one case where an explicit description of $\mathcal{G}_{\text{ev}}(r)^G$ will be important to us, namely, $G = \text{GL}(r) \subset \text{Sp}(2r)$, where $\{a^i\}$ spans a copy of the standard module \mathbb{C}^r and $\{b^i\}$ spans a copy of $(\mathbb{C}^r)^*$.

Theorem 4.6. $\mathcal{G}_{\text{ev}}(r)^{\text{GL}(r)}$ has a minimal strong generating set

$$\nu^j = \sum_{i=1}^r : a^i \partial^j b^i :, \quad 0 \leq j \leq r^2 + 4r - 1,$$

and is therefore of type $\mathcal{W}(3, 4, \dots, r^2 + 4r + 2)$. Moreover, $\mathcal{G}_{\text{ev}}(r)$ is completely reducible as a $\mathcal{G}_{\text{ev}}(r)^{\text{GL}(r)}$ -module, and all irreducible modules in this decomposition are highest-weight and C_1 -cofinite according to Miyamoto's definition.

Proof. The method of [LI] for studying the $\mathcal{W}_{1+\infty, -r}$ -algebra via its realization as the $\mathrm{GL}(r)$ -invariants in the rank r $\beta\gamma$ -system can be applied in this case. \square

Theorem 4.7. $\mathcal{G}_{\mathrm{odd}}(s)^{\mathcal{O}(s)}$ has a minimal strong generating set

$$\omega^{2j+1} = \frac{1}{2} \sum_{i=1}^s : \varphi^i \partial^{2j+1} \varphi^i :, \quad 0 \leq j \leq 2s - 1,$$

and is therefore of type $\mathcal{W}(4, 6, \dots, 4s + 2)$. Moreover, $\mathcal{G}_{\mathrm{odd}}(s)$ is completely reducible as a $\mathcal{G}_{\mathrm{odd}}(s)^{\mathcal{O}(s)}$ -module, and all irreducible modules in this decomposition are highest-weight and C_1 -cofinite according to Miyamoto's definition.

Proof. This can be reduced to showing that, in the notation of Equation 11.1 of [LV], $R_s(I, J) \neq 0$ for $I = (1, 1, \dots, 1)$ and $J = (2, 2, \dots, 2)$, where both lists have length $s + 1$. This follows easily from the recursive formula given by Equation 11.5 of [LV]. \square

Corollary 4.8. For any reductive group $G \subset \mathcal{O}(s)$, $\mathcal{G}_{\mathrm{odd}}(s)^G$ is strongly finitely generated.

Recall that the full automorphism groups of $\mathcal{H}(n)$ and $\mathcal{A}(m)$ are $\mathrm{O}(n)$ and $\mathrm{Sp}(2m)$, respectively. Suppose that G is a reductive group of automorphisms of

$$\mathcal{H}(n) \otimes \mathcal{A}(m) \otimes \mathcal{G}_{\mathrm{ev}}(r) \otimes \mathcal{G}_{\mathrm{odd}}(s)$$

which preserves the tensor factors, so that

$$G \subset \mathrm{O}(n) \times \mathrm{Sp}(2m) \times \mathrm{Sp}(2r) \times \mathrm{O}(s).$$

By the same argument as the proof of Theorem 4.2 of [CLII], we obtain

Corollary 4.9. $(\mathcal{H}(n) \otimes \mathcal{A}(m) \otimes \mathcal{G}_{\mathrm{ev}}(r) \otimes \mathcal{G}_{\mathrm{odd}}(s))^G$ is strongly finitely generated.

If $G \subset G_0^{\natural}$ is any reductive group of automorphisms of $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$, the orbifold $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})^G$ is also a deformable family and

$$\lim_{k \rightarrow \infty} \mathcal{W}^k(\mathfrak{g}, e_{-\theta})^G \cong \mathcal{T} \otimes (\mathcal{H}(n) \otimes \mathcal{A}(m) \otimes \mathcal{G}_{\mathrm{ev}}(r) \otimes \mathcal{G}_{\mathrm{odd}}(s))^G.$$

The argument is the same as the proof of Lemma 5.1 and Corollary 5.2 of [CLII]. Since the right hand side is strongly finitely generated, it follows from Theorem 4.1 that $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})^G$ is strongly finitely generated for generic values of k .

Finally, suppose that $\mathfrak{g}' \subset \mathfrak{g}^{\natural}$ is a reductive Lie subalgebra of dimension d , i.e., a sum of abelian and simple summands. Let $V^\ell(\mathfrak{g}')$ be the corresponding affine vertex subalgebra, and let

$$\mathcal{C}^k = \mathrm{Com}(V^\ell(\mathfrak{g}'), \mathcal{W}^k(\mathfrak{g}, e_{-\theta})).$$

Note that $\lim_{k \rightarrow \infty} V^\ell(\mathfrak{g}') \cong \mathcal{H}(d)$ and

$$\lim_{k \rightarrow \infty} \mathcal{W}^k(\mathfrak{g}, e_{-\theta}) \cong \mathcal{H}(d) \otimes \mathcal{H}(n - d) \otimes \mathcal{A}(m) \otimes \mathcal{G}_{\mathrm{ev}}(r) \otimes \mathcal{G}_{\mathrm{odd}}(s).$$

Moreover, $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ is *good* in the sense of [CLII], so by Lemma 6.2 of [CLII], G acts on $\mathcal{H}(n - d) \otimes \mathcal{A}(m) \otimes \mathcal{G}_{\mathrm{ev}}(r) \otimes \mathcal{G}_{\mathrm{odd}}(s)$, and we have

$$\lim_{k \rightarrow \infty} \mathcal{C}^k \cong (\mathcal{H}(n - d) \otimes \mathcal{A}(m) \otimes \mathcal{G}_{\mathrm{ev}}(r) \otimes \mathcal{G}_{\mathrm{odd}}(s))^G.$$

Since G preserves the tensor factors, this algebra is strongly finitely generated. By Theorem 4.1, we obtain

Theorem 4.10. For any \mathfrak{g}' as above, $\mathcal{C}^k = \text{Com}(V^\ell(\mathfrak{g}'), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$ is strongly finitely generated for generic values of k .

5. GENERIC STRUCTURE OF $\text{COM}(V^{k'}(\mathfrak{g}^\natural), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$: SOME EXPLICIT RESULTS

In this section, we find minimal strong generating sets for $\text{Com}(V^{k'}(\mathfrak{g}^\natural), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$ for generic values of k in a few cases.

The case $\mathfrak{g} = \mathfrak{sl}_n$. Recall from [KWI] that $(\mathfrak{sl}_n)^\natural = \mathfrak{gl}_{n-2}$, and

$$(\mathfrak{sl}_n)_{-1/2} \cong \mathbb{C}^{n-2} \oplus (\mathbb{C}^{n-2})^*$$

as \mathfrak{gl}_{n-2} -modules. The weight one subspace generates a copy of $V^{k+1}(\mathfrak{gl}_{n-2}) = \mathcal{H} \otimes V^{k+1}(\mathfrak{sl}_{n-2})$, where \mathcal{H} is a rank one Heisenberg algebra. It follows that

$$\lim_{k \rightarrow \infty} \mathcal{W}^k(\mathfrak{sl}_n, e_{-\theta}) \cong \mathcal{H}((n-2)^2) \otimes \mathcal{T} \otimes \mathcal{G}_{\text{ev}}(n-2),$$

and

$$\lim_{k \rightarrow \infty} \text{Com}(V^{k+1}(\mathfrak{gl}_{n-2}), \mathcal{W}^k(\mathfrak{sl}_n, e_{-\theta})) \cong \mathcal{T} \otimes \mathcal{G}_{\text{ev}}(n-2)^{\text{GL}(n-2)}.$$

By Theorem 4.6, $\mathcal{G}_{\text{ev}}(n-2)^{\text{GL}(n-2)}$ is of type $\mathcal{W}(3, 4, \dots, n^2 - 2)$, so we obtain

Theorem 5.1. $\text{Com}(V^{k+1}(\mathfrak{gl}_{n-2}), \mathcal{W}^k(\mathfrak{sl}_n, e_{-\theta}))$ is of type $\mathcal{W}(2, 3, \dots, n^2 - 2)$ for generic values of k .

In the case $n = 3$, this generalizes the result of [ACL] that $\text{Com}(\mathcal{H}, \mathcal{W}(\mathfrak{sl}_3, e_{-\theta}))$ is generically of type $\mathcal{W}(2, 3, 4, 5, 6, 7)$.

The case $\mathfrak{g} = \mathfrak{sp}_{2n}$. Recall from [KWI] that for $n \geq 2$, $(\mathfrak{sp}_{2n})^\natural \cong \mathfrak{sp}_{2n-2}$ and that

$$(\mathfrak{sp}_{2n})_{-1/2} \cong \mathbb{C}^{2n-2}$$

as \mathfrak{sp}_{2n-2} -modules. The weight one subspace generates a copy of $V^{k+1/2}(\mathfrak{sp}_{2n-2})$. It follows that

$$\lim_{k \rightarrow \infty} \mathcal{W}^k(\mathfrak{sp}_{2n}, e_{-\theta}) \cong \mathcal{H}(d) \otimes \mathcal{T} \otimes \mathcal{G}_{\text{ev}}(n-2), \quad d = \dim(\mathfrak{sl}_{2n-2}),$$

and that

$$\lim_{k \rightarrow \infty} \text{Com}(V^{k+1/2}(\mathfrak{sp}_{2n-2}), \mathcal{W}^k(\mathfrak{sp}_{2n}, e_{-\theta})) \cong \mathcal{T} \otimes \mathcal{G}_{\text{ev}}(n-2)^{\text{Sp}(2n-2)}.$$

By Theorem 4.4, $\mathcal{G}_{\text{ev}}(n-2)^{\text{Sp}(2n-2)}$ is of type $\mathcal{W}(4, 6, \dots, 2n^2 + 2n - 2)$. We obtain

Theorem 5.2. $\text{Com}(V^{k+1/2}(\mathfrak{sp}_{2n-2}), \mathcal{W}^k(\mathfrak{sp}_{2n}, e_{-\theta}))$ is of type $\mathcal{W}(2, 4, \dots, 2n^2 + 2n - 2)$ for generic values of k .

The case $\mathfrak{g} = \mathfrak{sl}(2|n)$, $n \neq 2$. In this case, $\mathfrak{g}^\natural \cong \mathfrak{gl}_n$ and

$$\mathfrak{g}_{-1/2} \cong \mathbb{C}^n \oplus (\mathbb{C}^n)^*,$$

regarded as an odd vector space. The weight one subspace generates a copy of $V^{k-1}(\mathfrak{gl}_n) = \mathcal{H} \otimes V^{k-1}(\mathfrak{sl}_n)$, and

$$\lim_{k \rightarrow \infty} \mathcal{W}^k(\mathfrak{sl}(2|n), e_{-\theta}) \cong \mathcal{H}(n^2) \otimes \mathcal{T} \otimes \mathcal{G}_{\text{odd}}(2n).$$

Letting $\mathcal{C}^k = \text{Com}(V^{k-1}(\mathfrak{gl}_n), \mathcal{W}^k(\mathfrak{sl}(2|n), e_{-\theta}))$, we have

$$\lim_{k \rightarrow \infty} \mathcal{C}^k \cong \mathcal{T} \otimes \mathcal{G}_{\text{odd}}(2n)^{\text{GL}(n)}.$$

By the same argument as Theorem 4.3 of [CLIII], it can be shown that $\mathcal{G}_{\text{odd}}(2n)^{\text{GL}(n)}$ is purely even and of type $\mathcal{W}(3, 4, \dots, 3n+2)$. We obtain

Theorem 5.3. *For $n \neq 2$, $\text{Com}(V^{k-1}(\mathfrak{gl}_n), \mathcal{W}^k(\mathfrak{sl}(2|n), e_{-\theta}))$ is even and is generically of type $\mathcal{W}(2, 3, \dots, 3n+2)$.*

This generalizes the case $n = 1$, where $\mathcal{W}^k(\mathfrak{sl}(2|1), e_{-\theta})$ is just the $N = 2$ superconformal vertex algebra. In this case, the above commutant is known to be of type $\mathcal{W}(2, 3, 4, 5)$ and is isomorphic to the universal parafermion algebra of \mathfrak{sl}_2 .

The case $\mathfrak{g} = \mathfrak{psl}(2|2)$. Recall that in this case, $\mathcal{W}^k(\mathfrak{psl}(2|2), e_{-\theta})$ is isomorphic to the small $N = 4$ superconformal vertex algebra. We have $\mathfrak{g}^\natural = \mathfrak{sl}_2$ and

$$\mathfrak{g}_{-1/2} \cong \mathbb{C}^2 \oplus (\mathbb{C}^2)^*,$$

regarded as an odd vector space. The weight one subspace generates a copy of $V^{k-1}(\mathfrak{sl}_2)$. Letting $\mathcal{C}^k = \text{Com}(V^{k-1}(\mathfrak{sl}_2), \mathcal{W}^k(\mathfrak{psl}(2|2), e_{-\theta}))$, we have

$$\lim_{k \rightarrow \infty} \mathcal{C}^k \cong \mathcal{T} \otimes \mathcal{G}_{\text{odd}}(4)^{\text{SL}(2)}.$$

It is not difficult to show that $\mathcal{G}_{\text{odd}}(4)^{\text{SL}(2)}$ is even and of type $\mathcal{W}(3^3, 4, 5^3, 6, 7^3, 8)$. In other words, a minimal strong generating set consists of three even fields in each weight 3, 5, 7, and one even field in weights 4, 6, 8. We obtain

Theorem 5.4. *For $\mathfrak{g} = \mathfrak{psl}(2|2)$, $\text{Com}(V^{k-1}(\mathfrak{sl}_2), \mathcal{W}^k(\mathfrak{psl}(2|2), e_{-\theta}))$ is even and is generically of type $\mathcal{W}(2, 3^3, 4, 5^3, 6, 7^3, 8)$.*

Note that in this case, $\text{Com}(V^{k-1}(\mathfrak{sl}_2), \mathcal{W}^k(\mathfrak{psl}(2|2), e_{-\theta}))$ contains a subalgebra of type $\mathcal{W}(2, 3, 4, 5, 6, 7, 8)$.

The case $\mathfrak{g} = \mathfrak{osp}(1|4)$. In all the above examples, \mathfrak{g}^\natural is a Lie algebra. The case $\mathfrak{g} = \mathfrak{osp}(1|4)$ is an example where \mathfrak{g}^\natural is the Lie superalgebra $\mathfrak{osp}(1|2)$, but we can still describe

$$\mathcal{C}^k = \text{Com}(V^{k+1/2}(\mathfrak{osp}(1|2)), \mathcal{W}^k(\mathfrak{osp}(1|4), e_{-\theta})).$$

We have $\mathfrak{g}_{-1/2} \cong \mathbb{C}^{2|1}$, i.e., the superspace whose even part has dimension 2 and odd part has dimension 1. Since $\text{Osp}(1|2)$ acts completely reducibly on $\mathcal{W}^k(\mathfrak{osp}(1|4), e_{-\theta})$, it can be shown using the methods of [CLII] that

$$\lim_{k \rightarrow \infty} \mathcal{C}^k \cong \mathcal{T} \otimes (\mathcal{G}_{\text{ev}}(1) \otimes \mathcal{G}_{\text{odd}}(1))^{\text{Osp}(1|2)}.$$

By the same argument as Theorem 12 of [LV], which is based on Sergeev's first and second fundamental theorems of invariant theory for $\text{Osp}(1|2)$ [SI, SII], it can be shown that $(\mathcal{G}_{\text{ev}}(1) \otimes \mathcal{G}_{\text{odd}}(1))^{\text{Osp}(1|2)}$ is purely even and of type $\mathcal{W}(4, 6, \dots, 18)$. We obtain

Theorem 5.5. *For $\mathfrak{g} = \mathfrak{osp}(1|4)$, $\text{Com}(V^{k+1/2}(\mathfrak{osp}(1|2)), \mathcal{W}^k(\mathfrak{osp}(1|4), e_{-\theta}))$ is even and is generically of type $\mathcal{W}(2, 4, \dots, 18)$.*

6. THE STRUCTURE OF THE NONGENERIC SET

Given a coset of the form $\mathcal{C}^k = \text{Com}(V^{k'}(\mathfrak{g}^\natural), \mathcal{W}^k(\mathfrak{g}, e_{-\theta}))$, suppose that \mathcal{C} is a deformable family depending on a parameter κ such that $\mathcal{C}/(\kappa - \sqrt{k}) \cong \mathcal{C}^k$ for generic values of k . There are values of k where the \mathcal{C}^k is larger than the specialization $\mathcal{C}/(\kappa - \sqrt{k})$, but it can be shown in a more general setting that these values consist of a discrete set of negative rational numbers [CLIV]. Suppose that S is a minimal strong generating set for \mathcal{C} . We call a value of k *S-generic* if $\mathcal{C}/(\kappa - \sqrt{k}) \cong \mathcal{C}^k$ and S is also a strong generating set for \mathcal{C}^k . Often we write *generic* instead of *S-generic* when no confusion can arise.

In [ACL], it was shown that in the case $\mathfrak{g} = \mathfrak{sl}_3$, $\mathcal{C}^k = \text{Com}(\mathcal{H}, \mathcal{W}^k(\mathfrak{sl}_3, e_{-\theta}))$ is generically of type $\mathcal{W}(2, 3, 4, 5, 6, 7)$, and we gave an explicit description of the nongeneric set; it consists only of $\{-1, -3/2\}$. This is easier than general situation because cosets of Heisenberg algebras are better behaved than cosets of general affine vertex algebras. In this section, we analyze the structure of the nongeneric set for \mathcal{C}^k in two examples where \mathfrak{g}^\natural is non-abelian in order to illustrate how this problem can be handled computationally.

The case $\mathfrak{g} = \mathfrak{sp}_4$. The generators of $\mathcal{W}^k(\mathfrak{sp}_4, e_{-\theta})$ are T, H, X, Y, G^\pm , where X, Y, H generate a copy of $V^{k+1/2}(\mathfrak{sl}_2)$, T is a Virasoro of central charge $c = -\frac{3(1+k)(1+2k)}{(3+k)}$, and G^\pm are primary of weight $3/2$ and satisfy

$$\begin{aligned} H(x)G^\pm(w) &\sim G^\pm(w)(z-w)^{-1}, & X(z)G^-(w) &\sim G^+(w)(z-w)^{-1}, & Y(z)G^+(w) &\sim G^-(w)(z-w)^{-1}, \\ G^+(z)G^+(w) &\sim (8+4k)X(w)(z-w)^{-2} + (4+2k)\partial X(w)(z-w)^{-1}, \\ G^-(z)G^-(w) &\sim -(8+4k)Y(w)(z-w)^{-2} - (4+2k)Y(w)(z-w)^{-1}, \\ G^+(z)G^-(w) &\sim -(4+10k+4k^2)(z-w)^{-3} - (4+2k)H(w)(z-w)^{-2} \\ &\quad + ((6+2k)T - 4 : XY : - : HH : -k\partial H)(w)(z-w)^{-1}. \end{aligned}$$

We use the notation $a = \lim_{k \rightarrow \infty} \frac{1}{k}G^+$ and $b = \lim_{k \rightarrow \infty} \frac{1}{k}G^-$ for the generators of $\mathcal{G}_{\text{ev}}(1)$, which satisfy

$$a(z)b(w) \sim -4(z-w)^{-3}.$$

Note that this normalization is different from the one used earlier. Recall that by classical invariant theory, the orbifold $\mathcal{G}_{\text{ev}}(1)^{Sp(2)}$ has generators

$$\omega_{i,j} = : \partial^i a \partial^j b : - : \partial^j a \partial^i b :, \quad 0 \leq i < j,$$

and that the smaller set $\{\omega_{0,n} \mid n = 1, 3, 5, \dots\}$ suffices.

By Theorem 5.2, $\mathcal{C}^k(2) = \text{Com}(V^{k+1/2}(\mathfrak{sp}_2), \mathcal{W}^k(\mathfrak{sp}_4, e_{-\theta}))$ is of type $\mathcal{W}(2, 4, 6, 8, 10)$ for generic k , and has Virasoro element

$$L = T - \frac{1}{10+4k} (: HH : +4 : XY : -2\partial H)$$

with central charge

$$c = -\frac{6(2+k)^2(1+2k)}{(3+k)(5+2k)}.$$

In order to write down additional elements of $\mathcal{C}^k(2)$, we first consider

$$u_{0,n} = : G^+ \partial^n G^- : - : (\partial^j G^+) G^- : \in \mathcal{W}^k(\mathfrak{sp}_4, e_{-\theta}), \quad n = 1, 3, 5, \dots$$

Clearly $u_{0,n}$ has weight $n+3$ and

$$\lim_{k \rightarrow \infty} \frac{1}{k^2} u_{0,n} = \omega_{0,n},$$

but $u_{0,n} \notin \mathcal{C}^k$. We shall construct suitable corrections

$$(6.1) \quad U_{0,n} = u_{0,n} + P_n \in \mathcal{C}^k,$$

such that P_n is a normally ordered polynomial in $T, H, X, Y, u_{0,1}, \dots, u_{0,n-2}$ and their derivatives, and $\lim_{k \rightarrow \infty} \frac{1}{k^2} P_n = 0$. First, for $k \neq -9/2$, we define

$$\begin{aligned} U_{0,1} = & u_{0,1} + \frac{2(2+k)(3+k)}{9+2k} \partial^2 T + \frac{13+8k+k^2}{3(9+2k)} \partial^3 H \\ & - \frac{2}{9+2k} : (\partial^2 H) H : - \frac{2+k}{2(9+2k)} : (\partial H) \partial H : - \frac{2(2+k)}{9+2k} : (\partial X) \partial Y : \\ & + \frac{2}{9+2k} : T H H : + \frac{8}{9+2k} : T X Y : - \frac{4}{9+2k} : T \partial H : + \frac{4}{9+2k} : H (\partial X) Y : \\ & + \frac{4}{9+2k} : H X \partial Y : - \frac{2(8+3k)}{9+2k} : T T : + \frac{2}{9+2k} : (\partial H) H H : \\ & - \frac{2(3+k)}{9+2k} : (\partial T) H : + \frac{2}{9+2k} : H G^+ G^- : + \frac{2}{9+2k} : X G^- G^- : - \frac{2}{9+2k} : Y G^+ G^+ : \end{aligned}$$

This lies in $\mathcal{C}^k(2)$ but is not primary with respect to L . Instead, it satisfies

(6.2)

$$\begin{aligned} L(z) U_{0,1}(w) \sim & -\frac{12(2+k)^2(1+2k)(18+7k)}{(3+k)(9+2k)} (z-w)^{-6} + \frac{4(5+2k)(3+17k+7k^2)}{(3+k)(9+2k)} L(w) (z-w)^{-4} \\ & + \frac{2(5+2k)(18+7k)}{9+2k} L(w) (z-w)^{-3} + 4U_{0,1}(w) (z-w)^{-2} + \partial U_{0,1}(w) (z-w)^{-1}. \end{aligned}$$

Whenever k is not a root of $p(x) = 30x^3 + 113x^2 + 59x - 105$, $U_{0,1}$ can be corrected uniquely to a primary element by adding

$$\frac{2(5+2k)^2(-147-32k+14k^2)}{(9+2k)(-105+59k+113k^2+30k^3)} : LL : - \frac{(3+k)(5+2k)(-273+22k+157k^2+42k^3)}{(9+2k)(-105+59k+113k^2+30k^3)} \partial^2 L,$$

but it is easier for our purposes to work with $U_{0,1}$.

Lemma 6.1. *For all $n = 1, 3, 5, \dots$, we have*

$$(6.3) \quad (U_{0,1})_{(1)}(u_{0,n}) = f(k, n) u_{0,n+2} + P_n,$$

where

$$(6.4) \quad f(k, n) = -\frac{2(3+k)(5+n)(7+2k+5n+2kn)(14+4k+5n+2kn)}{(9+2k)(1+n)(2+n)},$$

and P_n is a normally ordered polynomial in generators $T, X, Y, H, u_{0,1}, u_{0,3}, \dots, u_{0,n}$, and their derivatives.

Proof. The argument is similar to the proof of Theorem 6.3 of [AL], and is omitted. \square

Let

$$S = \left\{-\frac{9}{2}, -3\right\} \cup \left\{-\frac{7+5n}{2(1+n)} \mid n \geq 1\right\} \cup \left\{-\frac{14+5n}{2(2+n)} \mid n \geq 1\right\},$$

which is the set of values of k for which $f(n, k) = 0$ or is undefined.

We would like the corrections $U_{0,n}$ in (6.1) to satisfy

$$(U_{0,1})_{(1)}(U_{0,n}) = f(k, n)U_{0,n+2} + Q_n,$$

where Q_n is a normally ordered polynomial in $L, U_{0,1}, U_{0,3}, \dots, U_{0,n-2}$, and this can be achieved by a bootstrap procedure as we now demonstrate.

One can check by computer that

$$(U_{0,1})_{(1)}(U_{0,1}) = f(k, 1)u_{0,3} + P_1,$$

where P_1 depends on $T, H, X, Y, u_{0,1}$ and their derivatives. As a first attempt to correct $u_{0,3}$ to an element of $\mathcal{C}^k(2)$, define

$$\tilde{U}_{0,3} = u_{0,3} + \frac{1}{f(k, 1)}P_1,$$

which certainly lies in $\mathcal{C}^k(2)$ and satisfies $\lim_{k \rightarrow \infty} \frac{1}{k^2}\tilde{U}_{0,3} = \omega_{0,3}$. However, $\tilde{U}_{0,3}$ is not the desired correction since

$$(U_{0,1})_{(1)}(\tilde{U}_{0,3}) \neq f(k, 3)u_{0,5} + P_3$$

for any normally ordered polynomial P_3 in $T, H, X, Y, u_{0,1}, u_{0,3}$ and their derivatives. The reason is that the terms $:T(\partial G^+)G^-:$ and $:TG^+(\partial G^-):$ appear in P_1 with nonzero coefficient, and if we express $(U_{0,1})_{(1)}(:T(\partial G^+)G^-:)$ and $(U_{0,1})_{(1)}(:TG^+(\partial G^-):)$ as normally ordered polynomials in $T, H, X, Y, u_{0,1}, u_{0,3}, u_{0,5}$, the coefficient of $u_{0,5}$ is nonzero. It is easy to check the only normally ordered monomials ν in $T, H, X, Y, u_{0,1}$ which have the property that $(U_{0,1})_{(1)}\nu$ has a nontrivial coefficient of $u_{0,5}$ are $:T(\partial G^+)G^-:$ and $:TG^+(\partial G^-):$. Moreover, these terms can be eliminated as follows. We define

$$U_{0,3} = \tilde{U}_{0,3} - \frac{6}{19+6k} :LU_{0,1} : .$$

Then the terms $:T(\partial G^+)G^-:$ and $:TG^+(\partial G^-):$ do not appear in $U_{0,3}$, and

$$(6.5) \quad (U_{0,1})_{(1)}(U_{0,3}) = f(k, 3)u_{0,5} + P_3,$$

where P_3 is a normally ordered polynomial in $T, H, X, Y, u_{0,1}, u_{0,3}$ and their derivatives. Also, note that

$$(U_{0,1})_{(1)}(U_{0,1}) = f(k, 1)U_{0,3} + \frac{6f(k, 1)}{19+6k} :LU_{0,1} : .$$

As above, we attempt to correct $u_{0,5}$ by defining

$$\tilde{U}_{0,5} = u_{0,5} + \frac{1}{f(k, 3)}P_3,$$

which lies in $\mathcal{C}^k(2)$ and satisfies $\lim_{k \rightarrow \infty} \frac{1}{k^2}\tilde{U}_{0,5} = \omega_{0,5}$. However,

$$(U_{0,1})_{(1)}(\tilde{U}_{0,5}) \neq f(k, 5)u_{0,7} + P_5$$

for any P_5 depending only on $T, H, X, Y, u_{0,1}, u_{0,3}, u_{0,5}$ and their derivatives. As before, the reason is that P_3 contains terms ν of the form

$$(6.6) \quad : TT(\partial G^+)G^- :, \quad : TTG^+(\partial G^1) :, \quad : (\partial^i T)(\partial^j G^+)(\partial^\ell G^-) :, \quad i + j + \ell = 3,$$

which all have the property that the coefficient of $u_{0,7}$ in $(U_{0,1})_{(1)}\nu$ is nonzero. However, we can correct $\tilde{U}_{0,5}$ as follows:

$$\begin{aligned} U_{0,5} = & \tilde{U}_{0,5} - \frac{30(-1 + 5k + 2k^2)}{(3+k)(11+4k)(19+6k)(29+10k)} : LLU_{0,1} : \\ & - \frac{5(42 + 21k + 2k^2)}{(3+k)(11+4k)(29+10k)} : LU_{0,3} : \\ & - \frac{5(12234 + 13450k + 5169k^2 + 788k^3 + 36k^4)}{2(3+k)^2(11+4k)(19+6k)(29+10k)} : (\partial^2 L)U_{0,1} : \\ & - \frac{5(9523 + 8130k + 2186k^2 + 180k^3)}{4(3+k)(11+4k)(19+6k)(29+10k)} : (\partial L)\partial U_{0,1} : \\ & - \frac{5(5+2k)(491 + 256k + 30k^2)}{2(3+k)(11+4k)(19+6k)(29+10k)} : L\partial^2 U_{0,1} : . \end{aligned}$$

Then $U_{0,5}$ has no terms of the form (6.6), and

$$(U_{0,1})_{(1)}(U_{0,5}) = f(k, 5)u_{0,7} + P_5,$$

where P_5 is a normally ordered polynomial in $T, X, Y, u_{0,1}, u_{0,3}, u_{0,5}$ and their derivatives. Also, it is immediate from the above formula for $U_{0,5}$ that

$$(U_{0,1})_{(1)}(U_{0,3}) = f(k, 3)U_{0,5} + Q_3,$$

where Q_3 is a linear combination of

$$: LLU_{0,1} :, \quad : LU_{0,3} :, \quad : (\partial^2 L)U_{0,1} :, \quad : (\partial L)(\partial U_{0,1}) :, \quad : L(\partial^2 U_{0,1}) : .$$

Finally, note that the only values of k where the denominators of any terms appearing in $U_{0,3}$ and $U_{0,5}$ vanishes are elements of S . This is a consequence of (6.2) together with the fact that $U_{0,3}$ and $U_{0,5}$ lie in the algebra generated by L and $U_{0,1}$.

More generally, we can continue this process and construct elements $U_{0,n}$ for all $n = 1, 3, 5, \dots$, with the property that

$$(U_{0,1})_{(1)}(U_{0,n}) = f(k, n)U_{0,n+2} + Q_n$$

where Q_n is a linear combination of elements of the form

$$: (\partial^{i_1} L) \cdots (\partial^{i_r} L)(\partial^k U_{0,m}) :, \quad m = 1, 3, \dots, n, \quad i_1, \dots, i_r, k \geq 0.$$

Since $U_{0,n}$ all lie in the algebra generated by L and $U_{0,1}$, all points where the denominator of any term appearing in $U_{0,n}$ vanishes lie in S . It follows that

$$\{L, U_{0,n} \mid n = 1, 3, 5, \dots\}$$

closes under OPE. Since $\mathcal{C}^k(2) = \mathcal{C}/(\kappa - \sqrt{k})$ for all real numbers $k > -5/2$ and S does not intersect this set, we obtain

Corollary 6.2. *For all real numbers $k > -5/2$, $\mathcal{C}^k(2)$ is strongly generated by*

$$\{L, U_{0,n} \mid n = 1, 3, 5, \dots\}.$$

Next, we consider normally ordered relations among the generators. Recall that the first normally ordered relation among generators $\omega_{i,j}$ of $\mathcal{G}_{\text{ev}}(1)^{Sp(2)}$ occurs at weight 12, and is a quantum correction of a classical Pfaffian relation. The leading term is

$$: \omega_{0,1} \omega_{2,3} : - : \omega_{0,2} \omega_{1,3} : + : \omega_{0,3} \omega_{1,2} :,$$

and the subleading terms can all be expressed as normally ordered monomials in $\omega_{0,n}$ for $n = 1, 3, 5, 7, 9$. In fact, the relation can be written as follows.

$$(6.7) \quad \frac{1}{6} \omega_{0,9} = : \omega_{0,1} \omega_{0,5} : - 3 : \omega_{0,1} \partial^2 \omega_{0,3} : + 2 : \omega_{0,1} \partial^4 \omega_{0,1} : - : (\partial \omega_{0,1}) \partial^3 \omega_{0,1} : + : (\partial \omega_{0,1}) \partial \omega_{0,3} : \\ + : \omega_{0,3} \partial^2 \omega_{0,1} : - : \omega_{0,3} \omega_{0,3} : - \frac{5}{3} \partial^2 \omega_{0,7} + \frac{19}{2} \partial^4 \omega_{0,5} - \frac{587}{30} \partial^6 \omega_{0,3} + \frac{119}{10} \partial^8 \omega_{0,1}.$$

This shows that $\omega_{0,9}$ can be expressed as a normally ordered polynomial in $\omega_{0,1}, \dots, \omega_{0,7}$ and their derivatives. Starting with (6.7), and using the fact that

$$(\omega_{0,1})_{(1)}(\omega_{0,n}) = -4(n+5)\omega_{0,n+2} + \nu_n, \quad n = 1, 3, 5, \dots,$$

where ν_n is a linear combination of $\partial^2 \omega_{0,n}, \partial^4 \omega_{n-2}, \dots, \partial^{n+1} \omega_{0,1}$, one can construct decoupling relations

$$\omega_{0,n} = Q_n(\omega_{0,1}, \omega_{0,3}, \omega_{0,5}, \omega_{0,7}), \quad n = 11, 13, 15, \dots$$

This shows that $\mathcal{G}_{\text{ev}}(1)^{Sp(2)}$ is of type $\mathcal{W}(4, 6, 8, 10)$ with minimal strong generating set $\{\omega_{0,1}, \dots, \omega_{0,7}\}$, and is a special case of Theorem 4.4.

Recall the elements $U_{0,n}$ which satisfy $\lim_{k \rightarrow \infty} \frac{1}{k^2} U_{0,n} = \omega_{0,n}$. The above relation (6.7) can be deformed to a relation

$$\lambda(k) U_{0,9} = R_9(L, U_{0,1}, U_{0,3}, U_{0,5}, U_{0,7}),$$

and it can be verified by a lengthy computer calculation that

$$(6.8) \quad \lambda(k) = \frac{(2+k)(39+14k)(49+18k)}{756(9+2k)}.$$

Note that

$$\lim_{k \rightarrow \infty} \frac{1}{k^2} \lambda(k) = \frac{1}{6},$$

as expected by (6.7). It is clear from (6.8) that the set of values of k where the numerator or denominator of $\lambda(k)$ vanishes lies in S . Using the fact that $(U_{0,1})_{(1)}(U_{0,n}) = f(k, n)U_{0,n+2} + Q_n$, we can construct similar decoupling relations

$$U_{0,n} = R_n(L, U_{0,1}, U_{0,3}, U_{0,5}, U_{0,7}),$$

for all $n = 11, 13, 15, \dots$ and $k \notin S$. This implies

Theorem 6.3. *For all real numbers $k > -5/2$, $\mathcal{C}^k(2)$ is of type $\mathcal{W}(2, 4, 6, 8, 10)$ with minimal strong generating set $\{L, U_{0,1}, U_{0,3}, U_{0,5}, U_{0,7}\}$.*

The case $\mathfrak{g} = \mathfrak{sl}_4$. The generators of $\mathcal{W}^k(\mathfrak{sl}_4, e_{-\theta})$ are $J, X, Y, H, T, G^{1,\pm}, G^{2,\pm}$, where T is a Virasoro element of central charge $c = -\frac{3k(3+2k)}{4+k}$, J, H, X, Y are primary of weight one, and $G^{1,\pm}, G^{2,\pm}$ are primary of weight $3/2$. Moreover, H, X, Y generate a copy of $V^{k+1}(\mathfrak{sl}_2)$, J commutes with H, X, Y and generates a Heisenberg algebra, and we have the following OPE relations:

$$\begin{aligned}
J(z)J(w) &\sim 4(2+k)(z-w)^{-2}, & J(z)X^+(w) &\sim 2X^+(w), \\
G^{1,-}(z)G^{1,+}(w) &\sim -2(k+2)X(w)(z-w)^{-2} + (: JX : -(k+2)\partial X)(w)(z-w)^{-1}, \\
G^{2,-}(z)G^{2,+}(w) &\sim -2(k+2)Y(w)(z-w)^{-2} + (: JY : -(k+2)\partial Y)(w)(z-w)^{-1}, \\
G^{1,-}(z)G^{2,+}(w) &\sim -2(k+1)(k+2)(z-w)^{-3} + ((k+1)J - (k+2)H)(w)(z-w)^{-2} \\
&+ \left((4+k)T - \frac{3}{8} : JJ : + \frac{1}{2} : HJ : - \frac{1}{2} : HH : - 2 : XY : + \frac{1+k}{2}\partial J - \frac{k}{2}\partial H \right)(w)(z-w)^{-1}, \\
G^{1,+}(z)G^{2,-}(w) &\sim 2(k+1)(k+2)(z-w)^{-3} + ((k+1)J + (k+2)H)(w)(z-w)^{-2} \\
&+ \left(-(4+k)T + \frac{3}{8} : JJ : + \frac{1}{2} : HJ : + \frac{1}{2} : HH : + 2 : XY : + \frac{1+k}{2}\partial J + \frac{k}{2}\partial H \right)(w)(z-w)^{-1}.
\end{aligned}$$

We use the notation

$$a^i = \lim_{k \rightarrow \infty} \frac{1}{k} G^{i,+}, \quad b^i = \lim_{k \rightarrow \infty} \frac{1}{k} G^{i,-}, \quad i = 1, 2,$$

for the generators of $\mathcal{G}_{\text{ev}}(2)$, which satisfy

$$a^1(z)b^2(w) \sim 2(z-w)^{-3}, \quad b^1(z)a^2(w) \sim -2(z-w)^{-3}.$$

Note that this normalization is different from the one used earlier. By classical invariant theory, the orbifold $\mathcal{G}_{\text{ev}}(2)^{GL(2)}$ has strong generators

$$\omega_{i,j} = : \partial^i a^1 \partial^j b^2 : + : \partial^j a^2 \partial^i b^1 :, \quad i, j \geq 0,$$

and the smaller set $\{\omega_{0,n} | n \geq 0\}$ suffices. Recall that the coset

$$\mathcal{C}^k(4) = \text{Com}(V^{k+1}(\mathfrak{gl}_2), \mathcal{W}^k(\mathfrak{sl}_4, e_{-\theta}))$$

is generically of type $\mathcal{W}(2, 3, \dots, 14)$ and has Virasoro element

$$L = T - \frac{1}{8(2+k)} : JJ : - \frac{1}{4(3+k)} : HH : - \frac{1}{3+k} : XY : + \frac{1}{2(3+k)} \partial H$$

with central charge

$$c = -\frac{(1+k)(3+2k)(8+3k)}{(3+k)(4+k)}.$$

Next, consider

$$u_{0,n} = : G^{1,-} \partial^n G^{2,+} : + : (\partial^n G^{1,+}) G^{2,-} :, \quad n \geq 0.$$

These satisfy $\lim_{k \rightarrow \infty} \frac{1}{k^2} u_{0,n} = \omega_{0,n}$, but do not lie in $\mathcal{C}^k(4)$ and we would like to find suitable corrections $U_{0,n}$ in $\mathcal{C}^k(4)$. First, define

$$\begin{aligned}
U_{0,0} = u_{0,0} &- \frac{14+5k}{24(2+k)^2} : JJJ : - \frac{1}{2(2+k)} : JHH : - \frac{2}{2+k} : JXY : - \frac{1}{2} : (\partial J)H : \\
&- \frac{k}{2(2+k)} : J(\partial H) : + \frac{4+k}{2+k} : TJ : - \frac{16+9k+2k^2}{6(2+k)} \partial^2 J.
\end{aligned}$$

It is not difficult to verify that $U_{0,0}$ lies in \mathcal{C}^k and is primary with respect to L .

Lemma 6.4. For all $k \geq 0$, we have

$$(6.9) \quad (U_{0,0})_{(1)}(u_{0,n}) = f(n, k)u_{0,n+1} + P_n, \quad f(n, k) = -\frac{(k+4)(n+4)(5+k+3n+kn)}{(n+1)},$$

where P_n can be expressed as a normally ordered polynomial in $T, J, H, X, Y, u_{0,0}, \dots, u_{0,n}$ and their derivatives.

Proof. This is similar to the proof of Theorem 6.3 of [AL] and is omitted. \square

Let

$$S = \{-4\} \cup \left\{ -\frac{5+3n}{1+n} \mid n \geq 0 \right\},$$

which is the set of values of k where $f(n, k)$ is either zero or undefined.

We have $(U_{0,0})_{(1)}(U_{0,0}) = f(0, k)u_{0,1} + P_0$. Clearly $\tilde{U}_{0,1} = u_{0,1} + \frac{1}{f(0,k)}P_0$ lies in \mathcal{C}^k and satisfies $\lim_{k \rightarrow \infty} \frac{1}{k^2}\tilde{U}_{0,1} = \omega_{0,1}$, but $\tilde{U}_{0,1}$ is not the desired correction of $u_{0,1}$ since

$$(U_{0,0})_{(1)}(\tilde{U}_{0,1}) \neq f(1, k)u_{0,2} + P_1,$$

for any P_1 depending only on $J, H, X, Y, T, u_{0,0}$. The problem is that P_0 contains the terms $:TG^{1,-}G^{2,+}:$ and $:TG^{1,+}G^{2,-}:$ which have the property that $(U_{0,0})_{(1)}(:TG^{1,-}G^{2,+}:)$ and $(U_{0,0})_{(1)}(:TG^{1,+}G^{2,-}:)$ have a nonzero coefficient of $u_{0,1}$ when expressed as a normally ordered polynomial in $J, H, X, Y, T, u_{0,0}, u_{0,1}$. This can be corrected as follows:

$$U_{0,1} = \tilde{U}_{0,1} + :LU_{0,0} : .$$

This has the property that $(U_{0,0})_{(1)}(U_{0,1}) = f(1, k)u_{0,2} + P_1$ where P_1 depends only on $J, H, X, Y, T, u_{0,0}$. Moreover,

$$(U_{0,0})_{(1)}(U_{0,1}) = U_{0,1} + :LU_{0,0} : .$$

As above, we can construct elements $U_{0,n}$ for all $n \geq 0$ with the property that

$$(U_{0,0})_{(1)}(U_{0,n}) = f(k, n)U_{0,n+1} + Q_n,$$

where Q_n is a linear combination of the fields

$$: (\partial^{i_1} L) \cdots (\partial^{i_r} L) (\partial^k U_{0,m}) :, \quad m = 0, 1, \dots, n, \quad i_1, \dots, i_r, k \geq 0.$$

Moreover, since $U_{0,n}$ all lie in the algebra generated by L and $U_{0,0}$, all points where the denominator of any term appearing in $U_{0,n}$ vanishes lie in S . It follows that

$$\{L, U_{0,n} \mid n \geq 0\}$$

closes under OPE, and we obtain

Theorem 6.5. For all real numbers $k > -3$, $\mathcal{C}^k(4)$ is strongly generated by $\{L, U_{0,n} \mid n \geq 0\}$.

Now we consider relations among these generators. Recall that in $\mathcal{G}_{\text{ev}}(2)^{GL(2)}$, the first normally ordered relation among the generators $\omega_{i,j}$ occurs at weight 15 and has leading term

$$\begin{aligned} & : \omega_{0,0}\omega_{1,1}\omega_{2,2} : - : \omega_{0,0}\omega_{1,2}\omega_{2,1} : - : \omega_{0,1}\omega_{1,0}\omega_{2,2} : \\ & + : \omega_{0,1}\omega_{1,2}\omega_{2,0} : - : \omega_{0,2}\omega_{1,1}\omega_{2,0} : + : \omega_{0,2}\omega_{1,0}\omega_{2,1} :, \end{aligned}$$

which is a normally ordering of a classical determinantal relation in $\text{gr}(\mathcal{G}_{\text{ev}}(2))^{GL(2)}$. The subleading terms can all be expressed as normally ordered monomials in $\omega_{0,n}$ for $n = 0, 1, \dots, 12$. In fact, the relation can be rewritten in the form

$$\lambda\omega_{0,12} = P(\omega_{0,0}, \omega_{0,1}, \dots, \omega_{0,11}),$$

where P is a normally ordered polynomial in $\omega_{0,0}, \omega_{0,1}, \dots, \omega_{0,11}$ and their derivatives, and $\lambda \neq 0$. This relation deforms to a normally ordered relation among the generators $L, U_{0,0}, U_{0,1}, \dots, U_{0,11}$ for $\mathcal{C}^k(4)$, which has the form

$$\lambda(k)U_{0,12} = Q(L, U_{0,0}, U_{0,1}, \dots, U_{0,11}).$$

Here $\lambda(k)$ is a rational function of k satisfying $\lim_{k \rightarrow \infty} \frac{1}{k^2}\lambda(k) = \lambda$. All coefficients appearing in this relation are rational functions of k whose denominators lie in the set S . Unfortunately, it is too difficult to compute the numerator of $\lambda(k)$, although it has finitely many roots. An immediate consequence is

Theorem 6.6. *For all but finitely many real numbers $k > -3$, $\mathcal{C}^k(4)$ is of type $\mathcal{W}(2, 3, \dots, 14)$ and has a minimal strong generating set $\{L, U_{0,n} \mid n = 0, 1, \dots, 11\}$.*

By analogy with the case of \mathfrak{sp}_4 above, we conjecture that the set of distinct roots of the numerator of $\lambda(k)$ lies in S . If so, this implies that $\mathcal{C}^k(4)$ is of type $\mathcal{W}(2, 3, \dots, 14)$ for all real numbers $k > -3$.

7. COSETS OF $\mathcal{W}_k(\mathfrak{g}, e_{-\theta})$ AT NONGENERIC LEVELS

Of much more interest than cosets \mathcal{C}^k of $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ for generic values of k , are cosets \mathcal{C}_k of the simple quotient $\mathcal{W}_k(\mathfrak{g}, e_{-\theta})$ for special values of k when $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ is not simple. In [ACL], one such family was considered, namely,

$$\mathcal{C}_{p/2-3} = \text{Com}(\mathcal{H}, \mathcal{W}_{p/2-3}(\mathfrak{sl}_3, e_{-\theta})).$$

It was shown in [ArIII] that for $p = 5, 7, 9, \dots$, $\mathcal{W}_{p/2-3}(\mathfrak{sl}_3, e_{-\theta})$ is C_2 -cofinite and rational, and the main result of [ACL] is that $\mathcal{C}_{p/2-3}$ is isomorphic to the principal, rational $\mathcal{W}(\mathfrak{sl}_{p-3})$ -algebra with central charge $c = -\frac{3}{p}(p-4)^2$. Recall that

$$\mathcal{C}^{p/2-3} = \text{Com}(\mathcal{H}, \mathcal{W}^{p/2-3}(\mathfrak{sl}_3, e_{-\theta}))$$

is of type $\mathcal{W}(2, 3, 4, 5, 6, 7)$ for all p as above. Since the natural map $\mathcal{C}^{p/2-3} \rightarrow \mathcal{C}_{p/2-3}$ is surjective, this family of $\mathcal{W}(\mathfrak{sl}_{p-3})$ -algebras has the following *uniform truncation property*; it is of type $\mathcal{W}(2, 3, 4, 5, 6, 7)$ for all $p > 9$, even though the universal $\mathcal{W}(\mathfrak{sl}_{p-3})$ -algebra is of type $\mathcal{W}(2, 3, \dots, p-3)$. In fact, $\mathcal{W}_{p/2-3}(\mathfrak{sl}_3, e_{-\theta})$ is a simple current extension of $V_L \otimes \mathcal{W}(\mathfrak{sl}_{p-3})$, where V_L is the lattice vertex algebra with $L = \sqrt{3p-9}\mathbb{Z}$. This is a surprising coincidence, and one of the goals of this paper is to find other coincidences of this kind.

More generally, suppose that k is a value for which $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ is not simple. Let \mathcal{I}_k be the maximal proper ideal of $\mathcal{W}^k(\mathfrak{g}, e_{-\theta})$ graded by conformal weight, so that

$$\mathcal{W}_k(\mathfrak{g}, e_{-\theta}) = \mathcal{W}^k(\mathfrak{g}, e_{-\theta})/\mathcal{I}_k$$

is simple. Let $\mathfrak{g}' \subset \mathfrak{g}_0^{\mathfrak{h}}$ be a simple Lie subalgebra, so that $V^{k'}(\mathfrak{g}') \cong V^\ell(\mathfrak{g}')$ for some ℓ . Let \mathcal{J} denote the kernel of the map $V^\ell(\mathfrak{g}') \rightarrow \mathcal{W}_k(\mathfrak{g}, e_{-\theta})$, and suppose that \mathcal{J} is maximal so that $V^\ell(\mathfrak{g}')/\mathcal{J} \cong L_\ell(\mathfrak{g}')$. Finally, let

$$\mathcal{C}^k = \text{Com}(V^\ell(\mathfrak{g}'), \mathcal{W}^k(\mathfrak{g}, e_{-\theta})), \quad \mathcal{C}_k = \text{Com}(L_\ell(\mathfrak{g}'), \mathcal{W}_k(\mathfrak{g}, e_{-\theta})).$$

By Theorem 8.1 of [CLII], if $\ell + h^\vee$ is a positive real number, where h^\vee is the dual Coxeter number of \mathfrak{g}' , then $\pi_k : \mathcal{C}^k \rightarrow \mathcal{C}_k$ is surjective. A similar result holds if \mathfrak{g}' is any reductive Lie subalgebra such that this condition holds for each simple summand. If π_k is surjective and we know strong generators for \mathcal{C}^k , they will descend to strong generators for \mathcal{C}_k . This shows that determining both the generic behavior of \mathcal{C}^k and the structure of the nongeneric set provides a powerful approach to studying \mathcal{C}_k . Using this method, we describe \mathcal{C}_k in a few interesting examples, and we conjecture some new coincidences of the above kind.

Type C series at positive half-integer levels. Let $\mathfrak{g} = \mathfrak{sp}_{2n}$ for $n \geq 2$, and let k be a half-integer such that $k + 1/2$ is a positive integer. It was shown in [ArIV] that the associated variety of $L_k(\mathfrak{g})$ is the minimal nilpotent orbit closure of \mathfrak{g} , and hence [ArIV], $\mathcal{W}^k(\mathfrak{sp}_{2n}, e_{-\theta})$ is C_2 -cofinite. By [DM2] this implies that the maximal proper graded ideal $\mathcal{I}_k \subset \mathcal{W}^k(\mathfrak{sp}_{2n}, e_{-\theta})$ contains $(J^e)^{k+3/2}$, where $e \in \mathfrak{sp}_{2n-2}$ denotes the highest root. Therefore we have an embedding

$$L_{k+1/2}(\mathfrak{sp}_{2n-2}) \hookrightarrow \mathcal{W}_k(\mathfrak{sp}_{2n}, e_{-\theta}).$$

By Corollary 2.2,

$$\mathcal{C}_k(n) = \text{Com}(L_{k+1/2}(\mathfrak{sp}_{2n-2}), \mathcal{W}_k(\mathfrak{sp}_{2n}, e_{-\theta}))$$

is simple. The vertex algebra $\mathcal{W}_k(\mathfrak{sp}_{2n}, e_{-\theta})$ is conjecturally rational [ArIV], so one expects that $\mathcal{C}_k(n)$ is C_2 -cofinite and rational as well. However, since $\mathcal{W}_k(\mathfrak{sp}_{2n}, e_{-\theta})$ is not generally a simple current extension of $L_{k+1/2}(\mathfrak{sp}_{2n-2}) \otimes \mathcal{C}_k(n)$, this is out of reach at the moment.

We now specialize to the case $n = 2$. Recall that for all real numbers $k > -5/2$, $\mathcal{C}^k(2)$ is of type $\mathcal{W}(2, 4, 6, 8, 10)$ and the map $\pi_k : \mathcal{C}^k(2) \rightarrow \mathcal{C}_k(2)$ is surjective. We use the same notation $\{L, U_{0,1}, U_{0,3}, U_{0,5}, U_{0,7}\}$ for the images of the generators in $\mathcal{C}_k(2)$. Since π_k is surjective, this set strongly generates $\mathcal{C}_k(2)$, but it need not be minimal.

It was shown by Kawasetsu [Ka] that $\mathcal{C}_{1/2}(2)$ is isomorphic to the rational Virasoro algebra $L(-25/7, 0)$. Here we give an alternative proof of this result. It is easy to verify that $U_{0,1} + \frac{132}{5} : LL : - \frac{21}{2} \partial^2 L$ lies in the ideal $\mathcal{I}_{1/2} \subset \mathcal{C}^{1/2}(2)$, so we have the relation

$$U_{0,1} = -\frac{132}{5} : LL : + \frac{21}{2} \partial^2 L$$

in $\mathcal{C}_{1/2}(2)$. By applying $(U_{0,1})_{(1)}$ successively to this relation, and using the fact that

$$(U_{0,1})_{(1)}(U_{0,n}) = \frac{14(5+n)(4+3n)(8+3n)}{5(1+n)(2+n)} U_{0,n+2} + Q_n, \quad n = 1, 3, 5, \dots,$$

we obtain relations

$$U_{0,n} = P_n(L), \quad n = 3, 5, 7$$

in $\mathcal{C}_{1/2}(2)$, where $P_n(L)$ is a normally ordered polynomial in L and its derivatives. Therefore $\mathcal{C}_{1/2}(2)$ is strongly generated by L , and since $\mathcal{C}_{1/2}(2)$ is simple we recover Kawasetsu's result.

Theorem 7.1. $\mathcal{C}_{3/2}(2)$ is isomorphic to the rational $\mathcal{W}(\mathfrak{sp}_4, f_{prin})$ -algebra with central charge $c = -49/6$.

Proof. In weight 6, it can be checked by computer that the element

$$U_{0,3} - \frac{39664}{1701} : LLL : + \frac{117533}{5103} : (\partial^2 L)L : + \frac{1679}{972} : (\partial L)(\partial L) :$$

$$-\frac{3}{2} : LU_{0,1} : + \frac{34801}{20412} \partial^4 L + \frac{37}{4536} \partial^2 U_{0,1}$$

in $\mathcal{C}^{3/2}(2)$ lies in the ideal $\mathcal{I}_{3/2}$. Therefore the corresponding element in $\mathcal{C}_{3/2}(2)$ is a relation expressing $U_{0,3}$ as a normally ordered polynomial $P_3(L, U_{0,1})$ and $L, U_{0,1}$ and their derivatives. By applying $(U_{0,1})_{(1)}$ successively to this relation, and using the fact that

$$(U_{0,1})_{(1)}(U_{0,n}) = \frac{6(5+n)(5+2n)(5+4n)}{(1+n)(2+n)} U_{0,n+2} + Q_n,$$

we obtain relations $U_{0,5} = R_5(L, U_{0,1})$ and $U_{0,7} = R_7(L, U_{0,1})$, where R_5 and R_7 are normally ordered polynomials in $L, U_{0,1}$ and their derivatives. Therefore $\mathcal{C}_{3/2}(2)$ is of type $\mathcal{W}(2, 4)$ with strong generators $\{L, U_{0,1}\}$. Checking that it is actually a $\mathcal{W}(\mathfrak{sp}_4, f_{prin})$ is straightforward by computer using the explicit formulas for $\mathcal{W}(\mathfrak{sp}_4, f_{prin})$ appearing in [Zhi].

Since $\mathcal{C}_{3/2}(2)$ is simple and the simple quotient of $\mathcal{W}(\mathfrak{sp}_4, f_{prin})$ at this central charge is rational ([ArIV], see below), the claim follows. \square

We thus have that $\mathcal{W}_{3/2}(\mathfrak{sp}_4, e_{-\theta})$ is an extension of a rational VOA. The VOA extension problem has a precise categorical formulation and especially if the representation category of a VOA is semi-simple, then this is also true for any extension provided the extended VOA is a simple VOA [HKL, CKM]. In other words:

Corollary 7.2. *$\mathcal{W}_{3/2}(\mathfrak{sp}_4, e_{-\theta})$ is rational.*

The central charge of $\mathcal{C}_k(n)$ is

$$-\frac{(1+2k)(2+2k+n)(2+3k+2n)}{(1+k+n)(1+2k+2n)}.$$

This is the same as the central charge of the principal \mathcal{W} -algebra $\mathcal{W}_s(\mathfrak{sp}_{2m}, f_{prin})$, where

$$m = k + 1/2, \quad s + (k + 3/2) = p/q, \quad (p, q) = (n + k + 1/2, 2n + 2k + 2).$$

Since $q = 2(n + k + 1/2) + 1$ is odd, $p \geq m + 1$, $q \geq 2m$, the level s is a nondegenerate admissible number, so that $\mathcal{W}_s(\mathfrak{sp}_{2m}, f_{prin})$ is rational and C_2 -cofinite [ArIV, ArV].

Based on this observation, we make the following conjecture.

Conjecture 7.3. *For all $n \geq 2$ and k such that $k + 1/2$ is a positive integer, $\mathcal{C}_k(n)$ is isomorphic to the C_2 -cofinite, rational principal \mathcal{W} -algebra $\mathcal{W}_s(\mathfrak{sp}_{2m}, f_{prin})$ as above.*

Note that this conjecture would imply the rationality of $\mathcal{W}_{k+1/2}(\mathfrak{sp}_{2n}, e_{-\theta})$ for all $n \geq 2$ and all k such that $k + 1/2$ is a positive integer.

Type A series at positive integer levels. Let $\mathfrak{g} = \mathfrak{sl}_n$ for $n \geq 4$, and let k be a non-negative integer. One sees as in the proof of Theorem 7.2 of [AM1] that the maximal proper graded ideal $\mathcal{I}_k \subset \mathcal{W}^k(\mathfrak{sl}_n, e_{-\theta})$ contains $:(J^e)^{k+2}:$, where $e \in \mathfrak{sl}_{n-2}$ denotes the highest root. Therefore we have an embedding

$$L_{k+1}(\mathfrak{gl}_{n-2}) = \mathcal{H} \otimes L_{k+1}(\mathfrak{sl}_{n-2}) \hookrightarrow \mathcal{W}_k(\mathfrak{sl}_n, e_{-\theta}),$$

and by Corollary 2.2,

$$\mathcal{C}_k(n) = \text{Com}(L_{k+1}(\mathfrak{gl}_{n-2}), \mathcal{W}_k(\mathfrak{sl}_n, e_{-\theta}))$$

is simple. Since the projection $\pi_k : \mathcal{C}^k(n) \rightarrow \mathcal{C}_k(n)$ is surjective, $\mathcal{C}_k(n)$ is the simple quotient of $\mathcal{C}^k(n)$.

The \mathcal{W} -algebra $\mathcal{W}_k(\mathfrak{sl}_n, e_{-\theta})$ should be “small” in the following sense.

Conjecture 7.4. *For all integers $n \geq 4$ and $k \geq -1$, the associated variety of $\mathcal{W}_k(\mathfrak{sl}_n, e_{-\theta})$ is isomorphic to \mathbb{A}^1 . In particular, it is one-dimensional.*

Here the associated variety X_V of a vertex algebra V is defined as $X_V = \text{Specm}(R_V)$, where R_V is the Zhu’s C_2 -algebra of V [ArII]. Conjecture 7.4 holds for $k = -1$ by Theorem 7.4 of [AM2].

We specialize to the case $n = 4$. Recall that for all real numbers $k > -3$, $\mathcal{C}_k(4)$ is strongly generated by the fields $\{L, U_{0,n} \mid n \geq 0\}$ and $\pi_k : \mathcal{C}^k(4) \rightarrow \mathcal{C}_k(4)$ is surjective.

Theorem 7.5. *$\mathcal{C}_0(4)$ is isomorphic to the simple Zamolodchikov \mathcal{W}_3 -algebra with $c = -2$ [Zam], which we denote by $\mathcal{W}_{3,-2}$.*

Proof. It is not difficult to verify by computer that the element

$$U_{0,1} + \frac{16}{5} : LL : -\frac{3}{5} \partial^2 L$$

in $\mathcal{C}^0(4)$ lies in the ideal \mathcal{I}_0 , and hence gives rise to the relation $U_{0,1} = -\frac{16}{5} : LL : +\frac{3}{5} \partial^2 L$ in $\mathcal{C}_0(4)$. By applying $(U_{0,0})_{(1)}$ successively to this relation, and using the fact that

$$(U_{0,0})_{(1)}(U_{0,n}) = -\frac{4(4+n)(5+3n)}{1+n} U_{0,n+1} + Q_n,$$

we obtain relations

$$U_{0,n} = P_n(L, U_{0,0}), \quad n \geq 0.$$

It follows that $\mathcal{C}_0(4)$ is strongly generated by $\{L, U_{0,0}\}$. Finally, it is not difficult to check by computer that they satisfy the OPE relations of $\mathcal{W}_{3,-2}$. \square

Remark 7.6. *$\mathcal{C}_0(4)$ has a one-dimensional associated variety, and admits a well-known extension called a triplet algebra which is C_2 -cofinite but non-rational.*

Theorem 7.7. *$\mathcal{C}_1(4)$ is isomorphic to the simple parafermion algebra*

$$K_{-6/5}(\mathfrak{sl}_2) = \text{Com}(\mathcal{H}, L_{-6/5}(\mathfrak{sl}_2)),$$

which has central charge $c = -11/2$.

Proof. It can be checked by computer that there is a relation in $\mathcal{C}_1(4)$ at weight 6 of the form

$$U_{0,3} = P_3(L, U_{0,0}, U_{0,1}, U_{0,2}),$$

where P_3 is a normally ordered polynomial in $L, U_{0,0}, U_{0,1}, U_{0,2}$, and their derivatives. Applying $(U_{0,0})_{(1)}$ successively to this relation, we obtain relations

$$U_{0,n} = P_n(L, U_{0,0}, U_{0,1}, U_{0,2}), \quad n \geq 0,$$

so $\mathcal{C}_1(4)$ is of type $\mathcal{W}(2, 3, 4, 5)$ with strong generators $\{L, U_{0,0}, U_{0,1}, U_{0,2}\}$. It is then a straightforward but lengthy computer calculation to verify that these generators satisfy the OPE relations of the above parafermion algebra. \square

Remark 7.8. *One might guess that $K_{-6/5}(\mathfrak{sl}_2)$ is isomorphic to the algebra $\mathcal{W}(\mathfrak{sl}_5, f_{\text{prin}})$ with $c = -11/2$. However, this turns out to be false, which can be shown using the explicit OPE relations for $\mathcal{W}(\mathfrak{sl}_5, f_{\text{prin}})$ given in [ZhI].*

For $n \geq 3$, let $\mathcal{W}^\ell(\mathfrak{sl}_n, f_{\text{subreg}})$ be the \mathcal{W} -algebra associated with a subregular nilpotent element f_{subreg} at level ℓ [KRW]. It is freely generated of type $\mathcal{W}(1, 2, \dots, n-1, n/2, n/2)$ and the weight one field generates a Heisenberg algebra \mathcal{H} . The central charge is

$$c = -\frac{((\ell+n)(n-1)-n)((\ell+n)(n-2)n-n^2+1)}{\ell+n}.$$

It was recently shown by Genra [G] that $\mathcal{W}^\ell(\mathfrak{sl}_n, f_{\text{subreg}})$ is isomorphic to the vertex algebra $\mathcal{W}_{n,\ell}^{(2)}$ introduced by Feigin and Semikhatov [FS]. Note that for $n=1$ and $n=2$, $\mathcal{W}_{n,\ell}^{(2)}$ is well-defined and coincides with the rank one $\beta\gamma$ -system and the affine vertex algebra $V^\ell(\mathfrak{sl}_2)$, respectively. We denote by $\mathcal{W}_\ell(\mathfrak{sl}_n, f_{\text{subreg}})$ the unique simple quotient of $\mathcal{W}^\ell(\mathfrak{sl}_n, f_{\text{subreg}})$.

Observe that $\mathcal{C}_k(n)$ has central charge

$$c = -\frac{(1+k)(-1+2k+n)(3k+2n)}{(-1+k+n)(k+n)},$$

which coincides with the central charge of $\text{Com}(\mathcal{H}, \mathcal{W}_\ell(\mathfrak{sl}_{k+1}, f_{\text{subreg}}))$ for $\ell = -\frac{1+k^2+kn}{k+n}$. In the cases $k=0$ and $k=1$, this coincidence still holds where $\mathcal{W}_\ell(\mathfrak{sl}_{k+1}, f_{\text{subreg}})$ is now replaced by the $\beta\gamma$ -system and $L_\ell(\mathfrak{sl}_2)$, respectively. Based on this observation, and the belief that vertex algebras with this central charge and one-dimensional associated variety are rare, we make the following conjecture.

Conjecture 7.9. *For all integers $n \geq 4$ and $k \geq 0$, such that $n^2 - k^2 - 1 \geq n$,*

$$\mathcal{C}_k(n) \cong \text{Com}(\mathcal{H}, \mathcal{W}_\ell(\mathfrak{sl}_{k+1}, f_{\text{subreg}})), \quad \ell = -\frac{1+k^2+kn}{k+n}.$$

For $k=0$ and $k=1$, we replace $\mathcal{W}_\ell(\mathfrak{sl}_{k+1}, f_{\text{subreg}})$ above with the $\beta\gamma$ -system and the simple affine vertex algebra $L_\ell(\mathfrak{sl}_2)$, respectively.

Remark 7.10. *Conjecture 7.9 fails if k is large relative to n . Suppose that $n^2 - k^2 - 1 \geq n$. As $n^2 - k^2 - 1$ and $k+n$ are mutually prime, ℓ is a nondegenerate admissible number, that is, the associated variety of $L_1(\mathfrak{sl}_n)$ is the nilpotent cone of \mathfrak{sl}_n [ArIV]. Therefore, the associated variety of the simple Feigin-Semikhatov algebra at level ℓ , or the subregular \mathcal{W} -algebra of \mathfrak{sl}_n at level ℓ , is the intersection of the nilpotent cone with the Slodowy slice at a subregular nilpotent element, which is isomorphic to the A_n -singularity [ArIV]. In particular, it has the symplectic singularity and has dimension 2.*

Since $\mathcal{W}_{1,\ell}^{(2)}$ is isomorphic to the rank one $\beta\gamma$ -system \mathcal{S} for all ℓ , and $\text{Com}(\mathcal{H}, \mathcal{S}) \cong \mathcal{W}_{3,-2}$, this conjecture holds for $n=4$ and $k=0$ by Theorem 7.5. Similarly, since the simple Feigin-Semikhatov algebra $\mathcal{W}_{2,\ell}^{(2)} \cong L_\ell(\mathfrak{sl}_2)$, our conjecture holds for $n=4$ and $k=1$ by Theorem 7.7. In the case, $n=4$ and $k=2$, $\text{Com}(\mathcal{H}, \mathcal{W}_\ell(\mathfrak{sl}_3, f_{\text{subreg}}))$ is of type $\mathcal{W}(2, 3, 4, 5, 6, 7)$ [ACL], and many (but not all) OPE relations have been checked by computer to coincide with the OPE relations in $\mathcal{C}_2(4)$.

We now prove a more general statement using Theorem 3.1 that implies our conjecture for $k=0$ and all $n \geq 4$. Let $\mathcal{E}(n)$ be the rank n bc -system with generators b^i, c^i , and let $\mathcal{S}(m)$ be the rank m $\beta\gamma$ -system with generators β^i, γ^i . Let $\mathcal{A}(1)$ be the rank one symplectic fermion algebra with generators X^\pm . Define a $U(1)$ action on $\mathcal{E}(n) \otimes \mathcal{S}(m) \otimes \mathcal{A}(1)$ by

$$X^\pm \mapsto \lambda^{\pm 1} X^\pm, \quad b^i \mapsto \lambda b^i, \quad c^i \mapsto \lambda^{-1} c^i, \quad \beta^j \mapsto \lambda \beta^j, \quad \gamma^j \mapsto \lambda^{-1} \gamma^j$$

Here the $1 + n + m$ vectors $\{X^+, b^i, \beta^j\}$ and $\{X^-, c^i, \gamma^j\}$ should be viewed as carrying the standard and respectively conjugate representation of $\mathfrak{gl}(n+1|m)$. With this notation, we have the following result.

Theorem 7.11. *For all $(n, m) \neq (0, 2)$, $\mathcal{W}_0(\mathfrak{sl}(n+2|m), e_{-\theta}) \cong (\mathcal{E}(n) \otimes \mathcal{S}(m) \otimes \mathcal{A}(1))^{U(1)}$.*

Proof. In [CKLR] it is proven that $(\mathcal{E}(n) \otimes \mathcal{S}(m))^{U(1)} \cong L_1(\mathfrak{gl}(m|n))$ (see also [AP, KWII]). Furthermore, $\mathcal{A}(1)^{U(1)}$ is well known to be Zamolodchikov's \mathcal{W}_3 algebra $\mathcal{W}_{3,-2}$. It is clear that the generators of $L_1(\mathfrak{gl}(m|n))$ and $\mathcal{W}_{3,-2}$, together with the fields $:c^i X^+:$ and $:b^i X^-:$, form a strong generating set for $(\mathcal{E}(n) \otimes \mathcal{S}(m))^{U(1)}$. Also, it is easy to see that the dimension 3 field of $\mathcal{W}_{3,-2}$ (one can take $:X^+ \partial X^-:$ for it) is a normally ordered polynomial in the other strong generators and their derivatives. So all conditions for Theorem 3.1 are satisfied. Finally, by [DLM] the orbifold is simple. \square

Remark 7.12. *If $(n, m) = (0, 2)$ then the orbifold $\mathcal{S}(2)^{U(1)}$ is larger than just $L_{-1}(\mathfrak{gl}_2)$. Since $\mathfrak{gl}(2|2)$ is not reductive, it is better to consider the simple quotient $\mathfrak{psl}(2|2)$. In that case, the minimal \mathcal{W} -algebra is the small $N = 4$ super Virasoro algebra and constructions of it are given in [CKLR, Ad].*

Corollary 7.13. *Let*

$$\mathcal{D}_0(n, m) = \text{Com}(L_1(\mathfrak{gl}(n|m)), \mathcal{W}_0(\mathfrak{sl}(n+2|m), e_{-\theta})).$$

Then for all $(n, m) \neq (0, 2)$,

$$\mathcal{D}_0(n, m) \cong \mathcal{A}(1)^{U(1)} \cong \mathcal{W}_{3,-2}.$$

In particular, $\mathcal{C}_0(n) = \mathcal{D}_0(n-2, 0) \cong \mathcal{W}_{3,-2}$ for all $n \geq 4$, so Conjecture 7.9 holds for all $n \geq 4$ and $k = 0$.

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