

DECIDABILITY AND INDEPENDENCE OF CONJUGACY PROBLEMS IN FINITELY PRESENTED MONOIDS

JOÃO ARAÚJO, MICHAEL KINYON, JANUSZ KONIECZNY, AND ANTÓNIO MALHEIRO

ABSTRACT. There have been several attempts to extend the notion of conjugacy from groups to monoids. The aim of this paper is study the decidability and independence of conjugacy problems for three of these notions (which we will denote by \sim_p , \sim_o , and \sim_c) in certain classes of finitely presented monoids. We will show that in the class of polycyclic monoids, p -conjugacy is “almost” transitive, \sim_c is strictly included in \sim_p , and the p - and c -conjugacy problems are decidable with linear complexity. For other classes of monoids, the situation is more complicated. We show that there exists a monoid M defined by a finite complete presentation such that the c -conjugacy problem for M is undecidable, and that for finitely presented monoids, the c -conjugacy problem and the word problem are independent, as are the c -conjugacy and p -conjugacy problems.

2010 Mathematics Subject Classification: 68Q42, 20F10, 3D35, 3D15.

Keywords and phrases: Conjugacy; finitely presented monoids; polycyclic monoids; decision problem; decidability; independence; complexity.

1. INTRODUCTION

The well-known notion of conjugacy from group theory can be extended to monoids in many different ways. The authors dealt with four notions of conjugacy in monoids in [1, 2]. The present paper can be considered an extension of this work. Any generalization of the conjugacy relation to general monoids must avoid inverses. One of the possible formulations, spread by Lallement [20] for a free monoid M , was the following relation:

$$(1.1) \quad a \sim_p b \Leftrightarrow \exists_{u,v \in M} a = uv \text{ and } b = vu.$$

(Lallement credited the idea of the relation \sim_p to Lyndon and Schützenberger [22].) If M is a free monoid, then \sim_p is an equivalence relation on M [20, Corollary 5.2], and so it can be regarded as a conjugacy in M . In a general monoid M , the relation \sim_p is reflexive and symmetric, but not transitive. The transitive closure \sim_p^* of \sim_p has been defined as a conjugacy relation in a general semigroup [13, 18, 19]. (If $a \sim_p b$ in a general monoid, we say that a and b are *primarily conjugate* [19], hence our subscript in \sim_p).

Another relation that can serve as a conjugacy in any monoid is defined as follows:

$$(1.2) \quad a \sim_o b \Leftrightarrow \exists_{g,h \in M} ag = gb \text{ and } bh = ha.$$

This relation was defined by Otto for monoids presented by finite Thue systems [30], but it is an equivalence relation in any monoid. Its drawback – as a candidate for a conjugacy for general monoids – is that it reduces to the universal relation $M \times M$ for any monoid M that has a zero.

To remedy the latter problem, three authors of the present paper introduced a new notion of conjugacy [2], which retains Otto’s concept for monoids without zero, but does not reduce to $M \times M$ if M has a zero. The main idea was to restrict the set from which conjugators can be chosen. For a monoid M with zero and $a \in M \setminus \{0\}$, let $\mathbb{P}(a)$ be the set $\{g \in M : (\forall m \in M) mag = 0 \Rightarrow ma = 0\}$, and define $\mathbb{P}(0)$ to be $\{0\}$. If M has no zero, we agree that $\mathbb{P}(a) = M$, for every $a \in M$. Following [2], we define a relation \sim_c on any monoid M by

$$(1.3) \quad a \sim_c b \Leftrightarrow \exists_{g \in \mathbb{P}(a)} \exists_{h \in \mathbb{P}(b)} ag = gb \text{ and } bh = ha.$$

The relation \sim_c is an equivalence relation on an arbitrary monoid M . Moreover, if M is a monoid without zero, then $\sim_c = \sim_o$; and if M is a free monoid, then $\sim_c = \sim_o = \sim_p$. In the case when M has

a zero, the conjugacy class of 0 with respect to \sim_c is $\{0\}$. Throughout the paper we shall refer to \sim_i , where $i \in \{p, o, c\}$, as i -conjugacy.

The aim of this paper is to study the decidability and independence of the i -conjugacy problems in some classes of finitely presented monoids.

It is well-known that the conjugacy problem for finitely presented groups is undecidable; that is, there exists a finitely presented group for which the conjugacy problem is undecidable [27]. The relations \sim_p , \sim_o , and \sim_c reduce to group conjugacy when a monoid is a group. It follows that the i -conjugacy problem, for $i \in \{p, o, c\}$, is also undecidable. However, it is of interest to study decidability of the i -conjugacy problems in particular classes of finitely presented monoids.

First, we consider the class of polycyclic monoids, which are finitely presented monoids with zero. The polycyclic monoids P_n , with $n \geq 2$, were first introduced by Nivat and Perrot [26], and later rediscovered by Cuntz in the context of the theory of C^* -algebras [11, Section 1]. (Within the theory of C^* -algebras, the polycyclic monoids are often referred to as Cuntz inverse semigroups.) The polycyclic monoids appear to be related to the idea of self-similarity [14]. For example, the polycyclic monoid P_2 can be represented by partial injective maps on the Cantor set: its two generators, p_1 and p_2 , map, respectively, the left and right hand sides of the Cantor set, to the whole Cantor set. These monoids can also be characterized as the syntactic monoid of the restricted Dyck language on a set of cardinality n , that is, the language that consists of all correct bracket sequences of n types of brackets. The study of representations of the polycyclic monoids naturally connects with the study of its conjugacy relations [17, 21]. In [21], the classification of the ‘proper closed inverse submonoids’ of P_n depends on the study of its conjugacy classes.

In Section 3, we characterize p -conjugacy and c -conjugacy in the polycyclic monoids, and conclude that $\sim_c \subset \sim_p$. (For sets A and B , we write $A \subset B$ if A is a proper subset of B .) We then show that the p -conjugacy and c -conjugacy problems are decidable for polycyclic monoids, and that, given words a and b , testing whether or not $a \sim_i b$, for $i \in \{p, c\}$, can be done linearly on the lengths of a and b . Note that in a polycyclic monoid P_n , the relation \sim_o is universal since P_n has a zero.

These positive results obtained for polycyclic monoids concerning the decidability and complexity of the conjugacy problems cannot be extended to the general finitely presented monoids.

In Section 4, we study decidability results. In particular, we show that there exists a monoid M defined by a finite complete presentation such that the c -conjugacy problem for M is undecidable (Proposition 4.2).

In Section 5, we study independence results. The word problem for groups is undecidable [23, 28, 31]. However, for groups, the word problem is reducible to the conjugacy problem [30, page 225], hence if the conjugacy problem for a group G is decidable, then the word problem for G is also decidable. Therefore, the word problem and the conjugacy problem for groups are not independent. The situation for monoids is different. Osipova [29] has proved that for finitely presented monoids, the word problem, the p -conjugacy problem, and the o -conjugacy problem are pairwise independent. We show that for finitely presented monoids, the word problem and the c -conjugacy problem are independent (Theorem 5.2), and that the p -conjugacy problem and the c -conjugacy problem are also independent (Theorem 5.3). We do not know if the c -conjugacy problem and the o -conjugacy problem are independent.

We conclude the paper with Section 6 that lists open problems regarding the conjugacies under discussion.

2. BACKGROUND

In this section we will formulate the main concepts needed in the following sections. For further background on the free monoid, see [15]; for presentations, see [12, 32]; and, for rewriting systems, see [7].

Alphabets and words. Let Σ be a non-empty set, called an *alphabet*. We denote by Σ^* the set of finite strings (called *words*) of elements of Σ , including the *empty word* 1. For $w \in \Sigma^*$ and $a \in \Sigma$, we denote by $|w|$ the *length* of the word w and by $|w|_a$ the number of occurrences of a in w . For example, if $\Sigma = \{a, b, c\}$ and $w = aabba \in \Sigma^*$, then $|w| = 5$, $|w|_a = 3$, and $|w|_c = 0$.

A non-empty word z is said to be a *factor* of $w \in \Sigma^*$ if $w = uzv$, for some words $u, v \in \Sigma^*$. If w, u , and v are words with $w = uv$, then u is called a *prefix* of w and v a *suffix* of w ; the word u is said to be a *proper prefix* of w if v is non empty (the notion of proper suffix is dual). Two words u and v are said to be *prefix-comparable* if either u is a prefix of v or v is a prefix of u .

Rewriting systems. Any subset R of $\Sigma^* \times \Sigma^*$ is called a *rewriting system* (or a *Thue system*) on Σ . An element (x, y) of R , also commonly denoted $x = y$, is called a *rewriting rule*. If $(x, y) \in R$ and $u, v \in \Sigma^*$, we say that uxv *reduces* to uyv and we write $uxv \rightarrow uyv$. A word w is said to be *irreducible* if there is no $w' \in \Sigma^*$, such that $w \rightarrow w'$. We denote by $\overset{*}{\rightarrow}$ the reflexive and transitive closure of \rightarrow .

A rewriting system R on Σ is *special* if every element of R is of the form $(x, 1)$ with $x \neq 1$; it is *monadic* if every element of R is of the form (x, y) with $y \in \Sigma \cup \{1\}$ and $|x| > |y|$; it is *length reducing* if $|x| > |y|$ for all $(x, y) \in R$; it is *noetherian* if there is no infinite sequence w_1, w_2, w_3, \dots of words in Σ^* such that $w_1 \rightarrow w_2 \rightarrow w_3 \rightarrow \dots$; it is *confluent* if for all $u, v, w \in \Sigma^*$, if $u \overset{*}{\rightarrow} v$ and $u \overset{*}{\rightarrow} w$, then there exists $z \in \Sigma^*$ such that $v \overset{*}{\rightarrow} z$ and $w \overset{*}{\rightarrow} z$; and R is *complete* if it is both noetherian and confluent. Note that if R is special or monadic, then it is length reducing, and if R is length reducing, then it is noetherian.

Monoid presentations. Every rewriting system R on Σ defines a monoid. The set Σ^* with concatenation of words as multiplication is a monoid, called the *free monoid* on Σ . Denote by ρ_R the smallest congruence on Σ^* containing R (called the *Thue congruence*). We denote by $M(\Sigma; R)$ the quotient monoid Σ^*/ρ_R . The elements of $M(\Sigma; R)$ are the congruence classes $[u]_M = \{w : w \rho_R u\}$, where $u \in \Sigma^*$. Whenever possible and when it is clear from the context, we shall omit the brackets to denote congruence classes, and thus identify words with the elements of the monoid that they represent.

Suppose M is any monoid such that $M \cong M(\Sigma; R)$ (that is, M is isomorphic to $M(\Sigma; R)$). Then the pair $(\Sigma; R)$ is a *presentation* of M with *generators* Σ and *defining relations* R , and we say that M is defined by $(\Sigma; R)$ or simply by R . A presentation $(\Sigma; R)$ is said to be *finite* if both Σ and R are finite. A monoid M defined by a finite presentation is called *finitely presented*.

Definition 2.1. Let $M = M(\Sigma; R)$ be a finitely presented monoid, and let \sim_i be one of the conjugacy relations under consideration ($i \in \{p, o, c\}$). We say that the i -conjugacy problem for M is *decidable* if there is an algorithm that given any pair (u, v) of words in Σ^* , returns YES if $[u]_M \sim_i [v]_M$ and NO otherwise. If such an algorithm does not exist, we say that the i -conjugacy problem for M is *undecidable*. We have an analogous definition of the decidability of the word problem for M , in which case we check if $[u]_M = [v]_M$.

Monoids with zero: rewriting systems and presentations. Consider a rewriting system R defined on a set $\Sigma_0 = \Sigma \cup \{0\}$, where 0 is a symbol not in Σ , and a set R_0 of rewriting rules of the form $(x0, 0)$, $(0x, 0)$ and $(00, 0)$, for any $x \in \Sigma$. The monoid $T = M(\Sigma_0; R \cup R_0)$ is a monoid with zero $[0]_T$. For simplicity, we refer to the pair $(\Sigma_0; R)$ as a *monoid-with-zero presentation* of T . Notice that the monoid presentation $(\Sigma_0; R \cup R_0)$ is finite or monadic when $(\Sigma_0; R)$ is finite or monadic, respectively.

If a monoid M is defined by a presentation $(\Sigma; R)$ then the monoid M^0 , obtained from M by adding a zero element, is defined by the monoid-with-zero presentation $(\Sigma; R)$. Observe that $[0]_{M^0}$ is the zero in M^0 and that $M^0 \neq \{[0]_{M^0}\}$. Regarding these presentations, we can deduce by [3, Proposition 3.1] that if the rewriting system R on Σ is complete, then so is the new rewriting system $R \cup R_0$ on Σ_0 .

Throughout the text we refer to a presentation as noetherian, confluent, complete, monadic, etc., whenever the associated rewriting system has the respective property.

3. CONJUGACY IN THE POLYCYCLIC MONOIDS

In this section, we study p -conjugacy and c -conjugacy in the class of polycyclic monoids, an important class of inverse monoids. A monoid M is called an *inverse monoid* if for every $a \in M$, there exists a unique $a^{-1} \in M$ (an *inverse* of a) such that $aa^{-1}a = a$ and $a^{-1}aa^{-1} = a^{-1}$ [15, p. 145].

In general, p -conjugacy is not transitive in inverse semigroups. For instance, by [9, Proposition 4.2], p -conjugacy is not transitive in free inverse monoids. We will show that in the polycyclic monoids, p -conjugacy is transitive for the elements not \sim_p -related to zero, and that $\sim_c \subset \sim_p$.

We note that in the polycyclic monoids, \sim_o is the universal relation since every polycyclic monoid has a zero.

3.1. General properties of the polycyclic monoids. Let $n \geq 2$. Consider a set $A_n = \{p_1, \dots, p_n\}$ and denote by A_n^{-1} a disjoint copy $\{p_1^{-1}, \dots, p_n^{-1}\}$. Let $\Sigma = A_n \cup A_n^{-1}$. Given any $x \in \Sigma$, we define x^{-1} to be p_i^{-1} if $x = p_i \in A_n$, and to be p_i if $x = p_i^{-1} \in A_n^{-1}$. This notation can be extended to Σ^* by setting $(xw)^{-1} = w^{-1}x^{-1}$, for every $x \in \Sigma$ and $w \in \Sigma^*$, and $1^{-1} = 1$.

Denote by R the set of rewriting rules on $\Sigma_0 = \Sigma \cup \{0\}$ of the form $p_i^{-1}p_i = 1$, for $i \in \{1, \dots, n\}$, and of the form $p_i^{-1}p_j = 0$, for $i, j \in \{1, \dots, n\}$ and $i \neq j$. Consider the monoid P_n defined by the monoid-with-zero presentation $(\Sigma_0; R)$. The monoid P_n is called the *polycyclic monoid* on n generators. Notice that the given presentation of P_n is monadic, and thus length reducing.

An irreducible element (with respect to R) cannot have a factor of the form $p_i^{-1}p_j$, for any $i, j \in \{1, \dots, n\}$. Thus, irreducible elements have the form yx^{-1} , where $y, x \in A_n^*$, or the form 0. It is well known (e.g., [21, subsection 9.3]) that every nonzero element w of P_n has a *unique* irreducible representation \bar{w} of the form yx^{-1} with $y, x \in A_n^*$. Therefore, irreducible elements are in one-to-one correspondence with elements of the polycyclic monoid. We deduce the following:

Lemma 3.1. *The monoid-with-zero presentation $(\Sigma_0; R)$ of the polycyclic monoid P_n is finite and complete.*

Whenever we write $a = yx^{-1}$, it will be understood that $x, y \in A_n^*$. Hereafter, we shall identify irreducible elements with the elements of the polycyclic monoid that they represent.

We will frequently use the following lemma, which follows from the unique representation of the nonzero elements of P_n .

Lemma 3.2. *Consider nonzero elements yx^{-1} and vu^{-1} of P_n . Then:*

- (1) $yx^{-1} \cdot vu^{-1} \neq 0$ iff x and v are prefix-comparable;
- (2) if $yx^{-1} \cdot vu^{-1} \neq 0$, then

$$yx^{-1} \cdot vu^{-1} = \begin{cases} yzu^{-1} & \text{if } v = xz, \\ y(uz)^{-1} & \text{if } x = vz. \end{cases}$$

- (3) $y = v$ in P_n iff $y = v$ in A_n^* , and $x^{-1} = u^{-1}$ in P_n iff $x = u$ in A_n^* .

An irreducible word is said to be *cyclically reduced* if it is empty or zero, or if its first letter c and its last letter d satisfy $c \neq d^{-1}$. Every nonzero irreducible word can be written in the form $ryx^{-1}r^{-1}$, with $r \in A_n^*$ and yx^{-1} a cyclically reduced word. From any irreducible word a we compute a cyclically reduced word \tilde{a} in the following way: if a is cyclically reduced, we let \tilde{a} be equal to a ; otherwise, $a = ryx^{-1}r^{-1}$ as above, so we let \tilde{a} be the (possibly empty) cyclically reduced word yx^{-1} . We obtain the following fact for any nonzero irreducible word $a \in P_n$:

$$(3.4) \quad a = r\tilde{a}r^{-1} \text{ for some word } r \in A_n^*.$$

For each nonzero element $a = yx^{-1} \in P_n$, denote by $\rho(a)$ the irreducible word obtained from $x^{-1}y$. We also set $\rho(0) = 0$. Let $a = yx^{-1} \in P_n$. We record the following facts about \tilde{a} and $\rho(a)$:

- (a) $\rho(a)$ is $x^{-1}y$ reduced in P_n ;
- (b) \tilde{a} is $x^{-1}y$ reduced in (Σ, R_1) , where $R_1 = \{(p_i^{-1}p_i, 1) : i \in \{1, \dots, n\}\}$;
- (c) $\rho(a)$ is cyclically reduced;
- (d) $\rho(a) = \tilde{a} \in A_n^*$ if x is a prefix of y ; $\rho(a) = \tilde{a} \in (A_n^{-1})^*$ if y is a prefix of x ; and $\rho(a) = \rho(\tilde{a}) = 0$ otherwise.

For example, if $a = p_1p_2p_3^{-1}p_1^{-1}$, then $\tilde{a} = p_2p_3^{-1}$ and $\rho(a) = 0$.

The following lemma can be easily deduced.

Lemma 3.3. *For all $a = pq^{-1} \cdot rs^{-1} \in P_n$, the cyclically reduced word \tilde{a} is given by*

$$\begin{cases} lt & \text{if } r = qt \text{ and } p = sl, \\ \widetilde{tl^{-1}} & \text{if } r = qt \text{ and } s = pl, \\ \widetilde{lt^{-1}} & \text{if } q = rt \text{ and } p = sl, \\ (lt)^{-1} & \text{if } q = rt \text{ and } s = pl. \end{cases}$$

3.2. *p*-conjugacy in P_n . We first observe that for every $a \in P_n$, $a \sim_p \tilde{a}$ and $a \sim_p \rho(a)$ (by the definitions of \tilde{a} and $\rho(a)$).

Lemma 3.4. *Let $a \in P_n$. Then $a \sim_p 0$ if and only if $\rho(a) = 0$.*

Proof. Suppose that $a \sim_p 0$. If $a = 0$ then $\rho(a) = 0$. Suppose $a \neq 0$. Then $0 \neq a = pq^{-1} \cdot rs^{-1}$ and $0 = rs^{-1} \cdot pq^{-1}$, for some $p, q, r, s \in A_n^*$. The latter equality implies that p and s are not prefix-comparable (by Lemma 3.2). And the former implies that $r = qt$ or $q = rt$, for some $t \in A_n^*$. Hence, $a = pts^{-1}$ (if $r = qt$) or $a = p(st)^{-1}$ (if $q = rt$). Suppose that $a = pts^{-1}$. If pt is a prefix of s , then also p is a prefix of s ; and if s is a prefix of pt , then either s is a prefix of p or p is a prefix of s . It follows that neither pt is a prefix of s nor s is a prefix of pt , which implies $\rho(a) = 0$. By a similar argument, we obtain $\rho(a) = 0$ if $a = p(st)^{-1}$. The converse follows from the fact that $a \sim_p \rho(a)$. \square

Lemma 3.5. *Let a and b be nonzero elements of P_n with $\rho(a) = \rho(b) = 0$. Then $a \sim_p b$ if and only if $\tilde{a} = \tilde{b}$.*

Proof. Suppose that $a \sim_p b$. Then there exist elements $x, y, u, v \in A_n^*$ such that $a = yx^{-1} \cdot vu^{-1}$ and $b = vu^{-1} \cdot yx^{-1}$. Since $a \neq 0$, we know that x and v are prefix-comparable. Similarly, since $b \neq 0$, u and y are prefix-comparable.

Suppose first that x is a prefix of v and u is a prefix of y , that is, $v = xp$ and $y = uq$ for some $p, q \in A_n^*$. Hence $a = uqx^{-1}xpu^{-1} = uqpu^{-1}$, and so $\rho(a) = qp \neq 0$, which is a contradiction. Similarly, we obtain a contradiction if we assume that v is a prefix of x and y is a prefix of u .

Suppose that x is a prefix of v and that y is a prefix of u , that is, $v = xp$ and $u = yq$ for some $p, q \in A_n^*$. Then $a = yx^{-1}xpq^{-1}y^{-1} = y(pq^{-1})y^{-1}$ and $b = xpq^{-1}y^{-1}yx^{-1} = x(pq^{-1})x^{-1}$. Thus $\tilde{a} = \widetilde{y(pq^{-1})y^{-1}} = \tilde{b}$ as required. In a similar way we obtain the intended result if v is a prefix of x and u is a prefix of y .

The converse follows easily by first noticing that $a = r\tilde{a}r^{-1}$ and $b = s\tilde{b}s^{-1}$, for some $r, s \in A_n^*$ by (3.4). If $\tilde{a} = \tilde{b}$ we get the required result since $a = r\tilde{a}s^{-1} \cdot sr^{-1}$ and $b = sr^{-1} \cdot r\tilde{b}s^{-1}$. \square

The following theorem characterizes *p*-conjugacy in P_n .

Theorem 3.6. *Let $a, b \in P_n$. Then $a \sim_p b$ if and only if one of the following conditions is satisfied:*

- (a) $a = \rho(b) = 0$ or $\rho(a) = b = 0$;
- (b) $\rho(a) = \rho(b) = 0$ and $\tilde{a} = \tilde{b}$;
- (c) $\tilde{a}, \tilde{b} \in A_n^*$ and $\tilde{a} \sim_p \tilde{b}$ in the free monoid A_n^* ; or
- (d) $\tilde{a}, \tilde{b} \in (A_n^{-1})^*$ and $\tilde{a} \sim_p \tilde{b}$ in the free monoid $(A_n^{-1})^*$.

Proof. Suppose that $a \sim_p b$. If $a = 0$ or $b = 0$, then (a) holds by Lemma 3.4. Now assume that a and b are nonzero elements. Then, there are $p, q, r, s \in A_n^*$ such that $a = pq^{-1} \cdot rs^{-1}$ and $b = rs^{-1} \cdot pq^{-1}$. By Lemma 3.3, we have: if $r = qt$ and $p = sl$, then $\tilde{a} = lt$ and $\tilde{b} = tl$, so (c) holds; if $q = rt$ and $s = pl$, then $\tilde{a} = (lt)^{-1}$ and $\tilde{b} = (tl)^{-1}$, so (d) holds; if $r = qt$ and $s = pl$, then $\tilde{a} = \tilde{b} = tl^{-1}$; and if $q = rt$ and $p = sl$, then $\tilde{a} = \tilde{b} = lt^{-1}$. In the last two cases, we have $\tilde{a} = \tilde{b}$, and so $\rho(a) = \rho(b)$. Thus, in those cases, either (c) or (d) holds (if $\rho(a) = \rho(b) \neq 0$), or (b) holds by Lemma 3.5 (if $\rho(a) = \rho(b) = 0$).

Conversely, if (a) holds then $a \sim_p b$ by Lemma 3.4; and if (b) holds then $a \sim_p b$ by Lemma 3.5. Suppose that (c) holds and let $\tilde{a} = uv$ and $\tilde{b} = vu$, where $u, v \in A_n^*$. Then $a = puvp^{-1}$ and $b = quvq^{-1}$ for some $p, q \in A_n^*$ by (3.4). Hence $a = puq^{-1} \cdot qvp^{-1}$ and $b = qvp^{-1} \cdot puq^{-1}$, and so $a \sim_p b$ as required. The proof in the case when (d) holds is similar. \square

It is worth noting that the nonzero idempotents of P_n form a single *p*-conjugacy class. Indeed, the nonzero idempotents have the form xx^{-1} , and $\tilde{a} = 1$ if and only if a is an idempotent. So, they form a single *p*-conjugacy class by Theorem 3.6.

We recall that \sim_p is transitive in any free monoid. For the polycyclic monoid, we have the following result.

Theorem 3.7. *In the polycyclic monoid P_n , we have:*

- (1) for all $a, b, c \in P_n$ with $b \neq 0$, if $a \sim_p b$ and $b \sim_p c$, then $a \sim_p c$;
- (2) $\sim_p \circ \sim_p = \sim_p^*$.

Proof. To prove (a), let $a, b, c \in P_n$ with $b \neq 0$. Suppose that $a \sim_p b \sim_p c$. If $\rho(b) \neq 0$ then, by Theorem 3.6, either $\tilde{a}, \tilde{b}, \tilde{c} \in A_n^*$ or $\tilde{a}, \tilde{b}, \tilde{c} \in (A_n^{-1})^*$, and $a \sim_p c$ follows since \sim_p is transitive in the free monoid. Suppose that $\rho(b) = 0$. Then, by Theorem 3.6, $\rho(a) = \rho(c) = 0$. Thus, if $a = 0$ or $c = 0$, then $a \sim_p c$. Suppose that $a \neq 0$ and $c \neq 0$. Then, by Theorem 3.6, $\tilde{a} = \tilde{b} = \tilde{c}$, and so, again by Theorem 3.6, $a \sim_p c$.

Statement (2) follows from (1). \square

The relations \sim_p and \sim_p^* are not equal in P_n . For example, consider the polycyclic monoid P_2 with $A_2 = \{x, y\}$. Then, for $a = xx^{-1}$ and $c = yy^{-1}$ in P_2 , $a \sim_p 0 \sim_p c$, so $a \sim_p^* c$, but $(a, c) \notin \sim_p$ by Theorem 3.6.

3.3. c -conjugacy in P_n . Referring to the definition of \sim_c , we begin with a description of the set from which the conjugators must be chosen.

Lemma 3.8. *For all $x, y \in A_n^*$, $\mathbb{P}(yx^{-1}) = \{rs^{-1} : r \text{ is a prefix of } x\}$.*

Proof. Let $rs^{-1} \in \mathbb{P}(yx^{-1})$. Then $yx^{-1} \cdot rs^{-1} \neq 0$, and so r and x are prefix-comparable. Suppose that x is a proper prefix of r , that is, $r = xp_i t$ for some $p_i \in A_n = \{p_1, \dots, p_n\}$ and $t \in A_n^*$. Let $j \in \{1, \dots, n\}$ with $j \neq i$. Then $(yp_j)^{-1} \cdot yx^{-1} = (xp_j)^{-1} \neq 0$, while $(yp_j)^{-1} \cdot yx^{-1} \cdot rs^{-1} = (xp_j)^{-1} \cdot rs^{-1} = 0$ since neither xp_j is a prefix of r , nor r is a prefix of xp_j . This contradicts the hypothesis that $rs^{-1} \in \mathbb{P}(yx^{-1})$. Therefore, r is a prefix of x .

Now, let rs^{-1} be an element of P_n and assume that r is a prefix of x . Then $x = rz$ for some $z \in A_n^*$, which gives $yx^{-1} \cdot rs^{-1} = y(sz)^{-1}$. Thus, for every $vu^{-1} \in P_n$, $vu^{-1} \cdot yx^{-1} \cdot rs^{-1} = vu^{-1} \cdot y(sz)^{-1} = 0$ iff $vu^{-1} \cdot yx^{-1} = 0$ (see Lemma 3.2). Thus $rs^{-1} \in \mathbb{P}(yx^{-1})$. \square

The following theorem characterizes c -conjugacy in P_n .

Theorem 3.9. *Let $a, b \in P_n$. Then $a \sim_c b$ if and only if one of the following conditions is satisfied:*

- (a) $a = b = 0$;
- (b) $\tilde{a} = \tilde{b}$; or
- (c) $\tilde{a}, \tilde{b} \in (A_n^{-1})^*$ and $\tilde{a} \sim_p \tilde{b}$ in the free monoid $(A_n^{-1})^*$.

Proof. Suppose that $a \sim_c b$. If $a = 0$ or $b = 0$, then (a) holds since $[0]_c = \{0\}$. Suppose $a, b \neq 0$. Then, there exist $x, y, u, v \in A_n^*$ such that $a = yx^{-1}$ and $b = vu^{-1}$; and there exist $r, s \in A_n^*$ such that $rs^{-1} \in \mathbb{P}(yx^{-1})$ and $yx^{-1} \cdot rs^{-1} = rs^{-1} \cdot vu^{-1}$. By Lemma 3.8, $x = rz$ for some $z \in A_n^*$. By Lemma 3.2, s and v are prefix-comparable.

Suppose $v = sw$ for some $w \in A_n^*$. Then, $yx^{-1} \cdot rs^{-1} = yz^{-1}r^{-1}rs^{-1} = y(sz)^{-1}$ and $rs^{-1} \cdot vu^{-1} = rs^{-1}swu^{-1} = rwu^{-1}$. Thus, since $yx^{-1} \cdot rs^{-1} = rs^{-1} \cdot vu^{-1}$, we have $y(sz)^{-1} = rwu^{-1}$, and so $y = rw$ and $u = sz$. Hence, $a = yx^{-1} = rw(rz)^{-1} = r(wz^{-1})r^{-1}$ and $b = vu^{-1} = sw(sz)^{-1} = s(wz^{-1})s^{-1}$, and so (b) holds.

Suppose $s = vw$ for some $w \in A_n^*$. Then, $yx^{-1} \cdot rs^{-1} = y(sz)^{-1}$ (as in the previous case) and $rs^{-1} \cdot vu^{-1} = rw^{-1}v^{-1}vu^{-1} = r(uw)^{-1}$. Thus, $y(sz)^{-1} = r(uw)^{-1}$, and so $y = r$ and $sz = uw$. Since $s = vw$, we have $vwz = uw$, which implies that $u = vt$ for some $t \in A_n^*$. Thus, $uw = vt w$, which implies $vwz = vt w$, and so $wz = tw$. By [20, Corollary 5.2], $tw = wz$ implies $t \sim_p z$ in A_n^* . Further, $\tilde{a} = \widetilde{yx^{-1}} = \widetilde{r(rz)^{-1}} = \widetilde{rz^{-1}r^{-1}} = z^{-1}$ and $\tilde{b} = \widetilde{vu^{-1}} = \widetilde{v(vt)^{-1}} = \widetilde{vt^{-1}v^{-1}} = t^{-1}$. Hence $\tilde{a}, \tilde{b} \in (A_n^{-1})^*$. Since $t \sim_p z$ in the free monoid A_n^* , we have $z^{-1} \sim_p t^{-1}$ in $(A_n^{-1})^*$, and so $\tilde{a} \sim_p \tilde{b}$ in $(A_n^{-1})^*$. Hence (c) holds.

Conversely, if (a) holds, then clearly $a \sim_c b$. Suppose that (b) holds, that is, $\tilde{a} = \tilde{b}$. Let $r, s \in A_n^*$ be such that $a = r\tilde{a}r^{-1}$ and $b = s\tilde{b}s^{-1}$. Then, by Lemma 3.8, $rs^{-1} \in \mathbb{P}(a)$, $sr^{-1} \in \mathbb{P}(b)$, and

$$\begin{aligned} a \cdot rs^{-1} &= r\tilde{a}r^{-1}rs^{-1} = r\tilde{a}s^{-1} = rs^{-1}s\tilde{a}s^{-1} = rs^{-1} \cdot b, \\ b \cdot sr^{-1} &= s\tilde{b}s^{-1}sr^{-1} = s\tilde{b}r^{-1} = sr^{-1}r\tilde{b}r^{-1} = sr^{-1} \cdot a. \end{aligned}$$

Hence $a \sim_c b$. Now, suppose that (c) holds. Since $\tilde{a}, \tilde{b} \in (A_n^{-1})^*$, then letting $t^{-1} = \tilde{a}$ and $z^{-1} = \tilde{b}$ we have $a = yt^{-1}y^{-1} = y(yt)^{-1}$ and $b = vz^{-1}v^{-1} = v(vz)^{-1}$ for some $y, v \in A_n^*$. Moreover, we have

$\tilde{a} \sim_p \tilde{b}$ in the free monoid $(A_n^{-1})^*$, and so $t \sim_p z$ in A_n^* as well. Hence $tw = wz$ and $w't = zw'$ for some $w, w' \in A_n^*$. By Lemma 3.8, $y(vw')^{-1} \in \mathbb{P}(a)$ and $v(yw)^{-1} \in \mathbb{P}(b)$. Further,

$$\begin{aligned} a \cdot y(vw')^{-1} &= y(yt)^{-1}y(vw')^{-1} = yt^{-1}(vw')^{-1} = y(vw't)^{-1} = \\ &= y(vzw')^{-1} = y(w')^{-1}(vz)^{-1} = y(vw')^{-1}v(vz)^{-1} = y(vw')^{-1} \cdot b, \\ b \cdot v(yw)^{-1} &= v(vz)^{-1}v(yw)^{-1} = vz^{-1}(yw)^{-1} = v(ywz)^{-1} = \\ &= v(ytw)^{-1} = vw^{-1}(yt)^{-1} = v(yw)^{-1}y(yt)^{-1} = v(yw)^{-1} \cdot a. \end{aligned}$$

Hence $a \sim_c b$, which concludes the proof. \square

As for p -conjugacy, the nonzero idempotents form a single c -conjugacy class (see Theorem 3.9 and the paragraph after Theorem 3.6). Moreover, we have the following strict inclusion between \sim_c and \sim_p .

Corollary 3.10. *In the polycyclic monoid P_n , $\sim_c \subset \sim_p$.*

Proof. The inclusion $\sim_c \subseteq \sim_p$ follows by Theorems 3.6 and 3.9. To show that \sim_c is properly contained in \sim_p , consider two distinct generators x and y in A_n . Let $a = xxyx^{-1}$ and $b = yxyx^{-1}$ in P_n . Then $\tilde{a} = xy$ and $\tilde{b} = yx$. Hence $\tilde{a} \sim_p \tilde{b}$ in the free monoid A_n^* , and so $a \sim_p b$ in P_n by Theorem 3.6. On the other hand, none of (a), (b), or (c) of Theorem 3.9 holds for a and b , and so $a \not\sim_c b$ in P_n . \square

3.4. Decidability and complexity of conjugacy in P_n . It is known that for free monoids, the p -conjugacy problem is decidable in linear time [4, Theorem 2.5]. We will show that the same result is true for the p -conjugacy and c -conjugacy problems for the polycyclic monoids.

The following lemma is a special case of [6, Theorem 4.1].

Lemma 3.11. *Let $(\Sigma_0; R)$ be the monoid-with-zero presentation of P_n , and let $w \in \Sigma_0^*$. Then the irreducible element $\bar{w} \in \Sigma_0^*$ such that $w \rightarrow \bar{w}$ in P_n can be computed in time $O(|w|)$.*

(For more details on the big-O notation used in Lemma 3.11, and more generally for basic notions on complexity theory, see [33, Section 7].)

Lemma 3.12. *Let a be an irreducible word of P_n . Then the words \tilde{a} and $\rho(a)$, can be computed in time $O(|a|)$.*

Proof. The result is obvious if $a = 0$. Let $a = yx^{-1}$. To compute \tilde{a} proceed as follows:

- (1) compute the word $x^{-1}y$;
- (2) reduce $x^{-1}y$ to an irreducible word $u^{-1}v$ in (Σ, R_1) (see (b) above);
- (3) output the word $\tilde{a} = vu^{-1}$.

To compute $\rho(a)$ proceed in the same way to obtain the word vu^{-1} , and next proceed as follows:

- (4) if v and u are non-empty, then output $\rho(a) = 0$, otherwise output $\rho(a) = \tilde{a}$.

We show that each stage of this algorithm uses $O(|a|)$ steps, and so the result holds. For the first stage, it is sufficient to scan through the word yx^{-1} (from left to right), detect the first symbol in $(A_n^{-1})^*$, and output the symbols of x^{-1} followed by the symbols of y . This requires $O(|a|)$ steps. The third stage is similar. For the second stage, since R_1 is length reducing, we conclude by [6, Theorem 4.1] that \tilde{a} can be computed in $O(|a|)$ steps. Checking if a word is empty can be done in constant time, and so $\rho(a)$ can be computed in linear time as well. \square

Theorem 3.13. *Let $(\Sigma_0; R)$ be the monoid-with-zero presentation of P_n , and let $i \in \{p, c\}$. Then, given two words $x, y \in \Sigma_0^*$, it can be tested in time $O(m)$, where $m = \max\{|x|, |y|\}$, whether or not $x \sim_i y$ holds in P_n .*

Proof. Let $x, y \in \Sigma_0^*$. By Lemma 3.11, the irreducible words $\bar{x} = a$ and $\bar{y} = b$ can be computed in time $O(m)$, where $m = \max\{|x|, |y|\}$. Note that $|a| \leq |x|$ and $|b| \leq |y|$. By Lemma 3.12, each of the words $\tilde{a}, \tilde{b}, \rho(a)$, and $\rho(b)$ can be computed in time $O(m)$.

According to Theorems 3.6 and 3.9, in order to check whether or not $x \sim_i y$ holds it suffices to compute $a, b, \tilde{a}, \tilde{b}, \rho(a)$, and $\rho(b)$, and check whether or not they are equal (as words) or p -conjugate (in the free monoid). Since the p -conjugacy problem in the free monoids is decidable in linear time, we deduce the desired result. \square

4. DECIDABILITY IN FINITELY PRESENTED MONOIDS

In this section, we discuss the decidability of i -conjugacy problems in some classes of finitely presented monoids.

Separation of conjugacies. Let M be a monoid without zero. Consider the monoid M^0 obtained from M by adjoining a zero element. It is immediate that \sim_o is the universal relation in M^0 , while \sim_c is not universal in M^0 . Now, M^0 has no zero divisors, and hence any two given elements a and b of M are c -conjugate in M^0 if and only if they are o -conjugate in M . Therefore, $\sim_c^{M^0} = \sim_o^M \cup \{(0, 0)\}$ in M^0 . Similarly, for p -conjugacy we have $\sim_p^{M^0} = \sim_p^M \cup \{(0, 0)\}$. Thus, if we identify a monoid M for which $\sim_o \neq \sim_p$ in M , we then immediately obtain an example of a monoid M^0 where $\sim_c \neq \sim_p$, $\sim_o \neq \sim_p$, and $\sim_o \neq \sim_c$. To find such a monoid (within a certain class of rewriting systems), consider the following example from [30, Example 2.2].

Example 4.1. Let M be the monoid defined by the monadic and confluent presentation $(\Sigma; R)$ with $\Sigma = \{a, b, c\}$ and $R = \{(ab, b), (cb, b)\}$. As explained in [30, Example 2.2], we have $bac \sim_p ba$, but clearly bac and ba are not o -conjugate. Therefore, $\sim_o \neq \sim_p$ in M , and hence the relations \sim_o , \sim_p and \sim_c are pairwise distinct in M^0 .

We deduce that for monoids defined by monadic presentations, the relations \sim_c , \sim_p and \sim_o may be different, even when such systems are also finite and confluent.

Finite complete presentations. Narendran and Otto [25, Lemma 3.6] constructed a finite complete presentation $(\Sigma; R)$ such that the o -conjugacy problem is undecidable for the monoid $M = M(\Sigma; R)$. Using the above observation, we obtain the following result.

Proposition 4.2. *There is a monoid defined by a finite complete presentation for which the c -conjugacy problem is undecidable.*

Proof. Consider the monoid M^0 obtained from the monoid M defined by Narendran and Otto in [25, page 35] which has undecidable o -conjugacy problem. Since M is defined by a finite complete presentation, the monoid M^0 is also defined by a finite complete presentation by [3, Proposition 3.1]. It can be seen that M does not have a zero. Thus $\sim_c^{M^0} = \sim_o^M \cup \{(0, 0)\}$, and hence M^0 has undecidable c -conjugacy problem. \square

Special presentations. It is easy to see that a monoid defined by a special presentation has a zero if and only if it is trivial. Hence, within this class we have $\sim_c = \sim_o$. Zhang [34, Theorem 3.2] proved that in every monoid M defined by a special presentation, the relations \sim_p and \sim_o also coincide. Otto [30, Theorem 3.8] proved that if M is a monoid defined by a finite, special, and confluent presentation, then the o -conjugacy problem for M is decidable (and so the p -conjugacy and c -conjugacy problems are also decidable for M).

One-relator monoids. A monoid M is called a *one-relator* monoid if it admits a finite presentation with one defining relation, which we will write as $(\Sigma; u = v)$ instead of $(\Sigma, \{(u, v)\})$. Many decision problems have been studied in the class of one-relator monoids. For example, it is decidable whether a one-relator monoid has a zero [8, Proposition 14]. Moreover, a one-relator monoid M containing a zero admits a presentation $(\{a\}; a^{k+1} = a^k)$, where k is a positive integer [8, the proof of Proposition 14]. It is easy to check that in this monoid $\sim_p = \sim_c = \{(x, x) : x \in M\}$ and $\sim_o = M \times M$.

By the foregoing argument, if M is a one-relator monoid with a zero, then the c -conjugacy and o -conjugacy problems for M are decidable. If M has no zero, then $\sim_c = \sim_o$. Therefore, the c -conjugacy problem for such an M is decidable if and only if the o -conjugacy problem for M is decidable.

Some specific results concerning the decidability of the o -conjugacy problem for this class can be found in [34, 35].

5. INDEPENDENCE IN FINITELY PRESENTED MONOIDS

In this section, we prove that for finitely presented monoids, the word problem and the c -conjugacy problem are independent, and that the p -conjugacy problem and the c -conjugacy problem are independent.

Definition 5.1. Decision problems P_1 and P_2 are *independent* if there exist finitely presented monoids M_1 and M_2 such that for M_1 , P_1 is decidable and P_2 is undecidable; and for M_2 , P_2 is decidable and P_1 is undecidable.

Theorem 5.2. *For finitely presented monoids, the word problem and the c -conjugacy problem are independent.*

Proof. First, there are finitely presented groups with decidable word problem but undecidable conjugacy problem [5, 10]. Let G be a finitely presented group. A finite group presentation of G can be effectively converted to a finite (special) monoid presentation $(\Sigma; R)$ such that $G \cong M(\Sigma; R)$. It follows that there is a monoid M defined by a finite presentation for which the word problem is decidable and the c -conjugacy problem is undecidable.

We will construct a finitely presented monoid for which the converse is true. Let $G = M(\Sigma; R)$ be a finitely presented group with undecidable word problem (see [28]), where $(\Sigma; R)$ is a monoid presentation. Let a and b be symbols not in Σ , and let $M = M(A; T)$ be the monoid defined by the presentation $(A; T)$, where

$$\begin{aligned} A &= \Sigma \cup \{a, b\}, \\ T &= R \cup \{(xa, ax) : x \in \Sigma\} \cup \{(bx, b) : x \in \Sigma \cup \{a\}\} \cup \{(xb, b) : x \in \Sigma\} \cup \{(aa, a)\}. \end{aligned}$$

Notice that G is a subgroup of M . The word problem for M is undecidable (since otherwise it would be decidable for G). It is easy to see that M has no zero and that each congruence class $[u] = [u]_M$ has a representative of the form b^p , aw , ab^p , or w , where p is a positive integer and $w \in \Sigma^*$.

Observe that whenever a rewriting rule from T is applied to a word in A^* , the number of occurrences of b does not change. Thus, for all $u_1, u_2 \in A^*$, if $[u_1] = [u_2]$, then $|u_1|_b = |u_2|_b$. Let $[u], [v] \in M$. Suppose $[u] \sim_c [v]$. Then $[u][t] = [t][v]$ for some $t \in A^*$. Thus $[ut] = [tv]$, and so $|u|_b = |v|_b$ by the foregoing observation.

Conversely, suppose $|u|_b = |v|_b$. If $|u|_b = |v|_b = 0$, then $[u] \sim_c [v]$ since $[u][ab] = [ab] = [ab][v]$ and $[v][ab] = [ab] = [ab][u]$. Suppose $|u|_b = |v|_b = p > 0$. If $[u] = [v]$, then $[u] \sim_c [v]$. Suppose $[u] \neq [v]$. Then $[u] = [b^p]$ and $[v] = [ab^p]$, or vice versa. We may assume that $[u] = [b^p]$ and $[v] = [ab^p]$. Then $[u] \sim_c [v]$ since $[u][b] = [b^{p+1}] = [b][v]$ and $[v][a] = [ab^p] = [a][u]$.

We have proved that for all $u, v \in A^*$, $[u] \sim_c [v]$ if and only if $|u|_b = |v|_b$. Hence the c -conjugacy problem for M is decidable. \square

Theorem 5.3. *For finitely presented monoids, the p -conjugacy problem and the c -conjugacy problem are independent.*

Proof. Let $M = M(A; T)$ be the monoid from the proof of Theorem 5.2. For $w \in \Sigma^*$, we will write $[w] = [w]_M$ for the element of the monoid M , and $[w]_G$ for the element of the group G .

Let $u, v \in \Sigma^*$. Suppose $[u] \sim_p [v]$, that is, $[u] = [s][t]$ and $[v] = [t][s]$ for some $s, t \in A^*$. The words s and t cannot contain b since in the presentation $(A; T)$ a word with b cannot be reduced to a word without b . But then s and t cannot contain a either since a word with a cannot be reduced to a word without a unless b is also present. It follows that $[u]_G = [s]_G[t]_G$ and $[v]_G = [t]_G[s]_G$, and so $[u]_G \sim_p [v]_G$.

We have proved that for all $u, v \in \Sigma^*$, if $[u] \sim_p [v]$ in M , then $[u]_G \sim_p [v]_G$ in G . The converse is clearly true. Since \sim_p in G is the group conjugacy and G has undecidable word problem (and so undecidable conjugacy problem), it follows that the p -conjugacy problem for M is undecidable. We have already established in the proof of Theorem 5.2 that the c -conjugacy problem for M is decidable.

We will now present a monoid that has decidable p -conjugacy problem and undecidable c -conjugacy problem. Osipova [29] showed that there exists a finitely presented monoid M that has decidable p -conjugacy problem and undecidable l -conjugacy problem, where the l -conjugacy stands for the following relation \sim_l : given $a, b \in M$, $a \sim_l b$ if and only if there exists $g \in M$ such that $ag = gb$. Osipova's proof follows the following steps (we use the original notation): (i) she considers a finitely presented monoid $\Pi_1 = M(\mathcal{U}_1; \mathcal{B}_0)$ with undecidable p -conjugacy problem; (ii) she extends the alphabet \mathcal{U}_1 to $\mathcal{U}_3 = \mathcal{U}_1 \cup \{c, d, e_1, \dots, e_m\}$, where $m = |\mathcal{U}_1| + 2|\mathcal{B}_0|$, and builds a new finitely presented monoid $\Pi_3 = M(\mathcal{U}_3; \mathcal{B}_3)$; (iii) she shows [29, Lemma 4] that for all words $Q, R \in \mathcal{U}_1^*$, $Q \sim_p R$ in Π_1 if and only if there exists $X \in \mathcal{U}_3^*$ such that $XcQd = cRdX$ in Π_3 ; (iv) she concludes [29, Theorem 2]

that the l -conjugacy problem for Π_3 is undecidable; (v) she shows [29, Theorem 3] that the p -conjugacy problem for Π_3 is decidable.

Now, notice that \sim_p is symmetric, and hence, by [29, Lemma 4], for all words $Q, R \in \mathcal{U}_1^*$, we have $Q \sim_p R$ in Π_1 if and only if there exist $X, Y \in \mathcal{U}_3^*$ such that $XcQd = cRdX$ and $YcRd = cQdY$ in Π_3 . Equivalently, $Q \sim_p R$ in Π_1 if and only if $cQd \sim_o cRd$ in Π_3 . Therefore, Π_3 has undecidable o -conjugacy problem.

The set \mathcal{B}_3 of Π_3 has rewriting rules of the form $(e_i c G_i, c e_i)$, $(e_i b_j, b_j e_i)$, and $(e_i d, G'_i d e_i)$, where $i = 1, \dots, m$ and $j = 1, \dots, n$, the b_j are the letters of the alphabet \mathcal{U}_1 , and the G_i and G'_i are fixed words in \mathcal{U}_1^* [29, pages 70 and 71]. From the form of these rules, we can easily deduce that any two words in \mathcal{U}_3^* that are equal in Π_3 have the same number of occurrences of the letter c . Therefore, Π_3 does not have a zero since the zero element, say $[z]$, would satisfy the identity $[z][c] = [z]$, contradicting the above observation. Hence $\sim_o = \sim_c$, and hence Π_3 has undecidable c -conjugacy problem. \square

We do not know if the c -conjugacy problem and the o -conjugacy problem are independent for finitely presented monoids. Consider a finitely presented monoid M without zero that has undecidable c -conjugacy problem. Let M^0 be the monoid M with a zero 0 adjoined. Then M^0 is finitely presented and the c -conjugacy problem for M^0 is undecidable (since for all $a, b \in M$, $a \sim_c b$ in M^0 if and only if $a \sim_c b$ in M). On the other hand, the o -conjugacy problem for M^0 is decidable since $\sim_o = M^0 \times M^0$.

Now, suppose M is a finitely presented monoid that has decidable c -conjugacy problem. Then, if we could prove that M has a zero, then the algorithm that always says YES would decide if $[u]_M \sim_o [v]_M$ for all $[u]_M, [v]_M \in M$. Further, if we could prove that M has no zero, then the algorithm that works for \sim_c would also work for \sim_o . However, suppose that the statement “ M has a zero” can neither be proved nor disproved. Then it is conceivable that no algorithm for o -conjugacy problem in M exists, that is, that o -conjugacy problem is undecidable for M .

6. OPEN PROBLEMS

We conclude this paper with some natural questions related to conjugacy and presentations. As we have noticed in Section 5, the independence of the c -conjugacy and o -conjugacy problems is related to the decidability of a monoid having a zero. Hence whether the o -conjugacy and c -conjugacy problems are independent hinges on the answer to the following question.

Problem 6.1. Does there exist a finitely presented monoid M for which it is undecidable if it has a zero, the o -conjugacy problem for M is undecidable, and the c -conjugacy problem for M is decidable?

The word problem is decidable for certain restricted classes of finitely presented monoids, in particular those admitting a finite complete presentation. It is then natural to consider this property as a useful tool in proving decidability results. In the class of monoids defined by finite, length-reducing, and confluent rewriting systems, the o -conjugacy problem is decidable [24, Corollary 2.7]. It is also decidable if such monoids have a zero. However, the p -conjugacy problem is undecidable in this class [25, Corollary 2.4].

Problem 6.2. Is the c -conjugacy problem decidable for the class of monoids defined by finite, length-reducing, and confluent rewriting systems?

This problem could be approached by first considering the class of finite monadic confluent rewriting systems, as it is the case of polycyclic monoids.

Problem 6.3. Is the c -conjugacy [p -conjugacy] problem decidable for the class of monoids defined by finite, monadic, and confluent rewriting systems?

7. ACKNOWLEDGMENTS

The first and second authors acknowledge that this work was developed within FCT projects CAUL (PEst-OE/MAT/UI0143/2014) and CEMAT-CIÊNCIAS (UID/Multi/04621/2013).

The second author was also supported by Simons Foundation Collaboration Grant 359872.

The fourth author was supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project UID/MAT/00297/2013 (Centro de Matemática e Aplicações).

Finally, the first, second and fourth authors were supported by FCT through project “Hilbert’s 24th problem” (PTDC/MHC-FIL/2583/2014).

REFERENCES

- [1] J. Araújo, M. Kinyon, J. Konieczny, and A. Malheiro, Four notions of conjugacy for abstract semigroups, to appear in *Proc. Roy. Soc. Edinburgh Sect. A*.
- [2] J. Araújo, J. Konieczny, and A. Malheiro, Conjugation in semigroups, *J. Algebra* **403** (2014), 93–134.
- [3] J. Araújo and A. Malheiro, On finite complete presentations and exact decompositions of semigroups, *Comm. Algebra* **39** (2011), 3866–3878.
- [4] J. Avenhaus and K. Madlener, String matching and algorithmic problems in free groups, *Rev. Columbiana Mat.* **15** (1980), 1-16.
- [5] L.A. Bokut, Degrees of unsolvability of the conjugacy problem for finitely presented groups, *Algebra i Logika* **7** (1968), no. 5, 4–70, no. 6, 4–52. (Russian)
- [6] R.V. Book, Confluent and other types of Thue systems, *J. Assoc. Comput. Mach.* **29** (1982), 171–182.
- [7] R.V. Book and F. Otto, *String-rewriting Systems*, Texts and Monographs in Computer Science, Springer-Verlag, New York, 1993.
- [8] A. J. Cain and V. Maltcev, Decision problems for finitely presented and one-relation semigroups and monoids, *Internat. J. Algebra Comput.* **19** (2009), 747–770.
- [9] C. Choffrut, Conjugacy in free inverse monoids, *Internat. J. Algebra Comput.* **3** (1993), 169–188.
- [10] D. J. Collins, Recursively enumerable degrees and the conjugacy problem, *Acta Math.* **122** (1969) 115–160.
- [11] J. Cuntz, Simple C^* -algebras generated by isometries, *Comm. Math. Phys.* **57** (1977), 173–185.
- [12] P.M. Higgins, *Techniques of Semigroup Theory*, Oxford Science Publications. Oxford University Press, New York, 1992.
- [13] P.M. Higgins, The semigroup of conjugates of a word, *Internat. J. Algebra Comput.* **16** (2006), 1015–1029.
- [14] P.M. Hines, An application of polycyclic monoids to rings, *Semigroup Forum* **56** (1998), 146–149.
- [15] J.M. Howie, *Fundamentals of Semigroup Theory*, Oxford Science Publications, Oxford, 1995.
- [16] G. Kudryavtseva, On conjugacy in regular epigroups, <http://arxiv.org/abs/math/0605698>.
- [17] D.G. Jones and M.V. Lawson, Strong representations of the polycyclic inverse monoids: cycles and atoms, *Period. Math. Hungar.* **64** (2012), 53–87.
- [18] G. Kudryavtseva and V. Mazorchuk, On conjugation in some transformation and Brauer-type semigroups, *Publ. Math. Debrecen* **70** (2007), 19–43.
- [19] G. Kudryavtseva and V. Mazorchuk, On three approaches to conjugacy in semigroups, *Semigroup Forum* **78** (2009), 14–20.
- [20] G. Lallement, *Semigroups and Combinatorial Applications*, John Wiley & Sons, New York, 1979.
- [21] M. V. Lawson, Primitive partial permutation representations of the polycyclic monoids and branching function systems, *Period. Math. Hungar.* **58** (2009), 189–207.
- [22] R.C. Lyndon and M.P. Schützenberger, The equation $a^m = b^n c^p$ in a free group, *Michigan Math. J.* **9** (1962), 289–298.
- [23] A. Markov, The impossibility of certain algorithms in the theory of associative systems. II, *Doklady Akad. Nauk SSSR (N.S.)* **58** (1947), 353–356. (Russian)
- [24] P. Narendran and F. Otto, Complexity results on the conjugacy problem for monoids, *Theoret. Comput. Sci.* **35** (1985), 227–243.
- [25] P. Narendran and F. Otto, The problems of cyclic equality and conjugacy for finite complete rewriting systems, *Theoret. Comput. Sci.* **47** (1986), 27–38.
- [26] M. Nivat and J.F. Perrot, Une généralisation du monoïde bicyclique, *C. R. Acad. Sci. Paris Sér. A-B* **271** (1970), A824–A827.
- [27] P.S. Novikov, Unsolvability of the conjugacy problem in the theory of groups, *Izv. Akad Nauk SSSR. Ser. Mat.* **18** (1954), 485–524. (Russian)
- [28] P.S. Novikov, On the algorithmic insolvability of the word problem in group theory, *American Mathematical Society Translations, Ser 2* **9** (1958), 1–122.
- [29] V.A. Osipova, On the conjugacy problem in semigroups, *Trudy Mat. Inst. Steklov.* **133** (1973), 169–182, 275. (Russian)
- [30] F. Otto, Conjugacy in monoids with a special Church-Rosser presentation is decidable, *Semigroup Forum* **29** (1984), 223–240.
- [31] E.L. Post, Recursive unsolvability of a problem of Thue, *J. Symbolic Logic* **12** (1947), 1–11.
- [32] N. Ruškuc, *Semigroup Presentations*, Ph.D. Thesis, University of St Andrews, 1995.
- [33] M. Sipser, *Introduction to the Theory of Computation*, International Thomson Publishing, 1996.
- [34] L. Zhang, Conjugacy in special monoids, *J. Algebra* **143** (1991), 487–497.
- [35] L. Zhang, On the conjugacy problem for one-relator monoids with elements of finite order, *Internat. J. Algebra Comput.* **2** (1992), 209–220.

UNIVERSIDADE ABERTA, R. ESCOLA POLITÉCNICA, 147, 1269-001 LISBOA, PORTUGAL

CEMAT-CIÊNCIAS UNIVERSIDADE DE LISBOA, 1749-016 LISBOA, PORTUGAL

E-mail address: `jjaraujo@fc.ul.pt`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DENVER, DENVER, CO 80208

E-mail address: `mkinyon@du.edu`

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF MARY WASHINGTON, FREDERICKSBURG, VA 22401.

E-mail address: `jkoniecz@umw.edu`

CENTRO DE MATEMÁTICA E APLICAÇÕES, FACULDADE DE CIÊNCIAS E TECNOLOGIA, UNIVERSIDADE NOVA DE LISBOA, 2829-516 CAPARICA, PORTUGAL

DEPARTAMENTO DE MATEMÁTICA, FACULDADE DE CIÊNCIAS E TECNOLOGIA, UNIVERSIDADE NOVA DE LISBOA, 2829-516 CAPARICA, PORTUGAL

E-mail address: `ajm@fct.unl.pt`