

# APPROXIMATIONS OF COMPACT METRIC SPACES BY FULL MATRIX ALGEBRAS FOR THE QUANTUM GROMOV-HAUSDORFF PROPINQUITY

KONRAD AGUILAR AND FRÉDÉRIC LATRÉMOLIÈRE

ABSTRACT. We prove that all the compact metric spaces are in the closure of the class of full matrix algebras for the quantum Gromov-Hausdorff propinquity. Our techniques are inspired from our work on AF algebras as quantum metric spaces.

## 1. INTRODUCTION

The quantum Gromov-Hausdorff propinquity [17, 13, 18] provides a natural framework to discuss finite dimensional approximations of quantum spaces in a metric sense by extending the Gromov-Hausdorff distance to noncommutative geometry. Thus, for this new metric, quantum tori are limits of fuzzy tori [11], spheres are limits of full matrix algebras [23, 24, 25], AF algebras are limits of any inductive sequence from which they are constructed [1, 2, 3], any separable nuclear quasi-diagonal  $C^*$ -algebra equipped with a quasi-Leibniz Lip-norm is the limit of finite dimensional  $C^*$ -algebras [12], noncommutative solenoids are limits of matrix algebras [19], among other examples of such finite dimensional approximations. In [11] and [25] in particular, certain classical metric spaces are limits of full matrix algebras, an intriguing phenomenon. This note answers the natural question of which classical compact metric spaces are limits of full matrix algebras for the quantum propinquity. We shall prove that indeed, any classical compact metric space is the limit, for the quantum propinquity, of a sequence of  $(2, 0)$ -quasi-Leibniz quantum compact metric spaces constructed on full matrix algebras. Our approximations are very different from the ones presented in the above references, as our focus is not to preserve any symmetry of the limit space, but rather to find a very general method to obtain such full matrix algebra approximations. In particular, it is generally difficult to compute the closure of a particular set of quantum metric spaces for the propinquity. This paper proves that all classical compact metric spaces do lie in the closure of full matrix algebras for the propinquity and give examples to further test the theory of noncommutative geometry and what properties pass, or do not pass, to the limit for convergent sequences of quasi-Leibniz quantum compact metric spaces.

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Quantum compact metric spaces are noncommutative generalizations of Lipschitz algebras introduced in [21, 22] by Rieffel, and inspired by Connes [4]. In [17, 15], additional requirements were placed on the original definition of Rieffel to accommodate the construction of the quantum propinquity. The resulting notion of a quasi-Leibniz quantum compact metric space will be the starting point for our work.

*Notation 1.1.* For any unital  $C^*$ -algebra  $\mathfrak{A}$ , we denote the unit of  $\mathfrak{A}$  by  $1_{\mathfrak{A}}$ , the norm of  $\mathfrak{A}$  by  $\|\cdot\|_{\mathfrak{A}}$ , the Jordan-Lie algebra of the self-adjoint elements of  $\mathfrak{A}$  by  $\mathfrak{sa}(\mathfrak{A})$ , and the state space of  $\mathfrak{A}$  by  $\mathcal{S}(\mathfrak{A})$ .

**Definition 1.2** ([21, 22, 17, 15]). A  $(C, D)$ -quasi-Leibniz quantum compact metric space  $(\mathfrak{A}, L)$ , for some  $C \geq 1$  and  $D \geq 0$ , consists of unital  $C^*$ -algebra  $\mathfrak{A}$  with unit  $1_{\mathfrak{A}}$  and a seminorm  $L$  defined on a dense Jordan-Lie subalgebra  $\text{dom}(L)$  of the space  $\mathfrak{sa}(\mathfrak{A})$  of self-adjoint elements in  $\mathfrak{A}$ , such that:

- (1)  $\{a \in \text{dom}(L) : L(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$ ,
- (2) the Monge-Kantorovich metric  $\text{mk}_L$  defined for any two states  $\varphi, \psi \in \mathcal{S}(\mathfrak{A})$  by:

$$\text{mk}_L(\varphi, \psi) = \sup \{|\varphi(a) - \psi(a)| : a \in \text{dom}(L), L(a) \leq 1\}$$

metrizes the weak\* topology on  $\mathcal{S}(\mathfrak{A})$ ,

- (3)  $L$  is lower semi-continuous for  $\|\cdot\|_{\mathfrak{A}}$ ,
- (4) for all  $a, b \in \text{dom}(L)$ , we have:

$$\max \{L(a \circ b), L(\{a, b\})\} \leq C (\|a\|_{\mathfrak{A}}L(b) + \|b\|_{\mathfrak{A}}L(a)) + DL(a)L(b),$$

$$\text{where } a \circ b = \frac{ab+ba}{2} \text{ and } \{a, b\} = \frac{ab-ba}{2i}.$$

The seminorm  $L$  is called an  $L$ -seminorm.

Rieffel provided in [21] the fundamental characterization of compact quantum metric spaces, which is a noncommutative form of the Arzéla-Ascoli theorem. We will use a version of this characterization found in [20] in this paper, which we now recall and adapt slightly to our setting.

**Theorem 1.3** ([20]). *Let  $\mathfrak{A}$  be a unital  $C^*$ -algebra,  $L$  a lower semi-continuous seminorm defined on some dense Jordan-Lie subalgebra  $\text{dom}(L)$  of  $\mathfrak{sa}(\mathfrak{A})$  such that:*

$$\{a \in \text{dom}(L) : L(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$$

and, for some  $C \geq 1, D \geq 0$ :

$$\max \{L(a \circ b), L(\{a, b\})\} \leq C (\|a\|_{\mathfrak{A}}L(b) + \|b\|_{\mathfrak{A}}L(a)) + DL(a)L(b).$$

The following assertions are equivalent:

- (1)  $(\mathfrak{A}, L)$  is a  $C, D$ -quasi-Leibniz quantum compact metric space,
- (2) there exists a state  $\mu \in \mathcal{S}(\mathfrak{A})$  such that the set:

$$\{a \in \text{dom}(L) : \mu(a) = 0, L(a) \leq 1\}$$

is compact for  $\|\cdot\|_{\mathfrak{A}}$ ,

- (3) for all states  $\mu \in \mathcal{S}(\mathfrak{A})$ , the set:

$$\{a \in \text{dom}(L) : \mu(a) = 0, L(a) \leq 1\}$$

is compact for  $\|\cdot\|_{\mathfrak{A}}$ .

Quasi-Leibniz quantum compact metric spaces form a category for several natural notions of morphisms [26, 16]. The noncompact theory is more involved [9, 10] and will not be used in this note.

Much research has been concerned with the development of a noncommutative analogue of the Gromov-Hausdorff distance, starting with the pioneering work of Rieffel in [26] on the quantum Gromov-Hausdorff distance (for which the question raised in this note was solved by the second author in [8]). We will work with the quantum Gromov-Hausdorff propinquity introduced by Latrémolière in [17] to address two inherent difficulties with the construction of such an analogue: working within a class of quantum compact metric spaces satisfying a given form of the Leibniz inequality and having the desirable property that distance zero would imply \*-isomorphism of the underlying C\*-algebras.

The construction of the quantum propinquity is involved, and we refer to [17, 13, 18, 15, 14, 12, 3, 19, 16] for a detailed discussion of this metric, its basic properties and some important applications. For our purpose, we will focus on a core ingredient of the construction of the quantum propinquity called a bridge, which enables us to appropriately relate two quasi-Leibniz quantum compact metric spaces and compute a quantity on which the propinquity is based.

**Definition 1.4** ([17]). A bridge  $\gamma = (\mathfrak{D}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, x)$  from a unital C\*-algebra  $\mathfrak{A}$  to a unital C\*-algebra  $\mathfrak{B}$  consists of a unital C\*-algebra  $\mathfrak{D}$ , two unital \*-monomorphisms  $\pi_{\mathfrak{A}} : \mathfrak{A} \hookrightarrow \mathfrak{D}$  and  $\pi_{\mathfrak{B}} : \mathfrak{B} \hookrightarrow \mathfrak{D}$ , and an element  $x \in \mathfrak{D}$  such that:

$$\mathcal{S}(\mathfrak{D}|x) = \{\varphi \in \mathcal{S}(\mathfrak{D}) : \forall d \in \mathfrak{D} \quad \varphi(xd) = \varphi(dx) = \varphi(d)\} \neq \emptyset.$$

We associate a quantity to any bridge which estimates, for that given bridge, how far apart the domain and co-domain of the bridge are.

*Notation 1.5.* The Hausdorff distance [6] on the space of all compact subspaces of a metric space  $(X, d)$  is denoted by  $\text{Haus}_d$ .

**Definition 1.6** ([17]). The length  $\lambda(\gamma|L_{\mathfrak{A}}, L_{\mathfrak{B}})$  of a bridge  $\gamma = (\mathfrak{D}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}}, x)$  from  $(\mathfrak{A}, L_{\mathfrak{A}})$  to  $(\mathfrak{B}, L_{\mathfrak{B}})$  is the maximum of the following two quantities:

$$\zeta(\gamma|L_{\mathfrak{A}}, L_{\mathfrak{B}}) = \max \left\{ \text{Haus}_{\text{mk}_{L_{\mathfrak{A}}}}(\mathcal{S}(\mathfrak{A}), \{\varphi \circ \pi_{\mathfrak{A}} : \varphi \in \mathcal{S}(\mathfrak{D}|x)\}), \right. \\ \left. \text{Haus}_{\text{mk}_{L_{\mathfrak{B}}}}(\mathcal{S}(\mathfrak{B}), \{\varphi \circ \pi_{\mathfrak{B}} : \varphi \in \mathcal{S}(\mathfrak{D}|x)\}) \right\}$$

and

$$\varrho(\gamma|L_{\mathfrak{A}}, L_{\mathfrak{B}}) = \max \left\{ \begin{array}{l} \sup_{\substack{a \in s_{\mathfrak{A}}(\mathfrak{A}) \\ L_{\mathfrak{A}}(a) \leq 1}} \inf_{\substack{b \in s_{\mathfrak{A}}(\mathfrak{B}) \\ L_{\mathfrak{B}}(b) \leq 1}} \text{bn}_{\gamma}(a, b) \\ \sup_{\substack{b \in s_{\mathfrak{A}}(\mathfrak{B}) \\ L_{\mathfrak{B}}(b) \leq 1}} \inf_{\substack{a \in s_{\mathfrak{A}}(\mathfrak{A}) \\ L_{\mathfrak{A}}(a) \leq 1}} \text{bn}_{\gamma}(a, b) \end{array} \right\},$$

where  $\text{bn}_{\gamma}(a, b) = \|\pi_{\mathfrak{A}}(a)x - x\pi_{\mathfrak{B}}(b)\|_{\mathfrak{D}}$  for all  $a \in \mathfrak{A}$  and  $b \in \mathfrak{B}$ .

We note that in the present paper, all our bridges will have the unit for pivot and thus will have height zero; however the more descriptive Definition (1.6) is useful to state the following characterization of the quantum propinquity which we will use as our definition for this work.

**Theorem-Definition 1.7** ([17]). *Let  $C \geq 1$  and  $D \geq 0$ , and let  $\mathfrak{QM}_{C,D}$  be the class of all  $(C, D)$ -quasi-Leibniz quantum compact metric spaces. There exists a class function  $\Lambda$  on  $\mathfrak{QM}_{C,D} \times \mathfrak{QM}_{C,D}$ , called the quantum propinquity, such that:*

(1) *for all  $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})$  in  $\mathfrak{QM}_{C,D}$ :*

$$0 \leq \Lambda_{C,D}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) = \Lambda_{C,D}((\mathfrak{B}, L_{\mathfrak{B}}), (\mathfrak{A}, L_{\mathfrak{A}})) \\ \leq \max \{ \text{diam}(\mathfrak{A}, L_{\mathfrak{A}}), \text{diam}(\mathfrak{B}, L_{\mathfrak{B}}) \}.$$

(2) *for all  $(\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})$  and  $(\mathfrak{D}, L_{\mathfrak{D}})$  in  $\mathfrak{QM}_{C,D}$ :*

$$\Lambda_{C,D}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{D}, L_{\mathfrak{D}})) \leq \Lambda_{C,D}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) + \Lambda_{C,D}((\mathfrak{B}, L_{\mathfrak{B}}), (\mathfrak{D}, L_{\mathfrak{D}})),$$

(3) *for all  $(\mathfrak{A}, L_{\mathfrak{A}})$  and  $(\mathfrak{B}, L_{\mathfrak{B}})$  in  $\mathfrak{QM}_{C,D}$  and for any bridge  $\gamma$  from  $\mathfrak{A}$  to  $\mathfrak{B}$ , we have:*

$$\Lambda_{C,D}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) \leq \lambda(\gamma | L_{\mathfrak{A}}, L_{\mathfrak{B}}),$$

(4)  $\Lambda_{C,D}((\mathfrak{A}, L_{\mathfrak{A}}), (\mathfrak{B}, L_{\mathfrak{B}})) = 0$  *if and only if there exists a \*-isomorphism  $\theta : \mathfrak{A} \rightarrow \mathfrak{B}$  such that  $L_{\mathfrak{B}} \circ \theta = L_{\mathfrak{A}}$ .*

Moreover, the quantum propinquity is the largest class function satisfying Assertions (1),(2), (3) and (4).

The quantum propinquity can be applied to compact metric spaces, using the following encoding of such spaces in our  $C^*$ -algebraic framework — this construction is in fact the original model for quantum compact metric spaces. We will employ the following notation all throughout this paper.

*Notation 1.8.* The Lipschitz seminorm  $\text{Lip}_d$  for a compact metric space  $(X, d)$  is defined for all functions  $f \in C(X)$  by:

$$\text{Lip}_d(f) = \sup \left\{ \frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y \right\},$$

allowing for the value  $\infty$ .

**Theorem 1.9** ([17]). *If  $(X, d)$  be a compact metric space, then  $(C(X), \text{Lip}_d)$  is a Leibniz quantum compact metric space. Moreover, for all compact metric spaces  $(X, d_X)$  and  $(Y, d_Y)$ , we have:*

$$\Lambda((C(X), \text{Lip}_{d_X}), (C(Y), \text{Lip}_{d_Y})) \leq \text{GH}((X, d_X), (Y, d_Y)),$$

where GH is the Gromov-Hausdorff distance [5, 7] and furthermore, the topology induced by  $\Lambda$  on the class of classical compact quantum metric space is the same as the topology induced by GH.

We now answer the question: when is a classical compact metric space the limit, not only of finite dimensional  $C^*$ -algebras, but actually full matrix algebras, for the quantum propinquity?

## 2. FULL MATRIX APPROXIMATIONS

The main result of this note provides a way to construct full matrix approximations of finite metric spaces in a rather general context.

**Lemma 2.1.** *If  $\mathfrak{B}$  is a finite dimensional  $C^*$ -subalgebra of a unital  $C^*$ -algebra  $\mathfrak{A}$  and  $1_{\mathfrak{A}} \in \mathfrak{B}$  and if  $\mathfrak{A}$  has a faithful tracial state  $\mu \in \mathcal{S}(\mathfrak{A})$  then there exists a unique  $\mu$ -preserving conditional expectation  $\mathbb{E} : \mathfrak{A} \rightarrow \mathfrak{B}$ .*

*Proof.* See [12, Step 1 of Theorem (3.5)]. □

**Theorem 2.2.** *Let  $(X, d)$  be a finite metric space and let:*

$$\delta = \min \{d(x, y) : x, y \in X, x \neq y\} > 0.$$

*If  $\mathfrak{A}$  is a finite dimensional  $C^*$ -algebra, if  $\tau$  is some faithful tracial state on  $\mathfrak{A}$ , and if  $\mathfrak{B}$  is a  $C^*$ -subalgebra of  $\mathfrak{A}$  such that:*

- (1)  $1_{\mathfrak{A}} \in \mathfrak{B}$ ,
- (2) *there exists a unital  $*$ -isomorphism  $\rho : C(X) \rightarrow \mathfrak{B}$ ,*

*then, for any  $\beta > 0$ , and setting for all  $a \in \mathfrak{A}$ :*

$$L(a) = \max \left\{ \frac{\|a - \mathbb{E}(a)\|_{\mathfrak{A}}}{\beta}, \text{Lip}_d \circ \rho^{-1}(\mathbb{E}(a)), \right\}$$

*where  $\mathbb{E} : \mathfrak{A} \rightarrow \mathfrak{B}$  is the conditional expectation such that  $\tau \circ \mathbb{E} = \tau$ , we conclude that the space  $(\mathfrak{A}, L)$  is a  $(D, 0)$ -quasi-Leibniz compact quantum metric space, where:*

$$D = \max \left\{ 2, 1 + \frac{\beta}{\delta} \right\}$$

*such that:*

$$\Lambda((\mathfrak{A}, L), (C(X), \text{Lip}_d)) \leq \beta.$$

*Proof.* If  $a \in \mathfrak{A}$  with  $L(a) = 0$  then  $a = \mathbb{E}(a)$ , and  $\text{Lip}_d(\rho^{-1}(\mathbb{E}(a))) = 0$ , so  $\mathbb{E}(a) = \lambda 1_{\mathfrak{A}}$  for some  $\lambda \in \mathbb{R}$ . Thus  $a \in \mathbb{R}1_{\mathfrak{A}}$ , as desired. We also note that  $L(1_{\mathfrak{A}}) = 0$  by assumption.

We also note that since  $X$  is finite,  $\text{dom}(\text{Lip}_d) = C(X)$  so  $\text{dom}(L) = \mathfrak{A}$ .

Since  $L$  is the maximum of two (lower semi-)continuous functions over  $\mathfrak{A}$ , we also have  $L$  is (lower semi-)continuous on  $\mathfrak{A}$ .

The map  $\tau_X = \tau \circ \rho$  is a state of  $C(X)$ , and thus  $\{f \in C(X) : \tau_X(f) = 0, \text{Lip}_d(f) \leq 1\}$  is compact — since  $X$  is finite, this set is actually closed and bounded in the finite dimensional space  $C(X)$ . Let  $B > 0$  so that if  $\text{Lip}_d(f) \leq 1$  and  $\tau_X(f) = 0$  then  $\|f\|_{C(X)} \leq B$ .

Now if  $a \in \mathfrak{sa}(\mathfrak{A})$  with  $L(a) \leq 1$  and  $\tau(a) = 0$  then  $\text{Lip}_d \circ \rho^{-1}(\mathbb{E}(a)) \leq 1$  and  $\tau_X(\rho^{-1}(\mathbb{E}(a))) = \tau \circ \mathbb{E}(a) = \tau(a) = 0$ . Thus  $\|\mathbb{E}(a)\|_{\mathfrak{A}} \leq B$ . Now,  $\|a\|_{\mathfrak{A}} \leq \|a - \mathbb{E}(a)\|_{\mathfrak{A}} + \|\mathbb{E}(a)\|_{\mathfrak{A}} \leq \beta + B$ . So:

$$\{a \in \mathfrak{sa}(\mathfrak{A}) : L(a) \leq 1, \tau(a) = 0\} \subseteq \{a \in \mathfrak{sa}(\mathfrak{A}) : \|a\|_{\mathfrak{A}} \leq \beta + B\},$$

and the right-hand side is compact since  $\mathfrak{A}$  is finite dimensional, so  $(\mathfrak{A}, L)$  is a compact quantum metric space by Theorem (1.3).

Last, we check the quasi-Leibniz property of  $L$ . Let  $a, b \in \text{dom}(L)$  and  $x, y \in X$ . Since  $\rho$  is a  $*$ -isomorphism, we now compute:

$$\begin{aligned} & \left| \rho^{-1}(\mathbb{E}(ab))(x) - \rho^{-1}(\mathbb{E}(ab))(y) \right| \\ & \leq \left| \rho^{-1}(\mathbb{E}(ab))(x) - \rho^{-1}(\mathbb{E}(a\mathbb{E}(b)))(x) \right| \end{aligned}$$

$$\begin{aligned}
& + \left| \rho^{-1}(\mathbb{E}(a\mathbb{E}(b)))(x) - \rho^{-1}(\mathbb{E}(\mathbb{E}(a)b))(y) \right| \\
& + \left| \rho^{-1}(\mathbb{E}(\mathbb{E}(a)b))(y) - \rho^{-1}(\mathbb{E}(ab))(y) \right| \\
& \leq \|\mathbb{E}(a(b - \mathbb{E}(b)))\|_{\mathfrak{A}} \\
& + \left| \rho^{-1}(\mathbb{E}(a))(x)\rho^{-1}(\mathbb{E}(b))(x) - \rho^{-1}(\mathbb{E}(a))(y)\rho^{-1}(\mathbb{E}(b))(y) \right| \\
& + \|\mathbb{E}((a - \mathbb{E}(a))b)\|_{\mathfrak{A}} \\
& \leq \|a\|_{\mathfrak{A}}\beta L(b) + \|b\|_{\mathfrak{A}}\beta L(b) \\
& + \left| \rho^{-1}(\mathbb{E}(a))(x)\rho^{-1}(\mathbb{E}(b))(x) - \rho^{-1}(\mathbb{E}(a))(y)\rho^{-1}(\mathbb{E}(b))(y) \right|.
\end{aligned}$$

Hence:

$$\begin{aligned}
& \text{Lip}_d \circ \rho^{-1}(\mathbb{E}(ab)) \\
& = \sup \left\{ \frac{|\rho^{-1}(\mathbb{E}(ab))(x) - \rho^{-1}(\mathbb{E}(ab))(y)|}{d(x,y)} : x, y \in X, x \neq y \right\} \\
& \leq \|a\|_{\mathfrak{A}} \frac{\beta}{\delta} L(b) + \|b\|_{\mathfrak{A}} \frac{\beta}{\delta} L(b) \\
& + \sup \left\{ \frac{|\rho^{-1}(\mathbb{E}(a))(x)\rho^{-1}(\mathbb{E}(b))(x) - \rho^{-1}(\mathbb{E}(a))(y)\rho^{-1}(\mathbb{E}(b))(y)|}{d(x,y)} \right. \\
(2.0.1) \quad & \left. : x, y \in X, x \neq y \right\} \\
& \leq \frac{\beta}{\delta} (\|a\|_{\mathfrak{A}} L(b) + L(a)\|b\|_{\mathfrak{A}}) + \text{Lip}_d(\mathbb{E}(a)\mathbb{E}(b)) \\
& \leq \frac{\beta}{\delta} (\|a\|_{\mathfrak{A}} L(b) + L(a)\|b\|_{\mathfrak{A}}) + \text{Lip}_d \circ \mathbb{E}(a)\|b\|_{\mathfrak{A}} + \|a\|_{\mathfrak{A}} \text{Lip}_d \circ \mathbb{E}(b) \\
& \leq \left(1 + \frac{\beta}{\delta}\right) (\|a\|_{\mathfrak{A}} L(b) + L(a)\|b\|_{\mathfrak{A}}).
\end{aligned}$$

From this and from [3, Lemma 3.2], it follows easily that  $(\mathfrak{A}, L)$  is indeed a  $(D, 0)$ -quasi-Leibniz quantum compact metric space with  $D = \max \left\{ 2, \left(1 + \frac{\beta}{\delta}\right) \right\}$ .

We now compute an upper bound for  $\Lambda((\mathfrak{A}, L), (C(X), \text{Lip}_d))$  by exhibiting a particular bridge from  $\mathfrak{A}$  to  $C(X)$ .

Let  $\gamma = (\mathfrak{A}, \text{id}, \rho, 1_{\mathfrak{A}})$  where  $\text{id}$  is the identity  $*$ -morphism of  $\mathfrak{A}$ . By Definition (1.4), the quadruple  $\gamma$  is a bridge of height 0, so its length equals to its reach.

If  $f \in C(X)$  and  $\text{Lip}_d(f) \leq 1$ , then:

$$\frac{\|\rho(f) - \mathbb{E}(\rho(f))\|_{\mathfrak{A}}}{\beta} = 0$$

and  $\text{Lip}_d(\rho^{-1}(\mathbb{E}(\rho(f)))) = \text{Lip}_d(f) \leq 1$ . So  $L(\rho(f)) \leq 1$ .

Now, it is immediate that  $\text{bn}_{\gamma}(\rho(f), f) = \|\rho(f) - \rho(f)\|_{\mathfrak{A}} = 0$ . So:

$$\sup_{\substack{f \in C(X) \\ \text{Lip}_d(f) \leq 1}} \inf_{\substack{a \in \text{sa}(\mathfrak{A}) \\ L(a) \leq 1}} \text{bn}_{\gamma}(a, f) = 0.$$

If  $a \in \mathfrak{A}$  with  $L(a) \leq 1$ , then set  $f = \rho^{-1}(\mathbb{E}(a))$ . First, by definition of  $L$ , we have  $\text{Lip}_d(f) = \text{Lip}_d(\rho^{-1}(\mathbb{E}(a))) \leq L(a) \leq 1$ . Second:

$$\|a - \rho(f)\|_{\mathfrak{A}} = \|a - \mathbb{E}(a)\|_{\mathfrak{A}} \leq \beta.$$

Thus

$$\sup_{\substack{a \in \text{sa}(\mathfrak{A}) \\ L(a) \leq 1}} \inf_{\substack{f \in C(X) \\ \text{Lip}_d(f) \leq 1}} \text{bn}_{\gamma}(a, f) \leq \beta.$$

Therefore, the reach, and thus the length, of  $\gamma$  is no more than  $\beta$ . Hence by Theorem-Definition (1.7), we conclude  $\Lambda((\mathfrak{A}, L), (C(X), \text{Lip}_d)) \leq \beta$  as desired.  $\square$

We now deduce from Theorem (2.2) the main result of this note: compact metric spaces are always limits of full matrix algebras for the quantum propinquity. A notable component of the following result is how the constant  $\beta$  of Theorem (2.2) are related to the actual geometry of the limit classical space.

**Corollary 2.3.** *If  $(X, d)$  is a compact metric space, if  $Y \subseteq X$  is a finite subset of  $X$ , and if  $\beta_Y \in (0, \infty)$  such that:*

$$\frac{\beta_Y}{\min\{d(x, y) : x, y \in Y, x \neq y\}} \leq 1$$

then there exists a  $(2, 0)$ -quasi-Leibniz quantum compact metric space  $(\mathfrak{A}, L)$  where:

- (1)  $\mathfrak{A}$  is the  $C^*$ -algebra of  $\#Y \times \#Y$ -matrices over  $\mathbb{C}$  and  $\tau$  is the unique tracial state on  $\mathfrak{A}$ ,
- (2) with  $C(Y)$  identified with the diagonal  $C^*$ -subalgebra of  $\mathfrak{A}$  given by a unital  $*$ -isomorphism  $\rho$  with domain  $C(Y)$  and  $\mathbb{E}_Y$ , the unique  $\tau$ -preserving conditional expectation of  $\mathfrak{A}$  onto  $\rho(C(Y))$ , the  $L$ -seminorm  $L$  is given for all  $a \in \mathfrak{A}$  by:

$$(2.0.2) \quad L(a) = \max \left\{ \frac{\|a - \mathbb{E}_Y(a)\|_{\mathfrak{A}}}{\beta_Y}, \text{Lip}_d \circ \rho^{-1}(\mathbb{E}_Y(a)) \right\},$$

and

- (3)  $\Lambda((\mathfrak{A}, L), (C(X), \text{Lip}_d)) \leq \text{Haus}_d(X, Y) + \beta_Y$ .

*Proof.* Set  $\delta = \min\{d(x, y) : x, y \in Y, x \neq y\}$ . By Theorem (2.2), the compact quantum metric space  $(\mathfrak{A}, L)$  is  $(2, 0)$ -quasi-Leibniz since  $1 + \frac{\beta_Y}{\delta} \leq 2$  and:

$$\Lambda((\mathfrak{A}, L), (C(Y), \text{Lip}_d)) \leq \beta_Y.$$

Thus:

$$\begin{aligned} \Lambda((\mathfrak{A}, L), (C(X), \text{Lip}_d)) &\leq \\ &\Lambda((\mathfrak{A}, L), (C(Y), \text{Lip}_d)) + \Lambda((C(Y), \text{Lip}_d), ((C(X), \text{Lip}_d))) \\ &\leq \beta_Y + \text{Haus}_d(X, Y). \end{aligned}$$

This concludes our proof.  $\square$

**Corollary 2.4.** *Any compact metric space  $(X, d)$  is the limit for the quantum propinquity of sequences of  $(2, 0)$ -quasi-Leibniz quantum compact metric spaces consisting of full matrix algebras.*

*Proof.* We simply apply Corollary (2.3) to any sequence  $(X_n)_{n \in \mathbb{N}}$  of finite subsets of  $X$  with  $\lim_{n \rightarrow \infty} \text{Haus}_d(X, X_n) = 0$ , which always exists since  $(X, d)$  is compact, and to  $(\beta_{X_n})_{n \in \mathbb{N}} = \left( \frac{\min\{d(x, y) : x, y \in X_n, x \neq y\}}{n} \right)_{n \in \mathbb{N}}$ .  $\square$

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*E-mail address:* konrad.aguilar@gmail.com

SCHOOL OF MATHEMATICAL AND STATISTICAL SCIENCES, ARIZONA STATE UNIVERSITY, TEMPE  
AZ 85281

*E-mail address:* frederic@math.du.edu

*URL:* <http://www.math.du.edu/~frederic>

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF DENVER, DENVER CO 80208