SEMIGROUPOID, GROUPOID AND GROUP ACTIONS ON LIMITS FOR THE GROMOV-HAUSDORFF PROPINQUITY

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Abstract. The Gromov-Hausdorff propinquity provides an analytical framework motivated by mathematical physics where quasi-Leibniz quantum compact metric spaces may be studied by means of metric approximations. A natural question in this setting, answered in this paper, is whether group actions pass to the limit for this new geometry: if a sequence of quasi-Leibniz quantum compact metric spaces is Cauchy for the propinquity and each entry carries a non-trivial compact group action, and if the resulting sequence of groups converges, does the Gromov-Hausdorff limit carry a non-trivial action of the limit group ? What about actions from groupoids, or inductive limits of groups? We establish a general result addressing all these matters. Our result provides a first example of a structure which passes to the limit of quantum metric spaces for the propinquity, as well as a new method to construct group actions, including from non-locally compact groups seen as inductive limits of compact groups, on unital C*-algebras.

Contents

The noncommutative Gromov-Hausdorff propinquity [16, 12, 18, 15, 17] is a fundamental component of an analytic framework for the study of the geometry of the hyperspace of quantum metric spaces, with sights on applications to mathematical physics.

A natural yet delicate question is to determine which properties do pass to the limit of a sequence of quantum metric spaces. Partial answers to this question can for instance help us determine the closure of a class of quantum metric spaces. Most importantly, the propinquity is intended to be used to construct models for physics from finite dimensional [10, 1, 29] or other form of approximations [11, 19], with the models obtained as limits, possibly using the fact that the propinquity is complete. Thus, we would like to know which properties of the approximate finite models would indeed carry to the eventual limit.

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The existence of group actions on limits of sequences of quantum compact metric spaces is thus of prime interest. The question can be informally asked: if a sequence of metrized compact groups act on a Cauchy sequence of quasi-Leibniz quantum compact metric spaces, and if these groups converge in some appropriate sense, then does the limit carry an action of the limit group? Asked in such a manner, the answer is trivially yes — one may involve trivial actions — but it is easy and rather natural to impose a form of nontriviality using the underlying quantum metrics. Indeed, we can control the dilation of the automorphisms [14] arising from the actions at each level to ensure that any sort of limit would also be non-trivial. We also note that much efforts have been dedicated to constructing full matrix approximations of certain symmetric spaces [25, 27, 28, 29] with the idea that finite dimensional quantum metric spaces are the noncommutative analogues of finite spaces, though they can carry actions of compact Lie groups such as $SU(n)$, and therefore have highly non-trivial symmetries — thus, offering an option to work with "finite spaces" with continuous symmetries. From this perspective, our present work addresses a sort of dual question: if we do use approximations of a physical model using, say, finite dimensional algebras with some specified symmetries, what are sufficient conditions for these symmetries to still be present at the limit for the propinquity?

We establish a formal answer to this problem, in a more general context: rather than only working with groups, we prove the result for an appropriate notion of action of semigroupoids — thus, our result applies to groups actions, groupoid actions on class of C*-algebras, and semigroups actions, among others. We also note that our result can be applied to prove the existence of non-trivial actions of inductive limits of groups, even if this inductive limit is not locally compact. While left for further research, the results of this paper hints at a possible application of techniques from noncommutative metric geometry to the study of actions of non-locally compact groups on C*-algebras.

We begin this paper with a review of our theory for the hyperspace of quantum metric spaces. We then prove our main result in the context of actions of countable semigroupoids, appropriately defined. Our result does not per say require an action of a semigroupoid, but rather an approximate form of it. We then turn to various applications of our main result, concerning actions of groups, groupoids, and semigroups.

The following notations will be used throughout this paper.

Notation. If (E,d) is a metric space, the diameter $\sup\{d(x,y) : x,y \in E\}$ of E is denoted by $\mathrm{diam}\,(E,d)$.

The norm of a normed vector space X is denoted by $\|\cdot\|_X$ by default. The unit of a unital C*-algebra $\mathfrak{A}$ is denoted by $1_{\mathfrak{A}}$. The state space of a C*-algebra $\mathfrak{A}$ is denoted by $\mathscr{S}(\mathfrak{A})$ while the real subspace of self-adjoint elements of $\mathfrak{A}$ is denoted by $\mathfrak{sa}\,(\mathfrak{A})$.

Given any two self-adjoint elements $a,b \in \mathfrak{sa}\,(\mathfrak{A})$ of a C*-algebra $\mathfrak{A}$, the *Jordan product* $\frac{ab+ba}{2}$, i.e. the real part of ab, is denoted by $a \circ b$, while the *Lie product* $\frac{ab-ba}{2i}$ — the imaginary part of ab — is denoted by $\{a,b\}$. The notation $\circ$ will often be used for composition of various maps, but will always mean the Jordan product when applied to elements of a C*-algebra, as is customary. The triple $(\mathfrak{sa}\,(\mathfrak{A}), \cdot\circ\cdot, \{\cdot,\cdot\})$ is a Jordan-Lie algebra [9].

Last, for any continuous linear map $\varphi : E \to F$ between two normed vector spaces E and F, the norm of φ is denoted by $|||\varphi|||_F^E$ — or simply $|||\varphi|||_E$ if $E = F$.

1. Background

Compact quantum metric spaces were introduced by Rieffel [22, 23], inspired by the work of Connes [4], as a noncommutative generalization of algebras of Lipschitz functions over compact metric space [31]. We added to this original approach the requirement that a form of the Leibniz inequality holds for the generalized Lipschitz seminorm in a quantum compact metric space; the upper bound of this inequality is given using a particular type of functions:

Definition 1.1. A function $F : [0,\infty]^4 \to [0,\infty]$ is *admissible* when:

(1) F is weakly increasing if $[0,\infty]^4$ is equipped with the product order,
(2) for all $x, y, l_x, l_y \geqslant 0$ we have $xl_y + yl_x \leqslant F(x, y, l_x, l_y)$.

The second condition in the above definition of an admissible function ensures that the usual Leibniz inequality is the "best" upper bound one might expect for our results to hold in general.

A quasi-Leibniz quantum compact metric space is thus defined as follows:

Definition 1.2 ([4, 22, 23, 13])**.** A *F–quasi-Leibniz quantum compact metric space* $(\mathfrak{A}, \mathsf{L})$, where F is an admissible function, is an ordered pair of a unital C*-algebra $\mathfrak{A}$ and a seminorm L defined on a dense Jordan-Lie subalgebra $\mathrm{dom}\,(\mathsf{L})$ of $\mathfrak{sa}\,(\mathfrak{A})$ such that:

(1) $\{a \in \mathrm{dom}\,(\mathsf{L}) : \mathsf{L}(a) = 0\} = \mathbb{R}1_{\mathfrak{A}}$,
(2) the *Monge-Kantorovich metric* mk_{L} defined between any two states $\varphi, \psi \in \mathscr{S}(\mathfrak{A})$ of $\mathfrak{A}$ by:
$$\mathsf{mk}_{\mathsf{L}}(\varphi, \psi) = \sup\{|\varphi(a) - \psi(a)| : a \in \mathrm{dom}\,(\mathsf{L}), \mathsf{L}(a) \leqslant 1\}$$
metrizes the weak* topology of $\mathscr{S}(\mathfrak{A})$,
(3) for all $a, b \in \mathrm{dom}\,(\mathsf{L})$ we have:
$$\max\{\mathsf{L}\,(a \circ b)\,, \mathsf{L}\,(\{a, b\})\} \leqslant F\,(\|a\|_{\mathfrak{A}}, \|b\|_{\mathfrak{A}}, \mathsf{L}(a), \mathsf{L}(b))\,,$$
(4) L is lower semi-continuous with respect to $\|\cdot\|_{\mathfrak{A}}$.

When $(\mathfrak{A}, \mathsf{L})$ is a quasi-Leibniz quantum compact metric space, the seminorm L is called an *L-seminorm.*

Notation 1.3. If $(\mathfrak{A}, \mathsf{L})$ is a quasi-Leibniz quantum compact metric space, then $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}})$ is a compact metric space, so it has finite diameter, which we denote by $\mathrm{diam}\,(\mathfrak{A}, \mathsf{L})$.

Example 1.4 (Classical Metric Spaces)*.* Let (X, d) be a compact metric space, and for all $f : X \to \mathbb{R}$, define:
$$\mathsf{L}(f) = \sup\left\{\frac{|f(x) - f(y)|}{d(x, y)} : x, y \in X, x \neq y\right\}.$$

The pair $(C(X), \mathsf{L})$ of the unital C*-algebra of $\mathbb{C}$-valued continuous functions on X and the seminorm L is a Leibniz quantum compact metric space.

Example 1.5 (Noncommutative Examples). Many examples of noncommutative quasi-Leibniz quantum compact metric spaces are known to exist, including quantum tori [22, 24], curved quantum tori [11], AF algebras [1], noncommutative solenoids [14], hyperbolic groups C*-algebras [21], nilpotent group C*-algebras [3], among others.

Quasi-Leibniz quantum compact metric spaces are objects for several natural categories. For this paper, we will use three different notions of morphisms. We begin with the weakest of these notions, which we will use to state our main theorem at a hopefully reasonable level of generality.

Definition 1.6. Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. A continuous linear map $\varphi : \mathfrak{A} \to \mathfrak{B}$ is *Lipschitz* when:

$$\forall a \in \mathrm{dom}\,(\mathsf{L}_{\mathfrak{A}}) \quad \varphi(a) \in \mathrm{dom}\,(\mathsf{L}_{\mathfrak{B}})$$

and $\varphi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.

Remark 1.7. A linear Lipschitz map, in particular, maps self-adjoint elements to self-adjoint elements, since it is continuous and maps the domain of an L-seminorm to self-adjoint elements — while domains of L-seminorms are dense in the self-adjoint part of C*-algebras.

The terminology of Definition (1.6) is explained by the following corollary of our work in [19]:

Proposition 1.8. *If* $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ *and* $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ *are quasi-Leibniz quantum compact metric spaces, and if* $\varphi : \mathfrak{A} \to \mathfrak{B}$ *is a linear map with* $\varphi(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$*, then the following are equivalent:*

(1) φ *is Lipschitz,*
(2) there exists $k > 0$ *such that* $\mathsf{L}_{\mathfrak{B}} \circ \varphi \leqslant k\mathsf{L}_{\mathfrak{A}}$.

Proof. This result follows by [19, Theorem 2.1]. □

It is straightforward to check that the identity of any quasi-Leibniz quantum compact metric spaces is a Lipschitz linear map, and that the composition of Lipschitz linear maps is itself Lipschitz, so we have defined a category, as announced.

Among all the Lipschitz linear functions, we find the *Lipschitz morphisms* [19]:

Definition 1.9. Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be quasi-Leibniz quantum compact metric spaces. A *Lipschitz morphism* $\varphi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a unital *-morphism which is also Lipschitz linear.

Using Lipschitz morphisms as arrows does form a category over the class of quasi-Leibniz quantum compact metric spaces, which is in fact the *natural*, default category for our theory. To be more explicit, one may argue that going from just linear maps to actual *-morphisms is the key reason to introduce the propinquity, versus other versions of the noncommutative Gromov-Hausdorff distance [30, 8, 26]: the propinquity was introduced specifically to allow us to work within the category of C*-algebras with their *-morphisms. Moreover, the dual maps induced by unital *-morphisms restricts to functions between state spaces, and these functions are Lipschitz for the Monge-Kantorovich metric if and only if the *-morphism is a Lipschitz morphism, thus completing the geometric picture.

The only reason for our digression via the Lipschitz linear maps is to be able to phrase our main theorem at a slightly higher level of generality, as certain examples, such as semigroups of positive linear maps, have appeared in the literature and present potential examples for applications of our present work.

There is a natural distance on the space of unital continuous linear maps between quasi-Leibniz quantum compact metric spaces, which generalizes the Monge-Kantorovich length of *-automorphisms discussed in [19]. We will use this distance for two purposes. It will quantify the idea that the composition of two Lipschitz linear maps almost equal to some other Lipschitz linear map — a tool used in the hypothesis of our main theorem — and it will also give us a non-triviality test, in the conclusion of our main theorem.

Definition 1.10. Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ be a quasi-Leibniz quantum compact metric spaces and $\mathfrak{B}$ be a unital C*-algebra. For any two linear maps φ, ψ from $\mathfrak{A}$ to $\mathfrak{B}$ which map $1_{\mathfrak{A}}$ to $1_{\mathfrak{B}}$, we define:

$$\mathrm{mkD}_{\mathfrak{B}}^{\mathsf{L}_{\mathfrak{A}}}(\varphi,\psi) = \sup\left\{ \|\varphi(a)-\psi(a)\|_{\mathfrak{B}} : a \in \mathrm{dom}\,(\mathsf{L}_{\mathfrak{A}}), \mathsf{L}_{\mathfrak{A}}(a) \leqslant 1 \right\}.$$

In particular, if $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) = (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ then we write $\mathrm{mk}\ell_{\mathsf{L}_{\mathfrak{A}}}(\varphi)$ for $\mathrm{mkD}_{\mathfrak{A}}^{\mathsf{L}_{\mathfrak{A}}}(\mathrm{id},\varphi)$ where id is the identity automorphism of $\mathfrak{A}$.

In [19, Proposition 5.2], we proved that for any quasi-Leibniz quantum compact metric space $(\mathfrak{A}, \mathsf{L})$, the function $\mathrm{mk}\ell_{\mathsf{L}}$ is a length function on the group of *-automorphisms of $\mathfrak{A}$ which metrizes the pointwise convergence topology. This result and its proof extends readily to the setting of Definition (1.10), so we sketch the proof rather briefly:

Proposition 1.11. *Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ be a quasi-Leibniz quantum compact metric spaces and $\mathfrak{B}$ be a unital C*-algebra. The function $\mathrm{mkD}_{\mathfrak{B}}^{\mathsf{L}_{\mathfrak{A}}}(\cdot,\cdot)$ defines a distance on the set of all continuous linear functions from $\mathfrak{A}$ to $\mathfrak{B}$ which map $1_{\mathfrak{A}}$ to $1_{\mathfrak{B}}$.*

Proof. Let $\mu \in \mathscr{S}(\mathfrak{A})$. By definition of the Monge-Kantorovich metric, for all $a \in \mathrm{dom}\,(\mathsf{L}_{\mathfrak{A}})$:

$$\|a - \mu(a)1_{\mathfrak{A}}\|_{\mathfrak{A}} \leqslant \mathrm{diam}\,(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})\mathsf{L}_{\mathfrak{A}}(a).$$

Thus for any two continuous linear maps $\varphi, \psi : \mathfrak{A} \to \mathfrak{B}$ mapping $1_{\mathfrak{A}}$ to $1_{\mathfrak{B}}$, and for all $a \in \mathrm{dom}\,(\mathsf{L}_{\mathfrak{A}})$ with $\mathsf{L}_{\mathfrak{A}}(a) \leqslant 1$:

$$\begin{aligned}\|\varphi(a)-\psi(a)\|_{\mathfrak{B}} &= \|\varphi(a-\mu(a)1_{\mathfrak{A}}) - \psi(a-\mu(a)1_{\mathfrak{A}})\|_{\mathfrak{B}} \\ &\leqslant \mathrm{diam}\,(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \left(|||\varphi|||_{\mathfrak{B}}^{\mathfrak{A}} + |||\psi|||_{\mathfrak{B}}^{\mathfrak{A}} \right).\end{aligned}$$

Thus $\mathrm{mkD}_{\mathfrak{B}}^{\mathsf{L}_{\mathfrak{A}}}(\cdot,\cdot)$ is bounded. The triangle inequality and the symmetry properties of $\mathrm{mkD}_{\mathfrak{B}}^{\mathsf{L}_{\mathfrak{A}}}(\cdot,\cdot)$ are obvious.

If $\mathrm{mkD}_{\mathfrak{B}}^{\mathsf{L}_{\mathfrak{A}}}(\varphi,\psi) = 0$ then φ and ψ agree on the total set $\{a \in \mathfrak{sa}\,(\mathfrak{A}) : \mathsf{L}_{\mathfrak{A}}(a) \leqslant 1\}$, and thus by linearity and continuity, they agree on $\mathfrak{A}$. Thus $\mathrm{mkD}_{\mathfrak{B}}^{\mathsf{L}_{\mathfrak{A}}}(\cdot,\cdot)$ is a distance as claimed.

The proof that $\mathrm{mkD}_{\mathfrak{B}}^{\mathsf{L}_{\mathfrak{A}}}(\cdot,\cdot)$ induces the topology of pointwise convergence follows the same argument as [19, Claim 5.4, Proposition 5.2], as the reader can easily checked (where pointwise convergence topology means the locally convex topology induced by the family of seminorms defined over the space of linear maps from $\mathfrak{A}$ to $\mathfrak{B}$ by $\varphi \mapsto \|\varphi(a)\|_{\mathfrak{B}}$ where a ranges over all of $\mathfrak{A}$). □

Remark 1.12. Definition (1.10) and Proposition (1.11) would make sense and hold if the codomain was some normed vector space, as long as we work on the set of continuous linear maps which map the unit to some fixed vector. This level of generality is not however needed here.

Now, among all the Lipschitz morphisms, the quantum isometries form a subcategory, and are essential tools in the construction of the propinquity. Moreover, our chosen notion of isomorphism between quasi-Leibniz quantum compact metric spaces will be given by the full quantum isometries — an important observation to understand the geometry of the propinquity. The notion of a quantum isometry is largely due to Rieffel, and essentially expresses that McShane theorem [20] on extension of real-valued Lipschitz functions holds in our noncommutative context. This justifies, a posteriori, the emphasis on self-adjoint elements in Definition (1.2). Our modification of Rieffel's notion consists of the requirement that quantum isometries be *-morphisms, and in particular, full quantum isometries be *-isomorphisms, rather than positive unital linear maps.

Definition 1.13. Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two quasi-Leibniz quantum compact metric spaces. A *quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \twoheadrightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-epimorphism $\pi : \mathfrak{A} \twoheadrightarrow \mathfrak{B}$ such that for all $b \in \mathfrak{sa}(\mathfrak{B})$:

$$\mathsf{L}_{\mathfrak{B}}(b) = \inf \{\mathsf{L}_{\mathfrak{A}}(a) : \pi(a) = b\}.$$

A *full quantum isometry* $\pi : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is a *-isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

A full quantum isometry is a quantum isometry which is a also a bijection and whose inverse map is a quantum isometry. Quantum isometries are in particular 1-Lipschitz morphisms between quasi-Leibniz quantum compact metric spaces [15] and they are morphisms for a category whose objects are quasi-Leibniz quantum compact metric spaces [30].

Our work focuses on the construction of geometries on the hyperspace of all quasi-Leibniz quantum compact metric spaces, or phrased a little more formally, on defining metrics on the class of all quasi-Leibniz quantum compact metric spaces, or some subclass therein. There are many subtleties involved in these constructions; our focus will be on the dual Gromov-Hausdorff propinquity as introduced in [12, 18].

The dual Gromov-Hausdorff propinquity is a complete distance, up to full quantum isometry, on the class of F–quasi-Leibniz quantum compact metric spaces, which induces the topology of the Gromov-Hausdorff distance [5, 6] on the subclass of all classical metric spaces. Its construction is summarized as follows.

We first generalize the idea of an isometric embedding of two quasi-Leibniz quantum compact metric spaces into a third one.

Definition 1.14 ([12])**.** Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two F–quasi-Leibniz quantum compact metric spaces. An *F-tunnel* $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ is given by a F–quasi-Leibniz quantum compact metric space $(\mathfrak{D}, \mathsf{L}_{\mathfrak{D}})$ as well as two quantum isometries $\pi_{\mathfrak{A}} : (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \twoheadrightarrow (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $\pi_{\mathfrak{B}} : (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}) \twoheadrightarrow (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$.

The *domain* of τ is $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and the *codomain* of τ is $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$.

We then associate a numerical value to every tunnel.

Notation 1.15. If $\pi : \mathfrak{A} \to \mathfrak{B}$ is a positive unital linear map from a unital C*-algebra $\mathfrak{A}$ to a unital C*-algebra $\mathfrak{B}$, then the map $\varphi \in \mathscr{S}(\mathfrak{B}) \to \varphi \circ \pi \in \mathscr{S}(\mathfrak{A})$ is denoted by π^*.

Notation 1.16. The Hausdorff distance [7, p. 293] induced on the hyperspace of compact subsets of a metric space (X, d) is denoted by Haus_d; if X is in fact a vector space and d is induced by some norm $\|\cdot\|_X$, then Haus_d is denoted by $\mathsf{Haus}_{\|\cdot\|_X}$.

Definition 1.17 ([18])**.** The *extent* of a tunnel $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_1, \pi_2)$ is:

$$\max\left\{\mathsf{Haus}_{\mathsf{mk}_{\mathsf{L}_{\mathfrak{D}}}}\left(\mathscr{S}(\mathfrak{D}), \pi_j^*(\mathscr{S}(\mathfrak{A}_j))\right) : j = 1, 2\right\}.$$

Now, the dual propinquity can in fact be "specialized", by which we mean that we may consider only certain tunnels with additional properties. For instance, one might want to work with tunnels involving only strongly Leibniz L-seminorms [27]. The following definition summarizes the properties needed on a class of tunnels to allow for the definition of a version of the dual propinquity. The reader may skip this matter, except for the definition of the inverse of a tunnel in item (2), and simply assume that we will work with all F-tunnels for a given choice of an admissible function F in this paper; however, it is worth to point out that our main result applies equally well to any specialization of the propinquity.

Definition 1.18 ([18])**.** Let F be an admissible function. A class $\mathcal{T}$ of tunnels is *appropriate* for a nonempty class $\mathcal{C}$ of F–quasi-Leibniz quantum compact metric spaces when the following properties hold.

(1) If $\tau \in \mathcal{T}$ then $\mathrm{dom}(\tau), \mathrm{codom}(\tau) \in \mathcal{C}$,
(2) If $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are in $\mathcal{C}$, then there exists $\tau \in \mathcal{T}$ such that $\mathrm{dom}(\tau) = (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $\mathrm{codom}(\tau) = (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$.
(3) If $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{B}}, \pi, \rho) \in \mathcal{T}$ then $\tau^{-1} = (\mathfrak{D}, \mathsf{L}_{\mathfrak{B}}, \rho, \pi) \in \mathcal{T}$.
(4) If there exists a full quantum isometry $h : (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \to (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ for any $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in \mathcal{C}$, then the tunnels $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}, \mathrm{id}_{\mathfrak{A}}, h)$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}, h^{-1}, \mathrm{id}_{\mathfrak{B}})$ are in $\mathcal{T}$, where $\mathrm{id}_{\mathfrak{E}}$ is the identity automorphism of any C*-algebra $\mathfrak{E}$.
(5) If $\tau, \gamma \in \mathcal{T}$ with $\mathrm{codom}(\tau) = \mathrm{dom}(\gamma)$ them, for all $\varepsilon > 0$, there exists $\tau \circ_\varepsilon \gamma \in \mathcal{T}$ such that $\mathrm{dom}(\tau) = \mathrm{dom}(\tau \circ_\varepsilon \gamma)$, $\mathrm{codom}(\gamma) = \mathrm{codom}(\tau \circ_\varepsilon \gamma)$ and:
$$\chi(\tau \circ_\varepsilon \gamma) < \chi(\tau) + \chi(\gamma) + \varepsilon.$$

The most important example of an appropriate class of tunnels is given by:

Proposition 1.19 ([18])**.** *If F is an admissible function, then the class of all F-tunnels is appropriate for the class of all F–quasi-Leibniz quantum compact metric spaces.*

We now have all the ingredients to define our propinquity.

Notation 1.20. Let $\mathcal{T}$ be a class of tunnels appropriate for a nonempty class $\mathcal{C}$ of F–quasi-Leibniz quantum compact metric spaces for some admissible function F. The class of all tunnels in $\mathcal{T}$ from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{B}}) \in \mathcal{C}$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}) \in \mathcal{C}$ is denoted by:

$$\mathscr{T}\!unnels\left[(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \xrightarrow{\mathcal{T}} (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right].$$

Definition 1.21. Let $\mathcal{T}$ be a class of tunnels appropriate for a nonempty class $\mathcal{C}$ of F–quasi-Leibniz quantum compact metric spaces for some admissible function F. Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ in $\mathcal{C}$.

The *dual $\mathcal{T}$-propinquity* $\Lambda^*_{\mathcal{T}}((\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}}))$ between $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ is the real number:

$$\inf\left\{\chi(\tau) : \tau \in \mathscr{T}\!unnels\left[(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}}) \xrightarrow{\mathcal{T}} (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})\right]\right\}.$$

Notation 1.22. Let F be some admissible function. If $\mathcal{C}$ is the class of all F–quasi-Leibniz quantum compact metric spaces and $\mathcal{T}$ is the class of all F tunnels. The dual $\mathcal{T}$-propinquity is simply denoted by $\mathsf{\Lambda}^*_F$.

If, moreover, $F(x,y,z,w) = xw + yz$ for all $x,y,z,w \geqslant 0$, i.e. if we work within the class of Leibniz quantum compact metric spaces, then we denote $\mathsf{\Lambda}^*_F$ simply as $\mathsf{\Lambda}^*$.

The main properties of our metric are summarized in the following:

Theorem 1.23 ([10, 18])**.** *Let F be an admissible function. If $\mathcal{T}$ is a class of tunnels admissible for a nonempty class $\mathcal{C}$ of F–quasi-Leibniz quantum compact metric spaces for some admissible function F, then the dual $\mathcal{T}$-propinquity $\mathsf{\Lambda}^*_{\mathcal{T}}$ is a metric up to full quantum isometry on $\mathcal{C}$.*

Moreover the dual propinquity $\mathsf{\Lambda}^*_F$ is complete.

Last, when restricted to the class of classical metric spaces, through the identification given in Example (1.4), the propinquity $\mathsf{\Lambda}_F$ induces the same topology as the Gromov-Hausdorff distance [5, 6].

Tunnels behave, in some ways, like morphisms, albeit their composition is only defined up to an arbitrary choice of a positive constant. We will use the morphism-like properties of tunnels in a fundamental way in this paper and so we now recall them. Lipschitz elements of a quasi-Leibniz quantum compact metric space have images by tunnels, although these images are themselves subsets of the codomain.

Definition 1.24. Let F be an admissible function, and let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two F–quasi-Leibniz quantum compact metric spaces. Let $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$ be an F-tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$. The *$l$-target set* of $a \in \mathrm{dom}(\mathsf{L}_A)$ for $l \geqslant \mathsf{L}_{\mathfrak{A}}(a)$ is:

$$\mathfrak{t}_\tau(a|l) = \left\{ \pi_{\mathfrak{B}}(d) \in \mathfrak{sa}(\mathfrak{B}) \middle| \begin{array}{l} d \in \mathfrak{sa}(\mathfrak{D}) \\ \mathsf{L}_{\mathfrak{D}}(d) \leqslant l \\ \pi_{\mathfrak{A}}(d) = a \end{array} \right\}.$$

Target sets are a form of set-valued morphisms:

Proposition 1.25 ([12, 13])**.** *Let F be an admissible function, and let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be F–quasi-Leibniz quantum compact metric spaces. Let τ be an F-tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$. For all $a, a' \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ and $l \geqslant \max\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{A}}(a')\}$, if $b \in \mathfrak{t}_\tau(a|l)$ and $b' \in \mathfrak{t}_\tau(a'|l)$ then:*

1. *for all $t \in \mathbb{R}$:*
$$b + tb' \in \mathfrak{t}_\tau(a + ta'|(1+|t|)l),$$
2. *if $K(a,a',l,\tau) = F(\|a\|_A + 2l\chi(\tau), \|a'\|_{\mathfrak{A}} + 2l\chi(\tau), l, l)$ then:*
$$b \circ b' \in \mathfrak{t}_\tau(a \circ a'|K(a,a',l,\tau)) \text{ and } \{b,b'\} \in \mathfrak{t}_\tau(\{a,a'\}|K(a,a',l,\tau)).$$

The observation on which many proofs, including proofs in this paper, rely, is that target sets size is controlled by the extend of the tunnel.

Proposition 1.26 ([12])**.** *Let F be an admissible function and let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be F–quasi-Leibniz quantum compact metric spaces. Let τ be an F-tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$. For all $a \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ and $l \geqslant \mathsf{L}_{\mathfrak{A}}(a)$, the target set $\mathfrak{t}_\tau(a|l)$ is a nonempty compact subset of $\mathfrak{sa}(\mathfrak{B})$, and moreover if $d \in \mathfrak{sa}(\mathfrak{D})$ is chosen so that $\pi_{\mathfrak{A}}(d) = a$ and $\mathsf{L}_{\mathfrak{D}}(d) \leqslant l$ then:*

$$\|d\|_{\mathfrak{D}} \leqslant \|a\|_{\mathfrak{A}} + l\chi(\tau),$$

so in particular, if $b \in \mathfrak{t}_{\tau}(a|l)$ then:

$$\|b\|_{\mathfrak{B}} \leqslant \|a\|_{\mathfrak{A}} + 2l\chi(\tau).$$

As a corollary, if $a, a' \in \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ and $l \geqslant \max\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{A}}(a')\}$ then:

$$\forall b \in \mathfrak{t}_{\tau}(a|l) \forall b' \in \mathfrak{t}_{\tau}(a'|l) \quad \|b - b'\|_{\mathfrak{B}} \leqslant \|a - a'\|_{\mathfrak{A}} + 2l\chi(\tau)$$

and therefore:

$$\mathrm{diam}(\mathfrak{t}_{\tau}(a|l), \|\cdot\|_{\mathfrak{B}}) \leqslant 4l\chi(\tau).$$

We conclude this section with a new lemma. A tunnel enables us to pick corresponding ε-dense finite subsets of its domain and codomain. This lemma will prove helpful in checking that the morphisms which our main theorem builds will be non-trivial, under appropriate conditions.

Lemma 1.27. *Let F be an admissible function. Let $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ be two F–quasi-Leibniz quantum compact metric spaces. Let $\varepsilon > 0$ and let τ be an F-tunnel from $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ to $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ such that $\chi(\tau) < \varepsilon$. Let E be a finite subset of $\{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}_{\mathfrak{A}}(a) \leqslant 1\}$ such that for all $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}_{\mathfrak{A}}(a) \leqslant 1$, there exists $a' \in E$ and $t \in \mathbb{R}$ such that:*

$$\|a - (a' + t1_{\mathfrak{A}})\|_{\mathfrak{A}} < \varepsilon.$$

There exists a finite subset $G \subseteq \{a \in \mathfrak{sa}(\mathfrak{B}) : \mathsf{L}_{\mathfrak{B}}(a) \leqslant 1\}$ such that:

1. *for all $a \in \mathfrak{sa}(\mathfrak{B})$ with $\mathsf{L}_{\mathfrak{B}}(a) \leqslant 1$, there exists $t \in \mathbb{R}$ and $a' \in G$ such that $\|a - (a' + t1_{\mathfrak{A}})\|_{\mathfrak{B}} \leqslant 3\varepsilon$,*
2. *for all $a \in G$ there exists $a' \in E$ such that $a \in \mathfrak{t}_{\tau}(a'|1)$, and conversely, for all $a \in E$ there exists $a' \in G$ such that $a \in \mathfrak{t}_{\tau^{-1}}(a'|1)$.*

Proof. We write $\tau = (\mathfrak{D}, \mathsf{L}_{\mathfrak{D}}, \pi_{\mathfrak{A}}, \pi_{\mathfrak{B}})$.

For all $a \in E$, choose $f(a) \in \mathfrak{t}_{\tau}(a|1)$. Note that $\mathsf{L}_{\mathfrak{B}}(f(a)) \leqslant 1$ by construction. Moreover, note that $a \in \mathfrak{t}_{\tau^{-1}}(f(a)|1)$.

Let $G = \{f(a) : a \in E\}$. Now, let $b \in \mathfrak{sa}(\mathfrak{B})$ with $\mathsf{L}_{\mathfrak{B}}(b) \leqslant 1$. Let $a \in \mathfrak{t}_{\tau^{-1}}(b|1)$. Since $\mathsf{L}_{\mathfrak{A}}(a) \leqslant 1$, there exists $a' \in E$ and $t \in \mathbb{R}$ such that $\|a - (a' + t1_{\mathfrak{A}})\|_{\mathfrak{A}} < \varepsilon$.

By Definition (1.24), there exists $d \in \mathrm{dom}(\mathsf{L}_{\mathfrak{D}})$ with $\pi_{\mathfrak{A}}(d) = a'$, $\pi_{\mathfrak{B}}(d) = f(a')$, and $\mathsf{L}_{\mathfrak{D}}(d) \leqslant 1$. Now, $\pi_{\mathfrak{A}}(d + t1_{\mathfrak{D}}) = a' + t1_{\mathfrak{A}}$, while $\pi_{\mathfrak{B}}(d + t1_{\mathfrak{B}}) = f(a') + t1_{\mathfrak{B}}$, and moreover:

$$\mathsf{L}_{\mathfrak{D}}(d + t1_{\mathfrak{D}}) = \mathsf{L}_{\mathfrak{D}}(d) \leqslant 1.$$

Therefore $f(a') + t1_{\mathfrak{B}} \in \mathfrak{t}_{\tau}(a' + t1_{\mathfrak{A}}|1)$, again by Definition (1.24). It follows by Proposition (1.26) that:

$$\|b - (f(a') + t1_{\mathfrak{B}})\|_{\mathfrak{B}} \leqslant \|a - (a' + t1_{\mathfrak{A}})\|_{\mathfrak{A}} + 2\chi(\tau) \leqslant 3\varepsilon.$$

This concludes our proof. □

We now turn to our main theorem in the next section.

2. Actions of Semigroupoids on Limits

We expand on the idea that tunnels are sort of set-valued morphisms by constructing a structure similar to a commutative diagram at the core of our main theorem. Thus, for the next few results, we will work with the following ingredients.

Hypothesis 2.1. Let F be an admissible function. Let $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$, $(\mathfrak{B}, \mathsf{L}_\mathfrak{B})$, $(\mathfrak{E}, \mathsf{L}_\mathfrak{E})$, and $(\mathfrak{F}, \mathsf{L}_\mathfrak{F})$ be F–quasi-Leibniz quantum compact metric spaces. Let τ be an F-tunnel from $(\mathfrak{A}, \mathsf{L}_\mathfrak{A})$ to $(\mathfrak{E}, \mathsf{L}_\mathfrak{E})$, and let γ be an F-tunnel from $(\mathfrak{F}, \mathsf{L}_\mathfrak{F})$ to $(\mathfrak{B}, \mathsf{L}_\mathfrak{B})$. Let $\varphi : \mathfrak{E} \to \mathfrak{F}$ be a Lipschitz linear map and let $D > 0$ such that $\mathsf{L}_\mathfrak{F} \circ \varphi \leqslant D\mathsf{L}_\mathfrak{E}$ and $|||\varphi|||_\mathfrak{F}^\mathfrak{E} \leqslant D$.

As we look at tunnels as a form of morphism and target sets as the mean to get an image for elements by such morphisms, it is natural to define the image of a set by a tunnel by simply setting, for a tunnel τ from a quasi-Leibniz quantum compact metric space $(\mathfrak{A}, \mathsf{L})$, and for any nonempty subset $A \subseteq \mathrm{dom}(\mathsf{L})$ and for any real $l > 0$:

$$\mathfrak{t}_\tau(A|l) = \bigcup_{a \in A} \mathfrak{t}_\tau(a|l).$$

This image of a set is not empty as long as there exists an element $a \in A$ such that $\mathsf{L}(a) \leqslant l$.

For our purpose, the following specific application of this idea will be central.

Definition 2.2. Given Hypothesis (2.1), for $a \in \mathrm{dom}(\mathsf{L}_\mathfrak{A})$ and $l \geqslant \mathsf{L}_\mathfrak{A}(a)$, the *$l$-image-target set* of a is defined by:

$$\mathfrak{i}_{\varphi,\tau}(a|l) = \varphi(\mathfrak{t}_\tau(a|l))$$

and the *l-forward-target set* of a is defined by:

$$\mathfrak{f}_{\tau,\varphi,\gamma}(a|l, D) = \mathfrak{t}_\gamma(\mathfrak{i}_{\varphi,\tau}(a|l)|Dl).$$

Remark 2.3. Using Hypothesis (2.1), we thus have, for $a \in \mathrm{dom}(\mathsf{L}_\mathfrak{A})$ and $l \geqslant \mathsf{L}_\mathfrak{A}(a)$:

$$\mathfrak{f}_{\tau,\varphi,\gamma}(a|l, D) = \bigcup_{c \in \mathfrak{i}_{\varphi,\tau}(a|l)} \mathfrak{t}_\gamma(c|Dl) = \bigcup_{b \in \mathfrak{t}_\tau(a|l)} \mathfrak{t}_\gamma(\varphi(b)|Dl).$$

We thus form a sort of commutative diagram, though using set-valued functions:

$$\begin{array}{ccc} a \in \mathfrak{sa}(\mathfrak{A}) & \longrightarrow & \mathfrak{f}_{\tau,\varphi,\gamma}(a|l, D) \subseteq \mathfrak{sa}(\mathfrak{B}) \\ {\scriptstyle \mathfrak{t}_\tau(\cdot|l)} \downarrow & & \uparrow {\scriptstyle \mathfrak{t}_\gamma(\cdot|Dl)} \\ \mathfrak{t}_\tau(a|l) \subseteq \mathfrak{sa}(\mathfrak{E}) & \xrightarrow[\varphi]{} & \mathfrak{i}_{\tau,\varphi}(a|l) \subseteq \mathfrak{sa}(\mathfrak{F}) \end{array}$$

By Definition (2.2), if $l' \geqslant l$ and $D' \geqslant D$, then with the notations of Hypothesis (2.1), we have:

$$\mathfrak{f}_\tau(a|l, D) \subseteq \mathfrak{f}_\tau(a|l', D').$$

The forward target sets enjoy properties akin to target sets, which we now establish. We begin with the linearity property.

Lemma 2.4. *Assume Hypothesis (2.1). If $a, a' \in \mathrm{dom}(\mathsf{L}_\mathfrak{A})$ and $l \geqslant \max\{\mathsf{L}_\mathfrak{A}(a), \mathsf{L}_\mathfrak{A}(a')\}$, then for all $t \in \mathbb{R}$, $f \in \mathfrak{f}_{\tau,\varphi,\gamma}(a|l, D)$ and $f' \in \mathfrak{f}_{\tau,\varphi,\gamma}(a'|l, D)$:*

$$f + tf' \in \mathfrak{f}_{\tau,\varphi,\gamma}(a + ta'|(1 + |t|)l, D).$$

Proof. Let $f \in \mathfrak{f}_{\tau,\varphi,\gamma}(a|l, D)$ and $f' \in \mathfrak{f}_{\tau,\varphi,\gamma}(a'|l, D)$. By Definition (2.2), there exists $b \in \mathfrak{t}_\tau(a|l)$ and $b' \in \mathfrak{t}_\tau(a'|l)$ such that, if $c = \varphi(b)$ and $c' = \varphi(b')$, then $f \in \mathfrak{t}_\gamma(c|Dl)$ and $f' \in \mathfrak{t}_\gamma(c'|Dl)$.

Now, $b + tb' \in \mathfrak{t}_\tau(a + ta'|(1 + |t|)l)$ by Proposition (1.25). So:

$$c + tc' = \varphi(b + tb') \in \varphi(\mathfrak{t}_\tau(a + ta'|(1 + |t|)l)).$$

In particular, $\mathsf{L}_{\mathfrak{F}}(c+tc') \leqslant D\mathsf{L}_{\mathfrak{E}}(b+tb') \leqslant Dl(1+|t|)$.

On the other hand, again by Proposition (1.25):

$$f + tf' \in \mathfrak{t}_{\gamma}\left(c + tc' | (1+|t|)Dl\right).$$

Thus $f + tf' \in \mathfrak{f}_{\tau,\varphi,\gamma}\left(a + ta' | (1+|t|)l, D\right)$. □

We now establish the main geometric property of forward-target sets — namely, their diameters in norm is controlled by the extend of the tunnels from which they are defined.

Lemma 2.5. *Assume Hypothesis (2.1). Let $a, a' \in \mathrm{dom}\left(\mathsf{L}_{\mathfrak{A}}\right)$ and $l \geqslant \max\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{A}}(a')\}$. If $c \in \mathfrak{i}_{\tau,\varphi}\left(a|l\right)$ then:*

$$\|c\|_{\mathfrak{F}} \leqslant D\left(\|a\|_{\mathfrak{A}} + 2l\chi\left(\tau\right)\right)$$

and therefore for all $c' \in \mathfrak{i}_{\tau,\varphi}\left(a'|l\right)$:

$$\|c - c'\|_{\mathfrak{F}} \leqslant D\left(\|a - a'\|_{\mathfrak{A}} + 4l\chi\left(\tau\right)\right).$$

If moreover $f \in \mathfrak{f}_{\tau,\varphi,\gamma}\left(a|l, D\right)$, then:

$$\|f\|_{\mathfrak{B}} \leqslant D\left(\|a\|_{\mathfrak{A}} + 2l\left(\chi\left(\tau\right) + \chi\left(\gamma\right)\right)\right)$$

an therefore for all $f' \in \mathfrak{f}_{\tau,\varphi,\gamma}\left(a'|l, D\right)$:

$$\|f - f'\|_{\mathfrak{B}} \leqslant D\left(\|a - a'\|_{\mathfrak{A}} + 4l\left(\chi\left(\tau\right) + \chi\left(\gamma\right)\right)\right)$$

so that, in particular:

$$\mathrm{diam}\left(\mathfrak{f}_{\tau,\varphi,\gamma}\left(a|l, D\right), \|\cdot\|_{\mathfrak{B}}\right) \leqslant 4Dl\left(\chi\left(\tau\right) + \chi\left(\gamma\right)\right).$$

Proof. Let $c \in \mathfrak{i}_{\tau,\varphi}\left(a|l\right)$. There exists, by Definition (2.2), an element $b \in \mathfrak{t}_{\tau}\left(a|l\right)$ such that $c = \varphi(b)$. Now by Proposition (1.26), we have:

$$\|c\|_{\mathfrak{F}} \leqslant D\|b\|_{\mathfrak{E}} \leqslant D(\|a\|_{\mathfrak{A}} + 2l\chi\left(\tau\right)), \tag{2.1}$$

as desired. If $c' \in \mathfrak{i}_{\tau,\varphi}\left(a'|l\right)$ then $c' = \varphi(b')$ with $b' \in \mathfrak{t}_{\tau}\left(a|l\right)$, and thus $c - c' = \varphi(b - b')$ with $b - b' \in \mathfrak{t}_{\tau}\left(a - a'|2l\right)$ by Proposition (1.25). We conclude:

$$\|c - c'\|_{\mathfrak{F}} \leqslant D\left(\|a - a'\|_{\mathfrak{A}} + 4l\chi\left(\tau\right)\right)$$

using Expression (2.1).

Let now $f \in \mathfrak{f}_{\tau,\varphi,\gamma}\left(a|l, D\right)$. By Definition (2.2), there exists $c \in \mathfrak{i}_{\tau,\varphi}\left(a|l\right)$ such that $f \in \mathfrak{t}_{\gamma}\left(c|Dl\right)$. By Proposition (1.26) and Expression (2.1), we thus have:

$$\|f\|_{\mathfrak{B}} \leqslant \|c\|_{\mathfrak{F}} + 2Dl\chi\left(\gamma\right) \leqslant D(\|a\|_{\mathfrak{A}} + 2l(\chi\left(\tau\right) + \chi\left(\gamma\right))). \tag{2.2}$$

Let $f' \in \mathfrak{f}_{\tau,\varphi,\gamma}\left(a'|l, D\right)$. By Lemma (2.4), we have:

$$f - f' \in \mathfrak{f}_{\tau,\varphi,\gamma}\left(a - a'|2l, D\right)$$

and therefore, by Expression (2.2):

$$\|f - f'\|_{\mathfrak{B}} \leqslant D\left(\|a - a'\|_{\mathfrak{A}} + 4l(\chi\left(\tau\right) + \chi\left(\gamma\right))\right) \tag{2.3}$$

as needed.

Setting $a = a'$ in Expression (2.3), we then get:

$$\mathrm{diam}\left(\mathfrak{f}_{\tau,\varphi,\gamma}\left(a|l, D\right), \|\cdot\|_{\mathfrak{B}}\right) \leqslant 4Dl(\chi\left(\tau\right) + \chi\left(\gamma\right)),$$

thus concluding our lemma. □

We now complete the algebraic properties of the forward target sets by proving the following lemma.

Lemma 2.6. *Assume Hypothesis (2.1). If $a, a' \in \operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$ and $l \geqslant \max\{\mathsf{L}_{\mathfrak{A}}(a), \mathsf{L}_{\mathfrak{A}}(a')\}$, if $f \in \mathfrak{f}_{\tau,\varphi,\gamma}(a|l,D)$ and $f' \in \mathfrak{f}_{\tau,\varphi,\gamma}(a'|l,D)$, and if moreover φ is a Jordan-Lie morphism, then, setting:*

$$M_{\tau,\gamma}(\|a\|_{\mathfrak{A}}, \|a'\|_{\mathfrak{A}}, l, D) = \max\left\{F\left(\|a\|_{\mathfrak{A}} + 2l\chi(\tau), \|a'\|_{\mathfrak{A}} + 2l\chi(\tau), l, l\right), \frac{1}{D}F\left(D\left(\|a\|_{\mathfrak{A}} + 2l\chi(\tau) + 2l\chi(\gamma)\right), D\left(\|a'\|_{\mathfrak{A}} + 2l\chi(\tau) + 2l\chi(\gamma)\right), Dl, Dl\right)\right\}$$

we conclude:

$$f \circ f' \in \mathfrak{f}_{\tau,\varphi,\gamma}\left(a \circ a' | M_{\tau,\gamma}(\|a\|_{\mathfrak{A}}, \|a'\|_{\mathfrak{A}}, l, D), D\right)$$

and

$$\{f, f'\} \in \mathfrak{f}_{\tau,\varphi,\gamma}\left(\{a, a'\} | M_{\tau,\gamma}(\|a\|_{\mathfrak{A}}, \|a'\|_{\mathfrak{A}}, l, D), D\right).$$

Proof. By Definition (2.2), there exists $b \in \mathfrak{t}_{\tau}(a|l)$ and $b' \in \mathfrak{t}_{\tau}(a'|l)$ such that, if $c = \varphi(b)$ and $c' = \varphi(b')$, then $f \in \mathfrak{t}_{\gamma}(c|Dl)$ and $f' \in \mathfrak{f}_{\gamma}(a'|Dl)$.

Assume φ is a Lie-Jordan morphism. We then have by Proposition (1.25):

$$b \circ b' \in \mathfrak{t}_{\tau}\left(a \circ a' | F\left(\|a\|_{\mathfrak{A}} + 2l\chi(\tau), \|a'\|_{\mathfrak{A}} + 2l\chi(\tau), l, l\right)\right)$$

and $c \circ c' = \varphi(b) \circ \varphi(b') = \varphi(b \circ b')$ so:

$$\mathsf{L}_{\mathfrak{F}}(c \circ c') \leqslant DF\left(\|a\|_{\mathfrak{A}} + 2l\chi(\tau), \|a'\|_{\mathfrak{A}} + 2l\chi(\tau), l, l\right) \leqslant M_{\tau,\gamma}(\|a\|_{\mathfrak{A}}, \|a'\|_{\mathfrak{A}}, l, D).$$

We also have, again by Proposition (1.25):

$$f \circ f' \in \mathfrak{t}_{\gamma}\left(c \circ c' | F\left(\|c\|_{\mathfrak{F}} + 2Dl\chi(\gamma), \|c'\|_{\mathfrak{F}} + 2Dl\chi(\gamma), Dl, Dl\right)\right)$$

with:

$$\|c\|_{\mathfrak{F}} \leqslant D\|b\|_{\mathfrak{E}} \leqslant D(\|a\|_{\mathfrak{A}} + 2l\chi(\tau))$$

and

$$\|c'\|_{\mathfrak{F}} \leqslant D\|b'\|_{\mathfrak{E}} \leqslant D(\|a'\|_{\mathfrak{A}} + 2l\chi(\tau)),$$

by Lemma (2.5), so:

$$\begin{aligned} &F\left(\|c\|_{\mathfrak{F}} + 2Dl\chi(\tau), \|c'\|_{\mathfrak{F}} + 2Dl\chi(\tau), Dl, Dl\right) \leqslant \\ &\quad F\left(D(\|a\|_{\mathfrak{A}} + 2l\chi(\tau)) + 2Dl\chi(\tau), D(\|a'\|_{\mathfrak{A}} + 2l\chi(\tau)) + 2Dl\chi(\tau), Dl, Dl\right) \\ &\qquad \leqslant DM_{\tau,\gamma}(\|a\|_{\mathfrak{A}}, \|a'\|_{\mathfrak{A}}, l, D). \end{aligned}$$

By Definition, we thus have: $f \circ f' \in \mathfrak{f}_{\tau,\varphi,\gamma}\left(a \circ a' | M_{\tau,\gamma}(\|a\|_{\mathfrak{A}}, \|a'\|_{\mathfrak{A}}, l, D), D\right)$. The same proof holds for the Lie product. □

Last, we identify the topological nature of forward target sets.

Lemma 2.7. *Assume Hypothesis (2.1). For all $a \in \operatorname{dom}(\mathsf{L}_{\mathfrak{A}})$ and $l \geqslant \mathsf{L}_{\mathfrak{A}}(a)$, the sets $\mathfrak{i}_{\tau,\varphi}(a|l)$ and $\mathfrak{f}_{\tau,\varphi,\gamma}(a|l,D)$ are nonempty and compact, respectively, in $\mathfrak{sa}(\mathfrak{F})$ and $\mathfrak{sa}(\mathfrak{B})$.*

Proof. By [12, Lemma 4.2], the set $\mathfrak{t}_{\tau}(a|l)$ is a nonempty compact subset of $\mathfrak{sa}(\mathfrak{E})$. Since φ is continuous, $\mathfrak{i}_{\tau,\varphi}(a|l)$ is not empty and compact in $\mathfrak{sa}(\mathfrak{F})$ as well.

Let now $b \in \mathfrak{i}_{\tau}(a|l)$. By construction, $\mathsf{L}_{\mathfrak{D}} \circ \varphi(b) \leqslant D\mathsf{L}_{\mathfrak{B}}(b) \leqslant Dl$. Hence again by [12, Lemma 4.2], the set $\mathfrak{t}_{\gamma}(b|Dl)$ is a nonempty compact set. In particular, $\mathfrak{f}_{\tau,\varphi,\gamma}(a|l,D)$ is not empty as well. Moreover, by Lemma (2.5), the following inclusion holds:

$$\mathfrak{f}_{\tau,\varphi,\gamma}(a|l,D) \subseteq \left\{b \in \mathfrak{sa}(\mathfrak{B}) : \mathsf{L}_{\mathfrak{B}}(b) \leqslant Dl, \|b\|_{\mathfrak{A}} \leqslant D\left(\|a\|_{\mathfrak{A}} + 2l\left(\chi(\tau) + \chi(\gamma)\right)\right)\right\},$$

and the set on the right hand-side is compact since L is an L-seminorm. So $\mathfrak{f}_{\tau}(a|l,D)$ is totally bounded in $\mathfrak{sa}(\mathfrak{B})$.

To prove that $\mathfrak{f}_{\tau,\varphi,\gamma}(a|l,D)$ is compact, it is thus sufficient to show that is is closed, since it is totally bounded and $\mathfrak{sa}(\mathfrak{B})$ is complete. To this end, we write the tunnel γ as $(\mathfrak{D},\mathsf{L}_{\mathfrak{D}},\pi,\rho)$.

Let $(f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathfrak{f}_{\tau,\varphi,\gamma}(a|l,D)$, converging in $\mathfrak{B}$ to some f. By Definition (2.2), for each $k\in\mathbb{N}$, there exists $c_k\in\varphi(\mathfrak{t}_{\tau}(a|l))$ such that $f_k\in\mathfrak{t}_{\gamma}(c_k|Dl)$. There exists $b_k\in\mathfrak{t}_{\tau}(a|l)$ such that $c_k=\varphi(b_k)$. Now, $\mathfrak{t}_{\tau}(a|l)$ is compact, so there exists a convergent subsequence $(b_{j(k)})_{k\in\mathbb{N}}$ converging in norm to some $b\in\mathfrak{t}_{\tau}(a|l)$. Therefore, $(\varphi(b_{j(k)}))_{k\in\mathbb{N}}$ converges to $\varphi(b)$, which we denote by c. Thus $c\in\mathfrak{i}_{\tau,\varphi}(a|l)$.

Now, for each $k\in\mathbb{N}$, since $f_{j(k)}\in\mathfrak{t}_{\gamma}\left(c_{j(k)}\middle|Dl\right)$, there exists $d_k\in\mathfrak{sa}(\mathfrak{D})$ such that $\mathsf{L}_{\mathfrak{D}}(d_k)\leqslant Dl$ while $\pi(d_k)=c_{j(k)}$ and $\rho(d_k)=f_{j(k)}$.

Once more, $(d_k)_{k\in\mathbb{N}}$ lies in the compact set:

$$\{w\in\mathfrak{sa}(\mathfrak{D}):\mathsf{L}_{\mathfrak{D}}(w)\leqslant Dl,\|w\|_{\mathfrak{D}}\leqslant D(\|a\|_{\mathfrak{A}}+2l(\chi(\tau)+\chi(\gamma)))\}$$

and thus we extract a convergent subsequence $(d_{m(k)})_{k\in\mathbb{N}}$ with limit d with $\mathsf{L}_{\mathfrak{D}}(d)\leqslant Dl$.

Moreover by continuity:

$$\lim_{k\to\infty}\rho(d_{m(k)})=\lim_{k\to\infty}f_{j\circ m(k)}=f \text{ and } \lim_{k\to\infty}\pi(d_{m(k)})=\lim_{k\to\infty}c_{j\circ m(k)}=c.$$

By Definition, we thus conclude that $f\in\mathfrak{t}_{\gamma}(c|Dl)$ and therefore $f\in\mathfrak{f}_{\tau,\varphi,\gamma}(a|l,D)$. Thus $\mathfrak{f}_{\tau,\varphi,\gamma}(a|l,D)$ is closed. This concludes our lemma. ☐

We now provide the setting for our main theorem. We will prove not only that families of morphisms pass to the limit, informally speaking, for the propinquity, but also that composition of morphisms also passes to the limit. To encode composition, we will use the structure of semigroupoid. A semigroupoid is, informally, a category without the requirement that is contains identities: it is a set endowed with a partially defined associative operation. Formally:

Definition 2.8. A *semigroupoid* $(\mathcal{S},\mathcal{S}^{(0)},d,c,\circ)$ is a pair of classes $\mathcal{S}$ and $\mathcal{S}^{(0)}$, two maps $c,d:\mathcal{S}\to\mathcal{S}^{(0)}$ and a map $\circ:\mathcal{S}^{(2)}\to\mathcal{S}$ where:

$$\mathcal{S}^{(2)}=\left\{(s,s')\in\mathcal{S}^2:d(s)=c(s')\right\},$$

such that:

(1) for all $(s,s')\in\mathcal{S}^{(2)}$, we have $d(s\circ s')=d(s')$ and $c(s\circ s')=c(s)$,
(2) if $s,s',s''\in\mathcal{S}$ with $(s,s'),(s',s'')\in\mathcal{S}^{(2)}$, then $(s\circ s')\circ s''=s\circ(s'\circ s'')$.

Categories are examples of semigroupoids, and a semigroupoid can be turned into a category by adding an element for each object which behaves as an identity:

Definition 2.9. Let $(\mathcal{S},\mathcal{S}^{(0)},d,c,\circ)$ be a semigroupoid. An element $e\in\mathcal{S}$ is a *unit* when $d(e)=c(e)$ and $s\circ e=s$, $e\circ t=t$ for all $s,t\in\mathcal{S}$ with $d(s)=c(t)=d(e)$.

Given these similarities, a morphism of semigroupoid will be called a functoid if it is not required to preserve units, and a functor if it must.

Definition 2.10. A *functoid* F from a semigroupoid $(\mathcal{S},\mathcal{O},d,c,\circ)$ to a semigroupoid $(\mathcal{C},\mathcal{U},s,t,\square)$ is a pair of maps (which by abuse of notations, we still write F), one from $\mathcal{O}$ to $\mathcal{U}$, and one from $\mathcal{S}$ to $\mathcal{C}$, such that:

$$\forall \varphi \in \mathcal{S} \quad F(d(\varphi)) = s(F(\varphi)) \quad F(c(\varphi)) = t(F(\varphi))$$

and

$$\forall (\varphi, \varphi') \in \mathcal{S}^{(2)} \quad F(\varphi \circ \varphi') = F(\varphi) \Box F(\varphi').$$

If moreover, for any unit $u \in \mathcal{S}$, the element $F(u)$ is a unit of $\mathcal{C}$, then F is called a *functor*.

We note that Definition (2.10) is meaningful, as we easily check, using the notations of this definition, that if $(s, s') \in \mathcal{S}^{(2)}$ then $(F(s), F(s')) \in \mathcal{C}^{(2)}$.

An important example of semigroupoid is a groupoid, which is a semigroupoid for which all elements are invertible:

Definition 2.11. Let $(\mathcal{S}, \mathcal{S}^{(0)}, d, c, \circ)$ be a semigroupoid. An element $u \in \mathcal{S}$ is *invertible* when there exists $u' \in \mathcal{S}$, such that:

(1) $d(u) = c(u')$ and $d(u') = c(u)$,
(2) $u \circ u'$ is a unit,
(3) $u' \circ u$ is a unit.

The element u' is called an inverse of u.

We note that a trivial argument shows that units in a semigroupoid with a given domain are unique, and that the inverse of an element of a semigroupoid is unique as well. Functors preserve inverse while functoids do not in general.

We have now set the framework for our main theorem. We will use repeatedly in its proof the following simple result, which we record as a lemma to help clarify our main argument.

Lemma 2.12. *If $(A_n)_{n\in\mathbb{N}}$ and $(B_n)_{n\in\mathbb{N}}$ are two sequences of compact subsets of a complete metric space (X, d) such that:*

(1) $\lim_{n\to\infty} \operatorname{diam}(B_n, d) = 0$,
(2) for all $n \in \mathbb{N}$, we have $A_n \subseteq B_n$,

then the sequence $(A_n)_{n\in\mathbb{N}}$ converges for Haus_d if and only if $(B_n)_{n\in\mathbb{N}}$ does, in which case they have the same limit, a singleton of X.

Proof. We begin by proving that if a sequence $(C_n)_{n\in\mathbb{N}}$ of closed subsets of X with $\lim_{n\to\infty} \operatorname{diam}(C_n, d) = 0$ converges for Haus_d then its limit C is a singleton. Since $\operatorname{diam}(\cdot, d)$ is a continuous function for Haus_d, it is immediate that $\operatorname{diam}(C, d) = 0$. On the other hand, for each $n \in \mathbb{N}$, let $c_n \in C_n$. Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $p, q \geqslant N$, we have $\mathsf{Haus}_d(C_p, C_q) < \frac{\varepsilon}{3}$ and $\operatorname{diam}(C_n, d) < \frac{\varepsilon}{3}$, so:

$$d(c_p, c_q) \leqslant \operatorname{diam}(C_p, d) + \mathsf{Haus}_d(C_p, C_q) + \operatorname{diam}(C_q, d) < \varepsilon,$$

and thus $(c_n)_{n\in\mathbb{N}}$ is a Cauchy sequence; since (X, d) is complete, $(c_n)_{n\in\mathbb{N}}$ converges to some $c \in X$. Moreover, $d(c, C_n) \leqslant \varepsilon$ for $n \geqslant N$ by continuity. On the other hand, if $x \in C_n$ then $d(x, c) < \frac{4\varepsilon}{3}$ for $n \geqslant N$. Hence $\mathsf{Haus}_d(C_n, \{c\}) < \frac{4\varepsilon}{3}$ for $n \geqslant N$ and thus $C = \{c\}$.

We now prove the rest of our lemma.

Let $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that for all $n \geqslant N$:

$$\operatorname{diam}(B_n, d) < \frac{\varepsilon}{2}.$$

Since $A_n \subseteq B_n$ for all $n \in \mathbb{N}$, we conclude that $\mathsf{Haus}_d(A_n, B_n) < \frac{\varepsilon}{2}$ for all $n \geqslant N$.

Now, if $(A_n)_{n\in\mathbb{N}}$ converges for Haus_d, and $\{a\}$ is its limit, then there exists $M \in \mathbb{N}$ such that for all $n \geqslant M$, we have:

$$\mathsf{Haus}_d(A_n, \{a\}) < \frac{\varepsilon}{2}.$$

Therefore, for all $n \geqslant \max\{N, M\}$:

$$\mathsf{Haus}_d(B_n, \{a\}) \leqslant \mathsf{Haus}_d(B_n, A_n) + \mathsf{Haus}_d(A_n, \{a\}) < \varepsilon,$$

and thus $(B_n)_{n\in\mathbb{N}}$ converges for Haus_d to the same limit $\{a\}$.

If instead, $(B_n)_{n\in\mathbb{N}}$ converges to $\{b\}$ for Haus_d, then similarly, there exists $M \in \mathbb{N}$ such that $\mathsf{Haus}_d(B_n, \{b\}) < \frac{\varepsilon}{2}$ and thus for all $n \geqslant \max\{N, M\}$, we conclude:

$$\mathsf{Haus}_d(A_n, \{b\}) < \varepsilon$$

thus concluding our lemma. ☐

We now state and prove our main theorem.

Theorem 2.13. *Let F be an admissible function. Let $\mathcal{T}$ be a class of tunnels appropriate for a nonempty class $\mathcal{C}$ of F–quasi-Leibniz quantum compact metric spaces.*

Let $\left(\mathcal{S}, \mathcal{S}^{(0)}, d, c, \circ\right)$ be a semigroupoid, where $\mathcal{S}$ and $\mathcal{S}^{(0)}$ are both countable sets.

For all $s \in \mathcal{S}^{(0)}$, let $(\mathfrak{A}^s, \mathsf{L}^s) \in \mathcal{C}$, and let $(\mathfrak{A}^s_n, \mathsf{L}^s_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}$ such that:

$$\lim_{n\to\infty} \Lambda^*_{\mathcal{T}}((\mathfrak{A}^s_n, \mathsf{L}^s_n), (\mathfrak{A}^s, \mathsf{L}^s)) = 0.$$

Let $D : \mathcal{S} \to [0, \infty)$.

If, for all $s \in \mathcal{S}$ and for all $n \in \mathbb{N}$, there exists a Lipschitz linear map:

$$\varphi^s_n : (\mathfrak{A}^{d(s)}_n, \mathsf{L}^{d(s)}_n) \to (\mathfrak{A}^{c(s)}_n, \mathsf{L}^{c(s)}_n)$$

with the property:

$$|||\varphi^s_n|||^{\mathfrak{A}^{d(s)}_n}_{\mathfrak{A}^{c(s)}_n} \leqslant D(s) \text{ and } \mathsf{L}^{c(s)}_n \circ \varphi^s_n \leqslant D(s)\mathsf{L}^{d(s)}_n,$$

while for all $(s,t) \in \mathcal{S}^{(2)}$:

$$\lim_{n\to\infty} \mathrm{mkD}^{\mathsf{L}^{d(t)}_n}_{\mathfrak{A}^{c(s)}} \left(\varphi^s_n \circ \varphi^t_n, \varphi^{s\circ t}_n\right) = 0,$$

then there exists a functoid Ψ from $\mathfrak{S}$ to the category whose objects are the elements of $\mathcal{C}$, and whose morphisms are the Lipschitz linear maps, such that:

$$\forall s \in \mathcal{S}^{(0)} \quad \Psi(s) = (\mathfrak{A}^s, \mathsf{L}^s)$$

and

$$\forall s \in \mathcal{S} \quad |||\Psi(s)|||^{\mathfrak{A}^{d(s)}}_{\mathfrak{A}^{c(s)}} \leqslant \sqrt{2}D(s) \text{ and } \mathsf{L}^{c(s)} \circ \Psi(s) \leqslant D(s)\mathsf{L}^{d(s)}.$$

Furthermore:

1. *If, for some $s \in \mathcal{S}$ and for all $n \in \mathbb{N}$, the map φ^s_n is positive, and $\{a \in \mathrm{dom}\left(\mathsf{L}^{d(s)}\right) : a \geqslant 0\}$ is dense in $\{a \in \mathfrak{sa}\left(\mathfrak{A}^{d(s)}\right) : a \geqslant 0\}$, then $\Psi(s)$ is positive as well.*
2. *If, for some $s \in \mathcal{S}$, we can choose $(\varphi^s_n)_{n\in\mathbb{N}}$ such that φ^s_n is a Lipschitz morphism for all $n \in \mathbb{N}$, then $\Psi(s)$ is a Lipschitz morphism — in particular, $|||\Psi(s)|||^{\mathfrak{A}^{d(s)}}_{\mathfrak{A}^{c(s)}} = 1$.*

(3) *If, for some* $s, s' \in \mathcal{S}$*, we have* $c(s) = c(s')$ *and* $d(s) = d(s')$*, then* $\mathrm{mkD}_{\mathfrak{A}^{c(s)}}^{\mathsf{L}^{d(s)}}(\Psi(s), \Psi(s'))$ *is a limit point of the sequence* $\left(\mathrm{mkD}_{\mathfrak{A}_n^{c(s)}}^{\mathsf{L}_n^{d(s)}}\left(\varphi_n^s, \varphi_n^{s'}\right)\right)_{n\in\mathbb{N}}$*, and so in particular:*

$$\liminf_{n\to\infty} \mathrm{mkD}_{\mathfrak{A}_n^{c(s)}}^{\mathsf{L}_n^{d(s)}}\left(\varphi_n^s, \varphi_n^{s'}\right) \leqslant \mathrm{mkD}_{\mathfrak{A}^{c(s)}}^{\mathsf{L}^{d(s)}}\left(\varphi^s, \varphi^{s'}\right) \leqslant \limsup_{n\to\infty} \mathrm{mkD}_{\mathfrak{A}_n^{c(s)}}^{\mathsf{L}_n^{d(s)}}\left(\varphi_n^s, \varphi_n^{s'}\right).$$

Moreover, if for some $s \in \mathcal{S}$*, we have* $d(s) = c(s)$*, then:*

$$\liminf_{n\to\infty} \mathrm{mk}\ell_{\mathsf{L}_n^{d(s)}}(\varphi_n^s) \leqslant \mathrm{mk}\ell_{\mathsf{L}^{d(s)}}(\Psi(s)) \leqslant \limsup_{n\to\infty} \mathrm{mk}\ell_{\mathsf{L}_n^{d(s)}}(\varphi_n^s).$$

In particular, if $u \in \mathcal{S}$ *with* $c(u) = d(u)$ *and if* $\lim_{n\to\infty} \mathrm{mk}\ell_{\mathsf{L}_n^u}(\varphi^u) = 0$*, then* $\Psi(u)$ *is the identity of* $\mathfrak{A}^{d(u)}$*.*

(4) *If for any unit* $u \in \mathcal{S}$ *we have:*

$$\lim_{n\to\infty} \mathrm{mk}\ell_{\mathsf{L}_n^u}(\varphi^u) = 0$$

then Ψ *is a functor.*

(5) *If some* $s \in \mathcal{S}$ *is invertible, with inverse* s'*, and if for all* $n \in \mathbb{N}$*, we have:*

$$\lim_{n\to\infty} \mathrm{mk}\ell_{\mathsf{L}_n^{c(s)}} \varphi_n^{s\circ s} = 0 \text{ and } \lim_{n\to\infty} \mathrm{mk}\ell_{\mathsf{L}_n^{d(s)}} \varphi_n^{s'\circ s} = 0,$$

then $\Psi(s)$ *is a bijection from* $\mathfrak{A}^{d(s)}$ *to* $\mathfrak{A}^{c(s)}$ *with inverse* $\Psi(s')$*.*

Proof. We will employ all the notations given in the assumption of our theorem.

Up to extracting subsequences, a standard diagonal argument shows that since $\mathcal{S}^{(0)}$ is countable, we may as well assume that:

$$\forall s \in \mathcal{S}^{(0)}\ \forall n \in \mathbb{N} \quad \mathsf{\Lambda}^*_{\mathcal{T}}\left((\mathfrak{A}_n^s, \mathsf{L}_n^s), (\mathfrak{A}^s, \mathsf{L}^s)\right) \leqslant \frac{1}{n+2}.$$

For each $s \in \mathcal{S}^{(0)}$, $n \in \mathbb{N}$, let $\tau_n^s \in \mathcal{T}$ be a tunnel from $(\mathfrak{A}^s, \mathsf{L}^s)$ to $(\mathfrak{A}_n^s, \mathsf{L}_n^s)$ with $\chi(\tau_n^s) \leqslant \frac{1}{n+1}$. Let $\gamma_n \in \mathcal{T}$ be the inverse tunnel $(\tau_n^s)^{-1}$.

For any $s \in \mathcal{S}^{(0)}$, let $\mathfrak{S}^s$ be a countable, dense subset of $\mathfrak{sa}(\mathfrak{A}^s)$ with $\mathfrak{S}^s \subseteq \mathrm{dom}(\mathsf{L}^s)$.

For each $s \in \mathcal{S}$, $n \in \mathbb{N}$, we denote the forward-image set (see Definition (2.2)):

$$\mathfrak{f}_{\tau_n^{d(s)}, \varphi_n^s, \gamma_n^{c(s)}}(\cdot|\cdot)$$

simply as $\mathfrak{f}_{s,n}(\cdot|\cdot)$.

We now prove our theorem in several steps. The first few steps prove the existence of a linear map which will eventually be denoted by $\Psi(s)$ for each $s \in \mathcal{S}$. The construction is of intrinsic interest, as it may be possible to use it to extract additional information on these maps; some of the later steps will be examples of this idea.

Step 1. *Let* $s \in \mathcal{S}$*. Let* $j_0 : \mathbb{N} \to \mathbb{N}$ *be a strictly increasing function. Let* $a \in \mathrm{dom}\left(\mathsf{L}^{d(s)}\right)$*. There exists* $j_1 : \mathbb{N} \to \mathbb{N}$*, strictly increasing, and* $f \in \mathfrak{sa}\left(\mathfrak{A}^{c(s)}\right)$*, such that:*

$$\mathsf{L}^{c(s)}(f) \leqslant D(s)\mathsf{L}^{d(s)}(a) \text{ and } \|f\|_{\mathfrak{A}^{c(s)}} \leqslant D(s)\|a\|_{\mathfrak{A}^{d(s)}},$$

and for all $l \geqslant \mathsf{L}^{d(s)}(a)$*, the sequence* $(\mathfrak{f}_{s,j_0\circ j_1(n)}(a|l, D(s)))_{n\in\mathbb{N}}$ *converges, for* $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}$*, to the singleton* $\{f\}$*.*

Write $D = D(s)$ for this step. Let $a \in \mathrm{dom}\left(\mathsf{L}^{d(s)}\right)$. By Lemmas (2.5) and (2.7), the sequence:

$$\left(\mathfrak{f}_{s,j_0(n)}\left(a\middle|\mathsf{L}^{d(s)}(a), D\right)\right)_{n\in\mathbb{N}}$$

is a sequence of nonempty compact subsets of:

$$\left\{b \in \mathfrak{sa}\left(\mathfrak{A}^{c(s)}\right) : \mathsf{L}^{c(s)}(b) \leqslant D\mathsf{L}^{d(s)}(a) \text{ and } \|b\|_{\mathfrak{A}^{c(s)}} \leqslant D(\|a\|_{\mathfrak{A}^{d(s)}} + 2\mathsf{L}^{d(s)}(a))\right\}$$

which is itself compact in $\mathfrak{sa}\left(\mathfrak{A}^{c(s)}\right)$ since $\mathsf{L}^{c(s)}$ is an L-seminorm. Thus, there exists a strictly increasing function j_1 such that $\left(\mathfrak{f}_{s,j_0\circ j_1(n)}\left(a\middle|\mathsf{L}^{d(s)}(a), D\right)\right)_{n\in\mathbb{N}}$ converges for the Hausdorff distance $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}$, since the Hausdorff distance induces a compact topology on the hyperspace of closed subsets of a compact metric spaces (Blaschke's theorem [2, Theorem 7.2.8]). Let $\mathfrak{f}(a)$ be the limit of $\left(\mathfrak{f}_{s,j_0\circ j_1(n)}\left(a\middle|\mathsf{L}^{d(s)}(a), D\right)\right)_{n\in\mathbb{N}}$ for $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}$. By Lemma (2.5), we have:

$$\lim_{n\to\infty} \mathrm{diam}\left(\mathfrak{f}_{s,j_0\circ j_1(n)}\left(a\middle|\mathsf{L}^{d(s)}(a), D\right), \|\cdot\|_{\mathfrak{A}^{c(s)}}\right) = 0$$

and thus $\mathfrak{f}(a)$ is a singleton, as desired. We write $\mathfrak{f}(a) = \{f\}$.

In particular, we thus note that f is the limit of any sequence $(b_n)_{n\in\mathbb{N}}$ with $b_n \in \mathfrak{f}_{s,j_0\circ j_1(n)}\left(a\middle|\mathsf{L}^{d(s)}(a), D\right)$ for all $n \in \mathbb{N}$. Thus $\|f\|_{\mathfrak{A}^{c(s)}} = \lim_{n\to\infty} \|b_n\|_{\mathfrak{A}^{c(s)}_{j_0\circ j_1(n)}} \leqslant D\|a\|_{\mathfrak{A}^{d(s)}}$; moreover since $\mathsf{L}^{d(s)}$ is lower semicontinuous, we also have:

$$\mathsf{L}^{c(s)}(f) \leqslant \liminf_{n\to\infty} \mathsf{L}^{c(s)}_{j_0\circ j_1(n)}(b_n) \leqslant D\mathsf{L}^{d(s)}(a).$$

To conclude this step, let $l \geqslant \mathsf{L}^{d(s)}(a)$. Since for all $n \in \mathbb{N}$, we have:

$$\mathfrak{f}_{s,n}\left(a\middle|\mathsf{L}^{d(s)}(a), D\right) \subseteq \mathfrak{f}_{s,n}\left(a|l, D\right),$$

so by Lemma (2.12), since by Lemma (2.5) we have:

$$\lim_{n\to\infty} \mathrm{diam}\left(\mathfrak{f}_{s,n}\left(a|l, D\right), \|\cdot\|_{\mathfrak{A}^{c(s)}_n}\right) = 0,$$

we conclude that:

$$\left(\mathfrak{f}_{s,j_0\circ j_1(n)}\left(a|l, D\right)\right)_{n\in\mathbb{N}}$$

converges to $\{f\}$ as well.

Step 2. *Let $s \in \mathcal{S}$ and $J : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function. There exists a strictly increasing function $j : \mathbb{N} \to \mathbb{N}$ such that, for all $a \in \mathfrak{S}^{d(s)}$, there exists $\varphi^s(a) \in \mathfrak{sa}\left(\mathfrak{A}^{c(s)}\right)$, with:*

$$\|\varphi^s(a)\|_{\mathfrak{A}^{c(s)}} \leqslant D(s)\|a\|_{\mathfrak{A}^{d(s)}} \text{ and } \mathsf{L}^{c(s)} \circ \varphi^s(a) \leqslant D(s)\mathsf{L}^{d(s)}(a),$$

such that for all $l \geqslant \mathsf{L}^{d(s)}(a)$, the sequence:

$$\left(\mathfrak{f}_{s,J\circ j(n)}\left(a|l, D(s)\right)\right)_{n\in\mathbb{N}}$$

converges for $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}$ to $\{\varphi^s(a)\}$.

We now apply Step (1) repeatedly, using a diagonal argument. Fix $s \in \mathcal{S}$ and again, just write $D = D(s)$. Write $\mathfrak{S}^{d(s)} = \{a_n : n \in \mathbb{N}\}$. By Step (1), there exists $j_0 : \mathbb{N} \to \mathbb{N}$ strictly increasing and $\varphi^s(a_0) \in \mathfrak{sa}\left(A^{c(s)}\right)$, such that for all $l \geqslant \mathsf{L}^{d(s)}(a_0)$, the sequence $\left(\mathfrak{f}_{s,J\circ j_0(n)}\left(a_0|l, D\right)\right)_{n\in\mathbb{N}}$ converges to $\{\varphi^s(a_0)\}$ for $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}$, and $\|\varphi^s(a_0)\|_{\mathfrak{A}^{c(s)}} \leqslant D\|a_0\|_{\mathfrak{A}^{d(s)}}$ while $\mathsf{L}^{c(s)} \circ \varphi^s(a_0) \leqslant D\mathsf{L}^{d(s)}(a_0)$.

Assume now that for some $k \in \mathbb{N}$, there exist strictly increasing functions $j_0, \ldots, j_k$ from $\mathbb{N}$ to $\mathbb{N}$ and $\varphi^s(a_0), \ldots, \varphi^s(a_k) \in \mathfrak{sa}\left(\mathfrak{A}^{c(s)}\right)$ such that for $m \in \{0, \ldots, k\}$ and for all $l \geqslant \mathsf{L}^{d(s)}(a_j)$, the sequence:

$$\left(\mathfrak{f}_{s,J\circ j_0\circ\cdots\circ j_m(n)}\left(a_j|l,D\right)\right)_{n\in\mathbb{N}}$$

converges to the singleton $\{\varphi^s(a_j)\}$ for $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}$, with $\|\varphi(a_j)\|_{\mathfrak{A}^{c(s)}} \leqslant D\|a_j\|_{\mathfrak{A}^{d(s)}}$ and $\mathsf{L}^{c(s)} \circ \varphi^s(a_j) \leqslant D\mathsf{L}^{d(s)}(a_j)$.

By Step (1), there exists a strictly increasing function $j_{k+1} : \mathbb{N} \to \mathbb{N}$ such that, for all $l \geqslant \mathsf{L}^{d(s)}(a_{k+1})$, the sequence $\left(\mathfrak{f}_{s,J\circ j_0\circ\cdots\circ j_{k+1}(n)}\left(a_{k+1}|l,D\right)\right)_{n\in\mathbb{N}}$ converges, for $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}$, to some singleton denoted by $\{\varphi^s(a_{k+1})\}$ with $\|\varphi^s(a_{k+1})\|_{\mathfrak{A}^{c(s)}} \leqslant D\|a_{k+1}\|_{\mathfrak{A}^{d(s)}}$ and $\mathsf{L}^{c(s)} \circ \varphi^s(a_{k+1}) \leqslant D\mathsf{L}^{d(s)}(a_{k+1})$. This completes our induction.

Now, let $j : m \in \mathbb{N} \longmapsto J \circ j_0 \circ \cdots \circ j_m(m)$. By construction, for all $k \in \mathbb{N}$ and for all $l \geqslant \mathsf{L}^{d(s)}(a_k)$, the sequence $\left(\mathfrak{f}_{s,j(n)}\left(a_k|l,D\right)\right)_{n\in\mathbb{N}}$ converges to $\{\varphi^s(a_k)\}$ for $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}$.

Step 3. *There exists a strictly increasing function $j : \mathbb{N} \to \mathbb{N}$ such that, for all $s \in \mathcal{S}$ and for all $a \in \mathfrak{S}^{d(s)}$, there exists $\varphi^s(a) \in \mathfrak{sa}\left(\mathfrak{A}^{c(s)}\right)$, with:*

$$\|\varphi^s(a)\|_{\mathfrak{A}^{c(s)}} \leqslant D(s)\|a\|_{\mathfrak{A}^{d(s)}} \text{ and } \mathsf{L}^{c(s)} \circ \varphi^s(a) \leqslant D(s)\mathsf{L}^{d(s)}(a),$$

such that for all $l \geqslant \mathsf{L}^{d(s)}(a)$, the sequence:

$$\left(\mathfrak{f}_{s,j(n)}\left(a|l,D(s)\right)\right)_{n\in\mathbb{N}}$$

converges for $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}$ to $\{\varphi^s(a)\}$.

Since $\mathcal{S}$ is countable, we write it as $\mathcal{S} = \{s_m : m \in \mathbb{N}\}$ for this step. We now apply Step (2) repeatedly. First, let $j_0 : \mathbb{N} \to \mathbb{N}$ and φ^{s_0} be given by Step (2) for s_0 and $J : n \in \mathbb{N} \mapsto n$.

Assume we have constructed, for some $k \in \mathbb{N}$, strictly increasing maps $j_0, \ldots, j_k$ from $\mathbb{N}$ to $\mathbb{N}$, and functions $\varphi^{s_0} : \mathfrak{S}^{d(s_0)} \to \mathfrak{sa}\left(\mathfrak{A}^{c(s_0)}\right), \ldots, \varphi^{s_k} : \mathfrak{S}^{d(s_k)} \to \mathfrak{sa}\left(\mathfrak{A}^{c(s_k)}\right)$ such that for all $m \in \{0, \ldots, k\}$, $a \in \mathfrak{S}^{d(s_m)}$, and $l \geqslant \mathsf{L}^{d(s_m)}(a)$, we have:

$$\mathsf{L}^{c(s_m)} \circ \varphi^{s_m}(a) \leqslant D(s_m)\mathsf{L}^{d(s)}(a) \text{ and } \|\varphi^{s_m}(a)\|_{\mathfrak{A}^{c(m)}} \leqslant D(s_m)\|a\|_{\mathfrak{A}^{d(s_m)}},$$

and the sequence $\left(\mathfrak{f}_{s_m,j_0\circ\ldots\circ j_m(n)}\left(a|l,D(s_m)\right)\right)_{n\in\mathbb{N}}$ converges to $\{\varphi^{s_m}(a)\}$ for $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}$.

We apply Step (2) with $J : n \in \mathbb{N} \to j_0 \circ \ldots \circ j_k$ and s_{k+1}: thus there exists $j_{k+1} : \mathbb{N} \to \mathbb{N}$ strictly increasing and $\varphi^{s_{k+1}} : \mathfrak{S}^{d(k+1)} \to \mathfrak{sa}\left(\mathfrak{A}^{c(k+1)}\right)$ satisfying the conclusions of Step (2). This completes our induction.

We conclude this step by setting $j : m \in \mathbb{N} \longmapsto j_0 \circ \cdots \circ j_m(m)$.

Step 4. *There exists a strictly increasing function $j : \mathbb{N} \to \mathbb{N}$ and, for all $s \in \mathcal{S}$, a function $\varphi^s : \operatorname{dom}\left(\mathsf{L}^{d(s)}\right) \to \mathfrak{sa}\left(\mathfrak{A}^{c(s)}\right)$ such that for all $a \in \operatorname{dom}\left(\mathsf{L}^{d(s)}\right)$ and for all $l \geqslant \mathsf{L}^{d(s)}(a)$, and $D \geqslant D(s)$, the sequence:*

$$\left(\mathfrak{f}_{s,j(n)}\left(a|l,D\right)\right)_{n\in\mathbb{N}}$$

converges to $\{\varphi^s(a)\}$ for $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}$; moreover:

$$\|\varphi^s(a)\|_{\mathfrak{A}^{c(s)}} \leqslant D(s)\|a\|_{\mathfrak{A}^{d(s)}} \text{ and } \mathsf{L}^{c(s)}(\varphi^s(a)) \leqslant D(s)\mathsf{L}^{d(s)}(a).$$

Let j and, for all $s \in \mathcal{S}$, let $\varphi^s : \mathfrak{S}^{d(s)} \to \mathfrak{sa}\left(\mathfrak{A}^{c(s)}\right)$ be constructed by Step (3). Fix $s \in \mathcal{S}$ and write $D \geqslant D(s)$.

Let $a \in \mathrm{dom}\left(\mathsf{L}^{d(s)}\right)$, and $\varepsilon > 0$. There exists $a' \in \mathfrak{S}^{d(s)}$ such that $\|a - a'\|_{\mathfrak{A}^{d(s)}} < \frac{\varepsilon}{4D}$. Let $l = \max\{\mathsf{L}^{d(s)}(a), \mathsf{L}^{d(s)}(a')\}$. Since by Lemma (2.5):

$$\lim_{n\to\infty} \mathrm{diam}\left(\mathfrak{f}_{s,n}\left(a'|l,D\right), \|\cdot\|_{\mathfrak{A}^{c(s)}}\right) = 0$$

and since $\left(\mathfrak{f}_{s,j(n)}\left(a'|l,D(s)\right)\right)_{n\in\mathbb{N}}$ converges to a singleton for $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}$, by Lemma (2.12), the sequence $\left(\mathfrak{f}_{s,j(n)}\left(a'|l,D\right)\right)_{n\in\mathbb{N}}$ also converges to the same singleton.

Since $\left(\mathfrak{f}_{s,j(n)}\left(a'|l,D\right)\right)_{n\in\mathbb{N}}$ is convergent for the Hausdorff distance $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}$ by Step (3), it is Cauchy for $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}$. Let $N \in \mathbb{N}$ such that, for all $p, q \geqslant N$, we have:

$$\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}\left(\mathfrak{f}_{s,j(p)}\left(a'|l,D\right), \mathfrak{f}_{s,j(q)}\left(a'|l,D\right)\right) < \frac{\varepsilon}{3}.$$

By Lemma (2.5), we have, for all $l \geqslant \max\{\mathsf{L}^{d(s)}(a), \mathsf{L}^{d(s)}(a')\}$, for all $n \in \mathbb{N}$, for all $b \in \mathfrak{f}_{s,j(n)}\left(a|l,D\right)$ and for all $b' \in \mathfrak{f}_{s,j(n)}\left(a'|l,D\right)$:

$$\|b - b'\|_{\mathfrak{A}^{c(s)}} \leqslant D\left(\|a - a'\|_{\mathfrak{A}^{d(s)}} + 2l\left(\frac{1}{j(n)+1} + \frac{1}{j(n)+1}\right)\right) \leqslant \frac{\varepsilon}{4} + \frac{4l}{j(n)+1}.$$

Let $M \in \mathbb{N}$ such that for all $n \geqslant M$, we have $\frac{4l}{j(n)+1} \leqslant \frac{\varepsilon}{12}$, so that:

$$\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}\left(\mathfrak{f}_{s,j(n)}\left(a|l,D\right), \mathfrak{f}_{s,j(n)}\left(a'|l,D\right)\right) \leqslant \frac{\varepsilon}{3}.$$

Hence for all $p, q \geqslant \max\{N, M\}$:

$$\begin{aligned}
\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}&\left(\mathfrak{f}_{s,j(p)}\left(a|l,D\right), \mathfrak{f}_{s,j(q)}\left(a|l,D\right)\right) \\
&\leqslant \mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}\left(\mathfrak{f}_{s,j(p)}\left(a|l,D\right), \mathfrak{f}_{s,j(p)}\left(a'|l,D\right)\right) \\
&\quad + \mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}\left(\mathfrak{f}_{s,j(p)}\left(a'|l,D\right), \mathfrak{f}_{s,j(q)}\left(a'|l,D\right)\right) \\
&\quad + \mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}\left(\mathfrak{f}_{s,j(q)}\left(a'|l,D\right), \mathfrak{f}_{j(q)}\left(a|l,D\right)\right) \\
&< \varepsilon.
\end{aligned}$$

Hence $\left(\mathfrak{f}_{s,j(n)}\left(a|l,D\right)\right)_{n\in\mathbb{N}}$ is a Cauchy sequence for $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}$. Since $\left(\mathfrak{sa}\left(\mathfrak{A}^{c(s)}\right), \|\cdot\|_{\mathfrak{A}^{c(s)}}\right)$ is complete, so is $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}$ and thus $\left(\mathfrak{f}_{s,j(n)}\left(a|l,D\right)\right)_{n\in\mathbb{N}}$ converges for $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}$ to some set $\mathfrak{f}(a)$; again since $\lim_{n\to\infty} \mathrm{diam}\left(\mathfrak{f}_{s,j(n)}\left(a|l,D\right), \|\cdot\|_{\mathfrak{A}^{c(s)}}\right) = 0$, the set $\mathfrak{f}(a)$ is a singleton which we denote by $\{\varphi^s(a)\}$. We observe that if in fact $a \in \mathfrak{S}^{d(s)}$, then we introduce no confusion with this notation since by construction, $\left(\mathfrak{f}_{s,j(n)}\left(a|l,D\right)\right)_{n\in\mathbb{N}}$ converges to $\{\varphi^s(a)\}$ as defined by Step (3).

Using Lemma (2.12) and the observation that for all $n \in \mathbb{N}$, and for all $l'' \geqslant l' \geqslant \mathsf{L}^{d(s)}(a)$, we have:

$$\mathfrak{f}_{s,j(n)}\left(a|l',D\right) \subseteq \mathfrak{f}_{s,j(n)}\left(a|l'',D\right)$$

we conclude that for all $l \geqslant \mathsf{L}^{d(s)}(a)$, the sequence:

$$\left(\mathfrak{f}_{s,j(n)}\left(a|l,D\right)\right)_{n\in\mathbb{N}}$$

converges for $\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{c(s)}}}$ to $\{\varphi^s(a)\}$.

In particular, if $a \in \mathrm{dom}\left(\mathsf{L}^{d(s)}\right)$, then $\left(\mathfrak{f}_{s,j(n)}\left(a\middle|\mathsf{L}^{d(s)}(a), D(s)\right)\right)_{n\in\mathbb{N}}$ converges to $\{\varphi^s(a)\}$, and thus as before, since $\mathsf{L}^{c(s)}$ is lower semi-continuous, we conclude that $\mathsf{L}^{c(s)} \circ \varphi^s(a) \leqslant D(s)\mathsf{L}^{d(s)}(a)$ for all $a \in \mathrm{dom}\left(\mathsf{L}^{d(s)}\right)$; Similarly $\|\varphi(a)\|_{\mathfrak{A}^{c(s)}} \leqslant D(s)\|a\|_{\mathfrak{A}^{d(s)}}$.

Step 5. *For all $s \in \mathcal{S}$, the map $\varphi^s : \mathrm{dom}\left(\mathsf{L}^{d(s)}\right) \to \mathrm{dom}\left(\mathsf{L}^{c(s)}\right)$ is linear and $\varphi^s(1_{\mathfrak{A}^{d(s)}}) = 1_{\mathfrak{A}^{c(s)}}$.*

Fix $s \in \mathcal{S}$ and write $D = D(s)$. Let $a, a' \in \mathrm{dom}\left(\mathsf{L}^{d(s)}\right)$, $t \in \mathbb{R}$, and $l \geqslant \max\{\mathsf{L}^{d(s)}(a), \mathsf{L}^{d(s)}(a')\}$. For each $n \in \mathbb{N}$, let $b_n \in \mathfrak{f}_{s,j(n)}\left(a|l, D\right)$ and $b'_n \in \mathfrak{f}_{s,j(n)}\left(a'|l, D\right)$. By construction, $(b_n)_{n\in\mathbb{N}}$ converges to $\varphi^s(a)$ while $(b'_n)_{n\in\mathbb{N}}$ converges to $\varphi^s(a')$ in $\mathfrak{sa}\left(\mathfrak{A}^{c(s)}\right)$. On the other hand, by Lemma (2.4):

$$\forall n \in \mathbb{N} \quad b_n + tb'_n \in \mathfrak{f}_{s,j(n)}\left(a + ta'|(1 + |t|)l, D(s)\right),$$

so $(b_n + tb'_n)_{n\in\mathbb{N}}$ converges to $\varphi^s(a + ta')$ in $\mathfrak{sa}\left(\mathfrak{A}^{c(s)}\right)$. By uniqueness of the limit, we conclude:

$$\varphi^s(a + ta') = \varphi^s(a) + t\varphi^s(a').$$

We also note that since φ^s_n maps unit to unit, we have $1_{\mathfrak{A}^{c(s)}} \in \mathfrak{f}_{s,n}\left(1_{\mathfrak{A}^{d(s)}}|0, D\right)$ for all $n \in \mathbb{N}$. Thus $\varphi^s(1_{d(s)}) = 1_{\mathfrak{A}_{c(s)}}$.

Step 6. *For all $s \in \mathcal{S}$, the map φ^s has a unique extension (still denoted by φ^s) as a continuous linear map from $\mathfrak{A}^{d(s)}$ to $\mathfrak{A}^{c(s)}$ with:*

- $|||\varphi^s|||_{\mathfrak{sa}\left(\mathfrak{A}^{c(s)}\right)}^{\mathfrak{sa}\left(\mathfrak{A}^{d(s)}\right)} \leqslant D(s)$,
- $|||\varphi^s|||_{\mathfrak{A}^{c(s)}}^{\mathfrak{A}^{d(s)}} \leqslant \sqrt{2}D(s)$,
- $\mathsf{L}^{c(s)} \circ \varphi^s \leqslant D(s)\mathsf{L}^{d(s)}$.
- $\forall a \in \mathfrak{A}^{d(s)} \quad \varphi^s(a^*) = \varphi^s(a)^*$.

Fix $s \in \mathcal{S}$. The map $\varphi^s : \mathrm{dom}\left(\mathsf{L}^{d(s)}\right) \to \mathrm{dom}\left(\mathsf{L}^{c(s)}\right)$ satisfies $\mathsf{L}^{c(s)} \circ \varphi^s \leqslant D(s)\mathsf{L}^{d(s)}$ by Step (4).

For all $a \in \mathrm{dom}\left(\mathsf{L}^{d(s)}\right)$, we have $\|\varphi^s(a)\|_{\mathfrak{A}^{c(s)}} \leqslant D(s)\|a\|_{\mathfrak{A}^{d(s)}}$ by construction, so $\varphi^s : \mathrm{dom}\left(\mathsf{L}^{d(s)}\right) \to \mathfrak{sa}\left(\mathfrak{A}^{c(s)}\right)$ is a continuous linear map. Hence it is uniformly continuous on the dense subspace $\mathrm{dom}\left(\mathsf{L}^{d(s)}\right)$ of $\mathfrak{sa}\left(\mathfrak{A}^{d(s)}\right)$ and therefore, it has a unique uniformly continuous extension as a linear map from $\mathfrak{sa}\left(\mathfrak{A}^{d(s)}\right)$ to $\mathfrak{sa}\left(\mathfrak{A}^{c(s)}\right)$ with norm at most $D(s)$. As a side note, if $\mathsf{L}^{d(s)}(a) = \infty$ (equivalently if $a \in \mathfrak{sa}\left(\mathfrak{A}^{d(s)}\right) \setminus \mathrm{dom}\left(\mathsf{L}^{d(s)}\right)$) then the inequality $\mathsf{L}^{c(s)} \circ \varphi^s(a) \leqslant D(s)\mathsf{L}^{d(s)}(a) = \infty$ is trivial.

For all $a \in \mathfrak{A}^{d(s)}$, we then set:

$$\varphi^s(a) = \varphi^s\left(\Re a\right) + i\varphi^s\left(\Im a\right),$$

where $\Re a = \frac{a+a^*}{2}$ and $\Im a = \frac{a-a^*}{2}$ are the respective real and imaginary parts of a.

First, we note that if $a \in \mathfrak{sa}\left(\mathfrak{A}^{d(s)}\right)$ then $\Im a = 0$ and thus our extension of φ^s to $\mathfrak{A}^{d(s)}$ agrees with φ^s on $\mathfrak{sa}\left(\mathfrak{A}^{d(s)}\right)$, which justifies that we keep the notation φ^s. Moreover, it is straightforward that φ^s is linear — owing to the linearity of φ^s restricted to $\mathfrak{sa}\left(\mathfrak{A}^{d(s)}\right)$.

A quick computation shows that for all $a \in A$:

$$\begin{aligned}
\|\varphi^s(a)\|^2_{\mathfrak{A}^{c(s)}} &= \|\varphi^s(a)^*\varphi^s(a)\|_{\mathfrak{A}^{c(s)}} \\
&\leqslant \|\varphi^s(\Re a)^2 + \varphi^s(\Im a)^2\|_{\mathfrak{A}^{c(s)}} \\
&\leqslant 2D(s)^2\|a\|^2_{\mathfrak{A}^{c(s)}}.
\end{aligned}$$

Last, by construction, $\varphi^s(a^*) = \varphi^s(\Re a - i\Im a) = \varphi^s(\Re a) - i\varphi^s(\Im a) = \varphi^s(a)^*$ so φ^s preserves the adjoint operation.

This concludes our step.

Step 7. *For all $s, s' \in \mathcal{S}$, if $d(s) = c(s')$, then $\varphi^s \circ \varphi^{s'} = \varphi^{s\circ s'}$.*

Let $s, s' \in \mathcal{S}$ such that $d(s) = c(s')$. By assumption, $(\mathfrak{A}^{d(s)}, \mathsf{L}^{d(s)}) = (\mathfrak{A}^{c(s')}, \mathsf{L}^{c(s')})$, so the statement of this step is at least meaningful. We also note that it is obviously sufficient to prove that $\varphi^s \circ \varphi^{s'}(a) = \varphi^{s\circ s'}(a)$ for all $a \in \mathfrak{sa}\left(\mathfrak{A}^{d(s)}\right)$.

Moreover, by assumption:

$$\forall n \in \mathbb{N} \quad (\mathfrak{A}_n^{c(s')}, \mathsf{L}_n^{c(s')}) = \operatorname{codom}\left(\varphi_n^{s'}\right) = \operatorname{dom}\left(\varphi_n^{s}\right) = (\mathfrak{A}_n^{d(s)}, \mathsf{L}_n^{d(s)}).$$

Let $a \in \mathfrak{sa}\left(\mathfrak{A}^{d(s')}\right)$ and write $l = \mathsf{L}^{d(s')}(a)$. For all $n \in \mathbb{N}$, let $a_n \in \mathfrak{t}_{\tau_{j(n)}^{d(s')}}(a|l)$, and write $b_n = \varphi_{j(n)}^{s'}(a_n)$. Note that $b_n \in \mathfrak{A}_{j(n)}^{c(s')} = \mathfrak{A}_{j(n)}^{d(s)}$.

Let $c_n \in \mathfrak{t}_{\gamma_{j(n)}^{c(s')}}(b_n|D(s')l)$ — in particular, $c_n \in \mathfrak{f}_{s',j(n)}(a|l, D(s')) \subseteq \mathfrak{A}^{c(s')}$.

In addition, let $e_n \in \mathfrak{t}_{\gamma_{j(n)}^{c(s)}}\left(\varphi_{j(n)}^{s\circ s'}(a_n)\middle|D(s\circ s')l\right)$, so that by Definition (2.2), we have $e_n \in \mathfrak{f}_{s\circ s',j(n)}(a|l, D(s\circ s'))$ (so in particular, $e_n \in \mathfrak{A}^{c(s)}$). By construction, $(e_n)_{n\in\mathbb{N}}$ converges to $\varphi^{s\circ s'}(a)$ in $\mathfrak{A}^{c(s)}$, while $(c_n)_{n\in\mathbb{N}}$ converges to $\varphi^{s'}(a)$ in $\mathfrak{A}^{d(s)}$.

Let $d_n \in \mathfrak{t}_{\gamma_{j(n)}^{c(s)}}\left(\varphi_{j(n)}^{s}(b_n)\middle|D(s)D(s')l\right)$ — so $d_n \in \mathfrak{A}^{c(s)}$. Now, since we chose $c_n \in \mathfrak{t}_{\gamma_{j(n)}^{d(s)}}(b_n|D(s')l)$, and since $\gamma_{j(n)}^{d(s)} = \left(\tau_{j(n)}^{d(s)}\right)^{-1}$, we also have by symmetry $b_n \in \mathfrak{t}_{\tau_{j(n)}^{d(s)}}(c_n|D(s')l)$. So by Definition (2.2), we conclude that $d_n \in \mathfrak{f}_{s,j(n)}(c_n|D(s')l, D(s))$.

Let $h_n \in \mathfrak{f}_{s,j(n)}\left(\varphi^{s'}(a)\middle|D(s')l, D(s)\right)$. By construction, $(h_n)_{n\in\mathbb{N}}$ converges to $\varphi^s \circ \varphi^{s'}(a)$ in $\mathfrak{A}^{c(s)}$. On the other hand, by Lemma (2.5):

$$\|d_n - h_n\|_{\mathfrak{A}^{c(s)}} \leqslant D(s)\left(\|c_n - \varphi^{s'}(a)\|_{\mathfrak{A}^{d(s)}} + 4lD(s')\chi\left(\tau_{j(n)}^s\right)\right)$$

so

$$\begin{aligned} 0 &\leqslant \limsup_{n\to\infty} \|d_n - h_n\|_{\mathfrak{A}^{c(s)}} \\ &\leqslant D(s)\left(\lim_{n\to\infty}\|c_n - \varphi^{s'}(a)\|_{\mathfrak{A}^{c(s')}} + 4lD(s')\lim_{n\to\infty}\chi\left(\tau_{j(n)}^s\right)\right) = 0 \end{aligned}$$

so $(d_n)_{n\in\mathbb{N}}$ converges to $\varphi^s \circ \varphi^{s'}(a)$ as well.

Let $\varepsilon > 0$ and let $D = \max\{D(s)D(s'), D(s\circ s')\}$.

Let $N_1 \in \mathbb{N}$ such that for all $n \geqslant N_1$, we have $\|d_n - \varphi^s \circ \varphi^{s'}(a)\|_{\mathfrak{A}^{c(s)}} < \frac{\varepsilon}{4}$.

Let $N_2 \in \mathbb{N}$ such that for all $n \geqslant N_2$, we have $\|e_n - \varphi^{s\circ s'}(a)\|_{\mathfrak{A}^{c(s)}} < \frac{\varepsilon}{4}$.

Let $N_3 \in \mathbb{N}$ such that for all $n \geqslant N_3$, we have $\chi\left(\tau_{j(n)}^s\right) < \frac{\varepsilon}{4Dl}$.

Last, by assumption, there exists $N_4 \in \mathbb{N}$ such that for all $n \geqslant N_4$ and for all $a' \in \mathfrak{sa}\left(\mathfrak{A}^{d(s')}\right)$ with $\mathsf{L}^{d(s')}(a') \leqslant l$, we have:

$$\left\|\varphi_{j(n)}^{s\circ s'}(a') - \varphi_{j(n)}^{s} \circ \varphi_{j(n)}^{s'}(a')\right\|_{\mathfrak{A}_{j(n)}^{c(s)}} < \frac{\varepsilon}{4}.$$

We record that by assumption, we have:

$$d_n \in \mathfrak{t}_{\gamma_{j(n)}^{c(s)}}\left(\varphi_{j(n)}^{s}(b_n)\middle|D(s)D(s')l\right) \subseteq \mathfrak{t}_{\gamma_{j(n)}^{c(s)}}\left(\varphi_{j(n)}^{s}(b_n)\middle|Dl\right)$$

and

$$e_n \in \mathfrak{t}_{\gamma_{j(n)}^{c(s)}}\left(\varphi_{j(n)}^{s\circ s'}(a)\middle|D(s\circ s')l\right) \subseteq \mathfrak{t}_{\gamma_{j(n)}^{c(s)}}\left(\varphi_{j(n)}^{s\circ s'}(a)\middle|Dl\right).$$

For all $n \geqslant \max\{N_1, N_2, N_3, N_4\}$, we thus have:

$$\begin{aligned} 0 &\leqslant \|\varphi^{s \circ s'}(a) - \varphi^s \circ \varphi^{s'}(a)\|_{\mathfrak{A}^{c(s)}} \\ &\leqslant \|\varphi^{s \circ s'}(a) - e_n\|_{\mathfrak{A}^{c(s)}} + \|e_n - d_n\|_{\mathfrak{A}^{c(s)}} + \|d_n - \varphi^s \circ \varphi^{s'}(a)\|_{\mathfrak{A}^{c(s)}} \\ &\leqslant \frac{2\varepsilon}{4} + \|e_n - d_n\|_{\mathfrak{A}^{c(s)}} \\ &\leqslant \frac{2\varepsilon}{4} + \|\varphi^{s \circ s'}_{j(n)}(a_n) - \varphi^s_{j(n)}(b_n)\|_{\mathfrak{A}^{c(s)}_{j(n)}} + 2lD\chi\left(\tau^{c(s)}_{j(n)}\right) \text{ by Prop (1.26)} \\ &\leqslant \frac{3\varepsilon}{4} + \|\varphi^{s \circ s'}_{j(n)}(a_n) - \varphi^s_{j(n)}(\varphi^{s'}_{j(n)}(a_n))\|_{\mathfrak{A}^{c(s)}_{j(n)}} \\ &\leqslant \varepsilon. \end{aligned}$$

Hence, as $\varepsilon > 0$ is arbitrary, we conclude:

$$\left\|\varphi^{s \circ s'}(a) - \varphi^s \circ \varphi^{s'}(a)\right\|_{\mathfrak{A}^{c(s)}} = 0 \text{ so } \varphi^{s \circ s'}(a) = \varphi^s \circ \varphi^{s'}(a).$$

This concludes this step.

Summary 2.14. By setting $\Psi(s) = (\mathfrak{A}^s, \mathsf{L}^s)$ for all $s \in \mathcal{S}^{(0)}$, and $\Psi(s) = \varphi^s$ for all $s \in \mathcal{S}$, we have proven that Ψ is a functoid from $(\mathcal{S}, \mathcal{S}^{(0)}, d, c, \circ)$ to the category of F–quasi-Leibniz quantum compact metric spaces whose arrows are Lipschitz linear maps.

We now turn to various additional properties for our maps φ^s ($s \in \mathcal{S}$) under more stringent conditions. We begin with positivity.

Step 8. *If, for some $s \in \mathcal{S}$ and for all $n \in \mathbb{N}$, the map φ^s_n is positive, and $\{a \in \mathrm{dom}\left(\mathsf{L}^{d(s)}\right) : a \geqslant 0\}$ is dense in $\{a \in \mathfrak{sa}\left(\mathfrak{A}^{d(s)}\right) : a \geqslant 0\}$, then φ^s is positive as well.*

Let $a \in \mathrm{dom}\left(\mathsf{L}^{d(s)}\right)$ with $a \geqslant 0$ and write $l = \mathsf{L}^{d(s)}(a)$; we also write $D = D(s)$. Let $\mu \in \mathscr{S}\left(\mathfrak{A}^{c(s)}\right)$. Let $\varepsilon > 0$. There exists $N_1 \in \mathbb{N}$ such that, for all $n \geqslant N$, we can find $f_n \in \mathfrak{A}^{c(s)}$ such that $\|f_n - \varphi^s(a)\|_{\mathfrak{A}^{c(s)}} < \varepsilon$, with $f_n \in \mathfrak{t}_{\gamma^{c(s)}_{j(n)}}(\varphi^s_n(b_n)|Dl)$, where $b_n \in \mathfrak{t}_{\tau^{d(s)}_{j(n)}}(a|l)$.

There exists $N_2 \in \mathbb{N}$ such that for all $n \geqslant N_2$ we have:

$$\chi\left(\tau^{d(s)}_{j(n)}\right) < \frac{\varepsilon}{3(l+1)} \text{ and } \chi\left(\gamma^{c(s)}_{j(n)}\right) < \frac{\varepsilon}{3(Dl+1)}.$$

Let $N = \max\{N_1, N_2\}$ and $n \geqslant N$.

We will need, for this step only, a notation for the tunnels involved in the following computation. We write $\tau^{d(s)}_{j(n)} = (\mathfrak{D}^{d(s)}_n, \mathsf{R}^{d(s)}_n, \pi^{d(s)}_n, \rho^{d(s)}_n)$ and $\gamma^{c(s)}_{j(n)} = (\mathfrak{D}^{c(s)}_n, \mathsf{R}^{c(s)}_n, \rho^{c(s)}_n, \pi^{c(s)}_n)$.

By Definition (1.17) and [18, Proposition 2.12], there exists $\eta_n \in \mathscr{S}\left(\mathfrak{A}^{c(s)}_{j(n)}\right)$ such that $\mathsf{mk}_{\mathsf{R}^{c(s)}_n}(\mu, \eta_n) < \frac{\varepsilon}{3(Dl+1)}$. Since $\varphi^s_{j(n)}$ is positive and unital, the map $\mu_n = \eta_n \circ \varphi^s_n$ is a state of $\mathfrak{A}^{d(s)}_{j(n)}$. Again by Definition (1.17), there exists $\nu_n \in \mathscr{S}(\mathfrak{A}^{d(s)})$ with $\mathsf{mk}_{\mathsf{R}^{d(s)}_n}(\nu_n, \mu_n) < \frac{\varepsilon}{3(l+1)}$.

Moreover, by Definition (1.24), since $b_n \in \mathfrak{t}_{\tau^{d(s)}_{j(n)}}(a|l)$, there exists $d_n \in \mathfrak{D}^{d(s)}_n$ with $\mathsf{R}^{d(s)}_n(d_n) \leqslant l$ such that $\pi^{d(s)}_n(d_n) = a$ and $\rho^{d(s)}_n(d_n) = b_n$. There also exists $d'_n \in \mathfrak{D}^{c(s)}_n$ such that $\mathsf{R}^{c(s)}_n(d'_n) \leqslant Dl$, $\pi^{c(s)}_n(d'_n) = f_n$ and $\rho^{d(s)}_n(d'_n) = \varphi^s_n(b_n)$.

We then compute:

(2.4)
$$\begin{aligned}
|\mu(\varphi^s(a)) - \nu_n(a)| &\leqslant |\mu(\varphi^s(a)) - \mu(f_n)| + |\mu(f_n) - \nu_n(a)| \\
&\leqslant \|\varphi^s(a) - f_n\|_{\mathfrak{A}} + |\mu(f_n) - \nu_n(a)| \\
&\leqslant \frac{\varepsilon}{3} + |\mu(f_n) - \nu_n(a)| \\
&\leqslant \frac{\varepsilon}{3} + |\mu(f_n) - \mu_n(b_n)| + |\mu_n(b_n) - \nu_n(a)| \\
&\leqslant \frac{\varepsilon}{3} + |\mu(f_n) - \eta_n(\varphi^s_n(b_n))| + |\mu_n \circ \rho_n^{d(s)}(d_n) - \nu_n \circ \pi_n^{d(s)}(d_n)| \\
&\leqslant \frac{\varepsilon}{3} + |\mu \circ \pi_n^{c(s)}(d'_n) - \eta_n \circ \rho_n^{c(s)}(d'_n)| + l\mathsf{mk}_{\mathsf{R}_n^{d(s)}}(\nu_n, \mu_n) \\
&\leqslant \frac{\varepsilon}{3} + Dl\mathsf{mk}_{\mathsf{R}_n^{c(s)}}(\mu, \eta_n) + l\mathsf{mk}_{\mathsf{R}_n^{d(s)}}(\nu_n, \mu_n) \\
&\leqslant \frac{\varepsilon}{3} + \frac{Dl\varepsilon}{3Dl} + \frac{l\varepsilon}{3l} = \varepsilon.
\end{aligned}$$

Now, since $a \geqslant 0$, for any $n \geqslant N$, we have $\nu_n(a) \geqslant 0$. As the limit of the sequence of nonnegative numbers $(\nu_n(a))_{n\in\mathbb{N}}$ by Expression (2.4), we conclude that $\mu(\varphi^s(a)) \geqslant 0$. As $\mu \in \mathscr{S}(\mathfrak{A}^{c(s)})$ was an arbitrary state, we conclude that $\varphi^s(a) \geqslant 0$.

By continuity of φ^s, since the space of positive elements in $\mathfrak{A}^{c(s)}$ is closed in norm, and by assumption that any positive element of $\mathfrak{A}^{d(s)}$ is the limit of positive elements in $\mathrm{dom}\left(\mathsf{L}^{d(s)}\right)$, we conclude that φ^s is a positive map, as claimed.

Step 9. *If, for some $s \in \mathcal{S}$ and for all $n \in \mathbb{N}$, the map φ^s_n is a unital *-morphism, then φ is a unital *-morphism; in particular $|||\varphi|||^{\mathfrak{A}}_{\mathfrak{B}} = 1$.*

Fix $s \in \mathcal{S}$ such that for all $n \in \mathbb{N}$, the map φ^s_n is a unital *-morphism. For this step, set $D = D(s)$.

We begin by proving that φ^s, restricted to $\mathrm{dom}\left(\mathsf{L}_{\mathfrak{A}}\right)$, is a Jordan-Lie morphism. Let $a, a' \in \mathrm{dom}\left(\mathsf{L}^{d(s)}\right)$ and $l \geqslant \max\{\mathsf{L}^{d(s)}(a), \mathsf{L}^{d(s)}(a')\}$. For each $n \in \mathbb{N}$, let $b_n \in \mathfrak{f}_{j(n)}\left(a'|l, D\right)$ and $b'_n \in \mathfrak{f}_{j(n)}\left(a'|l, D\right)$. By construction, $(b_n)_{n\in\mathbb{N}}$ converges to $\varphi^s(a)$ while $(b'_n)_{n\in\mathbb{N}}$ converges to $\varphi^s(a')$. On the other hand, by Lemma (2.6) (in particular, using the notations therein), for all $n \in \mathbb{N}$, we have:

$$b_n \circ b'_n \in \mathfrak{f}_{j(n)}\left(a \circ a' | M(\|a\|_{\mathfrak{A}}, \|a'\|_{\mathfrak{A}}, l, l), D\right)$$

so $(b_n \circ b'_n)_{n\in\mathbb{N}}$ converges to $\varphi^s\left(a \circ a'\right)$. By uniqueness of the limit, we conclude:

$$\varphi^s\left(a \circ a'\right) = \varphi^s(a) \circ \varphi^s(a').$$

The same results holds for the Lie product. Thus φ^s restricted to $\mathrm{dom}\left(\mathsf{L}^{d(s)}\right)$ is a Jordan-Lie morphism. By continuity, φ^s is a Jordan-Lie morphism on from $\mathfrak{sa}\left(\mathfrak{A}^{d(s)}\right)$ to $\mathfrak{sa}\left(\mathfrak{A}^{c(s)}\right)$.

So φ^s is a unital Jordan-Lie morphism on from $\mathfrak{sa}\left(\mathfrak{A}^{d(s)}\right)$ to $\mathfrak{sa}\left(\mathfrak{A}^{c(s)}\right)$.

We note that by construction, $\varphi^s(a^*) = \varphi^s(a)^*$.

Let now $a, b \in \mathfrak{A}^{d(s)}$. We fist note that:

$$\begin{aligned}
\varphi^s(a \circ b) &= \varphi^s(\Re a \circ \Re b) - \varphi^s(\Im a \circ \Im b) + i\left(\varphi^s(\Re a \circ \Im b) + \varphi^s(\Im a \circ \Re b)\right) \\
&= \varphi^s(\Re a) \circ \varphi^s(\Re b) - \varphi^s(\Im a) \circ \varphi^s(\Im b) \\
&\quad + i\left(\varphi^s(\Re a) \circ \varphi^s(\Im b) + \varphi^s(\Im a) \circ \varphi^s(\Re b)\right) \\
&= \varphi^s(a) \circ \varphi^s(b).
\end{aligned}$$

The same computation holds for the Lie product. Therefore:

$$\begin{aligned}\varphi^s(ab) &= \varphi^s(\Re(ab)) + i\varphi^s(\Im(ab))\\ &= \varphi^s(a\circ b) + i\varphi^s(\{a,b\})\\ &= \varphi^s(a)\circ\varphi^s(b) + i\{\varphi^s(a),\varphi^s(b)\}\\ &= \varphi^s(a)\varphi^s(b).\end{aligned}$$

Thus φ is a unital *-morphism. This concludes our step.

We now prove a non-triviality result.

Step 10. *If, for some $s,s'\in\mathcal{S}$, we have $c(s)=c(s')$ and $d(s)=d(s')$, then:*

$$\liminf_{n\to\infty}\mathrm{mkD}_{\mathfrak{A}_n^{c(s)}}^{\mathsf{L}_n^{d(s)}}\left(\varphi_n^s,\varphi_n^{s'}\right)\leqslant\mathrm{mkD}_{\mathfrak{A}^{c(s)}}^{\mathsf{L}^{d(s)}}\left(\varphi^s,\varphi^{s'}\right)\leqslant\limsup_{n\to\infty}\mathrm{mkD}_{\mathfrak{A}_n^{c(s)}}^{\mathsf{L}_n^{d(s)}}\left(\varphi_n^s,\varphi_n^{s'}\right).$$

Moreover, if for some $s\in\mathcal{S}$, we have $d(s)=c(s)$, then:

$$\liminf_{n\to\infty}\mathrm{mk}\ell_{\mathsf{L}_n}(\varphi_n^s)\leqslant\mathrm{mk}\ell_{\mathsf{L}}(\varphi^s)\leqslant\limsup_{n\to\infty}\mathrm{mk}\ell_{\mathsf{L}_n}(\varphi_n^s).$$

Fix $s,s'\in\mathcal{S}$ with the desired properties and write $D=\max\{1,D(s),D(s')\}$. Let $\varepsilon>0$. Let E be a finite subset of $\{a\in\mathfrak{sa}\left(\mathfrak{A}^{d(s)}\right):\mathsf{L}^{d(s)}(a)\leqslant 1\}$ such that for all $a\in\mathfrak{sa}\left(\mathfrak{A}^{d(s)}\right)$ with $\mathsf{L}^{d(s)}(a)\leqslant 1$ there exists $a'\in E$ and $t\in\mathbb{R}$ such that:

$$\|a-(a'+t1_{\mathfrak{A}})\|_{\mathfrak{A}^{d(s)}}<\frac{\varepsilon}{12D}.$$

Let $N\in\mathbb{N}$ such that for all $n\geqslant N$, we have for all $a'\in E$:

$$\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{d(s)}}}\left(\mathfrak{f}_{s,j(n)}(a'|1,D),\{\varphi^s(a')\}\right)<\frac{\varepsilon}{8}$$

and:

$$\mathsf{Haus}_{\|\cdot\|_{\mathfrak{A}^{d(s)}}}\left(\mathfrak{f}_{s',j(n)}(a'|1,D),\{\varphi^{s'}(a')\}\right)<\frac{\varepsilon}{8}$$

while:

$$\chi\left(\tau_{j(n)}^s\right)<\frac{\varepsilon}{12D}.$$

Note that N exists since E is finite and by construction of φ^s above.

Fix $n\geqslant N$.

Let $G_n\subseteq\mathfrak{A}_{j(n)}^{d(s)}$ be given by Lemma (1.27), so that:

- for all $a\in\mathfrak{sa}\left(\mathfrak{A}_{j(n)}^{d(s)}\right)$ with $\mathsf{L}_n^{d(s)}(a)\leqslant 1$, there exists $t\in\mathbb{R}$ and $a'\in G_n$ such that $\|a-(a'+t1_{\mathfrak{A}})\|_{\mathfrak{A}_n^{d(s)}}\leqslant\frac{\varepsilon}{4D}$,
- for all $a\in G_n$ there exists $a'\in E$ such that $a\in\mathfrak{t}_{\tau_{j(n)}^{d(s)}}(a'|1)$, and conversely, for all $a\in E$ there exists $a'\in G_n$ such that $a\in\mathfrak{t}_{\gamma_{j(n)}^{d(s)}}(a'|1)$.

Let $a \in \mathfrak{sa}\left(\mathfrak{A}^{d(s)}\right)$ with $\mathsf{L}^{d(s)}(a) \leqslant 1$. There exists $a' \in E$ and $t \in \mathbb{R}$ such that $\|a - (a' + t1_{\mathfrak{A}})\|_{\mathfrak{A}^{d(s)}} < \frac{\varepsilon}{12D} < \frac{\varepsilon}{4D}$. We then have:

$$\begin{aligned}\left\|\varphi^{s'}(a) - \varphi^{s}(a)\right\|_{\mathfrak{A}^{c(s)}} &\leqslant \left\|\varphi^{s'}\left(a - (a' + t1_{\mathfrak{A}})\right)\right\|_{\mathfrak{A}^{c(s)}} \\ &\quad + \left\|\varphi^{s'}(a' + t1_{\mathfrak{A}}) - \varphi^{s}(a' + t1_{\mathfrak{A}})\right\|_{\mathfrak{A}^{c(s)}} \\ &\quad + \|\varphi^{s}(a - (a' + t1_{\mathfrak{A}}))\|_{\mathfrak{A}^{c(s)}} \\ &\leqslant \frac{D\varepsilon}{4D} + \|\varphi^{s'}(a') - \varphi^{s}(a')\|_{\mathfrak{A}^{c(s)}} + \frac{D\varepsilon}{4D} \\ &\leqslant \frac{\varepsilon}{2} + \|\varphi^{s'}(a') - \varphi^{s}(a')\|_{\mathfrak{A}^{c(s)}}.\end{aligned}$$

Let $a_n \in G_n \cap \mathfrak{t}_{\tau^{s}_{j(n)}}(a'|1)$. Let $c_n = \varphi^{s}_{j(n)}(a_n)$ and $d_n = \varphi^{s'}_{j(n)}(a_n)$. Let $c \in \mathfrak{t}_{\gamma_{j(n)}}(c_n|D)$ and $d \in \mathfrak{t}_{\gamma_{j(n)}}(d_n|D)$. In particular, $c \in \mathfrak{f}_{s,j(n)}(a'|1, D)$ and $d \in \mathfrak{f}_{s',j(n)}(a'|1, D)$. Thus by construction, $\|\varphi^{s}(a') - c\|_{\mathfrak{A}^{d(s)}} \leqslant \frac{\varepsilon}{8}$.

We then compute:

$$\begin{aligned}\|\varphi^{s'}(a') - \varphi^{s}(a')\|_{\mathfrak{A}^{c(s)}} &\leqslant \frac{\varepsilon}{4} + \|d - c\|_{\mathfrak{A}^{d(s)}} \\ &\leqslant \frac{\varepsilon}{4} + 2\frac{\varepsilon}{12D} + \|\varphi^{s'}_{j(n)}(a_n) - \varphi^{s}_{j(n)}(a_n)\|_{\mathfrak{A}^{d(s)}_{j(n)}} \text{ by Proposition (1.26)} \\ &\leqslant \frac{\varepsilon}{2} + \mathrm{mkD}^{\mathsf{L}^{d(s)}_{j(n)}}_{\mathfrak{A}^{c(s)}_{j(n)}}\left(\varphi^{s}_{j(n)}, \varphi^{s'}_{j(n)}\right).\end{aligned}$$

Hence for all $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}^{d(s)}(a) \leqslant 1$:

$$\|\varphi^{s}(a) - \varphi^{s'}(a)\|_{\mathfrak{A}^{c(s)}} \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + \mathrm{mkD}^{\mathsf{L}^{d(s)}_{j(n)}}_{\mathfrak{A}^{c(s)}_{j(n)}}\left(\varphi^{s}_{j(n)}, \varphi^{s'}_{j(n)}\right),$$

i.e. $\mathrm{mkD}^{\mathsf{L}^{d(s)}}_{\mathfrak{A}^{c(s)}}\left(\varphi^{s}, \varphi^{s'}\right) \leqslant \mathrm{mkD}^{\mathsf{L}^{d(s)}_{j(n)}}_{\mathfrak{A}^{c(s)}_{j(n)}}\left(\varphi^{s}_{j(n)}, \varphi^{s'}_{j(n)}\right) + \varepsilon$.

We proceed symmetrically to show the converse inequality. Namely, let $a_n \in \mathfrak{A}^{d(s)}_{j(n)}$ with $\mathsf{L}^{d(s)}_{\mathfrak{A}_{j(n)}}(a_n) \leqslant 1$. There exists $t \in \mathbb{R}$ and $a'_n \in G_n$ such that $\|a_n - (a'_n + t1_{\mathfrak{A}})\|_{\mathfrak{A}^{d(s)}_{j(n)}} < \frac{\varepsilon}{4D}$. As before, we have:

$$\|\varphi^{s'}_{j(n)}(a_n) - \varphi^{s}_{j(n)}(a_n)\|_{\mathfrak{A}^{d(s)}_{j(n)}} \leqslant \frac{\varepsilon}{2} + \|\varphi^{s'}_{j(n)}(a'_n) - \varphi^{s}_{j(n)}(a'_n)\|_{\mathfrak{A}^{d(s)}_{j(n)}}.$$

Let $a \in E$ such that $a \in \mathfrak{t}_{\gamma^{d(s)}_{j(n)}}(a'_n|1)$. Let $c \in \mathfrak{t}_{\gamma^{d(s)}_{j(n)}}\left(\varphi^{s}_{j(n)}(a'_n)\middle|D\right)$. Note that by symmetry, we have $a_n \in \mathfrak{t}_{\tau^{d(s)}_{j(n)}}(a|1)$, so that by construction $c \in \mathfrak{f}_{s,j(n)}(a|1, D)$ and thus $\|\varphi^{s}(a) - c\|_{\mathfrak{A}^{d(s)}} < \frac{\varepsilon}{8}$.

Similarly, let $d \in \mathfrak{t}_{\gamma^{d(s)}_{j(n)}}\left(\varphi^{s'}_{j(n)}(a'_n)\middle|D\right)$, and note $d \in \mathfrak{f}_{s',j(n)}(a|1, D)$ so $\|\varphi^{s'}(a) = d\| < \frac{\varepsilon}{8}$.

From this, we then obtain as before that $\|\varphi^{s'}_{j(n)}(a'_n) - \varphi^{s}_{j(n)}(a'_n)\|_{\mathfrak{A}^{d(s)}_{j(n)}} \leqslant \frac{\varepsilon}{2} + \mathrm{mkD}^{\mathsf{L}^{d(s)}}_{\mathfrak{A}^{c(s)}}\left(\varphi^{s'}, \varphi^{s}\right)$. Thus we get:

$$\mathrm{mkD}^{\mathsf{L}^{d(s)}_{j(n)}}_{\mathfrak{A}^{c(s)}_{j(n)}}\left(\varphi^{s}_{j(n)}, \varphi^{s'}_{j(n)}\right) \leqslant \mathrm{mkD}^{\mathsf{L}^{d(s)}}_{\mathfrak{A}^{c(s)}}\left(\varphi^{s}, \varphi^{s'}\right) + \varepsilon.$$

Hence we have shown that, for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have:

$$\mathrm{mkD}_{\mathfrak{A}^{c(s)}}^{\mathsf{L}^{d(s)}}\left(\varphi^s, \varphi^{s'}\right) - \varepsilon \leqslant \mathrm{mkD}_{\mathfrak{A}_{j(n)}^{c(s)}}^{\mathsf{L}_{j(n)}^{d(s)}}\left(\varphi_{j(n)}^s, \varphi_{j(n)}^{s'}\right) \leqslant \mathrm{mkD}_{\mathfrak{A}^{c(s)}}^{\mathsf{L}^{d(s)}}\left(\varphi^s, \varphi^{s'}\right) + \varepsilon.$$

Thus:

$$\lim_{n\to\infty} \mathrm{mkD}_{\mathfrak{A}_{j(n)}^{c(s)}}^{\mathsf{L}_{j(n)}^{d(s)}}\left(\varphi_{j(n)}^s, \varphi_{j(n)}^{s'}\right) = \mathrm{mkD}_{\mathfrak{A}^{c(s)}}^{\mathsf{L}^{d(s)}}\left(\varphi^s, \varphi^{s'}\right).$$

In particular, we have:

$$\liminf_{n\to\infty} \mathrm{mkD}_{\mathfrak{A}_n^{c(s)}}^{\mathsf{L}_n^{d(s)}}\left(\varphi_n^s, \varphi_n^{s'}\right) \leqslant \mathrm{mkD}_{\mathfrak{A}^{c(s)}}^{\mathsf{L}^{d(s)}}\left(\varphi^s, \varphi^{s'}\right) \leqslant \limsup_{n\to\infty} \mathrm{mkD}_{\mathfrak{A}_n^{c(s)}}^{\mathsf{L}_n^{d(s)}}\left(\varphi_n^s, \varphi_n^{s'}\right).$$

If $c(s) = d(s)$ then our argument above can be applied as well when we replace $\varphi_n^{s'}$ and $\varphi^{s'}$ with the identity of $\mathfrak{A}_{j(n)}^{d(s)}$ and $\mathfrak{A}^{d(s)}$, and adjusting the proof accordingly, in which case we conclude:

$$\liminf_{n\to\infty} \mathrm{mk}\ell_{\mathsf{L}_n^{d(s)}}(\varphi_n^s) \leqslant \mathrm{mk}\ell_{\mathsf{L}^{d(s)}}(\varphi^s) \leqslant \limsup_{n\to\infty} \mathrm{mk}\ell_{\mathsf{L}_n^{d(s)}}(\varphi_n^s).$$

Now, assume $u \in \mathcal{S}$ such that $d(u) = c(u)$ and $\lim_{n\to\infty} \mathrm{mk}\ell_{\mathsf{L}_n^{s(u)}}(\varphi_n^u) = 0$. By this step, we conclude that $\mathrm{mk}\ell_{\mathsf{L}^{s(u)}}(\varphi^u) = 0$ so φ^u is the identity of $\mathfrak{A}^{d(u)}$.

From this, and by definition, we conclude that if all units u of $\mathcal{S}$ satisfy $\lim_{n\to\infty} \mathrm{mk}\ell_{\mathsf{L}_n^{d(u)}}(\varphi_n^u) = 0$ then Ψ is a functor.

Moreover, if $s \in \mathcal{S}$ is invertible and we set $u = s \circ s'$ and $v = s' \circ s$, where s' is the inverse of s, and if:

$$\lim_{n\to\infty} \mathrm{mk}\ell_{\mathsf{L}_n^{d(u)}}(\varphi_n^u) = \lim_{n\to\infty} \mathrm{mk}\ell_{\mathsf{L}_n^{d(u)}}(\varphi_n^{s\circ s'}) = 0$$

and

$$\lim_{n\to\infty} \mathrm{mk}\ell_{\mathsf{L}_n^{d(v)}}(\varphi_n^v) = \lim_{n\to\infty} \mathrm{mk}\ell_{\mathsf{L}_n^{d(v)}}(\varphi_n^{s'\circ s}) = 0$$

then φ^u and φ^v are the identities of $\mathfrak{A}^{d(u)}$ and $\mathfrak{A}^{d(v)}$. Since moreover $\varphi^u = \varphi^{s\circ s'} = \varphi^s \circ \varphi^{s'}$ and $\varphi^v = \varphi^{s'\circ s} = \varphi^{s'} \circ \varphi^s$, we conclude that φ^s is invertible with inverse $\varphi^{s'}$.

This concludes our step and our theorem. ☐

Remark 2.15. We note that the morphisms obtained by Theorem (2.13) are certainly not unique in general. There are many different choices being made in their construction: the choice of the tunnels and the choices of subsequences of target sets from compactness. The proof shows how all these choices can be made in a coherent fashion to preserve composition, and ensure that the limit of identity automorphisms is the identity — thus preserving inverse relationships.

Remark 2.16. We need an additional assumption regarding the density of the positive elements with finite L-seminorm within the positive elements of the underlying C*-algebra for Item (1) of Theorem (2.13) which does not appear for the rest of the theorem, and in particular for Item (3) regarding *-morphisms.

This assumption is needed for the proof of Item (1) as we have no multiplicative property for this step, and not for Item (3) because *-morphisms are automatically positive. We note that for classical metric space, this special assumption is always satisfied because the truncation of a Lipschitz function is Lipschitz. On the other hand, we also note that when L-seminorms are obtained from ergodic group actions as in [22], which is an often used construction, then again this special property is met, using a standard regularization argument akin to the proof of the density of the domain of such L-seminorms in [22]. However, in general, the situation can be more involved and the Leibniz property does not seem to ensure that this special condition will always hold.

We proved our main Theorem (2.13) with the construction of the dual propinquity in [18]. We could just as easily make the same argument, with trivial changes, adapted to the original construction in [12] based on journeys — this would technically allow us to use somewhat more general classes of tunnels. Among the possible choices of tunnels, we could then employ the tunnels obtained directly from bridges [16]. The reason one would wish to adapt the argument in such an easy manner is if it is possible to extract more information about the functor constructed by Theorem (2.13) based on a more stringent choice of tunnels — for instance, when spaces converge for the quantum propinquity [16].

3. Applications

As a first observation, we apply Theorem (2.13) to the construction of a single *-morphism. The result is interesting because it is not necessarily easy to construct morphisms between C*-algebras, yet this first corollary shows a new method based on metric approximations.

Corollary 3.1. *Let F be an admissible function. Let $(\mathfrak{A}, \mathsf{L}^{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}^{\mathfrak{B}})$ be two F–quasi-Leibniz quantum compact metric spaces, and let $(\mathfrak{A}_n, \mathsf{L}_n^{\mathfrak{A}})_{n\in\mathbb{N}}$ and $(\mathfrak{B}, \mathsf{L}_n^{\mathfrak{B}})_{n\in\mathbb{N}}$ be two sequences of F–quasi-Leibniz quantum compact metric spaces such that:*

$$\lim_{n\to\infty} \mathsf{\Lambda}^*_F((\mathfrak{A}_n, \mathsf{L}_n^{\mathfrak{A}}), (\mathfrak{A}, \mathsf{L}^{\mathfrak{A}})) = \lim_{n\to\infty} \mathsf{\Lambda}^*_F((\mathfrak{B}_n, \mathsf{L}_n^{\mathfrak{B}}), (\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})) = 0.$$

Let $D > 0$. If for each $n \in \mathbb{N}$, there exists a Lipschitz morphism (resp. a Lipschitz linear map) $\varphi_n : \mathfrak{A}_n \to \mathfrak{B}_n$ such that:

(1) $\forall n \in \mathbb{N} \quad \mathsf{L}_n^{\mathfrak{B}} \circ \varphi_n \leqslant D\mathsf{L}_n^{\mathfrak{A}}$,
(2) $|||\varphi_n|||_{\mathfrak{B}_n}^{\mathfrak{A}_n} \leqslant D$,

then there exists a Lipschitz morphism (resp. a Lipschitz linear map) $\varphi : \mathfrak{A} \to \mathfrak{B}$ such that:

(1) $|||\varphi|||_{\mathfrak{B}}^{\mathfrak{A}} \leqslant D$
(2) $\mathsf{L}_{\mathfrak{B}} \circ \varphi \leqslant \mathsf{L}_{\mathfrak{A}}$.

If moreover $(\mathfrak{A}_n, \mathsf{L}_n^{\mathfrak{A}}) = (\mathfrak{B}_n, \mathsf{L}_n^{\mathfrak{B}})$ for all $n \in \mathbb{N}$ then:

$$\liminf_{n\to\infty} \mathrm{mk}\ell_{\mathsf{L}_n^{\mathfrak{A}}}(\varphi_n) \leqslant \mathrm{mk}\ell_{\mathsf{L}_{\mathfrak{A}}}(\varphi) \leqslant \limsup_{n\to\infty} \mathrm{mk}\ell_{\mathsf{L}_{\mathfrak{A}}}(\varphi_n).$$

If furthermore, φ_n is a bijection for infinitely many $n \in \mathbb{N}$, then φ may be chosen to be a bijection as well.

Proof. All but the last statement result from applying Theorem (2.13) to the semigroupoid $(\{s\}, \{a, b\}, c, d, \circ)$ with $d(s) = a$ and $c(s) = b$ (thus $\circ$ is the empty map). Setting $(\mathfrak{A}^a, \mathsf{L}^a) = (\mathfrak{A}, \mathsf{L}^{\mathfrak{A}})$, $(\mathfrak{A}^b, \mathsf{L}^b) = (\mathfrak{B}, \mathsf{L}^{\mathfrak{B}})$ and for all $n \in \mathbb{N}$, setting $(\mathfrak{A}_n^a, \mathsf{L}_n^a) = (\mathfrak{A}_n, \mathsf{L}_n^{\mathfrak{A}})$, $(\mathfrak{A}_n^b, \mathsf{L}_n^b) = (\mathfrak{B}_n, \mathsf{L}_n^{\mathfrak{B}})$, and $\varphi_n^s = \varphi_n$, Theorem (2.13) applies.

If φ_n is invertible for infinitely many $n \in \mathbb{N}$, then up to extracting a subsequence, we may as well assume that φ_n is invertible for all $n \in \mathbb{N}$. Then we employ the groupoid $(\{s, s', a, b\}, \{a, b\}, d, c, \circ)$, with the same definitions as above, with a and b serving as the units for the homonymous domains, and s' the inverse of s, with $\varphi_n^{s'} = \varphi_n^{-1}$ for all $n \in \mathbb{N}$. Theorem (2.13) applies again. ∎

As an obvious remark, we point out that if we are given two convergent sequences $(\mathfrak{A}_n, \mathsf{L}_n^{\mathfrak{A}})_{n\in\mathbb{N}}$ and $(\mathfrak{B}_n, \mathsf{L}_n^{\mathfrak{B}})_{n\in\mathbb{N}}$ of quasi-Leibniz quantum compact metric spaces which are entry-wise fully quantum isometric, then by [12], their respective limits $(\mathfrak{A}, \mathsf{L}^{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}^{\mathfrak{B}})$ are also fully quantum isometric:

$$0 \leqslant \mathsf{\Lambda}((\mathfrak{A}, \mathsf{L}^{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}^{\mathfrak{B}})) = \lim_{n\to\infty} \mathsf{\Lambda}((\mathfrak{A}_n, \mathsf{L}_n^{\mathfrak{A}}), (\mathfrak{B}, \mathsf{L}_n^{\mathfrak{B}})) = \lim_{n\to\infty} 0 = 0.$$

This coincidence property was a strong motivation behind the construction of the propinquity, and is proven using in part the quasi-Leibniz property. Thus our present Theorem (2.13) provides another illustration of the same principle that using the quasi-Leibniz property and the definition of the propinquity enables us to build *-morphisms between C*-algebras, albeit Theorem (2.13) application in the single morphism case is only new and interesting for Lipschitz morphisms which are not full quantum isometries.

We now turn to the main motivation for this paper: the existence of a nontrivial action of a group on the limit of a sequence of quasi-Leibniz quantum compact metric spaces, when the group is the limit of groups acting on each term of the sequence. One way to see the importance of this result is with a sight on mathematical physics applications: a model obtained as a limit for the propinquity of covariant models for some group will bear the same covariance.

There are in fact different manners to formalize this problem. We will study the covariant propinquity in a subsequent paper, to keep the present work of manageable length, and also because we take the perspective that this is a work on the properties of the Gromov-Hausdorff propinquity. For this purpose, we encode covariance in a strong form, in terms of a naturally enhanced Gromov-Hausdorff convergence for semigroups. Most likely, this definition is not new. We choose to construct the Gromov-Hausdorff group distance in terms of ε-isometries to keep our presentation brief.

We will use the term semigroup to mean an associative magma — a unit is not required (in other words semigroups are naturally identified with semigroupoids with everywhere defined composition, though we shall use the usual multiplicative notation for semigroups). A *compact metric semigroup* (G, δ_G) is a semigroup G endowed with a distance δ_G which induces a compact topology on G, and for which the multiplication of G is continuous.

Definition 3.2. Let (G_1, δ_1) and (G_2, δ_2) be two compact metric semigroups. Let $\varepsilon > 0$. A *ε-equivariant isometry* $(\varsigma, \varkappa)$ *from* (G_1, δ_1) *to* (G_2, δ_2) is a pair of functions $\varsigma_1 : G_1 \to G_2$ and $\varsigma_2 : G_2 \to G_1$ such that for all $\{j, k\} = \{1, 2\}$:

(1) $\forall g, g' \in G_j \quad |\delta_k(\varsigma_j(g), \varsigma_j(g')) - \delta_j(g, g')| \leqslant \varepsilon$
(2) $\forall g \in G_j \quad |\delta_j(\varsigma_k \circ \varsigma_j(g), g)| \leqslant \varepsilon$
(3) $\forall g, g' \in G_j \quad \delta_k(\varsigma_j(gg'), \varsigma_j(g)\varsigma_j(g')) \leqslant \varepsilon$.

If, moreover, G_1 and G_2 are monoids with respective units e_1 and e_2, and if:

$$\delta_j(e_j, \varsigma_k(e_k)) < \varepsilon$$

for all $\{j, k\} = \{1, 2\}$, then $(\varsigma_1, \varsigma_2)$ is called a *ε-unital equivariant isometry.*

The set of all ε-equivariant isometries is denoted by $\mathrm{Iso}_\varepsilon((G_1, \delta_1), (G_2, \delta_2))$. The set of all ε-unital equivariant isometries is denoted by $\mathrm{UIso}_\varepsilon((G_1, \delta_1), (G_2, \delta_2))$.

Conditions (1) and (2) are used to define ε-isometries, which provide in turn a mean to construct the Gromov-Hausdorff distance [2]. Thus we simply add (3).

The unital condition is essentially encoding the idea of working with pointed metric spaces, choosing monoid units as base points.

Definition 3.3. Let (G, δ_G) and (H, δ_H) be compact metric semigroups. The *Gromov-Hausdorff semigroup distance* between (G, δ_G) and (H, δ_H) is the real number:

$$\Upsilon((G, \delta_G), (H, \delta_H)) = \inf \{\varepsilon \geqslant 0 : \mathrm{Iso}_\varepsilon((G, \delta_G), (H, \delta_H)) \neq \emptyset\}.$$

Definition 3.4. Let (G, δ_G) and (H, δ_H) be compact metric monoids. The *Gromov-Hausdorff monoid distance* between (G, δ_G) and (H, δ_H) is the real number:

$$\Upsilon_*((G, \delta_G), (H, \delta_H)) = \inf \{\varepsilon \geqslant 0 : \mathrm{UIso}_\varepsilon((G, \delta_G), (H, \delta_H)) \neq \emptyset\}.$$

Proposition 3.5. *The Gromov-Hausdorff semigroup distance Υ is a metric, up to isometric, semigroup isomorphisms on the class of all compact metric semigroups. The Gromov-Hausdorff monoid distance Υ_* is a metric, up to isometric, monoid isomorphisms on the class of all compact metric monoids.*

Moreover it is dominated by the Gromov-Hausdorff distance.

Proof. We first work with the metric Υ.

Fix any three compact metric semigroups (G, δ_G), (H, δ_H) and (J, δ_J).

We first prove that $\Upsilon((G, \delta_G), (H, \delta_H)) = 0$ implies the existence of an isometric semigroup isomorphism between (G, δ_G) and (H, δ_H).

Let S be a countable dense subset of G. Let G_0 be the subsemigroup generated in G by S, which is also countable as it is the set of all finite products of elements in S.

Similarly, let H_0 be a countable dense subsemigroup of H.

Assume first that $\Upsilon((G, \delta_G), (H, \delta_H)) = 0$. For each $n \in \mathbb{N}$, let:

$$(\varsigma_n, \varkappa_n) \in \mathcal{I}_{\frac{1}{n+1}}((G, \delta_G), (H, \delta_H)).$$

For each $g \in G_0$ and any strictly increasing sequence $j : \mathbb{N} \to \mathbb{N}$, any subsequence $(\varsigma_{j(n)}(g))$ admits a convergent subsequence $(\varsigma_{j\circ k(n)})$ in the compact space H. Thus, as G_0 is countable, a diagonal argument shows that there exists a strictly increasing sequence j of natural numbers such that, for all $g \in G_0$, the sequence $(\varsigma_{j(n)}(g))$ converges to a limit we denote by $\varsigma(g)$ (similar to Steps (2) and (3) in the proof of Theorem (2.13)).

Now, for all $g, g' \in G_0$, we have $\delta_H(\varsigma(g), \varsigma(g')) = \lim_{n\to\infty} \delta_H(\varsigma_{j(n)}(g), \varsigma_{j(n)}(g')) = 0$ so ς is an isometry (using Condition (1) of Definition (3.2)). Moreover, by Condition (3), we also have that for all $g, g' \in G_0$:

$$\delta_H(\varsigma(gg'), \varsigma(g)\varsigma(g')) = \lim_{n\to\infty} \delta_H(\varsigma_{j(n)}(gg'), \varsigma_{j(n)}(g)\varsigma_{j(n)}(g')) = 0.$$

So ς is a semigroup morphism from G_0 to H.

Now, as a uniformly continuous function over the dense subset G_0 of G, ς admits a unique uniformly continuous extension to G which we still denote by ς. It is immediate that ς is an isometry. Moreover, as the product of G is continuous as a map from $G \times G$ to G for the topology induced by δ_G on G and its product on $G \times G$, it is immediate that ς is a semigroup morphism as well.

Now, by the same argument, there exists an isometric map $\varkappa : H \to G$ and a strictly increasing $k : \mathbb{N} \to \mathbb{N}$ such that for all $h \in H_0$, the sequence $\varkappa_{j\circ k(n))}(h)$ converges to $\varkappa(h)$. By Condition (2) of Definition (3.2), we have $\varkappa \circ \varsigma$ is the identity of G_0 and $\varsigma \circ \varkappa$ is the identity of H_0. By continuity, this means that ς and $\varkappa$ are inverse of each others.

Now, $\varkappa$ is a semigroup morphism because it is the inverse of ς, itself a semigroup morphism. Alternatively, Condition (3) of Definition (3.2) proves that $\varkappa$ is a morphism.

Thus, $\Upsilon((G,\delta_G),(H,\delta_H)) = 0$ implies that there exists an isometric semigroup isomorphism $\varsigma : G \to H$ as desired.

Conversely, if there exists an isometric semigroup isomorphism ς from (G,δ_G) onto (H,δ_H) then $\left(\varsigma,\varsigma^{-1}\right) \in \mathcal{I}_0((G,\delta_G),(H,\delta_H))$ so $\Upsilon((G,\delta_G),(H,\delta_H)) = 0$.

We also note that we can always pick $g_0 \in G$, $h_0 \in H$ and define $\varsigma : g \in G \mapsto h_0$, $\varkappa : h \in H \mapsto g_0$; we easily check that $(\varsigma,\varkappa) \in \mathrm{Iso}_M$ where $M = \max\{\mathrm{diam}\,(G,\delta_G), \mathrm{diam}\,(H,\delta_H)\}$, so $\Upsilon((G,\delta_G),(H,\delta_H)) \leqslant M$.

Symmetry is obvious since:

$$\forall (\varsigma,\varkappa) \in \mathrm{Iso}_\varepsilon((G,\delta_G),(H,\delta_H)) \quad (\varkappa,\varsigma) \in \mathrm{Iso}_\varepsilon((H,\delta_H),(G,\delta_G))$$

by the symmetry in the conditions of Definition (3.2). This justifies that Condition (3) of Definition (3.2) is symmetric, as we saw that it is not needed for Υ to have the sought-after coincidence property.

Last, triangle inequality follows from the easy observation that if:

$$(\varsigma,\varkappa) \in \mathcal{I}_\varepsilon((G,\delta_G),(H,\delta_H)) \text{ and } (\varpi,\omega) \in \mathcal{I}_\eta((H,\delta_H),(J,\delta_J))$$

then:

$$(\varpi \circ \varsigma, \varkappa \circ \omega) \in Iso_{\epsilon+\eta}((G,\delta_G),(J,\delta_J)).$$

It is obvious by construction that Υ dominates the Gromov-Hausdorff distance.

Now, the above proof applies equally well to Υ_*; the only additional observation is that if $\Upsilon_*((G,\delta_G),(H,\delta_H)) = 0$ then the semigroup isomorphism $\varkappa : G \to H$ constructed above is in fact unital, per the definition of ε-equivariant unital isometries.

This completes our proof. □

Remark 3.6. Let G and H be two compact groups endowed with respective continuous length functions ℓ_G and ℓ_H. If $\Upsilon(G,H) = 0$ then G and H are isometrically isomorphic as semigroup — and therefore as groups as well. So when restricted to the class of metric compact groups, the Gromov-Hausdorff semigroup distance is a metric up to isometric, group isomorphism equivalence.

However, as defined, Υ is not a metric up to isometric, monoid isomorphism as it need not preserve units. Moreover, the monoid distance will prove helpful in our work with groups.

We are now ready to prove our main application of Theorem (2.13).

Theorem 3.7. *Let (G,δ_G) be a compact metric semigroup. Let $(G_n,\delta_n)_{n\in\mathbb{N}}$ be a sequence of compact metric semigroups, converging to G for the Gromov-Hausdorff semigroup distance Υ. For each $n \in \mathbb{N}$, choose:*

$$(\varsigma_n,\varkappa_n) \in \mathrm{Iso}_{2\Upsilon((G_n,\delta_n),(G,\delta))}((G_n,\delta_n),(G,\delta)).$$

Let $D : G \to [0,\infty)$ be a continuous function. Let $K : [0,\infty) \to [0,\infty)$ be continuous with $K(0) = 0$.

Let $\mathcal{C}$ be a nonempty class of F–quasi-Leibniz quantum compact metric spaces for some admissible function F, and let $\mathcal{T}$ be a class of tunnels compatible with $\mathcal{C}$. Let $(\mathfrak{A}_n,\mathsf{L}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}$, such that for each $n \in \mathbb{N}$, there exists an action α_n by Lipschitz linear endomorphisms of G_n on $\mathfrak{A}_n$, where:

$$\forall n \in \mathbb{N},\ g \in G_n \quad |||\alpha_n^g|||_{\mathfrak{A}_n} \leqslant D(\varsigma_n(g)) \text{ and } \mathsf{L}_n \circ \alpha_n^g \leqslant D(\varsigma_n(g))\mathsf{L}_n,$$

and

$$\forall n \in \mathbb{N},\ g, g' \in G_n,\ a \in \mathfrak{sa}(\mathfrak{A}_n) \quad \left\| \alpha_n^g(a) - \alpha_n^{g'}(a) \right\|_{\mathfrak{A}_n} \leqslant K\left(\delta_n(g, g')\right) \|a\|_{\mathfrak{A}_n}.$$

If there exists $(\mathfrak{A}, \mathsf{L}) \in \mathcal{C}$ *such that:*

$$\lim_{n\to\infty} \mathsf{\Lambda}^*_{\mathcal{T}}((\mathfrak{A}_n, \mathsf{L}_n), (\mathfrak{A}, \mathsf{L})) = 0$$

then there exists a strongly continuous action α *of the semigroup* G *by Lipschitz linear endomorphisms such that for all* $g \in G$*:*

$$|||\alpha^g|||_{\mathfrak{A}} \leqslant D(g) \text{ and } \mathsf{L} \circ \alpha^g \leqslant D(g)\mathsf{L}, \tag{3.1}$$

and for all $g, h \in G$*:*

$$\liminf_{n\to\infty} \mathrm{mkD}^{\mathsf{L}_n}_{\mathfrak{A}_n}\left(\alpha_n^{\varkappa_n(g)}, \alpha_n^{\varkappa_n(h)}\right) \leqslant \mathrm{mkD}^{\mathsf{L}}_{\mathfrak{A}}\left(\alpha^g, \alpha^h\right) \leqslant \limsup_{n\to\infty} \mathrm{mkD}^{\mathsf{L}_n}_{\mathfrak{A}_n}\left(\alpha_n^{\varkappa_n(g)}, \alpha_n^{\varkappa_n(h)}\right), \tag{3.2}$$

while for all $g, h \in G$*, and* $a \in \mathrm{dom}(\mathsf{L})$*, we have:*

$$\left\|\alpha^g(a) - \alpha^h(a)\right\|_{\mathfrak{A}} \leqslant \mathsf{L}(a)\mathrm{diam}(\mathfrak{A}, \mathsf{L}) K(\delta_G(g, h)).$$

If moreover for all $n \in \mathbb{N}$*, the actions* α_n *of the semigroup* G_n *is by positive unital linear maps, then* α *is a strongly continuous action of* G *by positive unital linear maps.*

Remark 3.8. With the notations of Theorem (3.7), if G is a group, and G_n is a group for all $n \in \mathbb{N}$, then of course $|||\alpha^g|||_{\mathfrak{A}} = 1$ as α^g is a *-automorphism for all $g \in G$.

Remark 3.9. We emphasize that while Theorem (3.7) requires convergence of the quantum metric spaces and of the groups or semigroups, there is not requirement of convergence involving the actions themselves.

Proof. The sequence $(\mathrm{diam}(\mathfrak{A}_n, \mathsf{L}_n))_{n\in\mathbb{N}}$ is convergent to $\mathrm{diam}(\mathfrak{A}, \mathsf{L})$, since the diameter function is 2-Lipschitz for the dual propinquity.

Since G is a compact metric space, it is separable. Let E be a countable dense subset of G. Let H be the sub-semigroup generated by E. Since H consists of all the *finite* products of elements of the countable set E, it is itself countable.

As a semigroup, H is also a semigroupoid in a trivial manner (formally, the semigroupoid is $(H, \{e\}, c, d, \cdot)$ with c and d the constant functions equal to e and $\cdot$ the multiplication of H, and e need not be an element of H — if it is then it should be the unit of H which is then in fact a monoid. With these notations, we would set $(\mathfrak{A}^e, \mathsf{L}^e) = (\mathfrak{A}, \mathsf{L})$ and $(\mathfrak{A}^e_n, \mathsf{L}^e_n) = (\mathfrak{A}_n, \mathsf{L}_n)$ for all $n \in \mathbb{N}$, though we will not use this heavier notation here).

Let $g \in H$ and $a \in \mathrm{dom}(\mathsf{L})$. For each $n \in \mathbb{N}$, we choose some state $\mu_n \in \mathscr{S}(\mathfrak{A}_n)$. We now compute:

$$\begin{aligned}
&\left\|\alpha_n^{\varkappa_n(g)} \circ \alpha_n^{\varkappa_n(g')}(a) - \alpha_n^{\varkappa_n(gg')}(a)\right\|_{\mathfrak{A}_n} \\
&\quad = \left\|\alpha_n^{\varkappa_n(g)\varkappa_n(g')}(a) - \alpha_n^{\varkappa_n(gg')}(a)\right\|_{\mathfrak{A}_n} \\
&\quad = \left\|\alpha_n^{\varkappa_n(g)\varkappa_n(g')}(a - \mu_n(a)1_{\mathfrak{A}_n}) - \alpha_n^{\varkappa_n(gg')}(a - \mu_n(a)1_{\mathfrak{A}_n})\right\|_{\mathfrak{A}_n} \\
&\quad \leqslant K\left(\delta_n\left(\varkappa_n(g)\varkappa_n(g'), \varkappa_n(gg')\right)\right) \|a - \mu_n(a)1_{\mathfrak{A}_n}\|_{\mathfrak{A}_n} \\
&\quad \leqslant K\left(\delta_n\left(\varkappa_n(g)\varkappa_n(g'), \varkappa_n(gg')\right)\right) \mathrm{diam}(\mathfrak{A}_n, \mathsf{L}_n).
\end{aligned}$$

By Definition (3.2), the sequence $(\delta_n(\varkappa_n(g)\varkappa_n(g'), \varkappa_n(gg')))_{n\in\mathbb{N}}$ converges to 0. So:

$$\lim_{n\to\infty} K(\delta_n(\varkappa_n(g)\varkappa_n(g'), \varkappa_n(gg'))) \operatorname{diam}(\mathfrak{A}_n, \mathsf{L}_n) = 0.$$

Thus we fit the hypothesis of Theorem (2.13). Therefore, there exists an action α of H on $\mathfrak{A}$, with the property that, for all $a \in \mathfrak{sa}(\mathfrak{A})$ with $\mathsf{L}(a) \leqslant 1$ and for all $g, h \in H$:

$$\begin{aligned}\|\alpha^g(a) - \alpha^h(a)\|_{\mathfrak{A}} &\leqslant \limsup_{n\to\infty} K(\delta_n(\varkappa_n(g), \varkappa_n(h)))\operatorname{diam}(\mathfrak{A}_n, \mathsf{L}_n)\\ &\leqslant K(\delta_G(g,h))\operatorname{diam}(\mathfrak{A}, \mathsf{L}),\end{aligned}$$

and in addition, Expressions (3.1) and (3.2) hold for $g \in H$.

It thus immediately follows by homogeneity that for all $a \in \operatorname{dom}(\mathsf{L})$ and $g \in H$:

$$\|\alpha^g(a) - \alpha^h(a)\|_{\mathfrak{A}} \leqslant \mathsf{L}(a)\operatorname{diam}(\mathfrak{A}, \mathsf{L})K(\delta_G(g,h)).$$

Therefore $g \in H \mapsto \alpha^g(a)$ is uniformly continuous for all $a \in \operatorname{dom}(\mathsf{L})$ since $\lim_0 K = 0$, and thus it can be extended uniquely to a uniformly continuous function $g \in G \mapsto \alpha^g(a)$, with the additional property that for all $g, h \in G$:

$$\left\|\alpha^g(a) - \alpha^h(a)\right\|_{\mathfrak{A}} \leqslant \mathsf{L}(a)\operatorname{diam}(\mathfrak{A}, \mathsf{L})K(\delta_G(g,h))$$

by continuity of K (and δ_G).

It is straightforward that for all $a \in \operatorname{dom}(\mathsf{L})$:

(1) for all $g, h \in G$, we have by continuity of the multiplication on G that $\alpha^g \circ \alpha^h(a) = \alpha^{gh}(a)$,
(2) for all $g \in G$, the lower semi-continuity of L and the continuity of D gives us $\mathsf{L} \circ \alpha^g(a) \leqslant D(g)\mathsf{L}(a)$,
(3) for all $g \in G$, we also have by continuity that $\|\alpha^g(a)\|_{\mathfrak{A}} \leqslant D(g)\|a\|_{\mathfrak{A}}$.

In particular, Expression (3.1) holds for all $g \in G$.

Therefore, α is a semigroup action of G via Lipschitz linear maps.

Let $g \in G$. Let $\varepsilon > 0$, and let $R > 0$ be some upper bound for the convergent sequence $(\operatorname{diam}(\mathfrak{A}_n, \mathsf{L}_n))_{n\in\mathbb{N}}$. Since K is continuous and $K(0) = 0$, there exists $\delta > 0$ such that for $t \in [0, \delta)$, we have $K(t) < \frac{\varepsilon}{2R}$.

Let $h \in H$ with $\delta_G(g,h) < \frac{\delta}{3}$.

Now, there exists $N \in \mathbb{N}$ such that for all $n \geqslant N$, we have $\Upsilon((G_n, \delta_n), (G, \delta_G)) < \frac{\delta}{6}$. Let $n \geqslant N$. By Definition (3.2), we have:

$$|\delta_n(\varkappa_n(g), \varkappa_n(h)) - \delta_G(g,h)| < \frac{\delta}{3}.$$

Hence $\delta_n(\varkappa_n(g), \varkappa_n(h)) < \delta$.

We now compute, for any $a \in \operatorname{dom}(\mathsf{L}_n)$ with $\mathsf{L}_n(a) \leqslant 1$:

$$\begin{aligned}\left| \left\|a - \alpha_n^{\varkappa_n(g)}(a)\right\|_{\mathfrak{A}_n} - \left\|a - \alpha_n^{\varkappa_n(h)}(a)\right\|_{\mathfrak{A}_n} \right| &\leqslant \left\|\alpha_n^{\varkappa_n(g)}(a) - \alpha_n^{\varkappa_n(h)}(a)\right\|_{\mathfrak{A}_n}\\ &\leqslant \operatorname{diam}(\mathfrak{A}_n, \mathsf{L}_n)K(\delta_n(\varkappa_n(g), \varkappa_n(h))) \leqslant \frac{\varepsilon}{2}.\end{aligned}$$

Therefore:

$$\limsup_{n\to\infty} \operatorname{mk}\ell_{\mathsf{L}_n}(\alpha^{\varkappa_n(g)}) \leqslant \limsup_{n\to\infty} \operatorname{mk}\ell_{\mathsf{L}_n}(\alpha^{\varkappa_n(h)}) + \frac{\varepsilon}{2},$$

and

$$\liminf_{n\to\infty} \operatorname{mk}\ell_{\mathsf{L}_n}(\alpha^{\varkappa_n(g)}) \geqslant \liminf_{n\to\infty} \operatorname{mk}\ell_{\mathsf{L}_n}(\alpha^{\varkappa_n(h)}) - \frac{\varepsilon}{2}.$$

Now:

$$\left|\|a-\alpha^g(a)\|_{\mathfrak{A}} - \left\|a-\alpha^h(a)\right\|_{\mathfrak{A}}\right| \leqslant \left\|\alpha^g(a)-\alpha^h(a)\right\|_{\mathfrak{A}} \leqslant \mathrm{diam}\,(\mathfrak{A},\mathsf{L})K(\delta_G(g,h)) < \frac{\varepsilon}{2}.$$

Thus:

$$\begin{aligned}\|a-\alpha^g(a)\|_{\mathfrak{A}} &\leqslant \frac{\varepsilon}{2} + \left\|a-\alpha^h(a)\right\|_{\mathfrak{A}}\\ &\leqslant \frac{\varepsilon}{2} + \limsup_{n\to\infty}\mathrm{mk}\ell_{\mathsf{L}_n}(\alpha^{\varkappa_n(h)}) \text{ since } h\in H,\\ &\leqslant \frac{\varepsilon}{2}+\frac{\varepsilon}{2}+\limsup_{n\to\infty}\mathrm{mk}\ell_{\mathsf{L}_n}(\alpha^{\varkappa_n(g)}) = \varepsilon + \limsup_{n\to\infty}\mathrm{mk}\ell_{\mathsf{L}_n}(\alpha^{\varkappa_n(g)}),\end{aligned}$$

and therefore:

$$\mathrm{mk}\ell_{\mathsf{L}}(\alpha^g) \leqslant \limsup_{n\to\infty}\mathrm{mk}\ell_{\mathsf{L}_n}(\alpha^{\varkappa_n(g)}) + \varepsilon.$$

Similarly:

$$\mathrm{mk}\ell_{\mathsf{L}}(\alpha^g) \geqslant \liminf_{n\to\infty}\mathrm{mk}\ell_{\mathsf{L}_n}(\alpha^{\varkappa_n(h)}) - \varepsilon.$$

Since $\varepsilon>0$ is arbitrary, we conclude that for all $g\in G$:

$$\liminf_{n\to\infty}\mathrm{mk}\ell_{\mathsf{L}_n}(\alpha_n^{\varkappa_n(g)}) \leqslant \mathrm{mk}\ell_{\mathsf{L}}(\alpha^g) \leqslant \limsup_{n\to\infty}\mathrm{mk}\ell_{\mathsf{L}_n}(\alpha^{\varkappa_n(g)}).$$

Thus Expression (3.2) now holds for all $g\in G$.

Let now $a\in\mathfrak{sa}(\mathfrak{A})$, $\varepsilon>0$ and $g\in G$. Since D is continuous, let $\eta>0$ such that if $\delta_G(g,h)<\eta$, then $|D(g)-D(h)|<\varepsilon$.

By density, there exists $a'\in\mathrm{dom}\,(\mathsf{L})$ such that $\|a-a'\| < \frac{\varepsilon}{3(D(g)+\varepsilon)}$. Now, there exists $\eta_2>0$ such that if $t\in[0,\eta_2)$ then $K(t)<\frac{\varepsilon}{3\max\{\mathsf{L}(a'),1\}}$. Let now $h\in G$ with $\delta_G(g,h)<\min\{\eta,\eta_2\}$. We compute:

$$\begin{aligned}&\left\|\alpha^g(a)-\alpha^h(a)\right\|_{\mathfrak{A}}\\ &\quad\leqslant \|\alpha^g(a-a')\|_{\mathfrak{A}} + \left\|\alpha^g(a')-\alpha^h(a')\right\|_{\mathfrak{A}} + \left\|\alpha^h(a-a')\right\|_{\mathfrak{A}}\\ &\quad\leqslant D(g)\|a-a'\|_{\mathfrak{A}} + \mathsf{L}(a')\mathrm{diam}\,(\mathfrak{A},\mathsf{L})K(\delta_G(g,h)) + D(h)\|a-a'\|_{\mathfrak{A}}\\ &\quad\leqslant \varepsilon.\end{aligned}$$

Therefore, for all $a\in\mathfrak{sa}(\mathfrak{A})$, the function $g\in G\mapsto\alpha^g(a)$ is continuous, as desired. ∎

As an important corollary of Theorem (3.7), we obtain:

Corollary 3.10. *Let (G,δ_G) be a compact metric group. Let $(G_n,\delta_n)_{n\in\mathbb{N}}$ be a sequence of compact metric groups, converging to G for the Gromov-Hausdorff monoid distance Υ_*. For each $n\in\mathbb{N}$, choose:*

$$(\varsigma_n,\varkappa_n)\in\mathrm{UIso}_{2\Upsilon((G_n,\delta_n),(G,\delta))}((G_n,\delta_n),(G,\delta)).$$

Let $D:G\to[0,\infty)$ be a continuous function. Let $K:[0,\infty)\to[0,\infty)$ be continuous with $K(0)=0$.

Let $\mathcal{C}$ be a nonempty class of F–quasi-Leibniz quantum compact metric spaces for some admissible function F, and let $\mathcal{T}$ be a class of tunnels compatible with $\mathcal{C}$.

Let $(\mathfrak{A}_n, \mathsf{L}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}$, such that for each $n \in \mathbb{N}$, there exists an action α_n by Lipschitz automorphisms of G_n on $\mathfrak{A}_n$, where:

$$\forall n \in \mathbb{N},\ g \in G_n \quad \mathsf{L}_n \circ \alpha_n^g \leqslant D(\varsigma_n(g))\mathsf{L}_n,$$

and

$$\forall n \in \mathbb{N},\ g, g' \in G_n,\ a \in \mathfrak{sa}\left(\mathfrak{A}_n\right) \quad \left\| \alpha_n^g(a) - \alpha_n^{g'}(a) \right\|_{\mathfrak{A}_n} \leqslant K\left(\delta_n(g, g')\right) \|a\|_{\mathfrak{A}_n}.$$

If there exists $(\mathfrak{A}, \mathsf{L}) \in \mathcal{C}$ such that:

$$\lim_{n\to\infty} \Lambda^*_{\mathcal{T}}((\mathfrak{A}_n, \mathsf{L}_n), (\mathfrak{A}, \mathsf{L})) = 0$$

then there exists a strongly continuous action α of the group G by Lipschitz automorphisms such that for all $g \in G$:

$$\mathsf{L} \circ \alpha^g \leqslant D(g)\mathsf{L},$$

and for all $g \in G$:

$$\liminf_{n\to\infty} \mathrm{mk}\ell_{\mathsf{L}_n}\left(\alpha_n^{\varkappa_n(g)}\right) \leqslant \mathrm{mk}\ell_{\mathsf{L}}\left(\alpha^g\right) \leqslant \limsup_{n\to\infty} \mathrm{mk}\ell_{\mathsf{L}_n}\left(\alpha_n^{\varkappa_n(g)}\right),$$

while for all $g, h \in G$, and $a \in \mathrm{dom}\left(\mathsf{L}\right)$, we have:

$$\left\| \alpha^g(a) - \alpha^h(a) \right\|_{\mathfrak{A}} \leqslant \mathsf{L}(a)\mathrm{diam}\left(\mathfrak{A}, \mathsf{L}\right) K(\delta_G(g, h)).$$

Proof. For each $n \in \mathbb{N}$, let $e_n \in G_n$ be the unit of G_n, and let e be the unit of G. Since for all $n \in \mathbb{N}$, we are given an action of a group, $\alpha_n^{e_n}$ is the identity of $\mathfrak{A}_n$.

Moreover, by Definition (3.2), we have $\lim_{n\to\infty} \delta_n(\varkappa_n(e), e_n) = 0$, so if, for any $n \in \mathbb{N}$, we take $a \in \mathfrak{A}_n$ with $\mathsf{L}_n(a) \leqslant 1$, then:

$$\begin{aligned}\left\| \alpha_n^{\varkappa_n(e)}(a) - a \right\|_{\mathfrak{A}_n} &= \left\| \alpha^{\varkappa_n(e)}(a) - \alpha_n^{e_n}(a) \right\|_{\mathfrak{A}_n} \\ &\leqslant \mathrm{diam}\left(\mathfrak{A}_n, \mathsf{L}_n\right) K(\delta_n(\varkappa_n(e), e_n)) \xrightarrow{n\to\infty} 0.\end{aligned}$$

Hence, we can now follow the proof of Theorem (3.7), only noting that we can apply Theorem (2.13) to obtain an action of G by *-automorphisms. ☐

Another application of Theorem (2.13) involves inductive limits of groups. Notably, the following result does not require that the inductive limit be locally compact. This is a step in a possibly new approach to construct actions of non-locally compact groups on C*-algebras.

An inductive sequence of groups, for our purpose, is given by a sequence of groups $(G_n)_{n\in\mathbb{N}}$ together with group monomorphisms $\theta_n : G_n \to G_{n+1}$. We will assume all the groups G_n to be topological groups with a separable, first countable topology — hence, sequences will prove sufficient to test continuity.

The inductive limit group $G = \varinjlim_{n\to\infty}(G_n, \theta_n)$ is the quotient of $\prod_{n\in\mathbb{N}} G_n$ by the equivalence relation:

$$(g_n)_{n\in\mathbb{N}} \equiv (g'_n)_{n\in\mathbb{N}} \iff \exists N \in \mathbb{N} \quad \forall n \geqslant N \quad \theta_n(g_n) = \theta_n(g'_n).$$

Let $N \in \mathbb{N}$. We define, for $g \in G_N$, the element $\eta_N(g)$ as the equivalence class in G of:

$$\left(\begin{cases} \text{identity of } G_n \text{ if } n < N, \\ g \text{ if } n = N, \\ \theta_n(g) \text{ if } n > N. \end{cases}\right)_{n\in\mathbb{N}}$$

Thus defined, the maps η_n are group monomorphisms for all $n \in \mathbb{N}$. We henceforth identify G_n with $\eta_n(G_n)$ in G for all $n \in \mathbb{N}$. Thus $G = \bigcup_{n\in\mathbb{N}} G_n$ with $G_n \subseteq G_{n+1}$ for all $n \in \mathbb{N}$.

The inductive topology on G is by definition, the final topology for the maps θ_n. With the above identification, it becomes the largest topology which makes, for all $n \in \mathbb{N}$, the inclusion of G_n in G continuous. In particular, $U \subseteq G$ is open in the inductive topology if and only if for all $N \in \mathbb{N}$, the set $U \cap G_N$ is open in G_N.

In our next theorem, we work directly with inductive limits seen as increasing unions of groups with the inductive space topology. As a side note, this topology does not turn G to a topological group in general.

Theorem 3.11. *Let F be an admissible function, $\mathcal{C}$ a nonempty class of F–quasi-Leibniz quantum compact metric spaces and $\mathcal{T}$ an appropriate class of tunnels for $\mathcal{C}$.*

Let G be a group with unit e and $(G_n)_{n\in\mathbb{N}}$ an increasing sequence, for inclusion, of subgroups of G such that $G = \bigcup_{n\in\mathbb{N}} G_n$. We assume that for all $n \in \mathbb{N}$, the group G_n is a separable first-countable topological group and that the inclusion map $G_n \hookrightarrow G_{n+1}$ is continuous. We endow G with the inductive limit topology.

*Let $(\mathfrak{A}_n, \mathsf{L}_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}$, and for each $n \in \mathbb{N}$, let α_n be an action of G_n by *-automorphisms of $\mathfrak{A}_n$.*

Let $D : G \to [0, \infty)$ be a continuous function such that for all $n \in \mathbb{N}$ and for all $g \in G_n$:

$$\mathsf{L}_n \circ \alpha_n^g \leqslant D(g)\mathsf{L}_n.$$

Let $K : G \times G \to [0, \infty)$ be continuous, such that $K(g, g) = 0$ for all $g \in G$ and:

$$\forall n \in \mathbb{N}, g, h \in G_n, a \in \mathrm{dom}\,(\mathsf{L}_n) \quad \left\|\alpha_n^h(a) - \alpha_n^g(a)\right\|_{\mathfrak{A}_n} \leqslant \|a\|_{\mathfrak{A}_n} K(h, g).$$

If $(\mathfrak{A}, \mathsf{L}) \in \mathcal{C}$ satisfies:

$$\lim_{n\to\infty} \Lambda^*_{\mathcal{T}}\left((\mathfrak{A}_n, \mathsf{L}_n), (\mathfrak{A}, \mathsf{L})\right) = 0,$$

then there exists an strongly continuous action α of G on $\mathfrak{A}$ such that:

1. *for all $g \in G$, we have $\mathsf{L} \circ \alpha^g \leqslant D(g)\mathsf{L}$,*
2. *for all $g, h \in G$, and $a \in \mathrm{dom}\,(\mathsf{L})$, we have:*
$$\left\|\alpha^g(a) - \alpha^h(a)\right\|_{\mathfrak{A}} \leqslant \mathsf{L}(a)\mathrm{diam}\,(\mathfrak{A}, \mathsf{L})K(\delta_G(g, h)),$$
3. *the following inequalities hold for all $g \in G$, and for all $N \in \mathbb{N}$ such that $g \in G_N$:*
$$\liminf_{\substack{n\to\infty \\ n\geqslant N}} \mathrm{mk}\ell_{\mathsf{L}_n}(\alpha_n^g) \leqslant \mathsf{mk}_{\mathsf{L}}(\alpha^g) \leqslant \limsup_{\substack{n\to\infty \\ n\geqslant N}}(\alpha_n^g),$$

Proof. For any $n \in \mathbb{N}$, the group G_n is assume separable, so it admits a dense countable subgroup H_n. If U is nonempty open in G then for some $N \in \mathbb{N}$, we have $U \cap G_N$ is open and not empty, and thus $H_N \cap U$ is not empty as well. Consequently, $H = \bigcup_{n\in\mathbb{N}} H_n$ is a dense subgroup $\bigcup_{n\in\mathbb{N}} G_n$. Notably, H is countable.

Let $g \in H$. For all $n \in \mathbb{N}$, if $g \in H_n$ then write $\varphi_n^g = \alpha_n^g$; otherwise set φ_n^g to be the identity of $\mathfrak{A}_n$.

If $g, g' \in H$, there exists $N \in \mathbb{N}$ such that $g, g' \in G_N$. For $n \geqslant N$ we then have:

$$\left\|\varphi_n^g \circ \varphi_n^{g'} - \varphi_n^{gg'}\right\|_{\mathfrak{A}_n} = \left\|\alpha_n^g \circ \alpha_n^{g'} - \alpha_n^{gg'}\right\| = 0$$

and thus, we meet Theorem (2.13) hypothesis (noting that $\mathrm{mk}\ell_{\mathsf{L}_n}(\alpha_n^e) = 0$ for all $n \in \mathbb{N}$ and α_n are actions by *-automorphisms). We thus conclude that there exists an action α of H on $\mathfrak{A}$ by *-automorphisms such that in particular:

$$\left\|\alpha^h(a) - \alpha^g(a)\right\|_{\mathfrak{A}} \leqslant \operatorname{diam}(\mathfrak{A}, \mathsf{L})\mathsf{L}(a)K(h,g),$$

reasoning in the same manner as in the proof of Theorem (3.7).

Let now $g \in G$, and $N \in \mathbb{N}$ such that $g \in G_N$. Since H_N is dense in G_N, there exists a sequence $(g_k)_{k\in\mathbb{N}}$ in H_N such that $\lim_{k\to\infty} g_k = g$. Thus in particular, if $a \in \operatorname{dom}(\mathsf{L})$ then:

$$\begin{aligned}\|\alpha^{g_p}(a) - \alpha^{g_q}(a)\|_{\mathfrak{A}} &\leqslant \operatorname{diam}(\mathfrak{A}, \mathsf{L})\mathsf{L}(a)K(g_p, g_q) \\ &\xrightarrow{p,q\to\infty} \operatorname{diam}(\mathfrak{A}, \mathsf{L})\mathsf{L}(a)K(g,g) = 0\end{aligned}$$

and thus $(\alpha^{g_k}(a))_{k\in\mathbb{N}}$ is Cauchy in $\mathfrak{sa}(\mathfrak{A})$, which is complete. It thus converges.

Note that moreover, if we had chosen $N' \in \mathbb{N}$ for which $g \in G_{N'}$ and some other sequence $(h_k)_{k\in\mathbb{N}}$ converging in $G_{N'}$ to g, then, without loss of generality, we may assume $N' \geqslant N$; the sequence $(z_k)_{k\in\mathbb{N}}$ obtained by interleaving $(g_k)_{k\in\mathbb{N}}$ and $(h_k)_{k\in\mathbb{N}}$ converges to g in $G_{N'}$ and thus as above, $(\alpha^{z_k}(a))_{k\in\mathbb{N}}$ must converge for all $a \in \operatorname{dom}(\mathsf{L})$; thus both subsequences $(\alpha^{g_k}(a))_{k\in\mathbb{N}}$ and $(\alpha^{h_k}(a))_{k\in\mathbb{N}}$ have the same limit. We denote this limit by $\alpha^g(a)$.

A simple argument shows that α^g is a *-morphism and $\alpha^g \circ \alpha^{g'}(a) = \alpha^{gg'}(a)$ for all $a \in \mathfrak{A}$, so α is an algebraic action of G on $\mathfrak{A}$ by *-automorphisms.

By construction, we then note that for all $a \in \operatorname{dom}(\mathsf{L})$, the following inequality holds:

$$\begin{aligned}\left\|\alpha^g(a) - \alpha^h(a)\right\|_{\mathfrak{A}} &= \lim_{k\to\infty} \|\alpha^{g_k}(a) - \alpha^{h_k}(a)\|_{\mathfrak{A}} \\ &\leqslant \limsup_{k\to\infty} \operatorname{diam}(\mathfrak{A}, \mathsf{L})\mathsf{L}(a)K(g_k, h_k) = \operatorname{diam}(\mathfrak{A}, \mathsf{L})\mathsf{L}(a)K(g,h).\end{aligned}$$

Let $a \in \mathfrak{sa}(\mathfrak{A})$. Let $\varepsilon > 0$. There exists $a' \in \operatorname{dom}(\mathsf{L})$ such that $\|a - a'\|_{\mathfrak{A}} < \frac{\varepsilon}{3}$.

Since K is continuous, there exists an open subset U of G such that $e \in U$ and for all $g \in U$ we have $K(g,e) < \frac{\varepsilon}{3(\operatorname{diam}(\mathfrak{A},\mathsf{L})\mathsf{L}(a')+1)}$. We then have for all $g \in U$:

$$\begin{aligned}\|\alpha^g(a) - a\|_{\mathfrak{A}} &= \|\alpha^g(a) - \alpha^e(a)\|_{\mathfrak{A}} \\ &\leqslant \|\alpha^g(a - a')\|_{\mathfrak{A}} + \|\alpha^g(a') - \alpha^e(a')\|_{\mathfrak{A}} + \|\alpha^e(a - a')\|_{\mathfrak{A}} \\ &\leqslant 2\|a - a'\|_{\mathfrak{A}} + \frac{\varepsilon}{3} \leqslant \varepsilon.\end{aligned}$$

We conclude that we have proven that for all $a \in \mathfrak{sa}(\mathfrak{A})$:

$$\lim_{g\to e} \|\alpha^g(a) - a\|_{\mathfrak{A}} = 0$$

and thus it follows immediately that α is strongly continuous.

Now, the same general techniques used in the proof of Theorem (3.7) applies to complete the theorem — with K is now a function over $G \times G$. □

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