# Cosets of the $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})$-algebra

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**Abstract**

Let $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})$ be the universal $\mathcal{W}$-algebra associated to $\mathfrak{sl}_4$ with its subregular nilpotent element, and let $\mathcal{W}_k(\mathfrak{sl}_4, f_{\text{subreg}})$ be its simple quotient. There is a Heisenberg subalgebra $\mathcal{H}$, and we denote by $\mathcal{C}^k$ the coset $\text{Com}(\mathcal{H}, \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}}))$, and by $\mathcal{C}_k$ its simple quotient. We show that for $k = -4 + (m+4)/3$ where $m$ is an integer greater than 2 and $m+1$ is coprime to 3, $\mathcal{C}_k$ is isomorphic to a rational, regular $\mathcal{W}$-algebra $\mathcal{W}(\mathfrak{sl}_m, f_{\text{reg}})$. In particular, $\mathcal{W}_k(\mathfrak{sl}_4, f_{\text{subreg}})$ is a simple current extension of the tensor product of $\mathcal{W}(\mathfrak{sl}_m, f_{\text{reg}})$ with a rank one lattice vertex operator algebra, and hence is rational.

## 1 Introduction

Given a vertex operator algebra $V$ and a vertex operator subalgebra $W$ the subalgebra of $V$ that commutes with $W$, $C = \text{Com}(W, V)$ is called a coset vertex operator algebra of $V$. This was introduced by Frenkel and Zhu in [FZ], generalizing earlier constructions in [KP] and [GKO], where it was used to construct the unitary discrete series representations of the Virasoro algebra. It is widely believed that if both $V$ and $W$ satisfy certain nice properties, then so does $C$. For example if $V$ and $W$ are both rational or $C_2$-cofinite then one expects $C$ to be rational or $C_2$-cofinite as well. However, general results are very difficult to obtain.

For $n \geq 4$, let $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{subreg}})$ denote the $\mathcal{W}$-algebra at level $k$ associated to $\mathfrak{sl}_n$ with its subregular nilpotent element [CM, KRW], and let $\mathcal{W}_k(\mathfrak{sl}_n, f_{\text{subreg}})$ be the simple quotient. It was recently shown by Genra [Gen] that $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{subreg}})$ coincides with the Feigin-Semikhatov algebra $\mathcal{W}_n^{(2)}$ [FS], and is strongly generated by $n+1$ fields of conformal weights $1, 2, \dots, n-1, n/2, n/2$. Note that $\mathcal{W}_n^{(2)}$ is well defined for $n = 2$ and $n = 3$; $\mathcal{W}_2^{(2)}$ coincides with the affine vertex operator algebra $V^k(\mathfrak{sl}_2)$, and $\mathcal{W}_3^{(2)}$ coincides with the Bershadsky-Polyakov algebra [Ber, Pol].

The weight one field of $\mathcal{W}^k(\mathfrak{sl}_n, f_{\text{subreg}})$ generates a Heisenberg algebra $\mathcal{H}$, and we are interested in the coset

$$\mathcal{C}^k = \text{Com}(\mathcal{H}, \mathcal{W}^k(\mathfrak{sl}_n, f_{\text{subreg}}))$$

for generic values of $k$, as well as the simple quotient

$$\mathcal{C}_k = \text{Com}(\mathcal{H}, \mathcal{W}_k(\mathfrak{sl}_n, f_{\text{subreg}}))$$

at certain special values of $k$. It was conjectured in the physics literature over 20 years ago [B–H] that there is a sequence of levels $k$ where $\mathcal{C}_k$ is isomorphic to a rational

$\mathcal{W}(\mathfrak{sl}_m, f_{\text{reg}})$-algebra, where $m$ depends linearly on $k$. In the cases $n = 2$ and $n = 3$, where $\mathcal{W}_k(\mathfrak{sl}_n, f_{\text{subreg}})$ is replaced by the simple affine algebra $L_k(\mathfrak{sl}_2)$ and the simple Bershadsky-Polyakov algebra, respectively, this conjecture was proven recently in [ALY] and [ACL1]. We remark that subregular $\mathcal{W}$-algebras of type A have recently become important due to there rôle in four-dimensional supersymmetric gauge theories as chiral algebras of Argyres-Douglas theories [BN, C, CS]. Interestingly, these are exactly those levels where the $\mathcal{W}$-algebra has the logarithmic singlet vertex operator algebra [Ka, AM] as Heisenberg coset [CRW].

In the case $n = 4$, the above conjecture states that for $k = -4 + (m+4)/3$, $\mathcal{C}_k$ is isomorphic to a rational $\mathcal{W}(\mathfrak{sl}_m, f_{\text{reg}})$-algebra. Here $m$ is an integer greater than 2 such that $m + 1$ is coprime to 3. Our main result is a proof of this conjecture. In fact, we prove the stronger statement that for these values of $k$, $\mathcal{W}_k(\mathfrak{sl}_4, f_{\text{subreg}})$ is a simple current extension of $V_L \otimes \mathcal{W}(\mathfrak{sl}_m, f_{\text{reg}})$, where $V_L$ is a certain rank one lattice VOA. As a corollary, we obtain the $C_2$-cofiniteness and rationality of $\mathcal{W}_k(\mathfrak{sl}_4, f_{\text{subreg}})$ for the above values of $k$. Additionally, we show that $\mathcal{C}^k$ is of type $\mathcal{W}(2,3,4,5,6,7,8,9)$ for all values of $k$ except for $k = -2, -5/2, -8/3$. In particular, this strong generating set works for the simple quotient $\mathcal{C}_k$ for the above values of $k$, so this family of rational $\mathcal{W}(\mathfrak{sl}_m, f_{\text{reg}})$-algebras has the following *uniform truncation property.* For $m \geq 9$ they are all of type $\mathcal{W}(2,3,4,5,6,7,8,9)$, even though the universal $\mathcal{W}(\mathfrak{sl}_m, f_{\text{reg}})$-algebra is of type $\mathcal{W}(2,3,\dots,m)$. This happens because a singular vector of weight 10 in the universal algebra gives rise to decoupling relations in the simple quotient expressing the generators of weights $10, 11, \dots, m$ as normally ordered polynomials in the ones up to weight 9.

Here is a brief sketch of the proof of our result.

1. Since $\mathcal{C}^k \otimes \mathcal{H} \cong \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}$, studying $\mathcal{C}^k$ is equivalent to studying the $U(1)$-orbifold $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}$. We start with an obvious infinite set of generators coming from classical invariant theory in weights $1, 2, 3, \dots,$ and we compute normally ordered relations among these generators starting in weight 10. We show that for generic values of $k$, all generators in weights $w \geq 10$ can be eliminated using these relations. This shows that $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}$ is of type $\mathcal{W}(1,2,3,4,5,6,7,8,9)$, and hence that $\mathcal{C}^k$ is of type $\mathcal{W}(2,3,4,5,6,7,8,9)$ for generic $k$.

2. Coefficients in the normally ordered relations are polynomial functions in $k$, and zeros of coefficients correspond to non-generic values of $k$. It turns out that this only happens for $k = -2, -5/2, -8/3$, so these are the only nongeneric values.

3. We prove the conjecture of [B–H] by showing that a certain rational $\mathcal{W}(\mathfrak{sl}_m, f_{\text{reg}})$-algebra times a rank one lattice vertex operator algebra allows for a simple current extension whose OPE algebra coincides with the one of $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})$. The main ingredient is that the Jacobi identity implies that the full OPE algebra is uniquely determined by a small amount of data which is easily shown to coincide in both algebras. Therefore the two vertex operator algebras must be isomorphic.

This paper is part of a broader program of the authors to study *deformable familes* of vertex operator algebras, i.e. vertex operator algebras that depend continuously on one or more parameters. Examples include universal affine vertex algebras and $\mathcal{W}$-algebras, where the parameter is the level $k$, as well as orbifolds and cosets of these algebras. In many situations, the question of finding a minimal strong generating set for an orbifold or coset of such a deformable family can be decided by passing to the limit as $k$ approaches infinity, which is often an orbifold of a free field algebra [CL1]. The structure of orbifolds of free field algebras can then be determined using ideas from classical invariant theory [CL3, L1, L2, L3, L4].

Besides our pure interest in vertex operator algebra invariant theory, our findings have quite some impact on important questions we are interested in. It is widely believed that $\mathcal{W}$-algebras that are certain quantum Hamiltonian reductions of affine vertex operator algebras, can also be realized as coset algebras. The most famous example is surely the regular $\mathcal{W}$-algebra of a simply-laced Lie algebra $\mathfrak{g}$ as a coset of the affine vertex operator algebra $V^{k+1}(\mathfrak{g})$ inside $V^k(\mathfrak{g})\otimes L_1(\mathfrak{g})$. Jointly with Tomoyuki Arakawa we are able to prove this coset realization [ACL2]. Other examples are the coset of $V^{k+1}(\mathfrak{so}_n)$ inside $V^k(\mathfrak{so}_{n+1}) \otimes \mathcal{F}(n)$ with $\mathcal{F}(n)$ the free field algebra of $n$ free fermions. This is believed to be a regular $\mathcal{W}$-algebra of type $\mathfrak{osp}(2n+1|2n)$ and Lemma 7.17 of [CL1] tells us that this conjecture is indeed consistent with minimal strong generating sets. These families of $\mathcal{W}$-algebras all carry an action of the $N=1$ super Virasoro algebra. The $N=2$ super Virasoro case corresponds to the cosets of $V^{k+1}(\mathfrak{sl}_n)$ inside $V^k(\mathfrak{sl}_{n+1})\otimes\mathcal{F}(2n)$ and the expected $\mathcal{W}$-algebra is a regular $\mathcal{W}$-algebra of type $\mathfrak{sl}(n+1|n)$. Lemma 7.12 of [CL1] confirms this on the level of minimal strong generating sets. It is a rather ambitious future goal to indeed prove these conjectures. Correctness of these conjectures is the starting assumption for the (super) higher spin gravity on $\mathrm{AdS}_3$ to two-dimensional (super) conformal field theory correspondence of [GG, CHR1, CHR2].

There are many more conjectures emerging from deformable families of vertex operator algebras that come from the physics of four-dimensional supersymmetric gauge theories [CGai]. These involve cosets of affine vertex operator algebras inside certain $\mathcal{W}$-algebras. Besides their apparent importance in physics [GR], they also relate to the quantum geometric Langlands correspondence [CGai, AFO]. Here the starting point is a series of conjectural deformable families of vertex operator algebras extending the tensor product of two affine vertex operator algebras with Langlands dual Lie algebras; see the introduction of [CGai] for a list of these conjectures.

# 2 Vertex algebras

In this section, we define vertex algebras, which have been discussed from various points of view in the literature (see for example [Bor, FLM, K]). We will follow the formalism developed in [LZ] and partly in [LiI]. Let $V = V_0 \oplus V_1$ be a super vector space over $\mathbb{C}$, and let $z, w$ be formal variables. By $\mathrm{QO}(V)$, we mean the space of linear maps

$$V \to V((z)) = \{\sum_{n\in\mathbb{Z}} v(n)z^{-n-1} | v(n) \in V,\ v(n) = 0 \text{ for } n \gg 0\}.$$

Each element $a \in \mathrm{QO}(V)$ can be represented as a power series

$$a = a(z) = \sum_{n\in\mathbb{Z}} a(n)z^{-n-1} \in \mathrm{End}(V)[[z, z^{-1}]].$$

We assume that $a = a_0 + a_1$ where $a_i : V_j \to V_{i+j}((z))$ for $i, j \in \mathbb{Z}/2\mathbb{Z}$, and we write $|a_i| = i$.

For each $n \in \mathbb{Z}$, we have a nonassociative bilinear operation on $\text{QO}(V)$, defined on homogeneous elements $a$ and $b$ by

$$a(w)_{(n)}b(w) = \text{Res}_z a(z)b(w)\ \iota_{|z|>|w|}(z-w)^n - (-1)^{|a||b|}\text{Res}_z b(w)a(z)\ \iota_{|w|>|z|}(z-w)^n.$$

Here $\iota_{|z|>|w|}f(z,w) \in \mathbb{C}[[z,z^{-1},w,w^{-1}]]$ denotes the power series expansion of a rational function $f$ in the region $|z|>|w|$. For $a,b \in \text{QO}(V)$, we have the following identity of power series known as the *operator product expansion* (OPE) formula.

$$a(z)b(w) = \sum_{n\geq 0} a(w)_{(n)}b(w)\ (z-w)^{-n-1} + :a(z)b(w):. \tag{2.1}$$

Here $:a(z)b(w): = a(z)_-b(w)\ +\ (-1)^{|a||b|}b(w)a(z)_+$, where $a(z)_- = \sum_{n<0} a(n)z^{-n-1}$ and $a(z)_+ = \sum_{n\geq 0} a(n)z^{-n-1}$. Often, (2.1) is written as

$$a(z)b(w) \sim \sum_{n\geq 0} a(w)_{(n)}b(w)\ (z-w)^{-n-1},$$

where $\sim$ means equal modulo the term $:a(z)b(w):$, which is regular at $z=w$.

Note that $:a(w)b(w):$ is a well-defined element of $\text{QO}(V)$. It is called the *Wick product* or *normally ordered product* of $a$ and $b$, and it coincides with $a_{(-1)}b$. For $n \geq 1$ we have

$$n!\ a(z)_{(-n-1)}b(z) = :(\partial^n a(z))b(z):, \qquad \partial = \frac{d}{dz}.$$

For $a_1(z),\dots,a_k(z) \in \text{QO}(V)$, the $k$-fold iterated Wick product is defined inductively by

$$:a_1(z)a_2(z)\cdots a_k(z): = :a_1(z)b(z):, \qquad b(z) = :a_2(z)\cdots a_k(z):. \tag{2.2}$$

We often omit the formal variable $z$ when no confusion can arise.

A subspace $\mathcal{A} \subset \text{QO}(V)$ containing 1 which is closed under all the above products will be called a *quantum operator algebra* (QOA). We say that $a,b \in \text{QO}(V)$ are *local* if

$$(z-w)^N[a(z),b(w)] = 0$$

for some $N \geq 0$. Here $[,]$ denotes the super bracket. This condition implies that $a_{(n)}b = 0$ for $n \geq N$, so (2.1) becomes a finite sum. Finally, a *vertex algebra* will be a QOA whose elements are pairwise local. This notion is well known to be equivalent to the notion of a vertex algebra in the sense of [FLM].

A vertex algebra $\mathcal{A}$ is said to be *generated* by a subset $S = \{a_i|\ i \in I\}$ if $\mathcal{A}$ is spanned by words in the letters $a_i$, and all products, for $i \in I$ and $n \in \mathbb{Z}$. We say that $S$ *strongly generates* $\mathcal{A}$ if $\mathcal{A}$ is spanned by words in the letters $a_i$, and all products for $n<0$. Equivalently, $\mathcal{A}$ is spanned by

$$\{:\partial^{k_1}a_{i_1}\cdots\partial^{k_m}a_{i_m}:\ |\ i_1,\dots,i_m \in I,\ k_1,\dots,k_m \geq 0\}.$$

As a matter of notation, we say that a vertex algebra $\mathcal{A}$ is of type

$$\mathcal{W}(d_1,\dots,d_r)$$

if it has a minimal strong generating set consisting of one field in each weight $d_1,\dots,d_r$.

Given fields $a, b, c$ in any vertex algebra $\mathcal{V}$, and integers $m, n \geq 0$, the following identities are known as *Jacobi relations* of type $(a, b, c)$.

$$a_{(r)}(b_{(s)}c) = (-1)^{|a||b|}b_{(s)}(a_{(r)}c) + \sum_{i=0}^{r} \binom{r}{i} (a_{(i)}b)_{(r+s-i)}c. \tag{2.3}$$

For a fixed choice of fields $a, b, c$, these identites are nontrivial for only finitely many integers $m, n$.

Given a vertex algebra $\mathcal{V}$ and a vertex subalgebra $\mathcal{A} \subset \mathcal{V}$, the coset (or commutant) of $\mathcal{A}$ in $\mathcal{V}$, denoted by $\text{Com}(\mathcal{A}, \mathcal{V})$, is the subalgebra of elements $v \in \mathcal{V}$ such that $[a(z), v(w)] = 0$ for all $a \in \mathcal{A}$. Equivalently, $v \in \text{Com}(\mathcal{A}, \mathcal{V})$ if and only if $a_{(n)}v = 0$ for all $a \in \mathcal{A}$ and $n \geq 0$.

# 3 The algebra $\mathcal{W}^k(\mathfrak{sl}_4, f_{\textbf{subreg}})$

Let $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})$ denote that $\mathcal{W}$-algebra at level $k$ associated to $\mathfrak{sl}_4$ with its subregular nilpotent element $f_{\text{subreg}}$. By a theorem of Genra [Gen], it coincides with the algebra $\mathcal{W}_4^{(2)}$ [FS] in the family $\mathcal{W}_n^{(2)}$ constructed by Feigin and Semikhatov [FS]. It is freely generated by fields $J, T, W, G^{\pm}$ of weights $1, 2, 3, 2, 2$, respectively, satisfying the following OPEs.

$$\begin{aligned}
T(z)T(w) &\sim -\frac{(8+3k)(17+8k)}{2(4+k)}(z-w)^{-4} + 2T(w)(z-w)^{-2} + \partial T(w)(z-w)^{-1}, \\
T(z)J(w) &\sim J(w)(z-w)^{-2} + \partial J(w)(z-w)^{-1}, \\
T(z)W(w) &\sim 3W(w)(z-w)^{-2} + \partial W(w)(z-w)^{-1}, \\
T(z)G^{\pm}(w) &\sim 2G^{\pm}(w)(z-w)^{-2} + \partial G^{\pm}(w)(z-w)^{-1}, \\
J(z)J(w) &\sim (2 + \frac{3k}{4})(z-w)^{-2}, \\
J(z)G^{\pm}(w) &\sim \pm G^{\pm}(w)(z-w)^{-1},
\end{aligned} \tag{3.1}$$

$$\begin{aligned}
W(z)G^{\pm}(w) \sim &\pm \frac{2(4+k)(7+3k)(16+5k)}{(8+3k)^2} G^{\pm}(w)(z-w)^{-3} \\
&+ \Big( \pm \frac{3(4+k)(16+5k)}{2(8+3k)} \partial G^{\pm} - \frac{6(4+k)(16+5k)}{(8+3k)^2} :JG^{\pm}: \Big)(w)(z-w)^{-2} \\
&+ \Big( -\frac{8(3+k)(4+k)}{(2+k)(8+3k)} :J\partial G^{\pm}: - \frac{4(4+k)(16+15k+3k^2)}{(2+k)(8+3k)^2} :(\partial J)G^{\pm}: \\
&\pm \frac{(3+k)(4+k)}{2+k} \partial^2 G^{\pm} \mp \frac{2(4+k)^2}{(2+k)(8+3k)} :TG^{\pm}: \\
&\pm \frac{4(4+k)(16+5k)}{(2+k)(8+3k)^2} :JJG^{\pm}: \Big)(w)(z-w)^{-1},
\end{aligned} \tag{3.2}$$

$$G^+(z)G^-(w) \sim (2+k)(5+2k)(8+3k)(z-w)^{-4} + 4(2+k)(5+2k)J(w)(z-w)^{-3}$$
$$+ \Big( -(2+k)(4+k)T + 6(2+k) :JJ: + 2(2+k)(5+2k)\partial J \Big)(w)(z-w)^{-2}$$
$$+ \Big( (k+2)W + \frac{8(2+k)(32+11k)}{3(8+3k)^2} :JJJ: - \frac{4(2+k)(4+k)}{8+3k} :TJ: + 6(2+k) :(\partial J)J:$$
$$- \frac{1}{2}(2+k)(4+k)\partial T + \frac{4(2+k)(26+17k+3k^2)}{3(8+3k)}\partial^2 J \Big)(w)(z-w)^{-1}. \tag{3.3}$$

$$W(z)W(w) \sim \frac{2(k+4)(2k+5)(3k+7)(5k+16)}{3k+8}(z-w)^{-6} + \cdots, \tag{3.4}$$

The remaining terms in the OPE of $W(z)W(w)$ have been omitted but can be found in the paper [FS]. As in [ACL1], a *weak increasing filtration* on a vertex algebra $\mathcal{A}$ is a $\mathbb{Z}_{\geq 0}$-filtration

$$\mathcal{A}_{(0)} \subset \mathcal{A}_{(1)} \subset \mathcal{A}_{(2)} \subset \cdots, \qquad \mathcal{A} = \bigcup_{d \geq 0} \mathcal{A}_{(d)} \tag{3.5}$$

such that for $a \in \mathcal{A}_{(r)}$, $b \in \mathcal{A}_{(s)}$, we have

$$a \circ_n b \in \mathcal{A}_{(r+s)}, \qquad n \in \mathbb{Z}. \tag{3.6}$$

Then the associated graded algebra $\text{gr}(\mathcal{A}) = \bigoplus_{d \geq 0} \mathcal{A}_{(d)}/\mathcal{A}_{(d-1)}$ is a vertex algebra, and a strong generating set for $\text{gr}(\mathcal{A})$ lifts to a strong generating set for $\mathcal{A}$; see Lemma 4.1 of [ACL1]. We define a filtration

$$\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})_{(0)} \subset \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})_{(1)} \subset \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})_{(2)} \subset \cdots \tag{3.7}$$

on $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})$ as follows: $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})_{(-1)} = \{0\}$, and $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})_{(r)}$ is spanned by iterated Wick products of the generators $J, T, W, G^{\pm}$ and their derivatives, such that at most $r$ of the fields $W, G^{\pm}$ and their derivatives appear. It is clear from the defining OPE relations that this is a weak increasing filtration.

# 4 The $U(1)$-orbifold of $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}$

The action of the zero mode $J_0$ integrates to a $U(1)$-action on $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})$, and the orbifold $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}$ is just the kernel of $J_0$. Since $J, T, W$ lie in $(\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}$ and $J_0(G^{\pm}) = \pm G^{\pm}$, it is immediate that $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}$ is spanned by all normally ordered monomials of the form

$$:(\partial^{a_1}T)\cdots(\partial^{a_i}T)(\partial^{b_1}J)\cdots(\partial^{b_j}J)(\partial^{c_1}W)\cdots(\partial^{c_r}W)(\partial^{d_1}G^+) \\ \cdots(\partial^{d_s}G^+)(\partial^{e_1}G^-)\cdots(\partial^{e_s}G^-):, \tag{4.1}$$

where $i, j, r, s \geq 0$ and $a_1 \geq \cdots \geq a_i \geq 0$, $b_1 \geq \cdots \geq b_j \geq 0$, $c_1 \geq \cdots \geq c_r \geq 0$, $d_1 \geq \cdots \geq d_s \geq 0$, and $e_1 \geq \cdots \geq e_s \geq 0$.

The filtration (3.7) on $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})$ restricts to a weak increasing filtration on $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}$ where

$$(\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)})_{(r)} = \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)} \cap \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})_{(r)}.$$

Define

$$U_{i,j} = :\partial^i G^+ \partial^j G^- :,$$

which lies in $(\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)})_{(2)}$ and has weight $i+j+4$. By the same argument as the proof of Lemma 5.1 of [ACL1], we have

**Lemma 4.1.** *$\mathcal{W}^k(\mathfrak{sl}_4, f_{subreg})^{U(1)}$ is strongly generated as a vertex algebra by*

$$\{J, T, W, U_{0,m} |\ m \geq 0\}. \tag{4.2}$$

However, $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)}$ is not freely generated by (4.2). There is a relation of weight 10 of the form

$$\frac{(2+k)(5+2k)(8+3k)}{360} U_{0,6} = :U_{0,0}U_{1,1}: - :U_{0,1}U_{1,0}: + \cdots,$$

where the remaining terms are normally ordered monomials in $T, J, W, U_{0,i}$ and their derivatives, for $i \leq 5$. We can regard $:U_{0,0}U_{1,1}: - :U_{0,1}U_{1,0}:$ as the analogue of a classical relation which does not vanish due to the nonassociativity of the Wick product, and the remaining terms provide the necessary corrections to make it a genuine relation. This relation is unique up to scalar multiples, and the coefficient of $U_{0,6}$ is canonical in the sense that it does not depend on any choices of normal ordering in the expression on the right side. In particular, we see that $U_{0,6}$ decouples for all $k \neq -2, -5/2, -8/3$.

Similarly, for all $n > 1$ we have relations

$$\frac{n(9+n)(2+k)(5+2k)(8+3k)}{120(4+n)(5+n)} U_{0,n+5} = :U_{0,0}U_{1,n}: - :U_{0,n}U_{1,0}: + \cdots$$

where the remaining terms are normally ordered monomials in $T, J, W, U_{0,i}$ and their derivatives, for $i \leq 5$. The proof is similar to the proof of Theorem 5.4 of [ACL1]. Again, the coefficient of $U_{0,n+5}$ is canonical, and this shows that $U_{0,n+5}$ can be decoupled for all $n > 1$ whenever $k \neq -2, -5/2, -8/3$. We obtain

**Theorem 4.2.** *For all $k \neq -2, -5/2, -8/3$, $\mathcal{W}^k(\mathfrak{sl}_4, f_{subreg})^{U(1)}$ has a minimal strong generating set*

$$\{J, T, W, U_{0,i} |\ i \leq 5\},$$

*and in particular is of type $\mathcal{W}(1,2,3,4,5,6,7,8,9)$.*

# 5 The Heisenberg coset of $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})$

Let $\mathcal{H} \subset \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})$ denote the copy of the Heisenberg vertex algebra generated by $J$, and let $\mathcal{C}^k$ denote the commutant $\text{Com}(\mathcal{H}, \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}}))$. Note that

$$\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})^{U(1)} \cong \mathcal{H} \otimes \mathcal{C}^k$$

and $\mathcal{C}^k$ has a Virasoro element

$$T^{\mathcal{C}} = T - \frac{2}{8+3k} :JJ:$$

of central charge

$$c = -\frac{4(5+2k)(7+3k)}{4+k}.$$

Also, it is clear from the OPE algebra that $W \in \mathcal{C}^k$. By a straightforward computer calculation, we obtain

**Theorem 5.1.** *For $0 \leq i \leq 5$, and $k \neq -2, -5/2, -8/3$, there exist correction terms $\omega_i \in \mathcal{W}^k(\mathfrak{sl}_4, f_{subreg})^{U(1)}$ such that $U_i^{\mathcal{C}} = U_{0,i} + \omega_i$ lies in $\mathcal{C}^k$. Therefore $\mathcal{C}^k$ has a minimal strong generating set $\{T^{\mathcal{C}}, W, U_i^{\mathcal{C}} | \ 0 \leq i \leq 5\}$, and is therefore of type $\mathcal{W}(2,3,4,5,6,7,8,9)$.*

Next, let $\mathcal{W}_k(\mathfrak{sl}_4, f_{\text{subreg}})$ denote the simple quotient of $\mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}})$ by its maximal proper ideal graded by conformal weight, and let $\mathcal{C}_k = \text{Com}(\mathcal{H}, \mathcal{W}^k(\mathfrak{sl}_4, f_{\text{subreg}}))$. Evidently we have a surjective map

$$\mathcal{C}^k \to \mathcal{C}_k,$$

so for $k \neq -2, -5/2, -8/3$, $\mathcal{C}_k$ is strongly generated by the fields above.

# 6 Simple current extensions and $\mathcal{W}_\ell(\mathfrak{sl}_n, f_{\text{reg}})$

Vertex operator algebra extensions of a given vertex algebra $V$ can be efficiently studied using commutative, associative algebras with injective unit in the representation category of $V$. This has been developed in [KO, HKL, CKM] and especially structure about parafermionic cosets, i.e. cosets by a Heisenberg or lattice vertex algebra, has been derived in [CKL, CKLR, CKM]. Here we use these ideas to construct simple current extensions of rational, regular $\mathcal{W}$-algebras of type $A$ tensored with certain lattice vertex operator algebras. Recall that a simple current is an invertible object in the tensor category of the vertex operator algebra.

Let $n, r$ be in $\mathbb{Z}_{>1}$ such that $n+1$ and $n+r$ are coprime (so that especially $nr$ is even) and define

$$\mathcal{W}(n,r) := \mathcal{W}_\ell(\mathfrak{sl}_n, f_{\text{reg}}), \qquad \ell + n = \frac{n+r}{n+1}.$$

By [Ar1], $\mathcal{W}(n,r)$ is rational and $C_2$-cofinite. Let $L = \sqrt{nr}\mathbb{Z}$ and $V_L$ the lattice vertex operator algebra of $L$. Modules and their fusion rules for $\mathcal{W}(n,r)$ are essentially known due to [FKW, AvE]. Modules are parameterized by modules of $L_r(\mathfrak{sl}_n)$, i.e. by integrable positive weights of $\widehat{\mathfrak{sl}}_n$ at level $r$. Fusion rules (Theorem 4.3, Proposition 4.3 of [FKW] together with Corollary 8.4 of [AvE]) imply that the group of simple currents is $\mathbb{Z}/n\mathbb{Z}$ and these simple currents correspond to the modules $\mathbb{L}_{r\omega_i}$ with $\omega_i$ the fundamental weights of $\mathfrak{sl}_n$. The question of extending a given regular vertex algebra by a group of simple currents to a larger one is entirely decided by conformal dimension and quantum dimension of the involved simple currents. One gets a vertex operator superalgebra if and only if conformal dimensions of a set of generators of the group of simple currents are in $\frac{1}{2}\mathbb{Z}$. Moreover the quantum dimension of generators of the group of simple currents decide whether this is even a vertex operator algebra. See [CKL] for details.

By the quantum dimension of a module $M$ we mean the categorical dimension of $M$. By Verlinde's formula [H1, H2] one has

$$\text{qdim}(M) = \frac{S_{M,V}}{S_{V,V}}$$

with $S$-matrix of the modular transformation of torus one-point functions $\text{ch}[M](v,\tau) := \text{tr}_M(o(v)q^{L_0 - c/24})$ ($v$ in $V$ of conformal weight $k$ and $o(v)$ the zero-mode of $v$)

$$\text{ch}[M](v, -1/\tau) = \tau^k \sum_N S_{M,N}\, \text{ch}[N](v,\tau)\,.$$

The sum here is over all inequivalent modules of $V$. See [CG] for a review on modular and categorical aspects of vertex algebras.

The quantum dimension and conformal dimension of $\mathbb{L}_{r\omega_1}$ are now easily computed using the recent results of van Ekeren and Arakawa [AvE]:

$$\begin{aligned} \text{qdim}\left(\mathbb{L}_{r\omega_1}\right) &= \frac{S_{r\omega_1,0}}{S_{0,0}} = e^{2\pi i r(\omega_1,\rho)}\text{qdim}\left(L_{r\omega_1}\right) = e^{2\pi i r(\omega_1,\rho)} \\ &= e^{2\pi i r\frac{n-1}{2}} = (-1)^{r(n-1)} = (-1)^r \end{aligned} \tag{6.1}$$

Here $\text{qdim}\left(L_{r\omega_1}\right)$ is the quantum dimension of the $L_r\left(\mathfrak{sl}_n\right)$ module $L_{r\omega_1}$. We firstly used that the modular $S$-matrices of $\mathcal{W}(n,r)$ and of $L_r(\mathfrak{sl}_n)$ only differ by the factor $e^{2\pi i r(\omega_1,\rho)}$ with $\rho$ the Weyl vector of $\mathfrak{sl}_n$. Secondly we used that $L_r(\mathfrak{sl}_n)$ is unitary and hence all quantum dimensions are positive and so every simple current of $L_r(\mathfrak{sl}_n)$ must have quantum dimension one. Finally, in the last equality we used that $nr$ is even. The conformal dimension is

$$\Delta(\mathbb{L}_{r\omega_1}) = \frac{(n+1)}{2(n+r)}\left(r\omega_1, r\omega_1+2\rho\right) - (\omega_1,\rho) = \frac{(n-1)r}{2n}$$

since $\omega_1^2 = \frac{n-1}{n}$ and $(\omega_1,\rho) = \frac{n-1}{2}$. We denote by $V_{L+\gamma}$ the $V_L$-module corresponding to the coset $L+\gamma$ of $L$ in the dual lattilce $L' = \frac{1}{\sqrt{nr}}\mathbb{Z}$. Then $V_{L+\frac{r}{\sqrt{rn}}}$ has conformal dimension $\frac{r}{2n}$ and quantum dimension one since $V_L$ is unitary. It follows from [CKL] (see the Theorems listed in the introduction of that work) that

$$A(n,r) \cong \bigoplus_{s=0}^{n-1} V_{L+\frac{rs}{\sqrt{rn}}} \otimes \underbrace{\mathbb{L}_{r\omega_1} \boxtimes_{\mathcal{W}(n,r)} \cdots \boxtimes_{\mathcal{W}(n,r)} \mathbb{L}_{r\omega_1}}_{s-\text{times}} \tag{6.2}$$

is a vertex operator algebra extending $V_L \otimes \mathcal{W}(n,r)$. If $r$ is even, this is a $\mathbb{Z}$-graded vertex operator algebra, while for odd $r$ it is only $\frac{1}{2}\mathbb{Z}$-graded. The subspace of lowest conformal weight in each of the $\mathbb{L}_{r\omega_1} \boxtimes_{\mathcal{W}(n,r)} \cdots \boxtimes_{\mathcal{W}(n,r)} \mathbb{L}_{r\omega_1}$ is one-dimensional, and we denote the corresponding vertex operators by $X_s$. By Proposition 4.1 of [CKL] the OPE of $X_s$ and $X_{n-s}$ has a non-zero multiple of the identity as leading term. Without loss of generality, we may rescale $X_1$ and $X_{n-1}$ so that

$$X_1(z)X_{n-1}(w) \sim \prod_{i=1}^{n-1}(i(k+n-1)-1)(z-w)^{-r} + \dots. \tag{6.3}$$

Let $J$ be the Heisenberg field of $V_L$ and we normalize it such that

$$J(z)J(w) \sim \left(\frac{(n-1)k}{n} + n - 2\right)(z-w)^{-2}. \tag{6.4}$$

Then we have

$$J(z)X_1(w) \sim X_1(w)(z-w)^{-1}, \qquad J(z)X_{n-1}(w) \sim -X_{n-1}(w)(z-w)^{-1}. \tag{6.5}$$

Let $\widetilde{A}(n,r)$ be the vertex algebra generated by $X_1$ and $X_{n-1}$ under operator products We now rephrase a physics conjecture [B–H],

**Conjecture 6.1.** *Let $n, r$ as above and $k$ defined by $k + r = \frac{n+r}{r-1}$. Then*

$$A(n,r) \cong \widetilde{A}(n,r) \cong \mathcal{W}_k(\mathfrak{sl}_r, f_{subreg}).$$

*In particular, $\mathcal{W}_k(\mathfrak{sl}_r, f_{subreg})$ is rational and $C_2$-cofinite.*

We remark that Conjecture 6.1 is true for $r = 2, 3$ by [ALY] and [ACL1] and we will now prove it for $r = 4$ under some extra condition on $n$. For this, we now assume that $n-1$ is co-prime to at least one of $n+1$ and $n+r$ so that especially $n$ even would work. Under this condition the formula for fusion rules is more explicit, and we know from the fusion rules of $\mathcal{W}(n,r)$ [AvE] that

$$\begin{aligned} A(n,r) &\cong \bigoplus_{s=0}^{n-1} V_{L+\frac{rs}{\sqrt{rn}}} \otimes \underbrace{\mathbb{L}_{r\omega_1} \boxtimes_{\mathcal{W}(n,r)} \cdots \boxtimes_{\mathcal{W}(n,r)} \mathbb{L}_{r\omega_1}}_{s-\text{times}} \\ &\cong \bigoplus_{s=0}^{n-1} V_{L+\frac{rs}{\sqrt{rn}}} \otimes \mathbb{L}_{r\omega_s}. \end{aligned} \tag{6.6}$$

The lowest conformal weight of the $s$-th summand is $\min\{\frac{sr}{2}, \frac{(n-s)r}{2}\}$ and so in this instance $\widetilde{A}(n,r)$ is strongly generated by $X_1, X_{n-1}$ together with the Heisenberg field $J$ and some fields of $\mathcal{W}(n,r)$.

**Theorem 6.2.** *Conjecture 6.1 holds for $r = 4$ and all $n$ such that $n-1$ is co-prime to at least one of $n+1$ and $n+4$.*

*Proof.* Let $L$ be the Virasoro field of $\mathcal{W}(n,r)$ and let $T = L + \frac{2}{8+3k} : JJ :$ be the Virasoro field of $V_L \otimes \mathcal{W}(n,r)$. Also, let $W$ be the weight 3 field of $\mathcal{W}(n,r)$ which is known to generate $\mathcal{W}(n,r)$. Since the OPE of $X_1(z)X_{n-1}(w)$ can be expressed in terms of $J, T, W$, the most general form is

$$\begin{aligned} X_1(z)X_{n-1}(w) \sim\ & (2+k)(5+2k)(8+3k)(z-w)^{-4} + a_1 J(w)(z-w)^{-3} \\ & + \Big(a_2 T + a_3 : JJ : + a_4 \partial J\Big)(w)(z-w)^{-2} \\ & + \Big(a_5 W + a_6 : JJJ : + a_7 : TJ : + a_8 : (\partial J)J : + a_9 \partial T + a_{10}\partial^2 J\Big)(w)(z-w)^{-1}, \end{aligned} \tag{6.7}$$

where the $a_i$ are constants. By imposing all Jacobi relations of the form $(J, X_1, X_{n-1})$ and $(T, X_1, X_{n-1})$ we obtain all the above coefficients uniquely except for $a_5$, that is,

$$\begin{aligned} X_1(z)X_{n-1}(w) \sim\ & (2+k)(5+2k)(8+3k)(z-w)^{-4} + 4(2+k)(5+2k)J(w)(z-w)^{-3} \\ & + \Big(-(2+k)(4+k)T + 6(2+k) : JJ : + 2(2+k)(5+2k)\partial J\Big)(w)(z-w)^{-2} \\ & + \Big(a_5 W + \frac{8(2+k)(32+11k)}{3(8+3k)^2} : JJJ : - \frac{4(2+k)(4+k)}{8+3k} : TJ : \\ & + 6(2+k) : (\partial J)J : - \frac{1}{2}(2+k)(4+k)\partial T \\ & + \frac{4(2+k)(26+17k+3k^2)}{3(8+3k)}\partial^2 J\Big)(w)(z-w)^{-1}. \end{aligned} \tag{6.8}$$

Using the OPE relations (6.5), and the Jacobi relations of type $(X_1, X_1, X_{n-1})$, we see that $a_5 \neq 0$. Since we are free to rescale the field $W$, we may assume without loss of generality that

$$a_5 = (k+2).$$

This completely determines $X_1(z)X_{n-1}(w)$. Also, since $W$ appears in $\widetilde{A}(n,4)$ and generates $\mathcal{W}(n,4)$ (see Proposition A.3 of [ALY]), we must have $\widetilde{A}(n,4) = A(n,4)$.

Next, imposing all Jacobi relations of type $(T, W, X_1)$, $(J, W, X_1)$, $(T, W, X_{n-1})$ and $(J, W, X_{n-1})$ uniquely determines the OPEs

$$W(z)X_1(w), \qquad W(z)X_{n-1}(w). \tag{6.9}$$

Finally, using (6.3)-(6.5) and (6.8)-(6.9) and imposing all Jacobi relations of type $(W, X_1, X_{n-1})$ uniquely determines the OPE of $W(z)W(w)$. In particular, these OPE relations are precisely the OPE relations in $\mathcal{W}_k(\mathfrak{sl}_4, f_{\text{subreg}})$ with $X_1, X_{n-1}$ replaced by $G^+, G^-$. Since $A(n,4)$ and $\mathcal{W}_k(\mathfrak{sl}_4, f_{\text{subreg}})$ are simple vertex algebras with the same strong generating set and OPE algebra, they must be isomorphic. □

**Corollary 6.3.** *Let $k$ be defined by $k + 4 = \frac{n+4}{3}$, and assume that $n-1$ is co-prime to at least one of $n+1$ and $n+4$. Then $\mathcal{W}(n,4)$ is strongly generated by the fields in weights $2,3,4,5,6,7,8,9$ even though the universal regular $\mathcal{W}$-algebra of $\mathfrak{sl}_n$ is of type $\mathcal{W}(2,3,\dots,n)$.*

*Proof.* This is immediate from Theorems 5.1 and 6.2 and the fact that the map $\mathcal{C}^k \to \mathcal{C}_k$ is surjective. □

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