

CONVEX AND SEQUENTIAL EFFECT ALGEBRAS

Stan Gudder
Department of Mathematics
University of Denver
Denver, Colorado 80208
sgudder@du.edu

Abstract

We present a mathematical framework for quantum mechanics in which the basic entities and operations have physical significance. In this framework the primitive concepts are states and effects and the resulting mathematical structure is a convex effect algebra. We characterize the convex effect algebras that are classical and those that are quantum mechanical. The quantum mechanical ones are those that can be represented on a complex Hilbert space. We next introduce the sequential product of effects to form a convex sequential effect algebra. This product makes it possible to study conditional probabilities and expectations.

1 Introduction

One of the most important problems in the foundations of physics is to justify the axioms of quantum mechanics on physical grounds. A simplified version of the main axioms of quantum mechanics is the following.

- (A1) The pure states of a quantum system are represented by unit vectors in a complex Hilbert space K and the observables are represented by self-adjoint operators on K .

- (A2) If the system is in the state ϕ then the expectation (or average value) of an observable A is $\langle \phi, A\phi \rangle$
- (A3) The dynamics of the system is described by a one-parameter unitary group U_t , $t \in \mathbb{R}$. If the initial state is ϕ_0 then the state at time t is $U_t\phi_0$.

Several immediate questions come to mind. Where does the complex Hilbert space come from? In particular, what do complex numbers have to do with a physical system? What is the physical meaning of the complex inner product $\langle \phi, \psi \rangle$? Two observables are said to be compatible if their corresponding operators A, B commute. This is reasonable because if A and B commute they are both functions of another self-adjoint operator so they can be measured simultaneously. But if A and B do not commute, there is no physical meaning for the operator sum $A + B$ and the operator product AB . There are many other problems and questions like these. We conclude that these axioms are based upon unphysical structures whose basic mathematical operations have no physical meaning.

In this article we present a mathematical framework for quantum mechanics in which the basic entities and operations have physical significance. In this framework the primitive concepts are states and effects. The states represent initial preparations that describe the condition of the system while the effects represent yes-no measurements that probe the system. The effects may be unsharp or as they are sometimes called, fuzzy [1, 5, 6]. A state applied to an effect produces the probability that the effect gives a yes value. Effects can also be thought of as true-false or 0-1 measurements. The resulting mathematical structure is a convex-effect algebra \mathcal{E} [7, 12]. The two mathematical operations in \mathcal{E} are an orthogonal sum $a \oplus b$ and a scalar product λa , $\lambda \in [0, 1] \subseteq \mathbb{R}$ both of which having physical interpretations. The sum $a \oplus b$ corresponds to a parallel measurement of a and b while λa corresponds to an attenuation of a by the factor λ [7, 12]. Section 2 presents these basic definitions in detail.

One advantage of employing physically motivated mathematical operations is that they lead up to physically useful theorems and results. Our main theorems in Section 3 characterize the convex effect algebras that are classical and those that are quantum mechanical. The quantum mechanical convex effect algebras are those that can be represented on a complex Hilbert space and this answers the question: Where does the Hilbert space come from?

The key to the representation theorem is a concept we call contextuality as explained in Section 3.

In Section 4 we introduce the sequential product $a \circ b$ of effects a and b . This product corresponds to first measuring a and then measuring b in sequence. This product makes it possible to study conditional probabilities and expectations which are treated in Section 4. The resulting structure is called a convex sequential effect algebra [8, 9, 10, 11].

2 Convex Effect Algebras

Most statistical theories for physical systems contain two basic primitive concepts, namely effects and states. The effects correspond to simple yes-no measurements or experiments and the states correspond to preparation procedures that specify the initial conditions of the system being measured. Usually, each effect a and state s experimentally determine a probability $F(a, s)$ that the effect a occurs (has answer yes) when the system has been prepared in the state s . For a given physical system, denote its set of possible effects by \mathcal{E} and its set of possible states by \mathcal{S} . In a reasonable statistical theory, the probability function satisfies three axioms that are given in the following definition [7].

An *effect-state space* is a triple $(\mathcal{E}, \mathcal{S}, F)$ where \mathcal{E} and \mathcal{S} are nonempty sets and $F : \mathcal{E} \times \mathcal{S} \rightarrow [0, 1] \subseteq \mathbb{R}$ satisfies:

- (ES1) There exist elements $0, 1 \in \mathcal{E}$ such that $F(0, s) = 0$, $F(1, s) = 1$ for every $s \in \mathcal{S}$.
- (ES2) If $F(a, s) \leq F(b, s)$ for every $s \in \mathcal{S}$, then there exists a unique $c \in \mathcal{E}$ such that $F(a, s) + F(c, s) = F(b, s)$ for every $s \in \mathcal{S}$.
- (ES3) If $a \in \mathcal{E}$ and $\lambda \in [0, 1] \subseteq \mathbb{R}$, then there exists an element $\lambda a \in \mathcal{E}$ such that $F(\lambda a, s) = \lambda F(a, s)$ for every $s \in \mathcal{S}$.

The elements $0, 1$ in (ES1) correspond to the null effect that never occurs and the unit effect that always occurs, respectively. Condition (ES2) postulates that if a has a smaller probability of occurring than b in every state, then there exists a unique effect c which when combined with a gives the probability that b occurs in every state. The element λa of condition (ES3) is interpreted as the effect a attenuated by the factor λ . It is shown in [7] that if $F(a, s) + F(b, s) \leq 1$ for every $s \in \mathcal{S}$, then there exists a unique $c \in \mathcal{E}$

such that $F(c, s) = F(a, s) + F(b, s)$ for every $s \in \mathcal{S}$. We then write $a \perp b$ and define $a \oplus b = c$.

We now consider a previously studied mathematical framework that exposes the basic axioms of an effect-state space. An *effect algebra* [2, 3, 13, 14] is an algebraic system $(\mathcal{E}, 0, 1, \oplus)$ where 0 and 1 are distinct elements of \mathcal{E} and \oplus is a partial binary operation on \mathcal{E} that satisfies the following conditions (we write $a \perp b$ when $a \oplus b$ is defined).

- (E1) If $a \perp b$, then $b \perp a$ and $b \oplus a = a \oplus b$.
- (E2) If $a \perp b$ and $(a \oplus b) \perp c$, then $b \perp c$, $a \perp (b \oplus c)$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.
- (E3) For every $a \in \mathcal{E}$ there exists a unique $a' \in \mathcal{E}$ such that $a \perp a'$ and $a \oplus a' = 1$.
- (E4) If $a \perp 1$, then $a = 0$.

If $a \perp b$, we call $a \oplus b$ the *orthogonal sum* of a and b . We define $a \leq b$ if there exists $c \in \mathcal{E}$ such that $a \oplus c = b$. It can be shown that $(\mathcal{E}, 0, 1, \leq)$ is a bounded poset and $a \perp b$ if and only if $a \leq b'$ [3]. It is also shown in [3] that $a'' = a$ and that $a \leq b$ implies $b' \leq a'$ for every $a, b \in \mathcal{E}$.

An effect algebra \mathcal{E} is *convex* [7, 12] if for every $a \in \mathcal{E}$ and $\lambda \in [0, 1] \subseteq \mathbb{R}$ there exists an element $\lambda a \in \mathcal{E}$ such that the following conditions hold.

- (C1) If $\alpha, \beta \in [0, 1]$ and $a \in \mathcal{E}$, then $\alpha(\beta a) = (\alpha\beta)a$.
- (C2) If $\alpha, \beta \in [0, 1]$ with $\alpha + \beta \leq 1$ and $a \in \mathcal{E}$, then $\alpha a \perp \beta a$ and $(\alpha + \beta)a = \alpha a \oplus \beta a$.
- (C3) If $a, b \in \mathcal{E}$ with $a \perp b$ and $\lambda \in [0, 1]$, then $\lambda a \perp \lambda b$ and $\lambda(a \oplus b) = \lambda a \oplus \lambda b$.
- (C4) If $a \in \mathcal{E}$, then $1a = a$.

It is shown in [7] that a convex effect algebra is “convex” in the sense that $\lambda a \oplus (1 - \lambda)b$ is defined for every $\lambda \in [0, 1]$ and $a, b \in \mathcal{E}$ and hence is an element of \mathcal{E} . If \mathcal{E} and \mathcal{F} are effect algebras, a map $\phi: \mathcal{E} \rightarrow \mathcal{F}$ is *additive* if $a \perp b$ implies $\phi(a) \perp \phi(b)$ and $\phi(a \oplus b) = \phi(a) \oplus \phi(b)$. An additive map ϕ that satisfies $\phi(1) = 1$ is called a *morphism*. A morphism $\phi: \mathcal{E} \rightarrow \mathcal{F}$ for which $\phi(a) \perp \phi(b)$ implies that $a \perp b$ is called a *monomorphism*. A surjective

monomorphism is an *isomorphism*. If \mathcal{E} and \mathcal{F} are convex effect algebras, a morphism $\phi: \mathcal{E} \rightarrow \mathcal{F}$ is an *affine morphism* if $\phi(\lambda a) = \lambda\phi(a)$ for every $\lambda \in [0, 1]$, $a \in \mathcal{E}$. If there exists an affine isomorphism $\phi: \mathcal{E} \rightarrow \mathcal{F}$ we say that \mathcal{E} and \mathcal{F} are *affinely isomorphic*.

The simplest example of a convex effect algebra is the unit interval $[0, 1] \subseteq \mathbb{R}$ with the usual addition (when $a+b \leq 1$) and scalar multiplication. A *state* on an effect algebra \mathcal{E} is a morphism $\omega: \mathcal{E} \rightarrow [0, 1]$. We interpret $\omega(a)$ as the probability that the effect a occurs when the system is prepared in the state ω . We denote the set of states on \mathcal{E} by $\Omega(\mathcal{E})$. We say that $S \subseteq \Omega(\mathcal{E})$ is *separating* if $\omega(a) = \omega(b)$ for every $\omega \in S$ implies that $a = b$. We say that $S \subseteq \Omega(\mathcal{E})$ is *order determining* if $\omega(a) \leq \omega(b)$ for all $\omega \in S$ implies that $a \leq b$. It is shown in [7] that every state on a convex effect algebra is affine. The next result, which is proved in [7] shows that an effect-state space is equivalent to a convex effect algebra with an order determining set of states. It is surprising that the physically motivated framework of an effect-state space with three simple axioms is equivalent to a seemingly more complicated structure of a convex effect algebra with an order determining set of states which has nine axioms.

Theorem 2.1. *If (\mathcal{E}, S, F) is an effect-state space and $\widehat{S} = \{F(\cdot, s) : s \in S\}$, then $(\mathcal{E}, 0, 1, \oplus)$ is a convex effect algebra with an order determining set of states \widehat{S} . Conversely, if $(\mathcal{E}, 0, 1, \oplus)$ is a convex effect algebra and S is an order determining set of states on \mathcal{E} , then (\mathcal{E}, S, F) is an effect-state space where $F: \mathcal{E} \times S \rightarrow [0, 1]$ is defined by $F(a, s) = s(a)$.*

We now consider a general type of convex effect algebra called a linear effect algebra. Let V be a real linear space with zero θ . A subset K of V is a *positive cone* if $\mathbb{R}^*K \subseteq K$, $K + K \subseteq K$ and $K \cap (-K) = \{\theta\}$. For $x, y \in V$ we define $x \leq y$ if $y - x \in K$. Then \leq is a partial order on V and we call (V, K) an *ordered linear space* with positive cone K . We say that K is *generating* if $V = K - K$. Let $u \in K$ with $u \neq \theta$ and form the interval

$$[\theta, u] = \{x \in K : x \leq u\}$$

For $x, y \in [\theta, u]$ we write $x \perp y$ if $x + y \leq u$ and in this case we define $x \oplus y = x + y$. It is clear that $([\theta, u], \theta, u, \oplus)$ is an effect algebra with $x' = u - x$ for every $x \in [\theta, u]$. It is easy to check that $[\theta, u]$ is a convex subset of V and that $\lambda x \in [\theta, u]$ for every $\lambda \in [0, 1]$, $x \in [\theta, u]$. It follows that $[\theta, u]$ is a convex effect algebra which we call a *linear effect algebra*. We say

that $[\theta, u]$ generates K if $K = \mathbb{R}^+[\theta, u]$ and say that $[\theta, u]$ is *generating* if $[\theta, 0]$ generates K and K generates V . The following representation theorem, which is proved in [12], shows that convex effect algebras and linear effect algebras are equivalent structures.

Theorem 2.2. *If $(\mathcal{E}, 0, 1, \oplus)$ is a convex effect algebra, then \mathcal{E} is affinely isomorphic to a linear effect algebra $[\theta, u]$ that generates an ordered linear space (V, K) .*

A linear functional $f: V \rightarrow \mathbb{R}$ on an ordered linear space (V, K) is *positive* if $f(x) \geq 0$ for all $x \in K$. We denote the set of positive linear functionals on V by V^p . If $[\theta, u]$ generates (V, K) and $f \in V^p$ satisfies $f(u) = 1$ we say that f is *unital*. We denote the set of unital elements of V^p as V_u^p . It is clear that if $f \in V_u^p$, the restriction of f to $[\theta, u]$ is a state. The next result, which is proved in [7] gives a converse.

Theorem 2.3. *Let $[\theta, u]$ be a generating interval for (V, K) .*

- (i) *If $\omega \in \Omega([\theta, u])$, then ω has a unique extension $\widehat{\omega} \in V_u^p$.*
- (ii) *The map $\widehat{\cdot}: \Omega([\theta, u]) \rightarrow V_u^p$ is a bijection that satisfies*

$$(\lambda\omega_1 + (1 - \lambda)\omega_2)^\wedge = \lambda\widehat{\omega}_1 + (1 - \lambda)\widehat{\omega}_2$$

for all $\lambda \in [0, 1]$, $\omega_1, \omega_2 \in \Omega([\theta, u])$.

- (iii) *A subset $S \subseteq \Omega([\theta, u])$ is order determining if and only if $\widehat{S} \subseteq V_u^p$ is order determining.*

Of course, \widehat{S} order determining means that $\widehat{\omega}(x) \leq \widehat{\omega}(y)$ for all $\omega \in S$ implies that $x \leq y$. We close this section with two important examples of convex effect algebras. The first example comes from the quantum theory formalism [15, 16]. Let H be a complex Hilbert space and let $\mathcal{E}(H)$ be the set of operators on H that satisfy $0 \leq A \leq I$ where we are using the usual ordering of bounded operators. For $A, B \in \mathcal{E}(H)$ we write $A \perp B$ if $A + B \in \mathcal{E}(H)$ and in this case we define $A \oplus B = A + B$. For $\lambda \in [0, 1]$ and $A \in \mathcal{E}(H)$, $\lambda A \in \mathcal{E}(H)$ is the usual scalar multiplication for operators. It is easy to check that $\mathcal{E}(H)$ is a convex effect algebra which we call a *Hilbertian effect algebra*. If $\phi \in H$ is a unit vector, define the state $\widehat{\phi}$ by $\widehat{\phi}(A) = \langle \phi, A\phi \rangle$ for all $A \in \mathcal{E}(H)$. It follows by definition that this set of states is order determining.

Our second example comes from fuzzy probability theory [1, 5]. Let (Ω, \mathcal{A}) be a measurable space in which singleton sets are measurable and let

$\mathcal{E}(\Omega, \mathcal{A})$ be the set of measurable functions on Ω with values in $[0, 1] \subseteq \mathbb{R}$. If we define \oplus and λf analogously as in the previous example, we see that $\mathcal{E}(\Omega, \mathcal{A})$ is a convex effect algebra. The elements of $\mathcal{E}(\Omega, \mathcal{A})$ are called *fuzzy events* and we call $\mathcal{E}(\Omega, \mathcal{A})$ a *classical effect algebra*. If μ is a probability measure on (Ω, \mathcal{A}) then the map $f \mapsto \int f d\mu$ gives a state on $\mathcal{E}(\Omega, \mathcal{A})$. This set of states is order determining. In particular, the set of Dirac measures δ_ω , $\omega \in \Omega$ is order determining.

3 Classical and Hilbertian Effect Algebras

This section characterizes the classical and Hilbertian effect algebras. Roughly speaking, these correspond to classical and quantum mechanics, respectively. For simplicity, we only treat the finite-dimensional case. Our theory generalizes to infinite dimensions but then we have to treat σ -effect algebras [6]. This would introduce measure theoretic and convergence details that detract from the main ideas. Besides there are important physical systems such as quantum information and computation that fall within the finite dimensional domain.

Let \mathcal{E} be a convex effect algebra. By Theorem 2.2 we can assume that \mathcal{E} is a linear effect algebra $[\theta, u]$ that generates an ordered linear space (V, K) . For $x, y \in V$ we sometimes retain the notation $x \oplus y$ if $x, y \in \mathcal{E} = [\theta, u]$ with $x \perp y$ and otherwise we use $x + y$ for the sum. An effect $a \in \mathcal{E}$ is *sharp* [6] if the greatest lower bound $a \wedge a' = \theta$. Sharp effects are thought of as effects that are precise or unfuzzy. The sharp effects in $\mathcal{E}(\Omega, \mathcal{A})$ are the measurable characteristic functions or equivalently the sets in \mathcal{A} . The sharp effects in $\mathcal{E}(H)$ are the projection operators on H . We denote the sharp effects in \mathcal{E} by $S(\mathcal{E})$. An $a \in S(\mathcal{E})$ is *one-dimensional* if $a \neq \theta$ and if $b \in \mathcal{E}$ with $b \leq a$ implies that $b = \lambda a$ for some $\lambda \in [0, 1]$. It is shown in [12] that if $a \in S(\mathcal{E})$ with $a \neq \theta$ then there exists a state $\hat{a} \in \Omega(\mathcal{E})$ such that $\hat{a}(a) = 1$. We denote the set of one-dimensional sharp elements by $S_1(\mathcal{E})$.

A *context* is a finite set $\{a_1, \dots, a_n\} \subseteq S_1(\mathcal{E})$ such that

$$a_1 \oplus a_2 \oplus \dots \oplus a_n = u \tag{3.1}$$

It follows from (3.1) that $\hat{a}_i(a_j) = \delta_{ij}$. We interpret a context as a finest sharp measurement. That is, one of the effects a_i must occur and there is no finer sharp measurement. We say that \mathcal{E} is *finite-dimensional* if there exists a context on \mathcal{E} . For the remainder of this section, we shall assume that \mathcal{E} is

finite-dimensional. We say that \mathcal{E} is *spectral* if for every $b \in \mathcal{E}$ there exists a context $\{a_1, \dots, a_n\}$ such that $b = \lambda_1 a_1 \oplus \dots \oplus \lambda_n a_n$, $\lambda_i \in [0, 1]$, $i = 1, \dots, n$. We now characterize a classical effect algebra $\mathcal{E}(\Omega, \mathcal{A})$. We say that $\mathcal{E}(\Omega, \mathcal{A})$ is *finite* if $\Omega = \{\omega_1, \dots, \omega_n\}$ is finite.

Theorem 3.1. *Let \mathcal{E} be a finite dimensional convex effect algebra. Then \mathcal{E} is affinely isomorphic to a finite classical effect algebra if and only if \mathcal{E} possesses exactly one context and \mathcal{E} is spectral.*

Proof. For sufficiency, we can assume that $\mathcal{E} = \mathcal{E}(\Omega, \mathcal{A})$ where $\Omega = \{\omega_1, \dots, \omega_n\}$ is finite. A function $f \in \mathcal{E}$ is sharp if and only if f has the values 0 or 1; that is, f is a characteristic function. Indeed, characteristic functions are clearly sharp. Conversely, suppose $f \in \mathcal{E}$ is sharp and $f(\omega_0) \neq 0, 1$ for some $\omega_0 \in \Omega$. Let $\lambda \in (0, 1)$ satisfy $\lambda < f(\omega_0)$, $\lambda < 1 - f(\omega_0)$. Define $g \in \mathcal{E}$ by $g(\omega_0) = \lambda$, $g(\omega) = 0$ if $\omega \neq \omega_0$. Then $g < f$ and $g < 1 - f = f'$. Since $g \neq 0$, $f \wedge (1 - f) \neq 0$. This gives a contradiction so f is a characteristic function. The functions in $S_1(\mathcal{E})$ are the characteristic functions of singleton sets $\chi_{\{\omega\}}$, $\omega \in \Omega$. Since

$$\chi_{\{\omega_1\}} \oplus \dots \oplus \chi_{\{\omega_n\}} = 1$$

we see that $\{\chi_{\{\omega\}} : \omega \in \Omega\}$ is the only context in \mathcal{E} . Also every $f \in \mathcal{E}$ has the form $f = \sum \lambda_i \chi_{\{\omega_i\}}$, $\lambda_i \in [0, 1]$ so \mathcal{E} is spectral. Conversely, suppose \mathcal{E} has a single context $\{a_1, \dots, a_n\}$ and \mathcal{E} is spectral. Let (Ω, \mathcal{A}) be a finite measurable space with $\Omega = \{\omega_1, \dots, \omega_n\}$. For $b = \lambda_1 a_1 \oplus \dots \oplus \lambda_n a_n \in \mathcal{E}$, define $J(b) \in \mathcal{E}(\Omega, \mathcal{A})$ by $J(b)(\omega_i) = \lambda_i$. Then $J: \mathcal{E} \rightarrow \mathcal{E}(\Omega, \mathcal{A})$ is bijective, $J(1) = 1$, $J(\lambda b) = \lambda J(b)$. If $b \perp c$ with $c = \mu_1 a_1 \oplus \dots \oplus \mu_n a_n \in \mathcal{E}$ we have

$$b \oplus c = (\lambda_1 + \mu_1) a_1 \oplus \dots \oplus (\lambda_n + \mu_n) a_n$$

and

$$J(b \oplus c)(\omega_i) = \lambda_i + \mu_i = J(b)(\omega_i) + J(c)(\omega_i)$$

$i = 1, \dots, n$, so $J(b \oplus c) = J(b) + J(c)$. Finally, if $J(b) \perp J(c)$ we have that $J(b)(\omega_i) + J(c)(\omega_i) \leq 1$, $i = 1, \dots, n$. Hence, $\lambda_i + \mu_i \leq 1$, $i = 1, \dots, n$ so $b \perp c$. We conclude that J is an affine isomorphism. \square

If $\mathcal{A} = \{a_i : i = 1, \dots, n\}$ is a context on the convex effect algebra \mathcal{E} , we form the set of states $\widehat{\mathcal{A}} = \{\widehat{a}_i : i = 1, \dots, n\}$. It follows from Theorem 2.3

that $\widehat{\mathcal{A}}$ can be thought of as a set of positive, unital, linear functionals on (V, K) . We now construct the complex linear space

$$\mathcal{H}(\mathcal{A}) = \left\{ \sum_{i=1}^n \alpha_i \widehat{a}_i : \alpha_i \in \mathbb{C} \right\}$$

For $x, y \in \mathcal{H}(\mathcal{A})$ with $x = \sum \alpha_i \widehat{a}_i$, $y = \sum \beta_i \widehat{a}_i$ we define the inner product $\langle x, y \rangle = \sum \bar{\alpha}_i \beta_i$. Thus, $\mathcal{H}(\mathcal{A})$ is a complex Hilbert space that we call the *state space for the context* \mathcal{A} . Of course, $\mathcal{H}(\mathcal{A})$ is n -dimensional with orthonormal basis $\widehat{\mathcal{A}} = \{\widehat{a}_i : i = 1, \dots, n\}$.

Now $\widehat{\mathcal{A}}$ naturally generates a real linear space of linear functions on (V, K) so why did we choose $\mathcal{H}(\mathcal{A})$ to be a complex rather than a real space? One reason is that we need to describe a dynamics for states in $\mathcal{H}(\mathcal{A})$. Since a dynamics must preserve norms and orthogonality, it is represented by a continuous group of unitary operators $U_t : \mathcal{H}(\mathcal{A}) \rightarrow \mathcal{H}(\mathcal{A})$, $t \in \mathbb{R}$, for context \mathcal{A} . It is assumed that $U_{t_1+t_2} = U_{t_1}U_{t_2}$ so the operators U_t commute. Thus, they are simultaneously diagonalizable and hence have common eigenvectors $\phi_i \in \mathcal{H}(\mathcal{A})$ so that

$$U_t \phi_i = \alpha_i(t) \phi_i$$

$i = 1, \dots, n$. If $\mathcal{H}(\mathcal{A})$ is a real Hilbert space, then $\alpha_i(t) \in \mathbb{R}$ and since U_t is unitary $\alpha_i(t) = \pm 1$. But then U_t cannot be continuous unless $U_t = I$ for all t . In the complex case, $\alpha_i(t) = e^{i\theta_i(t)}$, $\theta_i(t) \in \mathbb{R}$, which is continuous if $\theta_i(t)$ is continuous, $i = 1, 2, \dots, n$. In fact, we have $\alpha_i(t) = e^{i\theta_i t}$. In this case, denoting the one-dimensional projection onto \widehat{a}_i , by $P(\widehat{a}_i)$, we have the Hamiltonian $L = \sum \theta_i P(\widehat{a}_i)$ so that $U_t = e^{iLt}$. There are also other groups such as rotations that require unitary representations on a complex Hilbert space of states.

Notice that the one-dimensional effects are atoms among the sharp effects. Indeed, if a is one-dimensional and $b \in \mathcal{E}$ with $0 < b < a$, then $b = \lambda a$, $\lambda \in (0, 1)$. If $\mu < \lambda$, $\mu < 1 - \lambda$, then $\mu a < \lambda a$ and since

$$(u + \lambda)a < (\mu + \lambda)u < u$$

we have that

$$\mu a < u - \lambda a = (\lambda a)'$$

Hence,

$$b \wedge b' = (\lambda a) \wedge (\lambda a)' \neq 0 \quad \text{whether or not it exists.}$$

Since $b \notin S(\mathcal{E})$, there are no nonzero sharp elements strictly below a so a is an atom in $S(\mathcal{E})$.

If $\mathcal{A} = \{a_i: i = 1, \dots, n\}$ is a context and $b \in \mathcal{E}$ define the linear operator $b_{\mathcal{A}}$ on $\mathcal{H}(\mathcal{A})$ by

$$b_{\mathcal{A}} \sum \alpha_i \widehat{a}_i = \sum \alpha_i \widehat{a}_i(b) \widehat{a}_i$$

Lemma 3.2. *The map $J: \mathcal{E} \rightarrow \mathcal{E}(\mathcal{H}(\mathcal{A}))$ given by $J(b) = b_{\mathcal{A}}$ is an affine morphism.*

Proof. Since $J(b)\widehat{a}_i = \widehat{a}_i(b)\widehat{a}_i$, we see that $J(b)$ is a positive linear operator with eigenvalues $0 \leq \widehat{a}_i(b) \leq 1$ and corresponding eigenvectors \widehat{a}_i . Thus $J(b) \in \mathcal{E}(\mathcal{H}(\mathcal{A}))$. Also, $J(\theta) = 0$, $J(u) = I$ and we have

$$\begin{aligned} J(b \oplus c) \sum \alpha_i \widehat{a}_i &= \sum \alpha_i \widehat{a}_i(b \oplus c) \widehat{a}_i = \sum \alpha_i [\widehat{a}_i(b) + \widehat{a}_i(c)] \widehat{a}_i \\ &= (J(b) + J(c)) \sum \alpha_i \widehat{a}_i \end{aligned}$$

Hence, $J(b \oplus c) = J(b) + J(c)$ so J is a morphism. Since $J(\lambda b) = \lambda J(b)$, $\lambda \in [0, 1]$, J is affine. \square

The affine morphism $J(b) = b_{\mathcal{A}}$ of Lemma 3.2 gives a representation of \mathcal{E} into the Hilbertian effect algebra $\mathcal{E}(\mathcal{H}(\mathcal{A}))$. However, J need not be injective or surjective and J need not preserve sharpness. Moreover, all the $J(b)$, $b \in \mathcal{E}$, commute so they do not convey quantum interference. One can say that J gives a distorted partial view of \mathcal{E} . The reason for this is that we are only employing a single context \mathcal{A} . Unlike a classical effect algebra with only one context, a quantum effect algebra has many contexts. Each gives a partial view and in order to obtain a total view, they must all be considered.

In order to consider several contexts together, we introduce a method to compare them. A collection of contexts $\Gamma = \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots\}$ is *comparable* if for every $\mathcal{A}, \mathcal{B} \in \Gamma$ there exists a unitary transformation $U_{\mathcal{AB}}: \mathcal{H}(\mathcal{A}) \rightarrow \mathcal{H}(\mathcal{B})$ such that $U_{\mathcal{AA}} = I$, $U_{\mathcal{AB}} = U_{\mathcal{BA}}^*$ and if $a \in \mathcal{A}$, $c \in \mathcal{C}$ then

$$|\langle U_{\mathcal{AB}} \widehat{a}, U_{\mathcal{CB}} \widehat{c} \rangle|^2 = \widehat{a}(c) \tag{3.2}$$

We call $\widehat{a}(c)$ in (3.2) the *transition probability* from a to c . In particular, we can compare the elements of \mathcal{A} and \mathcal{B} together by

$$\left| \langle U_{\mathcal{AB}} \widehat{a}, \widehat{b} \rangle \right|^2 = \left| \langle U_{\mathcal{AB}} \widehat{a}, U_{\mathcal{BB}} \widehat{b} \rangle \right|^2 = \widehat{a}(b)$$

Notice that a unit vector ϕ in $\mathcal{H}(\mathcal{A})$ can be considered as a vector in the Hilbert space $\mathcal{H}(\mathcal{A})$ or as a state on \mathcal{E} , where the state corresponding to ϕ is $\widehat{\phi}$ given by

$$\widehat{\phi}(b) = \langle \widehat{\phi}, b_{\mathcal{A}} \widehat{\phi} \rangle$$

This is consistent with $\widehat{a}(b) = \langle \widehat{a}, b_{\mathcal{A}} \widehat{a} \rangle$ for all $a \in \mathcal{A}$. A collection of contexts $\Gamma = \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \dots\}$ is *complete* if they are comparable and if for any $\mathcal{B} \in \Gamma$ and any unit vector $\phi \in \mathcal{H}(\mathcal{B})$ there exists an $\mathcal{A} \in \Gamma$ and an $a \in \mathcal{A}$ such that $U_{\mathcal{A}\mathcal{B}}(\widehat{a}) = \phi$.

As an example, in the classical case there is only one context \mathcal{A} . Then \mathcal{A} is comparable with $U_{\mathcal{A}\mathcal{A}} = I$. But \mathcal{A} is not complete unless $\mathcal{A} = \{1\}$ and $\mathcal{H}(\mathcal{A}) = \mathbb{C}$; that is, $\mathcal{H}(\mathcal{A})$ is one-dimensional. Indeed, suppose \mathcal{A} is complete and $\mathcal{A} = \{a_1, \dots, a_n\}$. If $\phi \in \mathcal{H}(\mathcal{A})$ with $\phi = \frac{1}{\sqrt{2}}(\widehat{a}_1 + \widehat{a}_2)$ then there exists $a_j \in \mathcal{A}$ such that

$$\widehat{a}_j = U_{\mathcal{A}\mathcal{A}}(\widehat{a}_j) = \phi$$

But this is impossible unless $\mathcal{A} = \{a_j\}$ and so $a_j = 1$. We conclude that \mathcal{E} is affinely isomorphic to $[0, 1] \subseteq \mathbb{R}$ and $\mathcal{H}(\mathcal{A}) = \mathbb{C}$.

Theorem 3.3. *Let \mathcal{E} be a finite dimensional convex effect algebra. Then \mathcal{E} is affinely isomorphic to a Hilbertian effect algebra if and only if its set of contexts is complete and \mathcal{E} is spectral.*

Proof. To prove necessity we can assume that $\mathcal{E} = \mathcal{E}(H)$ for some Hilbert space H . The elements of $S_1(\mathcal{E})$ become one-dimensional projections and it follows from the spectral theorem that $\mathcal{E}(H)$ is spectral. Since \mathcal{E} is finite dimensional, every context has the form $\mathcal{A} = \{a_1, \dots, a_n\}$ where $a_i \in S_1(\mathcal{E})$. Thus, a_i is a projection onto the subspace of H spanned by a unit vector ϕ_i where $\{\phi_1, \dots, \phi_n\}$ is an orthonormal basis for H . We can then identify $\widehat{\mathcal{A}}$ with this basis. It is now straightforward to show that the set of contexts of \mathcal{E} is complete. Conversely, suppose that the set of contexts for \mathcal{E} is complete and \mathcal{E} is spectral. Letting \mathcal{B} be a fixed context we shall show that \mathcal{E} is affinely isomorphic to $\mathcal{E}(\mathcal{H}(\mathcal{B}))$. If $b \in \mathcal{E}$, since \mathcal{E} is spectral, we have that $b = \sum \lambda_i a_i$, $\lambda_i \in [0, 1]$ for some context $\mathcal{A} = \{a_i : i = 1, \dots, n\}$. Now $\{\widehat{a}_i : i = 1, \dots, n\}$ forms an orthonormal basis for $\mathcal{H}(\mathcal{A})$ and since $U_{\mathcal{A}\mathcal{B}}$ is unitary, $\{U_{\mathcal{A}\mathcal{B}}(\widehat{a}_i) : i = 1, \dots, n\}$ is an orthonormal basis for $\mathcal{H}(\mathcal{B})$. Let $P(a_i)$ be the one-dimensional projection onto the subspace of $\mathcal{H}(\mathcal{B})$ spanned by $U_{\mathcal{A}\mathcal{B}}(\widehat{a}_i)$. Define $J : \mathcal{E} \rightarrow \mathcal{E}(\mathcal{H}(\mathcal{B}))$ by $J(b) = \sum \lambda_i P(a_i)$. To show that J is additive, suppose $c \in \mathcal{E}$ with $c \perp b$ and $c = \sum \mu_i c_i$ for some context

$\mathcal{C} = \{c_i\}$. Since \mathcal{E} is spectral, $b \oplus c = \sum \gamma_i d_i$ for some context $\mathcal{D} = \{d_i\}$. We then have that

$$\sum \gamma_i d_i = \sum \lambda_i a_i + \sum \mu_i c_i \quad (3.3)$$

If ϕ is a unit vector in $\mathcal{H}(\mathcal{B})$, there exists a $d \in S_1(\mathcal{E})$ and a context \mathcal{F} with $d \in \mathcal{F}$ and $U_{\mathcal{F}\mathcal{B}}\widehat{d} = \phi$. Applying \widehat{d} to (3.3) gives

$$\sum \gamma_i \widehat{d}(d_i) = \sum \lambda_i \widehat{d}(a_i) + \sum \mu_i \widehat{d}(c_i) \quad (3.4)$$

Since the contexts are comparable, applying (3.4) and (3.2) gives

$$\sum \gamma_i \left| \left\langle U_{\mathcal{F}\mathcal{B}}\widehat{d}, U_{\mathcal{D}\mathcal{B}}\widehat{d}_i \right\rangle \right|^2 = \sum \lambda_i \left| \left\langle U_{\mathcal{F}\mathcal{B}}\widehat{d}, U_{\mathcal{A}\mathcal{B}}\widehat{a}_i \right\rangle \right|^2 + \sum \mu_i \left| \left\langle U_{\mathcal{F}\mathcal{B}}\widehat{d}, U_{\mathcal{C}\mathcal{B}}\widehat{c}_i \right\rangle \right|^2$$

Hence,

$$\begin{aligned} \sum \gamma_i \left\langle U_{\mathcal{F}\mathcal{B}}\widehat{d}, P(d_i)U_{\mathcal{F}\mathcal{B}}\widehat{d} \right\rangle &= \sum \lambda_i \left\langle U_{\mathcal{F}\mathcal{B}}\widehat{d}, P(a_i)U_{\mathcal{F}\mathcal{B}}\widehat{d} \right\rangle \\ &\quad + \sum \mu_i \left\langle U_{\mathcal{F}\mathcal{B}}\widehat{d}, P(c_i)U_{\mathcal{F}\mathcal{B}}\widehat{d} \right\rangle \end{aligned}$$

which gives

$$\langle \phi, J(b \oplus c)\phi \rangle = \langle \phi, J(b)\phi \rangle + \langle \phi, J(c)\phi \rangle$$

Since the pure states of $\mathcal{H}(\mathcal{B})$ are separating we conclude that $J(b \oplus c) = J(b) + J(c)$ so J is additive. To show that J is affined, let $b = \sum \lambda_i a_i$. Then $\lambda b = \sum \lambda \lambda_i a_i$, $\lambda \in [0, 1]$ and we obtain

$$J(\lambda b) = \sum \lambda \lambda_i P(a_i) = \lambda J(b)$$

It is clear that J has a unique linear extension to V . We leave it to the reader to show that J is injective. To show that J is surjective, let P_ϕ be a one-dimensional projection onto the subspace of $\mathcal{H}(\mathcal{B})$ spanned by the unit vector ϕ . By completeness, there is an $a \in S_1(\mathcal{E})$ with $J(a) = P_\phi$. If $A \in \mathcal{E}(\mathcal{H}(\mathcal{B}))$ has spectral decomposition $A = \sum \lambda_i P_{\phi_i}$ we have $a_i \in S_1(\mathcal{E})$ with $J(a_i) = P_{\phi_i}$ and since J is linear we obtain

$$J\left(\sum \lambda_i a_i\right) = \sum \lambda_i J(a_i) = A$$

Moreover, $a_1 \oplus \cdots \oplus a_n = u$ because

$$J(a_1 \oplus \cdots \oplus a_n) = P_{\phi_1} + \cdots + P_{\phi_n} = I = J(u)$$

and J is injective. Hence, $\sum \lambda_i a_i \in \mathcal{E}$ so J is surjective. \square

It follows that if \mathcal{E} satisfies the conditions of Theorem 3.3, then the transition probability has the usual form $\widehat{a}(b) = \left| \langle \widehat{a}, \widehat{b} \rangle \right|^2$. We then have the symmetry relation $\widehat{a}(b) = \widehat{b}(a)$ which need not hold for a general \mathcal{E} .

We have seen in Theorems 3.1 and 3.3 that classical convex effect algebras have a single context, while Hilbertian convex effect algebras have an uncountable complete set of contexts. Are there convex effect algebras between these two cases? That is, are there convex effect algebras with only a finite number greater than one, of contexts? We conjecture that the answer is no. Although we have not been able to prove this conjecture in general, we can show it holds for the first few cases. First notice that if $\mathcal{E} \neq [0, 1] \subseteq \mathbb{R}$ then a context in \mathcal{E} must have at least two distinct elements. Indeed, if $\{a\}$ is a context, then $a = 1$. If $b \in \mathcal{E}$, then $b \leq 1$ so $b = \lambda 1$ for some $\lambda \in [0, 1] \subseteq \mathbb{R}$. Hence, $\mathcal{E} = [0, 1] \subseteq \mathbb{R}$ which is a contradiction.

Theorem 3.4. *A spectral convex effect algebra \mathcal{E} does not have exactly two or three mutually disjoint contexts.*

Proof. Suppose that \mathcal{E} has exactly two disjoint contexts $\mathcal{A} = \{a_1, \dots, a_n\}$, $\mathcal{B} = \{b_1, \dots, b_m\}$ with $n, m \geq 2$. Then

$$c = \frac{1}{2}a_1 + \frac{1}{2}b_1 \leq \frac{1}{2}1 + \frac{1}{2}1 = 1$$

so $c \in \mathcal{E}$. Since \mathcal{E} is spectral we can assume without loss of generality that $c = \sum \lambda_i a_i$, $\lambda_i \in [0, 1]$. Now

$$\widehat{a}_1(c) = \frac{1}{2} + \frac{1}{2}\widehat{a}_1(b_1) = \lambda_1$$

so we have that $\lambda_1 \leq 1/2$. Hence,

$$\frac{1}{2}b_1 = \left(\lambda_1 - \frac{1}{2}\right) a_1 \oplus \lambda_2 a_2 \oplus \dots \oplus \lambda_n a_n \quad (3.5)$$

where at least one of the coefficients $\lambda_1 - \frac{1}{2}, \lambda_2, \dots, \lambda_n$ is nonzero. If $\lambda_j \neq 0$, $j \in \{2, \dots, n\}$, then $\lambda_j a_j \leq \frac{1}{2}b_1$. Since $2\lambda_j a_j \leq b_1$ and $b_1 \in S_1(\mathcal{E})$ we conclude that $2\lambda_j a_j = \mu b_1$ for some $\mu \in [0, 1]$. Let $\alpha = 2\lambda_j/\mu$ so $b_1 = \alpha a_j$. If $\alpha < 1$, letting $\beta = \min(\alpha, 1 - \alpha)$ we obtain

$$\beta a_j \leq \alpha a_j = b_1$$

and

$$\beta a_j \leq (1 - \alpha)a_j = a_j - \alpha a_j \leq 1 - \alpha a_j = b'_1$$

Since $\beta a_j \neq 0$, this contradicts the fact that $b_1 \in S(\mathcal{E})$. If $\alpha > 1$ we get a similar contradiction. Hence, $\alpha = 1$ and $b_1 = a_j$ which contradicts the fact that $\mathcal{A} \cap \mathcal{B} = \emptyset$. If $\lambda_1 \neq 1/2$, we obtain a similar contradiction. We conclude that \mathcal{E} does not contain two disjoint contexts.

Next suppose that \mathcal{E} has exactly three mutually disjoint contexts $\mathcal{A} = \{a_1, \dots, a_n\}$, $\mathcal{B} = \{b_1, \dots, b_m\}$, $\mathcal{C} = \{c_1, \dots, c_p\}$ with $n, m, p \geq 2$. Then $d = \frac{1}{3}a_1 + \frac{1}{3}b_1 + \frac{1}{3}c_1 \in \mathcal{E}$ and as before we can assume that $d = \sum \lambda_i a_i$, $\lambda_i \in [0, 1]$. Since

$$\widehat{a}_1(d) = \frac{1}{3} + \frac{1}{3}\widehat{a}_1(b_1) + \frac{1}{3}\widehat{a}_1(c_1) = \lambda_1$$

we have that $\lambda_1 \geq \frac{1}{3}$. Hence,

$$\frac{1}{3}b_1 + \frac{1}{3}c_1 = (\lambda_1 - \frac{1}{3})a_1 \oplus \lambda_2 a_2 \oplus \dots \oplus \lambda_n a_n \quad (3.6)$$

Now $e = \frac{1}{2}b_1 + \frac{1}{3}c_1 \in \mathcal{E}$ but e cannot be spectral relative to \mathcal{B} or \mathcal{C} because we would obtain an equation like (3.5) which we saw in the previous paragraph leads to a contradiction. Hence,

$$\frac{1}{2}b_1 + \frac{1}{3}c_1 = \mu_1 a_1 \oplus \dots \oplus \mu_n a_n \quad (3.7)$$

with $\mu_i \in [0, 1]$. Since

$$\mu_1 a_1 + \dots + \mu_n a_n = \frac{1}{2}b_1 + \frac{1}{3}c_1 \geq \frac{1}{3}b_1 + \frac{1}{3}c_1 = (\lambda_1 - \frac{1}{3})a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n$$

we have that

$$\mu_1 = \widehat{a}_1(\mu_1 a_1) \geq \widehat{a}_1 \left[(\lambda_1 - \frac{1}{3}) a_1 \right] = \lambda_1 - \frac{1}{3}$$

and similarly $\mu_j \geq \lambda_j$, $j = 2, \dots, n$. Subtracting (3.6) from (3.7) gives

$$\begin{aligned} \frac{1}{6}b_1 &= \left(\frac{1}{2}b_1 + \frac{1}{3}c_1 \right) - \left(\frac{1}{3}b_1 + \frac{1}{3}c_1 \right) \\ &= \left[\mu_1 - \left(\lambda_1 - \frac{1}{3} \right) \right] a_1 + (\mu_2 - \lambda_2) a_2 + \dots + (\mu_n - \lambda_n) a_n \end{aligned} \quad (3.8)$$

As with (3.5) in the previous paragraph, we obtain a contradiction. We conclude that \mathcal{E} does not have three mutually disjoint contexts. \square

4 Convex Sequential Effect Algebras

A convex effect algebra describes the parallel sum $a \oplus b$ and the attenuated scalar product λa for effects. However, there is an important missing ingredient which is the sequential product $a \circ b$. The product $a \circ b$ describes an

experiment in which a is measured first and b is measured second. We might say that $a \circ b$ is a measurement of the effect b conditioned by a previous measurement of the effect a . Such a temporal or sequential order does not seem to be considered in classical probability theory. For example, if A and B are events in a classical probability space then their intersection $A \cap B$ represents the event that A and B both occur and no consideration is taken for which occurs first. A little more subtle is the conditional probability of B given A described by $P(B|A) = P(A \cap B)/P(A)$. It may appear that A occurs first but we have that

$$P(A)P(B|A) = P(B)P(A|B)$$

and if it happens that $P(A) = P(B)$ then $P(B|A) = P(A|B)$.

In quantum mechanics $a \circ b$ is useful for describing quantum interference. Because of the sequential order for $a \circ b$, since a is measured first, a may interfere with the b measurement and since b is measured second, b will never interfere with the a measurement. If $a \circ b = b \circ a$ we write $a|b$ and say that a and b *do not interfere*. We now present our general definition.

A *convex sequential effect algebra* (convex SEA) is an algebraic system $(\mathcal{E}, 0, 1, \oplus, \circ)$ where $(\mathcal{E}, 0, 1, \oplus)$ is an effect algebra and $\circ: \mathcal{E} \times \mathcal{E} \rightarrow \mathcal{E}$ is a binary operation satisfying:

- (S1) $b \mapsto a \circ b$ is additive for all $a \in \mathcal{E}$.
- (S2) $1 \circ a = a$ for all $a \in \mathcal{E}$.
- (S3) If $a \circ b = 0$, then $a|b$.
- (S4) If $a|b$, then $a|b'$ and $a \circ (b \circ c) = (a \circ b) \circ c$ for all $c \in \mathcal{E}$.
- (S5) If $c|a$ and $c|b$, then $c|a \circ b$ and $c|(a \oplus b)$ whenever $a \perp b$.
- (S6) For all $\lambda \in [0, 1] \subseteq \mathbb{R}$, $a, b \in \mathcal{E}$, we have that $(\lambda a) \circ b = a \circ (\lambda b) = \lambda(a \circ b)$.

The next theorem which is proved in [8] shows that the sequential product has desirable properties.

Theorem 4.1. (i) $a \circ b \leq a$ for all $a, b \in \mathcal{E}$. (ii) If $a \leq b$, then $c \circ a \leq c \circ b$ for all $c \in \mathcal{E}$. (iii) $a \in S(\mathcal{E})$ if and only if $a \circ a = a$. (iv) For $a \in \mathcal{E}$, $b \in S(\mathcal{E})$, $a \circ b = 0$ if and only if $a \perp b$. (v) For $a \in \mathcal{E}$, $b \in S(\mathcal{E})$, $a \leq b$ if and only if $a \circ b = b \circ a = a$ and $b \leq a$ if and only if $a \circ b = b \circ a = b$.

A classical effect algebra $\mathcal{E}(\Omega, \mathcal{A})$ is a convex SEA under the usual function product $f \circ g = fg$. It is shown in [8] that a Hilbertian effect algebra $\mathcal{E}(H)$ is a convex SEA under the product

$$A \circ B = A^{1/2}BA^{1/2}$$

where $A^{1/2}$ is the unique positive square root of A . It is shown in [8] that $A|B$ if and only if $AB = BA$. Of course $\mathcal{E}(\Omega, \mathcal{A})$ is commutative while $\mathcal{E}(H)$ is not where *commutative* means $a \circ b = b \circ a$ for all a, b .

A convex SEA has stronger properties than a convex effect algebra. We begin to illustrate this in the following lemma.

Lemma 4.2. *Let \mathcal{E} be a convex SEA. (i) For $a, b \in S_1(\mathcal{E})$ we have $a|b$ if and only if $a = b$ or $a \circ b = 0$. (ii) For two contexts $\mathcal{A} = \{a_1, \dots, a_n\}$, $\mathcal{B} = \{b_1, \dots, b_m\}$ in \mathcal{E} we have $a_i|b_j$, $i = 1, \dots, n$, $j = 1, \dots, m$, if and only if $\mathcal{A} = \mathcal{B}$.*

Proof. (i) If $a = b$ or $a \circ b = 0$, then $a|b$ by Theorem 4.1(iv). Conversely, suppose that $a|b$. By Theorem 4.1(i) we have that $a \circ b \leq a, b$ and hence $a \circ b = \lambda a$ and $a \circ b = \mu b$ for some $\lambda, \mu \in [0, 1] \subseteq \mathbb{R}$. If $\lambda = 0$, then $a \circ b = 0$. Otherwise, we have that $a = \frac{\mu}{\lambda}b$ and squaring gives

$$a = \left(\frac{\mu}{\lambda}\right)^2 b = \frac{\mu}{\lambda}b$$

Since $\frac{\mu}{\lambda} \neq 0$ we conclude that $\mu = \lambda$. Hence, $a = b$.

(ii) Since $a_i \perp a_j$ for $i \neq j$, by Theorem 4.1(iv) $a_i \circ a_j = 0$ for $i \neq j$. Hence, $a_i|a_j$, $i, j = 1, \dots, n$, by (S3). We conclude that if $\mathcal{A} = \mathcal{B}$ then $a_i|b_j$. Conversely, suppose $a_i|b_j$, $i = 1, \dots, n$, $j = 1, \dots, m$. Since $b_1 \oplus \dots \oplus b_m = 1$, by (S1) we have

$$a_i = a_i \circ b_1 \oplus \dots \oplus a_i \circ b_m$$

If $a_i \circ b_j = 0$ for $j = 1, \dots, m$, then $a_i = 0$ which is a contradiction. Hence, $a_i \circ b_j \neq 0$ for some $j = 1, \dots, m$. By (i) of this lemma, $a_i = b_j$. It follows that $m = n$ and $\mathcal{A} = \mathcal{B}$. \square

In the sequel, we shall assume that \mathcal{E} is a finite dimensional convex SEA. If \mathcal{E} is commutative, then \mathcal{E} is classical. Indeed, it follows from Lemma 4.2(ii) that \mathcal{E} possesses exactly one context $\mathcal{A} = \{a_1, \dots, a_n\}$. Moreover, if $b \in \mathcal{E}$ then by Theorem 4.1(i) we have

$$b = b \circ a_1 \oplus \dots \oplus b \circ a_n = a_1 \circ b \oplus \dots \oplus a_n \circ b = \lambda_1 a_1 \oplus \dots \oplus \lambda_n a_n$$

for $\lambda_i \in [0, 1]$, $i = 1, \dots, n$. It follows that \mathcal{E} is spectral so by Theorem 3.1, \mathcal{E} is classical as an effect algebra. To show that \mathcal{E} is classical as a SEA, consider the isomorphism $J: \mathcal{E} \rightarrow \mathcal{E}(\Omega, \mathcal{A})$ of Theorem 3.1. If $b \in \mathcal{E}$ is given as before we have $J(b)(\omega_i) = \lambda_i$, $i = 1, \dots, n$. If $c \in \mathcal{E}$ with $c = \mu_1 a_1 \oplus \dots \oplus \mu_n a_n$, then

$$J(b \circ c)(\omega_i) = \lambda_i \mu_i = J(b)(\omega_i) J(c)(\omega_i) = J(b) J(c)(\omega_i)$$

Hence, J is a SEA isomorphism so \mathcal{E} is a classical SEA.

For $\mathcal{A} = \{a_1, \dots, a_n\}$ with $a_i \in S(\mathcal{E})$ and $\sum a_i = 1$, we say that $a \in \mathcal{E}$ is \mathcal{A} -measurable if

$$a = \sum_{i=1}^n \lambda_i a_i \tag{4.1}$$

It follows from Theorem 4.1(iv) that $a_i \circ a_j = 0$ for $i \neq j$. It also follows from Theorem 4.1(iv) that if $a, b \in S(\mathcal{E})$ with $a \perp b$, then $a \oplus b \in S(\mathcal{E})$. Hence, we can and will assume without loss of generality that $\lambda_i \neq \lambda_j$, $i \neq j$, in (4.1). For $a \in \mathcal{E}$, we define $a^0 = 1$ and

$$a^i = a \circ a \circ \dots \circ a \quad (i \text{ factors})$$

An effect $b \in \mathcal{E}$ is a *function of* $a \in \mathcal{E}$ if

$$b = \sum_{i=1}^n \alpha_i a^i, \quad \alpha_i \in \mathbb{R}$$

Notice that some of the α_i can be negative and we can even have $\alpha_i > 1$ or $\alpha_i < -1$, but the sum is still in \mathcal{E} . The individual terms in the sum can be thought of being in the encompassing ordered vector space (V, K) . For example, $b' = 1 - b$ and

$$(b')^2 = (1 - b) \circ (1 - b) = 1 - 2b + b^2$$

so b' and $(b')^2$ are functions of b . For another example, if $a \perp a$ then $b = a \oplus a = 2a$ so b is a function of a and

$$b' = 1 - b = 1 - 2a$$

is again a function of a . Notice that if $a|b$, then any function of a commutes with any function of b .

If b_1 and b_2 are functions of a , then $b_1 \circ b_2 = b_2 \circ b_1$ is a function of a and $b_1 \oplus b_2$ is a function of a whenever $b_1 \perp b_2$. Also, $0, 1$ and λa , $\lambda \in [0, 1] \subseteq \mathbb{R}$

are functions of a . It follows that the functions of a form a commutative sub-convex SEA of \mathcal{E} . Suppose $a = \lambda_1 a_1 + \lambda_2 a_2$ is $\{a_1, a_2\}$ -measurable so that $a_1, a_2 \in S(\mathcal{E})$, $a_1 + a_2 = 1$, and $\lambda_1 \neq \lambda_2$. We now show that a_1 and a_2 are functions of a . Since $a_1 = 1 - a_2$ we have that

$$a = \lambda_1(1 - a_2) + \lambda_2 a_2 = \lambda_1 1 + (\lambda_2 - \lambda_1)a_2$$

Hence,

$$a_2 = \frac{a - \lambda_1 1}{\lambda_2 - \lambda_1}$$

and

$$a_1 = 1 - a_2 = \frac{\lambda_2 1 - a}{\lambda_2 - \lambda_1}$$

so a_1 and a_2 are functions of a . We now generalize this result.

Theorem 4.3. (i) If $a = \sum \lambda_i a_i$ is $\{a_1, \dots, a_n\}$ -measurable, then a_i is a function of a , $i = 1, \dots, n$. (ii) Also, if b is $\{b_i\}$ -measurable and $a|b$ then $a \circ b$ is $\{a_i \circ b_j\}$ -measurable and $a \oplus b$ is $\{a_i \circ b_j\}$ -measurable whenever $a \perp b$.

Proof. (i) If $a = \sum_{i=1}^n \lambda_i a_i$ we obtain the system of equations

$$\begin{aligned} a_1 + a_2 + \dots + a_n &= 1 \\ \lambda_1 a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n &= a \\ \lambda_1^2 a_1 + \lambda_2^2 a_2 + \dots + \lambda_n^2 a_n &= a^2 \\ &\vdots \\ \lambda_1^{n-1} a_1 + \lambda_2^{n-1} a_2 + \dots + \lambda_n^{n-1} a_n &= a^{n-1} \end{aligned}$$

the determinant for this system is the Vandermonde determinant

$$\begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \lambda_1^2 & \lambda_2^2 & \dots & \lambda_n^2 \\ \vdots & & & \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{vmatrix} = (-1)^{n(n-1)/2} \prod_{i < j} (\lambda_i - \lambda_j)$$

Since $\lambda_i \neq \lambda_j$, $i \neq j$, the determinant is nonzero. Hence, there is a unique solution to this system of equations for the unknowns a_i , $i = 1, \dots, n$. We conclude that a_i is a function of a , $i = 1, \dots, n$.

(ii) Suppose a and b are $\{a_i\}$ and $\{b_i\}$ -measurable and $a|b$. Then by (i) of this theorem we have $a = \sum \lambda_i a_i$, $b = \sum \mu_i b_i$ where the a_i are functions of a and the b_i are functions of b . Since $a|b$, any function of a commutes with any function of b . Hence, $a_i|b_j$ for all i, j . But then

$$a \circ b = \sum \lambda_i \mu_i a_i \circ b_j$$

where $\sum a_i \circ b_j = 1$ and $a_i \circ b_j \in S(\mathcal{E})$ by Theorem 4.1(iii). Hence, $a \circ b$ is $\{a_i \circ b_j\}$ -measurable. If we also have $a \perp b$, then

$$a \oplus b = \sum \lambda_i a_i + \sum \mu_j b_j = \sum_{i,j} \lambda_i a_i \circ b_j + \sum_{i,j} \mu_j a_i \circ b_j$$

Hence, $a \oplus b$ is $\{a_i \circ b_j\}$ -measurable. \square

We now apply Theorem 4.3 to obtain a strengthening of Theorem 3.4 for a convex SEA.

Corollary 4.4. *A spectral convex SEA \mathcal{E} does not have exactly two, three or four mutually disjoint contexts.*

Proof. Theorem 3.4 treats the two and three mutually disjoint contexts cases. Now suppose \mathcal{E} possesses exactly four mutually disjoint contexts $\mathcal{A} = \{a_i\}$, $\mathcal{B} = \{b_i\}$, $\mathcal{C} = \{c_i\}$ and $\mathcal{D} = \{d_i\}$. As in Theorem 3.4 we have that

$$e = \frac{1}{4} a_1 + \frac{1}{4} b_1 + \frac{1}{4} c_1 + \frac{1}{4} d_1 = \mathcal{E}$$

and we can assume without loss of generality that $e = \sum \lambda_i a_i$, $\lambda_i \in [0, 1]$ which gives

$$\frac{1}{4} b_1 + \frac{1}{4} c_1 + \frac{1}{4} d_1 = \left(\lambda_1 - \frac{1}{4}\right) a_1 + \lambda_2 a_2 + \dots + \lambda_n a_n \quad (4.2)$$

Now $\frac{1}{4} c_1 + \frac{1}{4} d_1$ cannot be spectral relative to \mathcal{C} , \mathcal{D} or \mathcal{A} because we would obtain a contradiction as with (3.5) in Theorem 3.4. We therefore have that

$$\frac{1}{4} c_1 + \frac{1}{4} d_1 = \sum \mu_i b_i$$

so by (4.2) we obtain

$$b = \left(\mu_1 + \frac{1}{4}\right) b_1 + \mu_2 b_2 + \cdots + \mu_m b_m = \left(\lambda_1 - \frac{1}{4}\right) a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n \quad (4.3)$$

By considering the coefficients in (4.3) that are different we can apply Theorem 4.3 to conclude that $b_i|a_j$ and $b_i \circ a_j \neq 0$ for some i and j . It follows from Lemma 4.2 that $b_i = a_j$. This contradicts the fact that $\mathcal{A} \cap \mathcal{B} = \emptyset$. \square

If $b = \sum \lambda_i a_i$ for a context $\mathcal{A} = \{a_i: i = 1, \dots, n\}$, then b is \mathcal{A} -measurable and the results of Theorem 4.3 hold. Moreover, b is \mathcal{A} -measurable if and only if $b|a_i$, $i = 1, \dots, n$. Indeed, if $b = \sum \lambda_i a_i$ then clearly, $b|a_i$, $i = 1, \dots, n$. Conversely, if $b|a_i$, $i = 1, \dots, n$, then

$$b = \sum b \circ a_i = \sum a_i \circ b = \sum \lambda_i a_i$$

We now discuss Theorems 3.1 and 3.3 in the case of a convex SEA.

Let \mathcal{E}, \mathcal{F} be convex SEA's with sequential products $a \circ b$ and $a \cdot b$, respectively. A SEA *isomorphism* for \mathcal{E} to \mathcal{F} is a convex effect algebra isomorphism $L: \mathcal{E} \rightarrow \mathcal{F}$ that satisfies $L(a \circ b) = (La) \cdot (Lb)$ for all $a, b \in \mathcal{E}$. As we have seen, the map J in Theorem 3.1 is a SEA isomorphism so that theorem characterizes convex SEA's that are isomorphic to a finite classical SEA. The situation for Hilbertian SEA's is more complicated. Let \mathcal{E} be a SEA satisfying the conditions of Theorem 3.3 and let $J: \mathcal{E} \rightarrow \mathcal{E}(\mathcal{H}(\mathcal{B}))$ be the convex effect algebra isomorphism of that theorem. Recall that for the chosen context \mathcal{B} if $b = \sum \lambda_i a_i$ where $\mathcal{A} = \{a_i: i = 1, \dots, n\}$ is some context, then

$$J(b) = \sum \lambda_i P(U_{\mathcal{A}\mathcal{B}} \hat{a}_i)$$

The next lemma shows that if $a|b$ then $J(a \circ b) = J(a)J(b)$ where $J(a)J(b)$ is the usual operator product.

Lemma 4.5. *We have $a|b$ if and only if $J(a)J(b) = J(b)J(a)$. Moreover, if $a|b$ then $J(a \circ b) = J(a)J(b)$.*

Proof. Suppose that $a|b$ where $a = \sum \lambda_i a_i$, $b = \sum \mu_j b_j$ for contexts $\mathcal{A} = \{a_i\}$, $\mathcal{C} = \{b_j\}$. It follows from Theorem 4.3(i) that $a_i|b$, for all i, j . Applying Lemma 4.2(ii) we conclude that $\mathcal{A} = \mathcal{C}$. By changing the order of the μ_i 's, we can assume that $b = \sum \mu_i a_i$. As in Theorem 4.3(ii) we have that $a \circ b = \sum \lambda_i \mu_i a_i$. Therefore,

$$J(a \circ b) = \sum \lambda_i \mu_i P(U_{\mathcal{A}\mathcal{B}}(\hat{a}_i)) = \sum \lambda_i P(U_{\mathcal{A}\mathcal{B}}(\hat{a}_i)) \sum \mu_i P(U_{\mathcal{A}\mathcal{B}}(\hat{a}_i))$$

$$= J(a)J(b) = J(b)J(a)$$

Conversely, suppose that $J(a)J(b) = J(b)J(a)$. Since

$$J(a) = \sum \lambda_i P(U_{\mathcal{A}\mathcal{B}}(\hat{a}_i)), J(b) = \sum \mu_i P(U_{\mathcal{C}\mathcal{B}}(\hat{b}_i))$$

as before, we have that

$$\{P(U_{\mathcal{A}\mathcal{B}}(\hat{a}_i))\} = \{P(U_{\mathcal{C}\mathcal{B}}(\hat{b}_i))\}$$

Since J is injective we conclude that $\mathcal{A} = \mathcal{C}$. Hence, $a|b$. \square

It follows from Lemma 4.5 that a is sharp if and only if $J(a)$ is sharp. We now define a product on $\mathcal{E}(\mathcal{H}(\mathcal{B}))$ induced by the sequential product on \mathcal{E} . If $A, B \in \mathcal{E}(\mathcal{H}(\mathcal{B}))$ are given by $A = J(a)$, $B = J(b)$ we define $A \cdot B = J(a \circ b)$. We then have

$$J(a \circ b) = J(a) \cdot J(b)$$

by definition. The next result shows that $A \cdot B$ is a sequential product.

Theorem 4.6. *With the product $A \cdot B$, $\mathcal{E}(\mathcal{H}(\mathcal{B}))$ is a convex SEA*

Proof. We assume that $J(a) = A$, $J(b) = B$, $J(c) = C$, $J(b_1) = B_1$ and $J(b_2) = B_2$. We now check the six axioms for a convex SEA.

(S1) Since $J(b_1 \oplus b_2) = J(b_1) \oplus J(b_2) = B_1 \oplus B_2$ we have

$$\begin{aligned} A \cdot (B_1 \oplus B_2) &= J(a \circ (b_1 \oplus b_2)) = J(a \circ b_1 \oplus a \circ b_2) = J(a \circ b_1) + J(a \circ b_2) \\ &= A \cdot B_1 \oplus A \cdot B_2 \end{aligned}$$

(S2) $I \cdot A = J(1 \circ a) = J(a) = A$

(S3) If $A \cdot B = 0$, the $J(a \circ b) = 0$. Since J is injective, $a = b = 0$ so $a|b$. Hence, $A \cdot B = B \cdot A$ by Lemma 4.5.

(S4) If $A \cdot B = B \cdot A$, then $A \cdot B' = B' \cdot A$. Moreover, since $a|b$ we have

$$\begin{aligned} A \cdot (B \cdot C) &= A \cdot J(b \circ c) = J[a \circ (b \circ c)] = J[(a \circ b) \circ c] = J(a \circ b) \cdot J(c) \\ &= [J(a) \cdot J(b)] \cdot J(c) = (A \cdot B) \cdot C \end{aligned}$$

(S5) If $C \cdot A = A \cdot C$ and $C \cdot B = B \cdot C$ then by Lemma 4.5, $c|a$ and $c|b$ so we have that $c|(a \circ b)$ and $c|(a \oplus b)$. Therefore, $J(c)|J(a \circ b)$ so $C|A \cdot B$ and $J(c)|J(a \oplus b)$ so $C|(A + B)$.

(S6) If $\lambda \in [0, 1] \subseteq \mathbb{R}$, then

$$(\lambda A) \cdot B = J(\lambda a \circ b) = \lambda J(a \circ b) = \lambda(A \cdot B)$$

and similarly, $A \cdot (\lambda B) = \lambda(A \cdot B)$. \square

It follows from Theorem 4.6 that J is a SEA isomorphism from \mathcal{E} to $\mathcal{E}(\mathcal{H}(\mathcal{B}))$. We have not proved that $A \cdot B$ is the standard sequential product $A \circ B = A^{1/2}BA^{1/2}$. A characterization of when $A \cdot B = A \circ B$ are the following physically justifiable conditions [10]:

(B1) For every density operator ρ and $A, B \in \mathcal{E}(\mathcal{H}(\mathcal{B}))$ we have

$$\text{tr}[(A \cdot \rho)B] = \text{tr}[\rho(A \cdot B)]$$

(B2) If P is a one-dimensional projection in $\mathcal{E}(\mathcal{H}(\mathcal{B}))$ and $A \in \mathcal{E}(\mathcal{H}(\mathcal{B}))$ with $A \cdot P \neq 0$ then $A \cdot P/\text{tr}(A \cdot P)$ is a one-dimensional projection.

It has been very important in our previous work that if \mathcal{E} is a convex effect algebra and $a \in S(\mathcal{E})$ then there exists a state $\hat{a} \in \Omega(\mathcal{E})$ such that $\hat{a}(a) = 1$. We now show that if \mathcal{E} is a convex SEA, then we can construct this state explicitly. For $b \in \mathcal{E}$, since $a \circ b \leq a$, there exists a $\lambda(a, b) \in [0, 1] \subseteq \mathbb{R}$ such that $a \circ b = \lambda(a, b)a$. Since $\lambda(a, 1) = 1$ and

$$\begin{aligned} \lambda(a, b_1 \oplus b_2)a &= a \circ (b_1 \oplus b_2) = a \circ b_1 \oplus a \circ b_2 = \lambda(a, b_1)a \oplus \lambda(a, b_2)a \\ &= [\lambda(a, b_1) + \lambda(a, b_2)]a \end{aligned}$$

we conclude that $\lambda(a, b_1 \oplus b_2) = \lambda(a, b_1) + \lambda(a, b_2)$. Hence, $b \mapsto \lambda(a, b)$ is a state satisfying $\lambda(a, a) = 1$. We then use the notation

$$\hat{a}(b) = \lambda(a, b)$$

for all $b \in \mathcal{E}$. Notice that $\hat{a}(a \circ b) = \hat{a}(b)$ for all $b \in \mathcal{E}$.

In the sequel, \mathcal{E} will denote a convex SEA with order determining set of states $\Omega(\mathcal{E})$. One of the advantages of working with a SEA is that it provides

a structure for defining conditional probabilities. If $\omega \in \Omega(\mathcal{E})$ and $a \in \mathcal{E}$ with $\omega(a) \neq 0$, then the state ω *conditioned by* a is

$$\omega(b|a) = \omega(a \circ b) / \omega(a)$$

for all $b \in \mathcal{E}$. Notice that $\omega(a \circ b) = \omega(a)\omega(b|a)$. When we write $\omega(b|a)$ we are implicitly assuming that $\omega(a) \neq 0$.

Lemma 4.7. (i) *For every $\omega \in \Omega(\mathcal{E})$ and $a \in S_1(\mathcal{E})$ we have that $\omega(b|a) = \hat{a}(b)$ for all $b \in \mathcal{E}$.* (ii) *$a \in S(\mathcal{E})$ if and only if $\omega(a|a) = 1$ for all $\omega \in \Omega(\mathcal{E})$.*

Proof. (i) For $a \in S_1(\mathcal{E})$ we have that

$$\omega(b|a) = \frac{\omega(a \circ b)}{\omega(a)} = \frac{\omega(\hat{a}(b)a)}{\omega(a)} = \hat{a}(b)$$

(ii) If $a \in S(\mathcal{E})$ then

$$\omega(a|a) = \frac{\omega(a \circ a)}{\omega(a)} = 1$$

for every $\omega \in \Omega(\mathcal{E})$. Conversely, if $\omega(a|a) = 1$ for all ω with $\omega(a) \neq 0$, then

$$\omega(a^2) = \omega(a)\omega(a|a) = \omega(a)$$

Clearly, $\omega(a^2) = \omega(a)$ if $\omega(a) = 0$. Since $\Omega(\mathcal{E})$ is separating $a^2 = a$ so $a \in S_1(\mathcal{E})$. \square

Lemma 4.7(i) shows that all states conditioned by an $a \in S_1(\mathcal{E})$ are the same. In this sense, \hat{a} is universal.

A *measurement* is a set $\mathcal{A} = \{a_1, \dots, a_n\} \subseteq \mathcal{E}$ satisfying $a_1 \oplus \dots \oplus a_n = 1$. We say that $b \in \mathcal{E}$ is *measurable relative to* \mathcal{A} if b has the form $b = \sum \lambda_i a_i$, $\lambda_i \in [0, 1]$. We say that \mathcal{A} is a *sharp measurement* if $a_i \in S(\mathcal{E})$, $i = 1, \dots, n$. We have already treated sharp measurements and in this case measurable relative to \mathcal{A} and \mathcal{A} -measurable are the same. The *law of total probability* for $\omega \in \Omega(\mathcal{E})$ says if $b \in \mathcal{E}$ and $\mathcal{A} = \{a_i: i = 1, \dots, n\}$ is a measurement, then

$$\omega(b) = \sum \omega(a_i \circ b) = \sum \omega(a_i)\omega(b|a_i)$$

This law holds for some ω , b and \mathcal{A} and not for others as the following lemma shows.

Lemma 4.8. (i) If $b|a_i$, $i = 1, \dots, n$, then $\omega(b) = \sum \omega(a_i \circ b)$ for every $\omega \in \Omega(\mathcal{E})$. (ii) If \mathcal{A} is sharp and $\omega(b) = \sum \omega(a_i \circ b)$ for every $\omega \in \mathcal{E}$, then $b|a_i$, $i = 1, \dots, n$.

Proof. (i) If $b|a_i$, $i = 1, \dots, n$, since $b = \sum b \circ a_i$ we have that

$$\omega(b) = \omega\left(\sum b \circ a_i\right) = \sum \omega(b \circ a_i) = \sum \omega(a_i \circ b)$$

(ii) Assume that \mathcal{A} is sharp and $\omega(b) = \sum \omega(a_i \circ b)$ for all $\omega \in \Omega(\mathcal{E})$. We then have that $\omega(b) = \omega(\sum a_i \oplus b)$ for all $\omega \in \Omega(\mathcal{E})$. Since $\Omega(\mathcal{E})$ is separating, we conclude that $b = \sum a_i \circ b$. Since

$$a_i \circ b \leq a_i \leq a'_j, \quad i \neq j$$

it follows from Theorem 4.1(v) that $a_i \circ b|a_i$ and $a_i \circ b|a'_j$ for $j \neq i$. Hence, $a_i \circ b|a_j$, $j = 1, \dots, n$. Therefore,

$$b \circ a_j = \left(\sum a_i \circ b\right) \circ a_j = a_j \circ \left(\sum a_i \circ b\right) = \sum_{i=1}^n (a_j \circ a_i) \circ b = a_j \circ b$$

$j = 1, \dots, n$. □

In a similar vein, *Bayes' Rule* for $\omega \in \Omega(\mathcal{E})$ says that if $b \in \mathcal{E}$ and $\mathcal{A} = \{a_i: i = 1, \dots, n\}$ is a measurement, then

$$\omega(a_i|b) = \frac{\omega(b|a_i)\omega(a_i)}{\omega(b)}$$

It immediately follows that Bayes' Rule holds for all $\omega \in \Omega(\mathcal{E})$ if and only if $b|a_i$, $i = 1, \dots, n$.

If $\omega \in \Omega(\mathcal{E})$ and $\mathcal{A} = \{a_i: i = 1, \dots, n\}$ is a measurement, the *conditional expectation of $b \in \mathcal{E}$ given \mathcal{A}* is an effect denoted by $E_\omega(b|\mathcal{A})$ that is measurable relative to \mathcal{A} and satisfies

$$\omega[a \circ E_\omega(b|\mathcal{A})] = \omega(a \circ b)$$

for all $a \in \mathcal{A}$. Notice that b is measurable relative to \mathcal{A} if and only if $E_\omega(b|\mathcal{A}) = b$.

Theorem 4.9. (i) The map $b \mapsto E_\omega(b|\mathcal{A})$ is affine and additive. (ii) If \mathcal{A} is sharp, $b \mapsto E_\omega(b|\mathcal{A})$ is a morphism. (iii) If $a_i \in S_1(\mathcal{E})$, $i = 1, \dots, n$, then $E_{\widehat{a}_i}(b|\mathcal{A}) = \widehat{a}_i(b|a_i)$ and

$$E_\omega(b|\mathcal{A}) = \sum \{\widehat{a}_i(b)a_i : \omega(a_i) \neq 0\}$$

Proof. (i) Since $E_\omega(b|\mathcal{A})$ is measurable relative to \mathcal{A} , clearly $\lambda E_\omega(b|\mathcal{A})$ is also for $\lambda \in [0, 1]$. Moreover, for $a \in \mathcal{A}$ we have

$$\begin{aligned} \omega[a \circ \lambda E_\omega(b|\mathcal{A})] &= \lambda \omega[a \circ E_\omega(b|\mathcal{A})] = \lambda \omega(a \circ b) \\ &= \omega(a \circ \lambda b) = \omega[a \circ E_\omega(\lambda b|\mathcal{A})] \end{aligned}$$

Hence, $E_\omega(\lambda b|\mathcal{A}) = \lambda E_\omega(b|\mathcal{A})$. If $b_1 \perp b_2$, then clearly $E_\omega(b_1|\mathcal{A}) \perp E_\omega(b_2|\mathcal{A})$. Moreover, for $a \in \mathcal{A}$ we have

$$\begin{aligned} \omega[a \circ E_\omega(b_1 \oplus b_2|\mathcal{A})] &= \omega[a \circ (b_1 \oplus b_2)] = \omega(a \circ b_1) + \omega(a \circ b_2) \\ &= \omega[a \circ E_\omega(b_1|\mathcal{A})] + \omega[a \circ E_\omega(b_2|\mathcal{A})] \\ &= \omega\{a \circ [E_\omega(b_1|\mathcal{A}) \oplus E_\omega(b_2|\mathcal{A})]\} \end{aligned}$$

We conclude that

$$E_\omega(b_1 \oplus b_2|\mathcal{A}) = E_\omega(b_1|\mathcal{A}) \oplus E_\omega(b_2|\mathcal{A})$$

(ii) Suppose \mathcal{A} is sharp and $E_\omega(b|\mathcal{A}) = \sum \lambda_i a_i$. We then have

$$\omega(a_j \circ b) = \omega[a_j \circ E_\omega(b|\mathcal{A})] = \omega\left(\sum \lambda_i a_j \circ a_i\right) = \lambda_j \omega(a_j)$$

Hence, $\lambda_j = \omega(b|a_j)$ and we have

$$E_\omega(b|\mathcal{A}) = \sum \omega(b|a_i) a_i \tag{4.4}$$

In particular $E_\omega(1|\mathcal{A}) = \sum a_i = 1$ so $E_\omega(\cdot|\mathcal{A})$ is a morphism.

(iii) This follows from (4.4). \square

Theorem 4.10. Let \mathcal{A} be a sharp measurement. (i) If c is measurable relative in \mathcal{A} , then for all $b \in \mathcal{E}$ we have

$$E_\omega(c \circ b|\mathcal{A}) = c \circ E_\omega(b|\mathcal{A})$$

(ii) $E_\omega(b|\mathcal{A}) = \sum E_\omega(a_i \circ b|\mathcal{A})$

Proof. (i) Clearly, $c \circ E_\omega(b|\mathcal{A})$ is measurable relative to \mathcal{A} . If $c = \sum \lambda_j a_j$, then by (4.4) we have

$$\begin{aligned} E_\omega(c \circ b|\mathcal{A}) &= \sum \omega(c \circ b|a_i) a_i = \sum \frac{\omega[(a_i \circ c) \circ b]}{\omega(a_i)} a_i \\ &= \sum \frac{\omega(\lambda_i a_i \circ b)}{\omega(a_i)} = \sum \lambda_i \omega(b|a_i) a_i \\ &= c \circ \left[\sum \omega(b|a_i) a_i \right] = c \circ E_\omega(b|\mathcal{A}) \end{aligned}$$

(ii) By (i) of this theorem, we have that

$$\begin{aligned} E_\omega(b|\mathcal{A}) &= E_\omega(b|\mathcal{A}) \circ \sum a_i = \sum E_\omega(b|\mathcal{A}) \circ a_i \\ &= \sum a_i \circ E_\omega(b|\mathcal{A}) = \sum E_\omega(a_i \circ b|\mathcal{A}) \quad \square \end{aligned}$$

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