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κ-STATIONARY SUBSETS OF $\mathscr{P}_{\kappa^+}\lambda$, INFINITARY GAMES, AND DISTRIBUTIVE LAWS IN BOOLEAN ALGEBRAS

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Abstract. We characterize the $(\kappa, \lambda, < \mu)$-distributive law in Boolean algebras in terms of cut and choose games $\mathscr{G}^{\kappa}_{<\mu}(\lambda)$, when $\mu \leq \kappa \leq \lambda$ and $\kappa^{<\kappa} = \kappa$. This builds on previous work to yield game-theoretic characterizations of distributive laws for almost all triples of cardinals κ, λ, μ with $\mu \leq \lambda$, under GCH. In the case when $\mu \leq \kappa \leq \lambda$ and $\kappa^{<\kappa} = \kappa$, we show that it is necessary to consider whether the κ-stationarity of $\mathscr{P}_{\kappa^+}\lambda$ in the ground model is preserved by $\mathbb{B}$. In this vein, we develop the theory of κ-club and κ-stationary subsets of $\mathscr{P}_{\kappa^+}\lambda$. We also construct Boolean algebras in which Player I wins $\mathscr{G}^{\kappa}_{\kappa}(\kappa^+)$ but the (κ, ∞, κ)-d.l. holds, and, assuming GCH, construct Boolean algebras in which many games are undetermined.

§1. Introduction. The investigation of relationships between games and distributive laws began with Jech's work in [10], where he characterized the (ω, ∞)-d.l. in terms of Player I not having a winning strategy in the descending sequence game of length ω. Later, he developed the theory of cut-and-choose games of length ω and related distributive laws in [11]. One of these games yields a property strictly intermediate between Axiom A and properness, and another of these games is used in Gray's Conjecture on von Neumann's Problem concerning measurable Boolean algebras (see [11]). In [5] we extended this work of Jech to more general distributive laws in Boolean algebras and related games of any cardinal length.

THEOREM 1 (Dobrinen, [5]). *Let $\mathbb{B}$ be a complete Boolean algebra.*

1. *If the (κ, λ)-d.l. fails in $\mathbb{B}$, then I has a winning strategy for $\mathscr{G}^{\kappa}_{1}(\lambda)$ in $\mathbb{B}$. This, in turn, implies that both the $(\lambda^{<\kappa}, \lambda)$-d.l. and the $(\kappa, \lambda^{<\kappa})$-d.l. fail in $\mathbb{B}$.*
2. *If the $(\kappa, \lambda, < \mu)$-d.l. fails in $\mathbb{B}$, then I has a winning strategy for $\mathscr{G}^{\kappa}_{<\mu}(\lambda)$ in $\mathbb{B}$. This, in turn, implies that the $((\lambda^{<\mu})^{<\kappa}, \lambda, < \mu)$-d.l. fails in $\mathbb{B}$.*

Under GCH, this gives a game-theoretic characterization of the (κ, λ)-d.l. whenever $\lambda < \kappa$ or $\mathrm{cf}(\lambda) \geq \kappa$, and a characterization of the $(\kappa, \lambda, < \mu)$-d.l. whenever $\lambda < \kappa$, or $\kappa = \lambda$ and is regular. However, these results left open the case when $\lambda > \kappa$.

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In this paper, we extend results of Kamburelis from [14] to regular uncountable cardinal length games. His results hinge on the property $V^{\mathbb{B}} \models$"$[\check{\lambda}]^{\omega} \cap V$ is stationary", which he showed to be the necessary and sufficient ingredient which in conjunction with the weak (ω, λ)-d.l. characterizes Player I not having a winning strategy for $\mathscr{G}_{\text{fin}}(\lambda)$ (see Theorems 7 and 8 below). When $\kappa > \omega$, we have found that instead of what would seem the obvious generalization, $V^{\mathbb{B}} \models$"$[\check{\lambda}]^{\leq\kappa} \cap V$ is stationary", the necessary and sufficient property is actually $V^{\mathbb{B}} \models$"$[\check{\lambda}]^{\leq\kappa} \cap V$ is κ-stationary" (see Definition 11).

In § 2 we give the necessary definitions, including the distributive laws and games, and discuss the background to our present work, including Kamburelis' results. § 3 hosts the general theory of $\geq \nu$-club and ν-club subsets of $\mathscr{P}_\kappa\lambda$ for regular $\nu < \kappa$. In § 4 the theory specific to κ-club subsets of $\mathscr{P}_{\kappa^+}\lambda$ is further developed. We give a combinatorial proof of Kueker's Theorem 21 for κ-club subsets of $\mathscr{P}_{\kappa^+}\lambda$ and investigate the κ-club filter. The main theorems regarding the relationship between general distributive laws and related games when $\lambda \geq \kappa$ are presented in § 5, as well as sufficient conditions for preserving the κ-stationarity of $\mathscr{P}_{\kappa^+}\lambda$ of the ground model. We arrive at conditions under which stronger general distributive laws are equivalent to weaker ones (see Corollaries 39 and 41). In § 6 we give several theorems which ensure preservation of all κ-stationary subsets of $\mathscr{P}_{\kappa^+}\lambda$, also investigating a game $\Gamma^\kappa_\kappa(\lambda)$ which naturally generalizes the properness game to uncountable lengths. We show that $\Gamma^\kappa_\kappa(\lambda)$ is strictly easier for Player II to win than $\mathscr{G}^\kappa_\kappa(\lambda)$, thus obtaining a Boolean algebra in which Player I wins $\mathscr{G}^\kappa_\kappa(\kappa^+)$ but the (κ, ∞, κ)-d.l. holds. Theorems giving sufficient conditions under which adding no new subsets of κ implies no new sequences of length κ are added appear in §§ 5 and 6. In § 7, we show that one can shoot a κ-club set through any κ-stationary subsets of $\mathscr{P}_{\kappa^+}\lambda$ without adding any new κ-length sequences, assuming $\kappa^{<\kappa} = \kappa$. This yields Boolean algebras in which many games are undetermined, improving, for these games, on the consistency result of [4] in the sense that weaker assumptions are used.

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§2. Background and definitions. Throughout this paper, we restrict ourselves to the class of complete Boolean algebras. We let $\mathbb{B}$ denote an arbitrary complete Boolean algebra and $\mathbb{B}^+$ denote $\mathbb{B} \setminus \{\mathbf{0}\}$. Basic set-theoretic notation is used throughout. We let $[\lambda]^\kappa = \{x \subseteq \lambda : |x| = \kappa\}$, $[\lambda]^{<\kappa} = \{x \subseteq \lambda : |x| < \kappa\} = \mathscr{P}_\kappa\lambda$, and $[\lambda]^{\leq\kappa} = \{x \subseteq \lambda : |x| \leq \kappa\} = \mathscr{P}_{\kappa^+}\lambda$. We let $(\lambda)^{<\kappa}$ denote the set (or tree ordered by end extension) of sequences from ordinals $\alpha < \kappa$ into λ.

Definition 2. [17] $\mathbb{B}$ satisfies the $(\kappa, \lambda, < \mu)$-*distributive law* ($(\kappa, \lambda, < \mu)$-d.l.) if for all families $\{b_{\alpha,\beta} : \alpha < \kappa,\ \beta < \lambda\} \subseteq \mathbb{B}$,

$$\bigwedge_{\alpha<\kappa} \bigvee_{\beta<\lambda} b_{\alpha,\beta} = \bigvee_{f:\kappa\to[\lambda]^{<\mu}} \bigwedge_{\alpha<\kappa} \bigvee_{\beta\in f(\alpha)} b_{\alpha,\beta}. \tag{2.1}$$

Notation. The $(\kappa, \lambda, < 2)$-d.l. is usually referred to as the (κ, λ)-d.l., and the $(\kappa, \lambda, < \omega)$-d.l. is usually referred to as the weak (κ, λ)-d.l. We say that the $(< \kappa, \lambda)$-d.l. holds if the (ρ, λ)-d.l. holds for all $\rho < \kappa$.

DEFINITION 3. [17] A *quasipartition of unity* $($*of* $a)$ is a collection $W \subseteq \mathbb{B}$ such that $\bigvee W = \mathbf{1}$ $(\bigvee W = a)$ and for all $b, c \in W$ with $b \neq c$, $b \wedge c = \mathbf{0}$. A *partition of unity* $($*of* $a)$ is a collection $W \subseteq \mathbb{B}^+$ which forms a quasipartition of unity (of a).

The following fact is well-known. A proof of (1) $\iff$ (2) can be found in [17]. A proof of (1) $\iff$ (3) for $\mu = 2$ can be found in [13], and a proof for the more general case for any $\mu \leq \lambda$ follows easily.

FACT 4. *The following are equivalent.*

1. $\mathbb{B}$ *satisfies the* $(\kappa, \lambda, < \mu)$*-d.l.*
2. *For any family* W_α, $(\alpha < \kappa)$, *of partitions of unity of* $\mathbb{B}$ *with each* $|W_\alpha| \leq \lambda$, *there exists a partition of unity* W *such that for each* $b \in W$, *for each* $\alpha < \kappa$, $b \wedge c \neq \mathbf{0}$ *for less than* μ*-many* $c \in W_\alpha$.
3. *For each* $\mathbb{B}$*-name* $\dot{g}$ *for a function from* $\check{\kappa}$ *into* $\check{\lambda}$ *and any generic filter* $G \subseteq \mathbb{B}^+$, *there is a function* $f : \kappa \to [\lambda]^{<\mu}$ *in* V *such that* $V[G] \models$*"*$\forall \alpha < \check{\kappa}$, $\dot{g}(\alpha) \in f(\alpha)$*".*

We now recall a game related to the $(\kappa, \lambda, < \mu)$-d.l., which we introduced in [5]. This game generalizes a game of Jech related to the weak (ω, λ)-d.l. in [11].

DEFINITION 5. [5] Let κ, λ be infinite cardinals and μ be a cardinal such that $2 \leq \mu \leq \lambda$. The game $\mathcal{G}^{\kappa}_{<\mu}(\lambda)$ is played between two players in a complete Boolean algebra $\mathbb{B}$ as follows: At the beginning of the game, Player I fixes some $a \in \mathbb{B}^+$. For $\alpha < \kappa$, the α-th round is played as follows: Player I chooses a partition W_α of a such that $|W_\alpha| \leq \lambda$; then Player II chooses some $E_\alpha \in [W_\alpha]^{<\mu}$. In this manner, the two players construct a sequence of length κ

$$\langle a, W_0, E_0, W_1, E_1, \ldots, W_\alpha, E_\alpha, \ldots : \alpha < \kappa \rangle \tag{2.2}$$

called a *play* of the game. Player I *wins the play* (2.2) if and only if

$$\bigwedge_{\alpha<\kappa} \bigvee E_\alpha = \mathbf{0}. \tag{2.3}$$

$\mathcal{G}^{\kappa}_{<\mu}(\infty)$ is the game played like $\mathcal{G}^{\kappa}_{<\mu}(\lambda)$, except now Player I can choose partitions of any size.

It is not hard to see that if II has a winning strategy for $\mathcal{G}^{\kappa}_{<\mu}(\lambda)$ in $\mathbb{B}$, then $\mathbb{B}$ satisfies the $(\kappa, \lambda, < \mu)$-d.l.

Notation. Jech's game $\mathcal{G}_{\text{fin}}(\lambda)$ in [11] is $\mathcal{G}^{\omega}_{<\omega}(\lambda)$ in our notation. If $\mu = \nu^+$, we often write $\mathcal{G}^{\kappa}_{\nu}(\lambda)$ instead of $\mathcal{G}^{\kappa}_{<\mu}(\lambda)$. In particular, we write $\mathcal{G}^{\kappa}_{1}(\lambda)$ for $\mathcal{G}^{\kappa}_{<2}(\lambda)$.

Remark. $\mathcal{G}^{\kappa}_{<\mu}(\infty)$ can be played on a partial ordering $\mathbb{P}$. We say that II wins the play iff there is a $p \in \mathbb{P}$ such that $p \leq a$ and $\forall \alpha < \kappa$, E_α is pre-dense below p. If $\mathbb{P}$ is a partial ordering, then Player I (II) has a winning strategy for $\mathcal{G}^{\kappa}_{<\mu}(\infty)$ in $\mathbb{P}$ iff Player I (II) has a winning strategy for $\mathcal{G}^{\kappa}_{<\mu}(\infty)$ in r.o.$(\mathbb{P})$.

By work of Cummings and Dobrinen, it is consistent with ZFC that for all cardinals κ, λ, μ with $\lambda \geq \mu$, there is a $\max(\kappa, \lambda)^+$-Suslin algebra which is $(\max(\kappa, \lambda), \infty)$-distributive and in which $\mathcal{G}^{\kappa}_{<\mu}(\lambda)$ is undetermined.

THEOREM 6 (Cummings/Dobrinen, [4]). *Let κ be any infinite cardinal, and let ν be any regular cardinal such that $\omega \leq \nu \leq \text{cf}(\kappa)$. Suppose that $\square_\kappa$ holds and $\diamondsuit_{\kappa^+}(S)$ holds for every stationary set $S \subseteq \{\alpha < \kappa^+ : \text{cf}(\alpha) = \nu\}$. Then there is a κ^+-Suslin algebra which contains a $<\nu$-closed dense subset, and in which for all ρ, λ, μ with $\nu \leq \rho \leq \text{cf}(\kappa)$ and $2 \leq \mu \leq \min(\lambda, \kappa)$, $\mathscr{G}^\rho_{<\mu}(\lambda)$ is undetermined.*

Jech showed in [11] that if the weak (ω, λ)-d.l fails in $\mathbb{B}$, then I has a winning strategy for $\mathscr{G}^\omega_{<\omega}(\lambda)$ in $\mathbb{B}$. In that paper, he asked whether the converse holds. Kamburelis gave a complete answer via Theorems 7 and 8 below, which we extend in Section 5.

Notation. Let λ be an ordinal. By $[\check{\lambda}]^\omega$ we mean the set of all (good) $\mathbb{B}$-names for ranges of functions from ω into $\check{\lambda}$.

THEOREM 7 (Kamburelis, [14]). *Assume $\mathbb{B}$ satisfies the weak (ω, λ)-d.l. and $V^{\mathbb{B}} \models$"$[\check{\lambda}]^\omega \cap V$ is stationary". Then Player I does not have a winning strategy for $\mathscr{G}^\omega_{<\omega}(\lambda)$ in $\mathbb{B}$.*

THEOREM 8 (Kamburelis, [14]). *Assume that $\|[\check{\lambda}]^\omega \cap V$ is non-stationary$\|_{\mathbb{B}} > \mathbf{0}$. Then Player I has a winning strategy for $\mathscr{G}^\omega_\omega(\lambda)$ in $\mathbb{B}$.*

COROLLARY 9 (Kamburelis, [14]). *Assume that $\mathbb{B}$ is weakly (ω, λ)-distributive. Then Player I has a winning strategy for $\mathscr{G}^\omega_{<\omega}(\lambda)$ in $\mathbb{B}$ iff*

$$\|[\check{\lambda}]^\omega \cap V \text{ is non-stationary} \|_{\mathbb{B}} > \mathbf{0}.$$

Our extensions of the preceding theorems of Kamburelis appear as Theorems 33 and 37 in Section 5. The following theorem of Kueker was essential to Kamburelis' proof of Theorem 8.

THEOREM 10 (Kueker, [18]). *If $C \subseteq [\lambda]^\omega$ is club, then there exists a function $f : [\lambda]^{<\omega} \to \lambda$ such that C_f is club and $C_f \subseteq C$, where $C_f = \{x \in [\lambda]^\omega : (\forall y \in [x]^{<\omega})\, f(y) \in x\}$.*

In Section 3 we present a generalization of Theorem 10, which we call the Strong κ-club Theorem 21. This theorem will be used extensively throughout this paper to lift results about $\mathscr{P}_{\aleph_1}\lambda$ to $\mathscr{P}_{\kappa^+}\lambda$, when $\kappa^{<\kappa} = \kappa$.

§3. $\geq \nu$-club, $\geq \nu$-stationary, ν-club, and ν-stationary subsets of $\mathscr{P}_\kappa\lambda$.

Along the way to generalizing Kamburelis's results to uncountable cardinals, the necessity of dealing with κ-club and κ-stationary subsets of $[\lambda]^{\leq\kappa}$ appeared, as will be seen in Section 5. We present here the more general notions of $\geq \nu$-club, $\geq \nu$-stationary, ν-club, and ν-stationary subsets of $[\lambda]^{<\kappa}$ and the basic theorems regarding such sets. (For more on ν-club and stationary subsets of κ, see [8], and [9], and [22]. $\geq \nu$-club subsets of κ^+ were used by Cummings and Dobrinen in [4] to obtain the aforementioned Theorem 6.)

DEFINITION 11. Suppose $\omega \leq \nu < \kappa \leq \lambda$ and ν, κ are regular. We say that a set $X \subseteq [\lambda]^{<\kappa}$ is *ν-closed* if for all increasing sequences $\langle x_\alpha : \alpha < \nu\rangle \subseteq X$, $\bigcup_{\alpha<\nu} x_\alpha \in X$. $C \subseteq [\lambda]^{<\kappa}$ is *ν-club* if it is ν-closed and unbounded in $[\lambda]^{<\kappa}$ (i.e. $(\forall y \in [\lambda]^{<\kappa})(\exists x \in C)\ y \subseteq x$). $S \subseteq [\lambda]^{<\kappa}$ is *ν-stationary* if $S \cap C \neq \emptyset$ for all ν-club $C \subseteq [\lambda]^{<\kappa}$. We say that a set $C \subseteq [\lambda]^{<\kappa}$ is *$\geq \nu$-club* if C is ρ-club for all regular ρ with $\nu \leq \rho < \kappa$. $S \subseteq [\lambda]^{<\kappa}$ is *$\geq \nu$-stationary* if $S \cap C \neq \emptyset$ for each $\geq \nu$-club $C \subseteq [\lambda]^{<\kappa}$.

Note. For any regular $\omega \leq \rho < \kappa$, a set $X \subseteq [\lambda]^{<\kappa}$ is closed under increasing sequences of length ρ iff X is closed under strictly increasing sequences of length ρ iff X is closed under increasing sequences of length with cofinality ρ.

FACT 12. *Suppose $\omega \leq \mu < \nu < \kappa \leq \lambda^+$ and μ, ν, κ are regular. On $[\lambda]^{<\kappa}$,*

1. *$\geq \omega$-club is the same as club; $\geq \omega$-stationary is the same as stationary.*
2. *$\geq \mu$-club $\Longrightarrow \geq \nu$-club $\Longrightarrow \geq \nu$-stationary $\Longrightarrow \geq \mu$-stationary.*
3. *$\geq \nu$-club $\Longrightarrow \nu$-club $\Longrightarrow \nu$-stationary $\Longrightarrow \geq \nu$-stationary.*

Remark. Suppose $\omega \leq \mu < \nu < \kappa \leq \lambda^+$ and μ, ν, κ are regular. The $\geq \nu$-club and $\geq \nu$-stationary sets form a strict hierarchy among the stationary sets of $[\lambda]^{<\kappa}$. However, there are μ-club and ν-club sets which are disjoint. $\{x \in [\lambda]^{<\kappa} : x \cap \kappa$ is an ordinal with $\mathrm{cf}(x \cap \kappa) \geq \nu\}$ is $\geq \nu$-club but not $\geq \mu$-club. If $\nu^+ < \kappa$, then $\{x \in [\lambda]^{<\kappa} : x \cap \kappa$ is an ordinal with $\mathrm{cf}(x \cap \kappa) = \nu\}$ is ν-club, but not $\geq \nu$-club, nor μ-club. By Theorem 17 below, there exist $\geq \nu$-stationary (ν-stationary) subsets of $[\lambda]^{<\kappa}$ which are not $\geq \nu$-club (ν-club). $\{x \in [\lambda]^{<\kappa} : x \cap \kappa$ is an ordinal with $\mathrm{cf}(x \cap \kappa) = \mu\}$ is μ-club, but not $\geq \nu$-stationary.

Fact 13 through Theorem 16 are easy generalizations of results about club and stationary subsets of $[\lambda]^{<\kappa}$ (see [13]).

FACT 13. *Suppose $\omega \leq \nu < \kappa \leq \lambda$ and ν, κ are regular. The intersection of less than κ-many $\geq \nu$-club (ν-club) subsets of $[\lambda]^{<\kappa}$ is $\geq \nu$-club (ν-club).*

PROOF. Let $\theta < \kappa$ and C_α ($\alpha < \theta$) be $\geq \nu$-club (ν-club) subsets of $[\lambda]^{<\kappa}$. Let $C = \bigcap_{\alpha<\theta} C_\alpha$. C is certainly $\geq \nu$-closed (ν-closed). Let $y \in [\lambda]^{<\kappa}$. Choose some $x_0 \in C_0$ such that $y \subseteq x_0$. In general, for $\beta = \theta \cdot \gamma + \alpha$, where $\gamma < \nu$ and $\alpha < \theta$, choose some $x_\beta \in C_\alpha$ such that $\bigcup\{x_\zeta : \zeta < \beta\} \subseteq x_\beta$. Let $x = \bigcup\{x_\beta : \beta < \theta \cdot \nu\}$. For each $\alpha < \theta$, $\langle x_{\theta\cdot\gamma+\alpha} : \gamma < \nu\rangle$ is an increasing sequence in C_α which is cofinal in $\langle x_\beta : \beta < \theta \cdot \nu\rangle$. Hence, $x = \bigcup\{x_{\theta\cdot\gamma+\alpha} : \gamma < \nu\} \in C_\alpha$, since C_α is $\geq \nu$-closed (ν-closed). ⊣

DEFINITION 14. [13] Suppose $\omega \leq \nu < \kappa \leq \lambda$ and ν, κ are regular. Given a collection C_α ($\alpha < \lambda$) of subsets of $[\lambda]^{<\kappa}$, define the *diagonal intersection of* $\{C_\alpha : \alpha < \lambda\}$ to be

$$\Delta_{\alpha<\lambda} C_\alpha = \{x \in [\lambda]^{<\kappa} : \forall \alpha \in x \ (x \in C_\alpha)\}. \tag{3.1}$$

LEMMA 15. *Suppose $\omega \leq \nu < \kappa \leq \lambda$ and ν, κ are regular. Let C_α ($\alpha < \lambda$) be a collection of $\geq \nu$-club (ν-club) subsets of $[\lambda]^{<\kappa}$. Then $\Delta_{\alpha<\lambda} C_\alpha$ is also $\geq \nu$-club (ν-club).*

The proof is similar to that of Lemma 8.23 in [13], but using closure under sequences of length ν in place of ω. The next theorem can be proved using the standard argument for Fodor's Theorem for stationary sets by substituting Lemma 15 where the version for club sets is normally used.

THEOREM 16 (Fodor's Theorem). *Suppose $\omega \leq \nu < \kappa \leq \lambda$, ν, κ are regular, and $S \subseteq [\lambda]^{<\kappa}$ is $\geq \nu$-stationary (ν-stationary). If $f : S \to \lambda$ satisfies $f(x) \in x$ for all $x \in S$, then there exists an $\alpha < \lambda$ such that $\{x \in S : f(x) = \alpha\}$ is $\geq \nu$-stationary (ν-stationary).*

The following theorem is a strengthening of Proposition 25.5 in [16]. The main ideas are due to Baumgartner.

THEOREM 17. *If $\nu \leq \kappa$ are regular and $\kappa < \lambda$, then every $\geq\nu$-stationary (ν-stationary) $X \subseteq [\lambda]^{\leq\kappa}$ can be decomposed into λ many disjoint $\geq\nu$-stationary (ν-stationary) subsets.*

PROOF. Let $S = \{x \in [\lambda]^\kappa : |x \cap \kappa^+| = \kappa\}$. S is club. Let $X \subseteq [\lambda]^{\leq\kappa}$ be $\geq \nu$-stationary. We can assume that $X \subseteq S$, since $X \cap S$ is again $\geq \nu$-stationary. For each $x \in X$, fix an injection $f_x : x \to x \cap \kappa^+$. For $\alpha < \lambda$, let $X'_\alpha = \{x \in X : \alpha \in x\}$ and define $g_\alpha : X'_\alpha \to \kappa^+$ by $g_\alpha(x) = f_x(\alpha)$. $X'_\alpha = X \cap \{x \in [\lambda]^{\leq\kappa} : \alpha \in x\}$ which is $\geq \nu$-stationary. Each g_α is regressive. By Theorem 16, there is an $\eta_\alpha < \kappa$ such that $X_\alpha = \{x \in X'_\alpha : g_\alpha(x) = \eta_\alpha\}$ is $\geq \nu$-stationary. Note that for $\alpha \neq \beta$ with $\eta_\alpha = \eta_\beta$, $X_\alpha \cap X_\beta = \emptyset$, since each f_x is 1–1.

We have two cases. First assume $\mathrm{cf}(\lambda) > \kappa$. Then there is an $\eta < \kappa$ such that $|\{\alpha < \lambda : \eta_\alpha = \eta\}| = \lambda$. Otherwise, $\mathrm{cf}(\lambda) \leq \kappa$, and there is an $\eta < \kappa$ such that $|\{\alpha < \kappa^+ : \eta_\alpha = \eta\}| = \kappa^+$. Letting $Y_\alpha = \{x \in X : g_\alpha(x) = \eta\}$ for each α with $\eta_\alpha = \eta$ gives us κ^+ disjoint $\geq \nu$-stationary subsets of X. List $\{\alpha < \kappa^+ : \eta_\alpha = \eta\}$ in increasing order as $\{\alpha_\delta : \delta < \kappa^+\}$. Let $\langle \mu_\delta : \delta < \mathrm{cf}(\lambda)\rangle$ be an increasing sequence of regular cardinals cofinal in λ such that $\mu_0 > \kappa$. For each $\delta < \mathrm{cf}(\lambda)$, repeat the procedure above to get μ_δ-many disjoint $\geq \nu$-stationary subsets of Y_{α_δ}.

Replacing each instance of $\geq \nu$-stationary by ν-stationary yields the theorem for ν-stationary sets. ⊣

More generally, at the suggestion of Piper, we have confirmed the following. The proof is almost identical to that of Proposition 25.5 in [16].

THEOREM 18. *Let $S = \{x \in [\lambda]^{<\kappa} : |x \cap \kappa| = |x|\}$. Suppose κ is regular, $\lambda > \kappa$, and $(*)$ is a closure property for sets on $[\lambda]^{<\kappa}$ such that every club set is $(*)$-club and the intersection of two $(*)$-clubs is again $(*)$-club. We say that a set is $(*)$-*stationary *if it has non-empty intersection with every $(*)$-club set. Suppose the Fodor Theorem holds for $(*)$-stationary sets. If S is $(*)$-stationary, then every $(*)$-stationary subset of S can be decomposed into λ disjoint $(*)$-stationary sets.*

The following holds for $\geq\nu$-club sets, since closure under a $<\nu$-ary function guarantees the set to be $\geq\nu$-closed.

LEMMA 19. *Suppose [either $(\omega \leq \nu < \kappa \leq \lambda)$ or $(\omega \leq \nu \leq \lambda$ and $\nu^+ = \kappa)$], ν, κ are regular, and $\theta^{<\nu} < \kappa$ for all $\theta < \kappa$.*

1. *For any $\delta < \kappa$ and any collection of functions $f_\alpha : [\lambda]^{<\nu} \to [\lambda]^{<\kappa}$, $(\alpha < \delta)$, letting $\mathscr{F} = \{f_\alpha : \alpha < \delta\}$, the set*
$$C_{\mathscr{F}} = \{x \in [\lambda]^{<\kappa} : (\forall\alpha < \delta)(\forall y \in [x]^{<\nu})\ f_\alpha(y) \subseteq x\} \tag{3.2}$$
is a $\geq \nu$-club subset of $[\lambda]^{<\kappa}$.
2. *For any $\delta < \kappa$ and any collection of functions $h_\alpha : [\lambda]^{<\nu} \to \lambda$, $(\alpha < \delta)$, letting $\mathscr{H} = \{h_\alpha : \alpha < \delta\}$, the set*
$$C_{\mathscr{H}} = \{x \in [\lambda]^{<\kappa} : (\forall\alpha < \delta)(\forall y \in [x]^{<\nu})\ h_\alpha(y) \in x\} \tag{3.3}$$
is a $\geq \nu$-club subset of $[\lambda]^{<\kappa}$.

PROOF. Let ρ be a regular cardinal satisfying $\nu \leq \rho < \kappa$, and let $\langle x_\beta : \beta < \rho\rangle$ be an increasing sequence in $C_{\mathscr{F}}$. Let $x = \bigcup_{\beta<\rho} x_\beta$. Given $y \in [x]^{<\nu}$ there is some $\beta < \rho$ such that $y \subseteq x_\beta$. $x_\beta \in C_{\mathscr{F}}$ implies $f_\alpha(y) \subseteq x_\beta$ for all $\alpha < \delta$. Hence, $x \in C_{\mathscr{F}}$.

Let $x_0 \in [\lambda]^{<\kappa}$. Define $x_1 = x_0 \cup \bigcup\{f_\alpha(y) : \alpha < \delta,\ y \in [x_0]^{<\nu}\}$. In general, for $\beta < \nu$, let

$$x_\beta = \bigcup_{\gamma<\beta} x_\gamma \cup \bigcup\Big\{f_\alpha(y) : \alpha < \delta,\ y \in \Big[\bigcup_{\gamma<\beta} x_\gamma\Big]^{<\nu}\Big\}. \tag{3.4}$$

Each $|x_\beta| < \kappa$, since we are assuming that $\theta^{<\nu} < \kappa$ for all $\theta < \kappa$. Let $x = \bigcup_{\beta<\nu} x_\beta$. Given $y \in [x]^{<\nu}$, there exists a $\beta < \nu$ such that $y \subseteq x_\beta$. For each $\alpha < \delta$, $f_\alpha(y) \subseteq x_{\beta+1} \subseteq x$. Thus, $x \in C_{\mathscr{F}}$.

By a similar argument, $C_{\mathscr{H}}$ is easily shown to be $\geq \nu$-club. ⊣

The following, essentially due to Kueker [18], is a natural generalization of Theorem 10.

THEOREM 20 (Strong Club). 1. *Suppose $\kappa \leq \lambda$ and κ is regular. For each club $C \subseteq [\lambda]^{<\kappa}$ there exists a function $f : [\lambda]^{<\omega} \to [\lambda]^{<\kappa}$ such that $C_f \subseteq C$, where*

$$C_f = \{x \in [\lambda]^{<\kappa} : (\forall y \in [x]^{<\omega})\ f(y) \subseteq x\}. \tag{3.5}$$

2. *Suppose $\kappa < \lambda$ and κ is regular. For each club $C \subseteq [\lambda]^{\leq\kappa}$, there exist functions $h_\alpha : [\lambda]^{<\omega} \to \lambda$, $(\alpha < \kappa)$, such that $C_{\mathscr{H}} \subseteq C$, where*

$$C_{\mathscr{H}} = \{x \in [\lambda]^{\leq\kappa} : (\forall \alpha < \kappa)(\forall y \in [x]^{<\omega})\ h_\alpha(y) \in x\}. \tag{3.6}$$

Moreover, C_f and $C_{\mathscr{H}}$ are club.

§4. κ-club subsets of $\mathscr{P}_{\kappa^+}\lambda$.

If $\kappa < \lambda$ and κ is regular, we shall refer to $\geq \kappa$-club subsets of $\mathscr{P}_{\kappa^+}\lambda$ simply as *κ-club* sets.

The requirement of κ-many functions in (2) of Theorem 20 can sometimes be a hinderence to extending results for club subsets of $[\lambda]^\omega$ to $[\lambda]^{\leq\kappa}$. Of course, if one is willing to restrict to the club set $\tilde{K} = \{x \in [\lambda]^{<\kappa} : x \cap \kappa \in \kappa\}$, then for any club $C \subseteq \tilde{K}$, there exists a function $f : [\lambda]^{<\omega} \to \lambda$ such that $\{x \in \tilde{K} : \forall y \in [x]^{<\omega},\ f(y) \in x\} \subseteq C$. However, the set $\tilde{K}$ is not absolute with respect to forcing; in some forcing extensions, the set $\tilde{K}$ of the ground model may not even be club in the $\mathscr{P}_\kappa\lambda$ of the forcing extension. This can produce problems when one needs to define club sets in the extension merely by functions, not modulo some club. However, the situation improves for κ-club sets. The following theorem is essentially due to Kueker. In [19], he states result (2) (without proof) in connection with infinitary logic. We present here a purely combinatorial proof of (2) and refine it to (3), which increases the usefulness of the theorem. Theorem 21 is used in many of the results in the following sections, as it allows us to lift results for club and stationary subsets of $[\lambda]^\omega$ to κ-club and κ-stationary subsets of $[\lambda]^{\leq\kappa}$, if $\kappa^{<\kappa} = \kappa$.

THEOREM 21 (Strong κ-Club). *Suppose $\kappa^{<\kappa} = \kappa \leq \lambda$ and $C \subseteq [\lambda]^{\leq\kappa}$ is κ-club. Then*

1. *There exist functions $f_\alpha : [\lambda]^{<\kappa} \to C$ $(\alpha < \kappa)$ such that $C_{\mathscr{F}} \subseteq C$.*
2. *There exists a function $f : [\lambda]^{<\kappa} \to [\lambda]^{\leq\kappa}$ such that $C_f \subseteq C$.*
3. *There exists a function $h : [\lambda]^{<\kappa} \to \lambda$ such that $C_h \subseteq C$, and moreover, $\forall x \in C_h$, $cf(o.t.(x)) = \kappa$.*

PROOF. 1. We will recursively define functions $f_\alpha : [\lambda]^{<\kappa} \to C$ so that for each $\alpha < \kappa$, for each $x \in [\lambda]^{<\kappa}$,

$$f_\alpha(x) \supseteq x \cup \bigcup\{f_\beta(y) : \beta < \alpha,\ y \in [x]^{<\kappa}\}. \tag{4.1}$$

For each $x \in [\lambda]^{<\kappa}$, let $f_0(x) \in C$ such that $x \subseteq f_0(x)$. Having defined f_β for all $\beta < \alpha$, for each $x \in [\lambda]^{<\kappa}$ let $f_\alpha(x) \in C$ such that $f_\alpha(x) \supseteq \bigcup\{f_\beta(y) : \beta < \alpha,\ y \in [x]^{<\kappa}\}$. Note that $x \subseteq f_0(x) \subseteq f_\alpha(x)$, since $x \in [x]^{<\kappa}$. The $f_\alpha : [\lambda]^{<\kappa} \to C$ $(\alpha < \kappa)$ satisfy (4.1).

Let $C_{\mathscr{F}} = \{x \in [\lambda]^{\leq\kappa} : (\forall\alpha < \kappa)(\forall y \in [x]^{<\kappa})\ f_\alpha(y) \subseteq x\}$. $C_{\mathscr{F}}$ is κ-club by Lemma 19, since $\kappa^{<\kappa} = \kappa$. Let $x \in C_{\mathscr{F}}$. Let $\langle u_\alpha : \alpha < \kappa\rangle$ be an increasing sequence of subsets of x such that $\bigcup_{\alpha<\kappa} u_\alpha = x$ and each $|u_\alpha| < \kappa$. If $\beta < \alpha < \kappa$, then (4.1) implies $f_\beta(u_\beta) \subseteq f_\alpha(u_\alpha)$. Since each $f_\alpha(u_\alpha) \in C$ and C is κ-closed, $\bigcup_{\alpha<\kappa} f_\alpha(u_\alpha) \in C$. Now $x = \bigcup_{\alpha<\kappa} u_\alpha \subseteq \bigcup_{\alpha<\kappa} f_\alpha(u_\alpha)$, by (4.1); and $x \supseteq \bigcup_{\alpha<\kappa} f_\alpha(u_\alpha)$, since $x \in C_{\mathscr{F}}$ and all $u_\alpha \in [x]^{<\kappa}$. Therefore, $x \in C$.

2. Using the f_α's from part 1, define $f : [\lambda]^{<\kappa} \to [\lambda]^{\leq\kappa}$ by $f(y) = \bigcup_{\alpha<\kappa} f_\alpha(y)$. Let $C_f = \{x \in [\lambda]^{\leq\kappa} : (\forall y \in [x]^{<\kappa})\ f(y) \subseteq x\}$. Then $C_f = C_{\mathscr{F}}$.

3. For each $x \in [\lambda]^{\leq\kappa}$ fix an enumeration $\langle (x)_\alpha : 0 < \alpha < \kappa\rangle$ of x (possibly with repetitions). For $0 < \alpha < \kappa$, define $h_\alpha : [\lambda]^{<\kappa} \to \lambda$ by $h_\alpha(y) = (f(y))_\alpha$, the α-th element of the enumeration of $f(y)$. Let $C_{\mathscr{H}} = \{x \in [\lambda]^{\leq\kappa} : (\forall 0 < \alpha < \kappa)(\forall y \in [x]^{<\kappa})\ h_\alpha(y) \in x\}$. Then $C_{\mathscr{H}} = C_f$.

Define $h_0 : [\lambda]^{<\kappa} \to \lambda$ by $h_0(y) = (\sup y) + 1$. Fix a surjection $\phi : \kappa \setminus \{0\} \to \kappa \times (\kappa \setminus \{0\})$ such that (a) $\phi(1) = \langle 0, 1\rangle$; (b) $\forall\alpha \in \kappa \setminus \{0\}$, $\phi(\alpha) = \langle\beta, \gamma\rangle$ implies $\gamma \leq \alpha$; and (c) $\forall\alpha \in \lim(\kappa)$, $\phi(\alpha) = \langle 0, \alpha\rangle$. Finally, define $h : [\lambda]^{<\kappa} \to \lambda$ as follows: For $y \in [\lambda]^{<\kappa}$, if $\langle y_\zeta : \zeta < \alpha\rangle$ enumerates y in increasing order and $\phi(\alpha) = \langle\beta, \gamma\rangle$, then $h(y) = h_\beta(\{y_\zeta : \zeta < \gamma\})$. We claim that $C_h \subseteq C_{\mathscr{H}}$.

Let $x \in C_h$. Let $\delta = \text{o.t.}(x)$ and $\langle x_\zeta : \zeta < \delta\rangle$ be the enumeration of x in increasing order. By (a), for each $\zeta < \delta$, $h(\{x_\zeta\}) = x_\zeta + 1 \in x$, so δ must be a limit ordinal. If $\text{cf}(\delta) < \kappa$, then let $z = \langle z_\zeta : \zeta < \text{cf}(\delta)\rangle$ be a cofinal subset of x. $z \in [x]^{<\kappa}$ implies $h(z) \in x$. By (c), $h(z) = h_0(z) = (\sup z) + 1 = (\sup x) + 1$. Contradiction. Thus, $\text{cf}(\delta) = \kappa$.

Now let $\beta < \kappa$, $y \in [x]^{<\kappa}$, and $\langle y_\zeta : \zeta < \gamma\rangle$ be the increasing enumeration of y. Take an $\alpha \in \kappa \setminus \{0\}$ such that $\phi(\alpha) = \langle\beta, \gamma\rangle$. By (b), $\alpha \geq \gamma$. $\text{cf}(x) = \kappa$ implies there is an increasing sequence $\langle x_\zeta : \gamma \leq \zeta < \alpha\rangle \subseteq x$ with $\sup(y) < x_\gamma$. Then $h_\beta(y) = h(y \cup \{x_\zeta : \gamma \leq \zeta < \alpha\}) \in x$. Therefore, $x \in C_{\mathscr{H}}$. ⊣

The following decomposition theorem follows easily from the argument of Proposition 25.11 in [16] of a result of Matsubara [21], using Theorem 21. We include the proof for the sake of completeness.

THEOREM 22. *Suppose $\kappa^{<\kappa} = \kappa$, $2^\kappa < \lambda^\kappa$, and $|(\lambda^\kappa)^{(\lambda^{<\kappa})}| = |\lambda^\kappa|$. Then every κ-stationary subset of $[\lambda]^{\leq\kappa}$ can be decomposed into λ^κ disjoint κ-stationary sets.*

PROOF. First note: if Y is cofinal in $[\lambda]^{\leq\kappa}$, then $|Y| = \lambda^\kappa$. By the hypotheses, there are λ^κ many functions from $[\lambda]^{<\kappa}$ into $[\lambda]^{\leq\kappa}$. Let $\langle f_\alpha : \alpha < \lambda^\kappa\rangle$ be an enumeration of all such functions so that each function appears cofinally often. For each $\alpha < \lambda^\kappa$, the set $C_{f_\alpha} := \{x \in [\lambda]^{\leq\kappa} : (\forall y \in [x]^{<\kappa})\ f_\alpha(y) \subseteq x\}$ is κ-club, since $\kappa^{<\kappa} = \kappa$. By Theorem 21, each κ-club contains a C_{f_α} for some $\alpha < \lambda^\kappa$.

Let X be a κ-stationary subset of $[\lambda]^{\leq\kappa}$. By recursion on $\alpha < \lambda^\kappa$, having picked $\langle \eta_\xi^\beta : \xi < \beta < \alpha \rangle$, choose $\langle \eta_\xi^\alpha : \xi < \alpha \rangle$ distinct members of $(X \cap C_{f_\alpha}) \setminus \{\eta_\xi^\beta : \xi < \beta < \alpha\}$.

For $\xi < \lambda^\kappa$, set $X_\xi = \{\eta_\xi^\alpha : \xi < \alpha < \lambda^\kappa\}$. These sets are pairwise disjoint. Let C be κ-club. Then $\exists \alpha < \lambda^\kappa$ such that $\alpha > \xi$ and $C_{f_\alpha} \subseteq C$. Then $\eta_\xi^\alpha \in X_\xi \cap C_{f_\alpha} \subseteq X_\xi \cap C$. Therefore, $C \cap X_\xi \neq \emptyset$. Hence, X_ξ is κ-stationary. ⊣

We now investigate the κ-club filter on $[\lambda]^{\leq\kappa}$. Krueger asked whether the κ-club filter is just the club filter restricted to some κ-club set. Foreman later asked the same question and showed that, indeed, it is when we are working in $[H(\lambda)]^{\leq\kappa}$. We basically give his argument below.

Foreman also noted that every stationary subset of the collection of internally approachable sets of length κ on $[H(\lambda)]^{\leq\kappa}$ is κ-stationary. We thank both Foreman and Krueger for helpful discussions.

Let $K_0 = \{x \in [H(\lambda)]^{\leq\kappa} : [x]^{<\kappa} \subseteq x\}$ and $K_1 = \{x \in [H(\lambda)]^{\leq\kappa} : (x)^{<\kappa} \subseteq x\}$. K_0 and K_1 are both κ-club in $[H(\lambda)]^{\leq\kappa}$. The following fact was pointed out (in a slightly different form) by Foreman.

Fact 23. *Suppose $\kappa^{<\kappa} = \kappa$. Given a function $f : [H(\lambda)]^{<\kappa} \to [H(\lambda)]^{\leq\kappa}$, there is a function $g : H(\lambda) \to [H(\lambda)]^{\leq\kappa}$ such that $C_g \cap K_0 \subseteq C_f$, where*

$$C_f = \{x \in [H(\lambda)]^{\leq\kappa} : \forall y \in [x]^{<\kappa},\ f(y) \subseteq x\} \tag{4.2}$$

and

$$C_g = \{x \in [H(\lambda)]^{\leq\kappa} : \forall y \in x,\ g(y) \subseteq x\}. \tag{4.3}$$

Proof. First, note that as defined above, C_f is κ-club and C_g is club. Given f, define g by $g(y) = f(y)$ if $|y| < \kappa$, and $g(y) = \emptyset$ otherwise. Let $x \in C_g \cap K_0$ and $y \in [x]^{<\kappa}$. Then $y \in x$, so $g(y) \subseteq x$. $|y| < \kappa$ implies $g(y) = f(y)$. Hence, $x \in C_f$. ⊣

A similar argument works for K_1. Hence, we have the following theorem.

Theorem 24. *For $i = 0, 1$, the κ-club filter on $\mathscr{P}_{\kappa^+}(H(\lambda))$ is generated by the club sets intersected with K_i.*

The proof follows immediately from Fact 23 and Theorems 20 and 21.

We now review what it means for an element of $[H(\lambda)]^{<\kappa}$ to be internally approachable.

Definition 25. [6] Let $N \in [H(\lambda)]^{<\kappa}$. N is *internally approachable of length β* if $N = \bigcup_{\alpha<\beta} N_\alpha$, where for all $\beta' < \beta$, $\langle N_\alpha : \alpha < \beta' \rangle \in N$.

Let IA denote the set of internally approachable elements of $[H(\lambda)]^{\leq\kappa}$. Let $K = K_0 \cap K_1$. That the set $K \subseteq IA$ is not hard to show: Let $x \in K \setminus \{\emptyset\}$. Then $|x| = \kappa$. Let $f : \kappa \to x$ be a bijection. For each $\alpha < \kappa$, let $N_\alpha = f''\alpha$. Then $x = \bigcup_{\alpha<\kappa} N_\alpha$. For each $\alpha < \kappa$, $N_\alpha \in x$. Hence, for each $\beta < \kappa$, $\langle N_\alpha : \alpha < \beta \rangle \in (x)^{<\kappa} \subseteq x$. Therefore, x is internally approachable.

It then follows from Theorem 24 that every κ-stationary subset of $[H(\lambda)]^{\leq\kappa}$ contains a stationary subset of IA (in fact, contains a κ-stationary subset of IA). This can also be seen indirectly from Corollary 57 in § 6 along with Lemma 2.5 of [6]. However, the converse does not hold, but does hold for IA stationary sets of length κ.

FACT 26. *$IA\setminus K_0$ and $IA\setminus K_1$ are stationary subsets of IA which are not κ-stationary.*

PROOF. Let $C \subseteq [H(\lambda)]^{\leq\kappa}$ be club. Without loss of generality, assume that for all $x \in C$, $x \prec \langle H(\theta), \in, \dots\rangle$ for some fixed suitably large θ. Let $N_0 \in C$. Then $\langle N_0\rangle \in H(\lambda)$. Given $N_0 \subseteq \dots \subseteq N_i$ in C, the sequence $\langle N_0, \dots, N_i\rangle \in H(\lambda)$. Let $N_{i+1} \in C$ such that $\langle N_0, \dots, N_i\rangle \in N_{i+1}$ and $N_i \subseteq N_{i+1}$. Let $N = \bigcup_{i<\omega} N_i$. Then $N \in C$ and $N \subseteq IA$, since for each $j < \omega$, $\langle N_i : i \leq j\rangle \in N_{j+1} \subseteq N$. However, $N \notin K_0$, since $\{N_i : i < \omega\} \notin N$. Similarly, $N \notin K_1$. ⊣

OPEN PROBLEM 27. If $\lambda^{<\lambda} > \lambda$, is the κ-club filter on $[\lambda]^{\leq\kappa}$ equal to the club filter restricted to some κ-club set?

We end this section by mentioning some results of Piper. In his PhD thesis [23], Piper showed there is a cardinal and cofinality-preserving forcing which adds a destructible stationary set. He has noted that a similar forcing will add μ-many disjoint destructible $(*)$-stationary subsets of $[\lambda]^{<\kappa}$ for any $\mu < \kappa$, where $(*)$-stationary here denotes any of fat stationary, $\geq \nu$-stationary for any regular $\nu < \kappa$, or r-stationary (see [24] for the definition) for κ Mahlo or the successor of a regular cardinal. We note that r-stationarity on $[\lambda]^{\leq\kappa}$ is equivalent to κ-stationarity for regular κ. We mention a couple of his theorems which apply to κ-stationary sets and refer the reader to his papers for r-stationary versions when κ is Mahlo.

THEOREM 28 (Piper, [24]). *If κ is regular and $\diamondsuit^r_{\kappa^+,\lambda,\subseteq}$, then there is a family of $2^{|\mathscr{P}_{\kappa^+}\lambda|}$-many κ-stationary subsets of $\mathscr{P}_{\kappa^+}\lambda$ such that the intersection of any two of them is not stationary.*

THEOREM 29 (Piper, [24]). *If κ is regular, then $\diamondsuit^r_{\kappa^+}\lambda$ implies $\clubsuit'^r_{\kappa^+,\lambda}$.*

THEOREM 30 (Piper, [25]). *Let κ be regular. If F is a strongly normal filter over $[\lambda]^{\leq\kappa}$ with $\{x \in [\lambda]^{\leq\kappa} : |x \cap \kappa^+| = |x|\} \in F$, then the κ-club filter is contained in F.*

§5. The game $\mathscr{G}^{\kappa}_{<\mu}(\lambda)$ and a characterization of the $(\kappa, \lambda, <\mu)$-distributive law when $\lambda \geq \kappa$. We now set out to extend Theorems 7 and 8 of Kamburelis. Unlike his work with $[\lambda]^{\omega}$, we now must deal with the facts that new subsets of λ of size less than κ may be added, and that stationary sets no longer play the decisive role, rather, κ-stationary sets do.

The following generalizes Lemma 1.1 in [14] to games of uncountable length and/or uncountable choice sets for Player II. The proof, being completely analogous to that given by Kamburelis for $\mathscr{G}^{\omega}_{<\omega}(\lambda)$, is omitted. Recall: $\mathbb{B}$ denotes a complete Boolean algebra.

LEMMA 31. *There is a natural correspondence $\sigma \mapsto \dot{f}_\sigma$ between strategies σ for Player I in $\mathscr{G}^{\kappa}_{<\mu}(\lambda)$ and $\mathbb{B}$-valued names $\dot{f}_\sigma$ such that $\|\dot{f}_\sigma : ([\check{\lambda}]^{<\check{\mu}})^{<\check{\kappa}} \to \check{\lambda}\| > \mathbf{0}$. Moreover, σ is a winning strategy for I iff for each branch x in $([\lambda]^{<\mu})^{<\kappa}$ (in V),*

$$\|(\forall\alpha < \check{\kappa})\ \dot{f}_\sigma(\check{x} \upharpoonright \alpha) \in \check{x}(\alpha)\| = \mathbf{0}. \tag{5.1}$$

Notation. By $[\check{\lambda}]^{\leq\check{\kappa}}$ we denote the set of (good) $\mathbb{B}$-names for ranges of functions from $\check{\kappa}$ into $\check{\lambda}$. Thus, in $V[G]$, $[\check{\lambda}]^{\leq\check{\kappa}}$ denotes the collection of ranges of functions in $V[G]$ from from the ordinal $\check{\kappa}$ into the ordinal $\check{\lambda}$, regardless of their cardinalities in $V[G]$. This is equal to the collection of subsets (in $V[G]$) of the ordinal $\check{\lambda}$ of cardinality less than or equal to the cardinality of $\check{\kappa}$ in $V[G]$. So there is no ambiguity about what we mean by $[\check{\lambda}]^{\leq\check{\kappa}}$ in $V[G]$. Similarly for $([\check{\lambda}]^{<\check{\mu}})^{<\check{\kappa}}$. Whenever $\mathbb{B}$ preserves κ as a regular cardinal, we write $[\check{\lambda}]^{\leq\kappa}$ instead of $[\check{\lambda}]^{\leq\check{\kappa}}$ and $([\check{\lambda}]^{<\check{\mu}})^{<\kappa}$ instead of $([\check{\lambda}]^{<\check{\mu}})^{<\check{\kappa}}$.

The following easy fact is useful for guaranteeing preservation of a regular cardinal.

FACT 32. *If κ is regular and $\mathbb{B}$ satisfies the $(\rho, \kappa, <\kappa)$-d.l. for all $\rho < \kappa$, then κ is a regular cardinal in any extension of V by $\mathbb{B}$.*

The next theorem basically follows Kamburelis' proof of Theorem 7. The main differences are the following. When dealing with $[\check{\lambda}]^{<\kappa}$ in $V^{\mathbb{B}}$, we must keep track of which sets are in V and which are new. The key set $\dot{C}$ in the proof is not necessarily club, but only κ-club. Also, we realized that it is only necessary to assume $\mathbb{B}$ is $(\kappa, \kappa, <\mu)$-distributive, not $(\kappa, \lambda, <\mu)$-distributive, thereby improving on Kamburelis' result in the case of $\kappa = \omega$. We note that for $\mu \leq \kappa$, the $(\kappa, \kappa^+, <\mu)$-d.l. holds iff the $(\kappa, \kappa, <\mu)$-d.l. holds and κ^+ is preserved.

THEOREM 33. *Suppose the following hold.*

1. $\mu \leq \kappa = \kappa^{<\kappa} \leq \lambda$;
2. *$\mathbb{B}$ satisfies the $(\kappa, \kappa, <\mu)$-d.l.;*
3. *If $\lambda > \kappa$, then $\mathbb{B}$ preserves κ^+ as a cardinal above κ;*
4. *$V^{\mathbb{B}} \models$ "$|\kappa^{<\kappa}| = \kappa$ and $[\check{\lambda}]^{\leq\kappa} \cap V$ is κ-stationary".*

Then Player I does not have a winning strategy for $\mathscr{G}^{\kappa}_{<\mu}(\lambda)$ in $\mathbb{B}$.

PROOF. Since κ is regular in V and $\mathbb{B}$ is $(\kappa, \kappa, <\mu)$-distributive, κ is again a regular cardinal in any generic extension by $\mathbb{B}$. Hence, we use κ throughout instead of $\check{\kappa}$, since no ambiguity arises. Let σ be a strategy for Player I, $\dot{f}_\sigma$ be the corresponding $\mathbb{B}$-name from Lemma 31, and $a = \|\dot{f}_\sigma : ([\check{\lambda}]^{<\check{\mu}})^{<\kappa} \to \check{\lambda}\|$. Let $\dot{C}$ be the set of "good" $\mathbb{B}$-names $\dot{X}$ for elements of $[\check{\lambda}]^{\leq\kappa}$ such that $\|\forall s \in ([\dot{X}]^{<\check{\mu}})^{<\kappa}\ \dot{f}_\sigma(s) \in \dot{X}\| \geq a$. Using the fact that $V^{\mathbb{B}} \models$ "$|\kappa^{<\kappa}| = \kappa$" and Lemma 19, it is routine to show that

$$a \leq \|\dot{C} \text{ is } \kappa\text{-club in } [\check{\lambda}]^{\leq\kappa}\|. \tag{5.2}$$

Note: $\dot{C}$ is not necessarily club. We can only guarantee closure under $\dot{f}_\sigma$ for sequences in $\dot{C}$ of length κ.

By assumption (4), $\mathbf{1} = \|[\check{\lambda}]^{\leq\kappa} \cap V$ is κ-stationary$\|$, so $a \leq \|\dot{C} \cap V \cap [\check{\lambda}]^{\leq\kappa} \neq \emptyset\|$. Hence, by (3) there exists $\mathbf{0} < b \leq a$ and an $X \in [\lambda]^{\leq\kappa}$ in V such that $b \leq \|\check{X} \in \dot{C}\|$. Let T denote the tree $([X]^{<\mu})^{<\kappa}$ in V. Since $\mathbb{B}$ satisfies the $(\kappa, \kappa, <\mu)$-d.l. and $b \leq \|\dot{f}_\sigma \restriction \check{T} : \check{T} \to \check{X}\|$, there exists $\mathbf{0} < c \leq b$ and an $F \in V$ such that $F : T \to [X]^{<\mu}$ and $c \leq \|(\forall s \in \check{T})\ \dot{f}_\sigma(s) \in \check{F}(s)\|$.

Let $x(0) = F(\langle\ \rangle)$. For $\alpha < \kappa$, let $x(\alpha) = F(x \restriction \alpha)$. Then x is a branch of T, and $x \in V$. $c \leq \|(\forall \alpha < \kappa)\ \dot{f}_\sigma(\check{x} \restriction \alpha) \in \check{x}(\alpha)\|$. By Lemma 31, σ is not a winning strategy for I in $\mathscr{G}^{\kappa}_{<\mu}(\lambda)$. ⊣

Remark. If $\mathbb{B}$ is $(\kappa, \kappa^+, <\mu)$-distributive, then κ^+ is not collapsed to κ, so (3) holds automatically.

Theorems 1 and 33 yield the following.

COROLLARY 34 (GCH). 1. *Suppose κ, λ are such that $\neg((\lambda \geq \kappa) \wedge$ (κ is singular) $\wedge$ ($cf(\lambda) < \kappa$)). Then $\mathbb{B}$ is (κ, λ)-distributive iff I does not have a winning strategy for $\mathscr{G}_1^\kappa(\lambda)$ in $\mathbb{B}$.*

2. *Suppose ($\lambda < \kappa$) or ($\kappa = \lambda$ and is regular) or (κ is regular, $\mu \leq \kappa$, and $V^{\mathbb{B}} \models$"$|\kappa^{<\kappa}| = \kappa$"). Then $\mathbb{B}$ is $(\kappa, \kappa, <\mu)$-distributive iff I does not have a winning strategy for $\mathscr{G}_{<\mu}^\kappa(\kappa)$.*

EXAMPLE 35. Suppose κ is regular and $\kappa \leq \lambda$. Let $\mathbb{B} = \text{Coll}(\kappa, \lambda)$. $\mathbb{B}$ satisfies the $(<\kappa, \infty)$-d.l. (hence preserves κ) but not the $(\kappa, \kappa, <\mu)$-d.l. for any $\mu \leq \kappa$. Trivially, $V^{\mathbb{B}} \models$ "$[\check{\lambda}]^{\leq\kappa} \cap V$ is club in $[\check{\lambda}]^{\leq\kappa}$". However, I wins $\mathscr{G}_{<\mu}^\kappa(\lambda)$, since the $(\kappa, \lambda, <\mu)$-d.l. fails in $\mathbb{B}$.

THEOREM 36. *Suppose $\kappa < \lambda$, and κ is regular. If $\|[\check{\lambda}]^{\leq\check{\kappa}} \cap V$ is non-stationary$\|_{\mathbb{B}} > \mathbf{0}$, then Player I has a winning strategy for $\mathscr{G}_\kappa^\kappa(\lambda)$ in $\mathbb{B}$.*

We omit the proof of Theorem 36, as it is similar to that of Theorem 37, using the Strong Club Theorem 20 in place of the Strong κ-Club Theorem 21.

The following theorem shows that the assumption $V^{\mathbb{B}} \models$"$[\check{\lambda}]^{\leq\kappa} \cap V$ is κ-stationary" was necessary for Theorem 33. Our proof basically follows that of Kamburelis for Theorem 8, using Theorem 21 (3) in place of his use of Kueker's Theorem 10.

THEOREM 37. *Suppose that $\kappa^{<\kappa} = \kappa \leq \lambda$ and $\mathbb{B}$ satisfies the $(<\kappa, \kappa)$-d.l. If $\|[\check{\lambda}]^{\leq\kappa} \cap V$ is not κ-stationary$\|_{\mathbb{B}} > \mathbf{0}$, then Player I has a winning strategy for $\mathscr{G}_\kappa^\kappa(\lambda)$ in $\mathbb{B}$.*

PROOF. The $(<\kappa, \kappa)$-distributivity of $\mathbb{B}$ implies some useful facts. First, all cardinals less than or equal to κ are preserved (so κ is still regular). Hence, we write κ instead of $\check{\kappa}$ throughout. Second, for any set $x \in V$ with $|x| \leq \kappa$, $[x]^{<\kappa}$ is the same in any extension of V by $\mathbb{B}$ as it is in V. Hence, $V^{\mathbb{B}} \models$"$|\kappa^{<\kappa}| = \kappa$".

Let $a = \|[\check{\lambda}]^{\leq\kappa} \cap V$ is not κ-stationary$\|$. Then there is some $\mathbb{B}$-name $\dot{C}$ such that

$$a = \|\dot{C} \subseteq [\check{\lambda}]^{\leq\kappa},\ \dot{C} \text{ is } \kappa\text{-club, and } \dot{C} \cap V = \emptyset\|. \tag{5.3}$$

By the Strong κ-Club Theorem 21 in $V^{\mathbb{B}}$, there exists a $\mathbb{B}$-name $\dot{h}$ such that

$$a = \|\dot{h} : [\check{\lambda}]^{<\kappa} \to \check{\lambda}, \text{ and } \dot{D} \subseteq \dot{C}\|, \tag{5.4}$$

where

$$\dot{D} = \{x \in [\check{\lambda}]^{\leq\kappa} : (\forall y \in [x]^{<\kappa})\ \dot{h}(y) \in x\}. \tag{5.5}$$

In V, fix a surjection $\langle \pi, \varphi \rangle : \kappa \setminus \{0\} \to \kappa \times \kappa$ such that for all $\alpha < \kappa$, $\pi(\alpha) < \alpha$. The following defines a winning strategy for Player I for $\mathscr{G}_\kappa^\kappa(\lambda)$ in $\mathbb{B}$. At the beginning of the game, I fixes a. On round 0, I plays

$$W_0 = \{a \wedge \|\dot{h}(\emptyset) = \beta\| : \beta < \lambda\}, \tag{5.6}$$

which is a partition of a of size $\leq \lambda$. Then II chooses some $K_0 \in [\lambda]^{\leq\kappa}$ (in V) and plays

$$E_0 = \{a \wedge \|\dot{h}(\emptyset) = \beta\| : \beta \in K_0\}. \tag{5.7}$$

Enumerate $[K_0]^{<\kappa}$ as $\langle F^0_\gamma : \gamma < \kappa\rangle$, allowing repetitions. (Recall: $|K_0| \leq \kappa$ and $(<\kappa, \kappa)$-distributivity of $\mathbb{B}$ imply $[K_0]^{<\kappa}$ is the same in V as in any extension of V by $\mathbb{B}$.) On round $0 < \alpha < \kappa$, let I play the partition of a

$$W_\alpha = \{a \wedge \|\dot{h}(\check{F}^{\pi(\alpha)}_{\varphi(\alpha)}) = \beta\| : \beta < \lambda\}. \tag{5.8}$$

II chooses some $K_\alpha \in [\lambda]^{\leq\kappa}$ and plays

$$E_\alpha = \{a \wedge \|\dot{h}(\check{F}^{\pi(\alpha)}_{\varphi(\alpha)}) = \beta\| : \beta \in K_\alpha\}. \tag{5.9}$$

Enumerate $[\bigcup_{\delta\leq\alpha} K_\delta]^{<\kappa}$ as $\langle F^\alpha_\gamma : \gamma < \kappa\rangle$; continue in this manner. Let

$$b = \bigwedge_{\alpha<\kappa} \bigvee E_\alpha. \tag{5.10}$$

Suppose $b > \mathbf{0}$. Let $K = \bigcup_{\alpha<\kappa} K_\alpha$. Note that $K \in V$. Fix an $F \in [K]^{<\kappa}$. There is some $\beta < \kappa$ for which $F \subseteq \bigcup_{\zeta\leq\beta} K_\zeta$. So $F = F^\beta_\gamma$ for some $\gamma < \kappa$. Take an $\alpha < \kappa$ satisfying $\langle \pi(\alpha), \varphi(\alpha)\rangle = \langle \beta, \gamma\rangle$. Then

$$b \leq \|\dot{h}(\check{F}^{\pi(\alpha)}_{\varphi(\alpha)}) \in \check{K}_\alpha\| = \|\dot{h}(\check{F}) \in \check{K}_\alpha\|. \tag{5.11}$$

Since F was arbitrary,

$$b \leq \|(\forall F \in [\check{K}]^{<\kappa})\ \dot{h}(F) \in \check{K}\|. \tag{5.12}$$

So $b \leq \|\check{K} \in \dot{D}\|$. $b \leq a$ implies $b \leq \|\dot{D} \cap V = \emptyset\|$, so $b \leq \|\check{K} \notin V\|$. Contradiction. Therefore, $b = \mathbf{0}$, so I wins $\mathcal{G}^\kappa_\kappa(\lambda)$. ⊣

From Theorems 33 and 37 we can extract the following corollaries.

COROLLARY 38. *Assume $\mu \leq \kappa = \kappa^{<\kappa} \leq \lambda$, $\mathbb{B}$ is $(<\kappa, \kappa)$-distributive, and if $\lambda > \kappa$ then $\mathbb{B}$ does not collapse κ^+. Then the following are equivalent.*

1. *$\mathbb{B}$ is $(\kappa, \lambda, <\mu)$-distributive and $V^{\mathbb{B}} \models$"$[\check{\lambda}]^{\leq\kappa} \cap V$ is κ-stationary".*
2. *$\mathbb{B}$ is $(\kappa, \kappa, <\mu)$-distributive and $V^{\mathbb{B}} \models$"$[\check{\lambda}]^{\leq\kappa} \cap V$ is κ-stationary".*
3. *Player I does not have a winning strategy for $\mathcal{G}^\kappa_{<\mu}(\lambda)$ in $\mathbb{B}$.*

For $\kappa = \omega$, $(1) \iff (3)$ is due to Kamburelis [14].

COROLLARY 39. *Assume the following.*

1. $\kappa = \kappa^{<\kappa} < \lambda$;
2. *$\mathbb{B}$ is $(<\kappa, \kappa)$-distributive*;
3. *$V^{\mathbb{B}} \models$"$[\check{\lambda}]^{\leq\kappa} \cap V$ is κ-stationary".*

Then $\forall\mu \leq \kappa$, $\mathbb{B}$ is $(\kappa, \kappa^+, <\mu)$-distributive iff $\mathbb{B}$ is $(\kappa, \lambda, <\mu)$-distributive. In particular, if (1) and (3) hold, then $\mathbb{B}$ is $(\kappa, 2)$-distributive iff $\mathbb{B}$ is (κ, λ)-distributive.

Remark. The case $\kappa = \omega$ follows from the proof of Proposition 4.1 in Jech's [12]: If $\mathbb{B}$ preserves stationarity of $[\check{\lambda}]^\omega \cap V$ for all cardinals λ and adds no new reals, then $\mathbb{B}$ adds no new ω-sequences.

One might naturally want to understand the relationships of the condition $V^{\mathbb{B}} \models$"$[\check{\lambda}]^{\leq\kappa} \cap V$ is κ-stationary" with neighboring properties. We first point out two trivial facts. The $(\kappa, \lambda, \kappa)$-d.l. holds in $\mathbb{B}$ iff the set $[\lambda]^{\leq\kappa}$ in V remains unbounded in the set $[\check{\lambda}]^{\leq\check{\kappa}}$ in $V^{\mathbb{B}}$; and whenever $\mathbb{B}$ is (κ, λ)-distributive, then $V^{\mathbb{B}} \models$"$[\check{\lambda}]^{\leq\kappa} \cap V$ is κ-stationary". The condition $V^{\mathbb{B}} \models$"$[\check{\lambda}]^{\leq\kappa} \cap V$ is κ-stationary" is strictly weaker than preserving all κ-stationary subsets of $[\check{\lambda}]^{\leq\kappa}$, but non-trivial. Shooting a club through a stationary/co-stationary subset of $\aleph_1$ is not proper, yet satisfies $V^{\mathbb{B}} \models$"$[\check{\lambda}]^{\omega} \cap V$ is stationary" for all λ since this forcing is (ω, ∞)-distributive. More generally, for $\kappa^{<\kappa} = \kappa$, shooting a κ-club through a κ-stationary/co-κ-stationary subset of $[\lambda]^{\leq\kappa}$ (see Theorem 63) gives a partial ordering which destroys a κ-stationary subset of $[\check{\lambda}]^{\leq\kappa}$, yet preserves κ-stationarity of $[\check{\lambda}]^{\leq\kappa} \cap V$ for all λ since this forcing is (κ, ∞)-distributive.

The next Theorem 40 gives general conditions under which $V^{\mathbb{B}} \models$"$[\check{\lambda}]^{\leq\kappa} \cap V$ is κ-stationary" holds. The proof uses ideas from Theorem 6 of Abraham and Shelah in [1], which gives the result for $\kappa = \omega$ and $\lambda = \aleph_2$. Note: when κ^+ is not collapsed to κ, $[\lambda]^{\leq\kappa} \cap V$ is the same as $([\lambda]^{\leq\kappa})^V$.

Theorem 40. *Let $2 \leq n < \omega$. Suppose $V \subseteq W$ are models of ZFC such that for all $m \leq n$, κ^{+m} is the same in V as in W, and $\kappa^{<\kappa} = \kappa$ in W. Then $[\kappa^{+n}]^{\leq\kappa} \cap V$ is κ-stationary in W.*

Proof. Let C be κ-club in $([\kappa^{+n}]^{\leq\kappa})^W$. In W, by the Strong κ-Club Theorem 21, there is a function $f : [\kappa^{+n}]^{<\kappa} \to \kappa^{+n}$ such that $C_f \subseteq C$, where $C_f = \{x \in ([\kappa^{+n}]^{\leq\kappa})^W : (\forall y \in [x]^{<\kappa})\ f(y) \in x\}$. First we note that κ^{+n} is closed under f in W. The rest of the proof follows by finitely many applications of the next claim. Its proof uses the fact that $\kappa^{<\kappa} = \kappa$ in W and all κ^{+m} are preserved $(m \leq n)$ imply that $(\kappa^{+m})^{<\kappa} = \kappa^{+m}$ $(m \leq n)$.

Claim. Let $m \leq n$. If $a \subseteq \kappa^{+n}$ in V such that $|a| = \kappa^{+m}$ and a is closed under f in W, then there exists $b \subseteq a$ such that $b \in V$, $|b| < \kappa^{+m}$, and b is closed under f in W.

$a \in V$ and $|a| = \kappa^{+m}$ imply there is a bijection $h : \kappa^{+m} \to a$ in V. We will show there is a $\beta < \kappa^{+m}$ such that $h''\beta$ is closed under f in W. Let $w_0 = h''1$, $z_0 = w_0 \cup \{f(y) : y \in [w_0]^{<\kappa}\}$, $x_0 = h^{-1}(z_0)$, and $\beta_0 = \sup(x_0) + 1$. In general, let $w_{\zeta+1} = h''\beta_\zeta$, $z_{\zeta+1} = w_{\zeta+1} \cup \{f(y) : y \in [w_{\zeta+1}]^{<\kappa}\}$, $x_{\zeta+1} = h^{-1}(z_{\zeta+1})$, and $\beta_{\zeta+1} = \sup(x_{\zeta+1}) + 1$. For limit $\zeta < \kappa$, let $\beta_\zeta = \sup\{\beta_\xi : \xi < \zeta\}$. Let $\beta = \sup\{\beta_\zeta : \zeta < \kappa\}$. Then $\beta < \kappa^{+m}$ and $\mathrm{cf}(\beta) = \kappa$. Let $y \in [h''\beta]^{<\kappa}$. Then $\exists\zeta < \kappa$ such that $y \subseteq h''\beta_\zeta$; so $f(y) \in z_{\zeta+1} = h''(x_{\zeta+1}) \subseteq h''\beta_{\zeta+1} \subseteq h''\beta$. Let $b = h''\beta$. Then $b \in V$ and $|b| = |\beta| < \kappa^{+m}$. b is closed under f in W, so $b \in C_f$. $\dashv$

From Corollary 39 and Theorem 40 we obtain the following.

Corollary 41. *Suppose $2 \leq n < \omega$, $\kappa^{<\kappa} = \kappa$, and $\mathbb{B}$ preserves κ^{+m} for all $m \leq n$.*

1. *If $\mathbb{B}$ is $(\kappa, 2)$-distributive, then $\mathbb{B}$ is (κ, κ^{+n})-distributive.*
2. *If $\mu \leq \kappa$, $\mathbb{B}$ is $(\kappa, \kappa, < \mu)$-distributive, then $\mathbb{B}$ is $(\kappa, \kappa^{+n}, < \mu)$-distributive.*

Example 42 (Kamburelis, [14]). There is a Boolean algebra in which the weak (ω, ∞)-d.l. holds, but I has a winning strategy for $\mathscr{G}^{\omega}_{\omega}(\omega_2)$.

Open Problem 43. For $\omega < \kappa \leq \lambda$ and $\mu \leq \kappa$, is there a Boolean algebra in which the $(\kappa, \lambda, < \mu)$-d.l. holds, yet I has a winning strategy for $\mathscr{G}^{\kappa}_{<\mu}(\lambda)$?

Such a winning strategy must be dependent on Player II's choices, for otherwise, the distributive law would fail. We will demonstrate later (Example 61) a solution for this problem when $\mu = \kappa^+$.

§6. Preservation of κ-stationarity and a generalized version of the proper game. We say that $\mathbb{B}$ *preserves κ-stationary subsets of* $[\lambda]^{\leq\kappa}$ if $\check{\kappa}$ is regular in $V^{\mathbb{B}}$ and for each κ-stationary $S \subseteq [\lambda]^{\leq\kappa}$ in V, $\check{S}$ is a $\check{\kappa}$-stationary subset of $[\check{\lambda}]^{\leq\check{\kappa}}$ in $V^{\mathbb{B}}$. We define a game $\Gamma^{\kappa}_{\kappa}(\lambda)$ which is a natural generalization and refinement of the proper game, due to Gray [7], and which is easier for Player II to win than $\mathscr{G}^{\kappa}_{\kappa}(\lambda)$. We map out the relationships between $\mathscr{G}^{\kappa}_{\kappa}(\lambda)$, $\Gamma^{\kappa}_{\kappa}(\lambda)$, and preservation of κ-stationary subsets of $[\lambda]^{\leq\kappa}$, leading to a, perhaps, surprising Corollary 59.

FACT 44. *If $\mathbb{P}$ preserves κ-stationary subsets of $[\lambda]^{\leq\kappa}$, then $\mathbb{P}$ does not collapse κ^+ to κ, nor does $\mathbb{P}$ reduce any cofinalities $\kappa < \nu \leq \lambda$ to κ.*

PROPOSITION 45. *Suppose $\kappa^{<\kappa} = \kappa \leq \lambda$, and $\mathbb{B}$ satisfies the $(\lambda^{<\kappa}, \lambda, \kappa)$-d.l. and the $(<\kappa, \lambda)$-d.l. Then every κ-club in $V^{\mathbb{B}}$ contains a κ-club in V. Hence, $\mathbb{B}$ preserves every κ-stationary subset of $[\lambda]^{\leq\kappa}$.*

PROOF. The $(<\kappa, \lambda)$-d.l. implies that all cardinals $\leq \kappa$ are preserved and that $\forall x \in V$ with $|x| \leq \lambda$, $\mathbb{B}$ adds no new subsets of x of cardinality less than κ. Let $S \subseteq [\lambda]^{\leq\kappa}$ be a κ-stationary set in V. Let G be a generic filter on $\mathbb{B}$, and let $\dot{C}$ be a $\mathbb{B}$-name for which $V[G] \models$ "$\dot{C}$ is a κ-club subset of $[\check{\lambda}]^{\leq\kappa}$". In $V[G]$, $\kappa^{<\kappa} = \kappa$, since $\kappa^{<\kappa} = \kappa$ in V and $\mathbb{B}$ is $(<\kappa, \kappa)$-distributive.

In what follows, all $\mathbb{B}$-names are evaluated in $V[G]$. By the Strong κ-Club Theorem 21, there exists an $\dot{h} : [\check{\lambda}]^{<\kappa} \to \check{\lambda}$ such that, letting

$$\dot{D} = \{x \in [\check{\lambda}]^{\leq\kappa} : (\forall y \in [x]^{<\kappa})\ \dot{h}(y) \in x\}, \tag{6.1}$$

$\dot{D}$ is κ-club in $[\check{\lambda}]^{\leq\kappa}$ and $\dot{D} \subseteq \dot{C}$. Since $\mathbb{B}$ satisfies the $(\lambda^{<\kappa}, \lambda, \kappa)$-d.l., there exists some $g \in V$ such that $g : [\lambda]^{<\kappa} \to [\lambda]^{\leq\kappa}$ and for all $x \in [\lambda]^{<\kappa}$, $\dot{h}(x) \in g(x)$. In V, let $C_g = \{x \in [\lambda]^{\leq\kappa} : (\forall y \in [x]^{<\kappa})\ g(y) \in x\}$. Then C_g is κ-club, by Lemma 19, so $S \cap C_g \neq \emptyset$. Moreover, $x \in C_g \Longrightarrow (\forall y \in [x]^{<\kappa})\ \dot{h}(y) \in g(y) \subseteq x \Longrightarrow x \in \dot{D}$. Hence, $\dot{D} \cap S \neq \emptyset$. Therefore, S is κ-stationary in $V[G]$. ⊣

FACT 46. *If $\mathbb{B}$ is κ^+-c.c., then $\mathbb{B}$ is (∞, ∞, κ)-distributive.*

The proof is straightforward, using the forcing equivalent of the (∞, ∞, κ)-d.l. (Fact 4 (3)).

COROLLARY 47. *Suppose $\kappa^{<\kappa} = \kappa \leq \lambda$, $\mathbb{B}$ is κ^+-c.c., and $\mathbb{B}$ is $(<\kappa, \lambda)$-distributive. Then $\mathbb{B}$ preserves all κ-stationary subsets of $[\lambda]^{\leq\kappa}$.*

We now present a game which is a natural extension of the properness game to uncountable lengths.

DEFINITION 48. $\mathscr{S}^{\kappa}_{1}(\lambda)$ is a game of length κ played on a partial ordering $(\mathbb{P}, \leq)$ as follows: At the beginning of the game, Player I fixes a $p \in \mathbb{P}$. On the α-th round, I plays $\dot{\zeta}_\alpha$ a $\mathbb{P}$-name for an ordinal less than λ; then II chooses an ordinal $\delta_\alpha < \lambda$. II *wins the play* iff there is a $q \leq p$ such that $\forall\alpha < \kappa$, $q \Vdash$ "$(\exists\beta < \check{\kappa})\ \dot{\zeta}_\alpha = \delta_\beta$".

$\mathscr{S}^{\kappa}_{1}(\lambda)$ has the following equivalent version $\mathscr{S}^{\kappa}_{\kappa}(\lambda)$: On each round $\alpha < \kappa$, instead of choosing one ordinal δ_α, II now chooses a set of ordinals $B_\alpha \subseteq \lambda$ of size $\leq \kappa$. II wins the play iff $\exists q \leq p$ such that $\forall\alpha < \kappa$, $q \Vdash$ "$\dot{\zeta}_\alpha \in \bigcup_{\beta<\kappa} B_\beta$".

Let $\mathscr{S}^{\kappa}_{1}(\infty)$ and $\mathscr{S}^{\kappa}_{1}(\infty)$ denote the games when the players are allowed to chose names for ordinals and ordinals of any size.

Fact 49. 1. (Gray, [7]) *II has a winning strategy for $\mathcal{S}_1^\omega(\infty)$ in $\mathbb{P}$ iff $\mathbb{P}$ is proper.*
2. [12] *II has a winning strategy for $\mathcal{S}_1^\omega(\omega_1)$ in $\mathbb{P}$ iff $\mathbb{P}$ is semiproper.*

Jech showed that II wins $\mathscr{G}_\omega^\omega(\infty)$ strictly implies properness, and mentioned that by examples of Baumgartner and Shelah, Axiom A strictly implies II wins $\mathscr{G}_\omega^\omega(\infty)$ [11]. Zapletal has shown that II wins $\mathscr{G}_1^\omega(2)$ implies II wins $\mathcal{S}_1^\omega(\omega_1)$. Somewhat surprisingly, he has also shown that, assuming the consistency of the existence of a measurable cardinal, it is consistent that there is a Boolean algebra which is not proper, but in which II wins $\mathscr{G}_1^\omega(2)$. Moreover, in ZFC there is a proper (ω, ∞)-distributive Boolean algebra of density $2^{\aleph_0}$ in which $\mathscr{G}_1^\omega(2)$ is undetermined. See [26] for more results on $\mathscr{G}_1^\omega(2)$.

The following $\Gamma_\kappa^\kappa(\lambda)$ is a Boolean algebraic version of $\mathcal{S}_\kappa^\kappa(\lambda)$. $\Gamma_\omega^\omega(\infty)$ can be found in [12] under the notation $\mathscr{G}$.

Definition 50. Let $\kappa < \lambda$. The game $\Gamma_\kappa^\kappa(\lambda)$ is played on a complete Boolean algebra as follows. First, Player I fixes some $a \in \mathbb{B}^+$. On round $\alpha < \kappa$, Player I chooses W_α a partition of a with $|W_\alpha| \leq \lambda$. Then for each $\beta \leq \alpha$, Player II chooses a $B_\beta^\alpha \subseteq W_\beta$ such that $|B_\beta^\alpha| \leq \kappa$. II *wins the play* iff

$$\bigwedge_{\alpha<\kappa} \bigvee_{\alpha\leq\gamma<\kappa} \left(\bigvee B_\alpha^\gamma\right) > \mathbf{0}. \tag{6.2}$$

$\Gamma_\kappa^\kappa(\infty)$ denotes the game where Player I is permitted to choose partitions of any cardinality.

Fact 51. *The games $\mathcal{S}_\kappa^\kappa(\lambda)$, $\mathcal{S}_1^\kappa(\lambda)$, and $\Gamma_\kappa^\kappa(\lambda)$ are equivalent, in that I (II) has a winning strategy for one of them iff I (II) has a winning strategy for each of them.*

Fact 52. *Let $\lambda > \kappa$.*

1. *If II wins $\mathscr{G}_\kappa^\kappa(\lambda)$, then II wins $\Gamma_\kappa^\kappa(\lambda)$; the reverse holds for Player I.*
2. *If Player II has a winning strategy for $\Gamma_\kappa^\kappa(\lambda)$ in $\mathbb{B}$, then $\mathbb{B}$ satisfies the $(\kappa, \lambda, \kappa)$-d.l.*

Remarks. 1. The property "II wins $\mathscr{G}_\kappa^\kappa(\lambda)$" is strictly stronger than "II wins $\Gamma_\kappa^\kappa(\lambda)$", as will be shown in Example 61. The reverse holds for Player I.
2. There is a κ-length version of Axiom A which implies that II wins $\mathscr{G}_\kappa^\kappa(\infty)$, analogously to Jech's result for ω-length games.

The following generalizes and refines the fact that if $\mathbb{P}$ is proper and adds no new reals, then $\mathbb{P}$ adds no new ω-sequences.

Proposition 53. *For $\lambda > \kappa$, if II wins $\mathcal{S}_\kappa^\kappa(\lambda)$ in $\mathbb{P}$ and r.o.$(\mathbb{P})$ is $(\kappa, 2)$-distributive, then r.o.$(\mathbb{P})$ is (κ, λ)-distributive.*

Proof. Let $\dot{\tau} = \langle \dot{\tau}_\alpha : \alpha < \kappa \rangle$ be a sequence of names for ordinals below λ. Let $p \in \mathbb{P}$. On round α, let I play $\dot{\tau}_\alpha$. Let II choose an ordinal $\xi_\alpha < \lambda$ according to II's winning strategy. Let $B = \{\xi_\alpha : \alpha < \kappa\}$. There exists a $q \leq p$ such that $q \Vdash (\forall \alpha < \check{\kappa})\, \dot{\tau}_\alpha \in B$. Since r.o.$(\mathbb{P})$ is $(\kappa, 2)$-distributive and $B \in V$, the evaluation of $\langle \dot{\tau}_\alpha : \alpha < \kappa \rangle$ in any generic extension of $\mathbb{P}$ is an element of V. Hence, $\mathbb{P}$ adds no new functions from $\check{\kappa}$ into $\check{\lambda}$. ⊣

We make the following definition, analogously to one given (but not named) by Jech in [12].

Definition 54. Let κ be regular and $\kappa < \lambda$. We say that $\mathbb{B}$ is *κ-club distributive over λ* if for each $b > \mathbf{0}$, for every collection $W_\alpha = \{b_{\alpha\beta} : \beta < \lambda\}$, $\alpha < \lambda$, such that each W_α is a quasipartition of b, there is a κ-club $C \subseteq [\lambda]^{\leq\kappa}$ such that for each $x \in C$,

$$\bigwedge_{\alpha\in x}\bigvee_{\beta\in x} b_{\alpha\beta} > \mathbf{0}. \tag{6.3}$$

The main ideas of the following Proposition 55 come from Jech's proof for the case $\kappa = \omega$, which can be found in [12]. Again, we use the Strong κ-Club Theorem 21 in place of Theorem 10.

Proposition 55. *Suppose $\kappa^{<\kappa} = \kappa < \lambda$.*

1. *If II has a winning strategy for $\Gamma^\kappa_\kappa(\lambda)$ played in $\mathbb{B}$, then $\mathbb{B}$ is κ-club distributive over λ.*
2. *Suppose also $\lambda = \lambda^{<\kappa}$, $\mathbb{B}$ is $(<\kappa, \kappa)$-distributive, and $\mathbb{B}$ is κ-club distributive over λ. Then $\mathbb{B}$ preserves all κ-stationary subsets of $[\lambda]^{\leq\kappa}$.*

Proof. To prove (1), suppose $\kappa^{<\kappa} = \kappa < \lambda$ and II has a winning strategy σ for $\Gamma^\kappa_\kappa(\lambda)$. Let $b > \mathbf{0}$ and $W_\alpha = \{b_{\alpha\beta} : \beta < \lambda\}$, $\alpha < \lambda$, be quasipartitions of b. Define a function $F : [\lambda]^{<\kappa} \to [\lambda]^{\leq\kappa}$ as follows. Let $e \in [\lambda]^{<\kappa}$. For each enumeration $\langle\alpha_j : j < \delta\rangle$ of e, let $\{B^j_i\}^{j<\delta}_{i\leq j}$ denote the moves of II using σ against $\langle W_{\alpha_j} : j < \delta\rangle$. Take $F(e) \in [\lambda]^{\leq\kappa}$ so that for each enumeration $\langle\alpha_j : j < \delta\rangle$ of e, $\forall i \leq j < \delta$, $(b_{\alpha_i,\beta} \in B^j_i \to \beta \in F(e))$. Let $C_F = \{x \in [\lambda]^{\leq\kappa} : \forall e \in [x]^{<\kappa},\ F(e) \subseteq x\}$. C_F is κ-club, by Lemma 19.

Let $x \in C_F$ and $\langle\alpha_i : i < \kappa\rangle$ be an enumeration of x. Consider the game $\Gamma^\kappa_\kappa(\lambda)$ when I fixes b and plays $\langle W_{\alpha_i} : i < \kappa\rangle$. For each $i < \kappa$, $\{\alpha_k : k \leq i\} \in [x]^{<\kappa}$, so $F(\{\alpha_k : k \leq i\}) \subseteq x$. Thus, for each fixed $i < \kappa$,

$$\begin{aligned} \textstyle\bigvee_{\beta\in x} b_{\alpha_i,\beta} &\geq \textstyle\bigvee\{b_{\alpha_i,\beta} : \beta \in F(e) \text{ for some } e \in [x]^{<\kappa}\} \\ &\geq \textstyle\bigvee_{i\leq j<\kappa}\left(\bigvee B^i_j\right). \end{aligned} \tag{6.4}$$

Since σ is a winning strategy for II,

$$\bigwedge_{\alpha\in x}\bigvee_{\beta\in x} b_{\alpha\beta} = \bigwedge_{i<\kappa}\bigvee_{\beta\in x} b_{\alpha_i,\beta} \geq \bigwedge_{i<\kappa}\bigvee_{i\leq j<\kappa}\left(\bigvee B^i_j\right) > \mathbf{0}. \tag{6.5}$$

Now assume the hypotheses of (2). Let $S \subseteq [\lambda]^{\leq\kappa}$ be a κ-stationary set. $(<\kappa, \kappa)$-distributivity of $\mathbb{B}$ and $\kappa^{<\kappa} = \kappa$ imply that $V^{\mathbb{B}} \models$ "$\check{\kappa}$ is a regular cardinal and $|\check{\kappa}^{<\check{\kappa}}| = \check{\kappa}$". So the Strong κ-Club Theorem 21 applies in $V^{\mathbb{B}}$. Hence, it suffices to show that for any $\mathbb{B}$-name $\dot{h} : [\check{\lambda}]^{<\kappa} \to \check{\lambda}$ and each $b > \mathbf{0}$, there is an $x \in S$ such that $\mathbf{0} < b \wedge \|\check{x}$ is closed under $\dot{h}\|$.

In V, let $\langle e_\alpha : \alpha < \lambda\rangle$ be a fixed enumeration of $[\lambda]^{<\kappa}$. Define $f : [\lambda]^{<\kappa} \to \lambda$ by $f(e) =$ the least α such that $e = e_\alpha$. $C_f = \{x \in [\lambda]^{\leq\kappa} : (\forall y \in [x]^{<\kappa})\ f(y) \in x\}$ is κ-club, by Lemma 19. Let $b > \mathbf{0}$. For each $\alpha, \beta < \lambda$, let $b_{\alpha\beta} = b \wedge \|\dot{h}(\check{e}_\alpha) = \beta\|$. Then for each $\alpha < \lambda$, $\{b_{\alpha\beta} : \beta < \lambda\}$ is a quasipartition of b. Since $\mathbb{B}$ is κ-club distributive over λ, there is a κ-club $C \subseteq [\lambda]^{\leq\kappa}$ such that for each $x \in C$, $\bigwedge_{\alpha\in x}\bigvee_{\beta\in x} b_{\alpha\beta} > \mathbf{0}$. Let $x \in S \cap C \cap C_f$ and let $a = \bigwedge_{\alpha\in x}\bigvee_{\beta\in x} b_{\alpha\beta}$. $x \in C_f$

implies for each $e \in [x]^{<\kappa}$ there is an $\alpha \in x$ such that $e = e_\alpha$. $\mathbb{B}$ is $(<\kappa,\kappa)$-distributive implies $[x]^{<\kappa}$ is the same in V as in $V^{\mathbb{B}}$. Therefore,

$$\|(\forall\alpha \in \check{x})(\exists\beta \in \check{x})\ \dot{h}(\check{e}_\alpha) = \beta\| \leq \|(\forall e \in [\check{x}]^{<\kappa})(\exists\beta \in \check{x})\ \dot{h}(e) = \beta\|. \qquad (6.6)$$

So, we have $\mathbf{0} < a \leq b \wedge \|\check{x}$ is closed under $\dot{h}\|$. Therefore, $\mathbb{B}$ preserves κ-stationarity of S. ⊣

Remark. By using arguments of Jech [11], one can directly show that if $\kappa^{<\kappa} = \kappa$, $\mathbb{B}$ is $(<\kappa,\kappa)$-distributive, and II wins $\mathscr{G}^\kappa_\kappa(\lambda)$ in $\mathbb{B}$, then $\mathbb{B}$ preserves all κ-stationary subsets of $[\lambda]^{\leq\kappa}$. This does not require $\lambda^{<\kappa} = \lambda$, as does (2) of Proposition 55.

The following extends a theorem of Menas to κ-club sets. (A proof of his result can be found in [13].) His argument works here, just replacing his use of Kueker's Theorem 10 with the Strong κ-Club Theorem 21.

LEMMA 56. *Let $A \subseteq B$ with $|A| \geq \kappa$. For $X \subseteq [A]^{\leq\kappa}$, let $X^* = \{y \in [B]^{\leq\kappa} : y \cap A \in X\}$. For $Y \subseteq [B]^{\leq\kappa}$, let $Y \restriction A = \{y \cap A : y \in Y\}$.*

1. *If $C \subseteq [A]^{\leq\kappa}$ is κ-club, then C^* is κ-club in $[B]^{\leq\kappa}$. Hence, if $S \subseteq [B]^{\leq\kappa}$ is κ-stationary, then $S \restriction A$ is κ-stationary in $[A]^{\leq\kappa}$.*
2. *Assume $\kappa^{<\kappa} = \kappa$. If $C \subseteq [B]^{\leq\kappa}$ is κ-club, then $C \restriction A$ contains a κ-club set in $[A]^{\leq\kappa}$. Hence, if $S \subseteq [A]^{\leq\kappa}$ is κ-stationary, then S^* is κ-stationary in $[B]^{\leq\kappa}$.*

COROLLARY 57. *Suppose $\kappa^{<\kappa} = \kappa$, $\mathbb{B}$ is $(<\kappa,\kappa)$-distributive, and Player II has a winning strategy for $\Gamma^\kappa_\kappa(\infty)$ in $\mathbb{B}$. Then $\mathbb{B}$ preserves all κ-stationary subsets of $[\lambda]^{\leq\kappa}$ for all cardinals $\lambda \geq \kappa$. Hence, $\kappa^{<\kappa} = \kappa$ and $<\kappa^+$-closure imply preservation of all κ-stationary subsets of $[\lambda]^{\leq\kappa}$ for all cardinals $\lambda \geq \kappa$.*

PROOF. Let $\lambda \geq \kappa$ be given. Let $\theta = \lambda^\kappa$. Then $\theta^{<\kappa} = \theta$. Let $S \subseteq [\lambda]^{\leq\kappa}$ be κ-stationary. By Lemma 56, $S^* = \{x \in [\theta]^{\leq\kappa} : x \cap \lambda \in S\}$ is κ-stationary in $[\theta]^{\leq\kappa}$. II wins $\Gamma^\kappa_\kappa(\theta)$ implies $\mathbb{B}$ preserves κ-stationarity of S^*, by Proposition 55. Let G be a $\mathbb{B}$-generic filter and evaluate the following in $V[G]$. Let $C \subseteq [\check{\lambda}]^{\leq\kappa}$ be κ-club, and let $C^* = \{x \in [\check{\theta}]^{\leq\kappa} : x \cap \check{\lambda} \in C\}$. C^* is κ-club, by Lemma 56. Let $x \in S^* \cap C^*$. Then $x \cap \check{\lambda} \in S \cap C$. Thus, S is a κ-stationary subset of $[\check{\lambda}]^{\leq\kappa}$ in $V[G]$. ⊣

EXAMPLES 58. Assume $\kappa^{<\kappa} = \kappa$. Let $\mathbb{P}(\kappa)$ denote perfect tree forcing on ${}^\kappa 2$. Kanamori investigated this forcing for $\kappa > \omega$ [15]. When $\kappa = \omega$ this is just Sacks forcing. Let $\mathbb{S}(\kappa)$ denote superperfect tree forcing on ${}^\kappa\kappa$, when $\kappa > \omega$. This has the same flavor as Miller forcing. Brown investigated $\mathbb{S}(\kappa)$ in [3]. Both $\mathbb{P}(\kappa)$ and $\mathbb{S}(\kappa)$ are $<\kappa$-closed; and by κ-length fusion, II has a winning strategy for $\mathscr{G}^\kappa_\kappa(\infty)$. Hence, they preserve all κ-stationary subsets of $[\lambda]^{\leq\kappa}$ for all $\lambda \geq \kappa$. In r.o.$(\mathbb{P}(\kappa))$, I wins $\mathscr{G}^\kappa_1(2)$, since a new function from κ into 2 is added. If κ is strongly inaccessible, then II has a winning strategy for $\mathscr{G}^\kappa_{<\kappa}(\infty)$. In r.o.$(\mathbb{S}(\kappa))$, I wins $\mathscr{G}^\kappa_{<\kappa}(\kappa)$, since the forcing adds a new unbounded function from κ into κ.

The next Corollary follows from Theorem 33, Proposition 55, and Corollary 57.

COROLLARY 59.
1. *Suppose $\mu \leq \kappa^{<\kappa} = \kappa < \lambda = \lambda^{<\kappa}$ and $\mathbb{B}$ is $(<\kappa,\kappa)$-distributive and $(\kappa,\kappa,<\mu)$-distributive. If II has a winning strategy for $\Gamma^\kappa_\kappa(\lambda)$, then I does not have a winning strategy for $\mathscr{G}^\kappa_{<\mu}(\lambda)$.*
2. *Suppose $\mu \leq \kappa^{<\kappa} = \kappa$ and $\mathbb{B}$ is $(<\kappa,\kappa)$-distributive and $(\kappa,\kappa,<\mu)$-distributive. If II has a winning strategy for $\Gamma^\kappa_\kappa(\infty)$, then I does not have a winning strategy for $\mathscr{G}^\kappa_{<\mu}(\infty)$.*

We now show that the games $\mathscr{G}^\kappa_\kappa(\lambda)$ and $\Gamma^\kappa_\kappa(\lambda)$ are not in general equivalent. What follows is a family of forcings which are variations on Baumgartner's adding a club to ω_1 with finite conditions. Letting $\nu \le \kappa$ be regular cardinals, we shall say that a function $f : \kappa^+ \to \kappa^+$ is $\ge \nu$-*normal* if f is strictly increasing and continuous at each ordinal $\zeta < \kappa^+$ such that $\mathrm{cf}(\zeta) \ge \nu$.

Definition 60. Let ν, κ be regular and $\omega \le \theta \le \nu \le \kappa$. Let $\mathbb{P}^\nu_\kappa(\theta)$ denote the following forcing notion. Conditions are partial functions p satisfying $|p| < \theta$, $\mathrm{dom}(p) \subseteq \kappa^+$, $p : \mathrm{dom}(p) \to \kappa^+$, and there is a $\ge \nu$-normal function $f : \kappa^+ \to \kappa^+$ such that $f \supseteq p$. $q \le p$ iff $q \supseteq p$.

Example 61. Suppose $\kappa^{<\theta} = \kappa$. $\mathbb{P}^\nu_\kappa(\theta)$ adds a new $\ge \nu$-club subset of κ^+. In r.o.$(\mathbb{P}^\nu_\kappa(\theta))$, I wins $\mathscr{G}^\nu_\kappa(\kappa^+)$ but II wins $\Gamma^\kappa_\kappa(\infty)$; hence, the (κ, ∞, κ)-d.l. holds.

Proof. The argument follows the basic outline of Jech's proofs for the case of $\kappa = \omega$, as given in [11] and [12]. We give our generalized version here for the sake of an unambiguous presentation.

By the usual density arguments, one can see that $\mathbb{P}^\nu_\kappa(\theta)$ adds a new $\ge \nu$-normal function from κ^+ into κ^+; its range yields a new $\ge \nu$-club subset of κ^+.

Claim 1. Player I wins $\mathscr{G}^\nu_\kappa(\kappa^+)$.

Let $W_0 = \{\{(0, \xi)\} : \xi < \kappa^+\}$. Let II choose some $E_0 \in [W_0]^{\le\kappa}$ and let $\gamma_0 = 0$. Let $\gamma_1 < \kappa^+$ be an ordinal satisfying $\gamma_1 = \kappa^{\gamma_1}$ and $\gamma_1 > \sup\{\xi < \kappa^+ : \{(0, \xi)\} \in E_0\}$. In general, given $\alpha < \nu$ and γ_α, let I play $W_\alpha = \{\{(\gamma_\alpha, \xi)\} : \{(\gamma_\alpha, \xi)\} \in \mathbb{P}^\nu_\kappa(\theta)\}$. II chooses some $E_\alpha \in [W_\alpha]^{\le\kappa}$. Pick a $\gamma_{\alpha+1} < \kappa^+$ satisfying $\gamma_{\alpha+1} = \kappa^{\gamma_{\alpha+1}}$ and $\gamma_{\alpha+1} > \sup\{\xi < \kappa^+ : \{(\gamma_\alpha, \xi)\} \in E_\alpha\}$. For limit ordinals $\alpha < \nu$, choose $\gamma_\alpha < \kappa^+$ satisfying $\gamma_\alpha = \kappa^{\gamma_\alpha}$ and $\gamma_\alpha > \sup_{\beta<\alpha} \gamma_\beta$.

Suppose I does not win $\mathscr{G}^\nu_\kappa(\kappa^+)$. Then there is a $p \in \mathbb{P}^\nu_\kappa(\theta)$ such that for each $\alpha < \nu$, $p \le \bigvee E_\alpha$. Let $\lambda = \sup_{\alpha<\nu} \gamma_\alpha$. Since $|p| < \theta \le \nu$ and ν is regular, there is an $\alpha < \nu$ such that $\mathrm{dom}(p) \cap \lambda = \mathrm{dom}(p) \cap \gamma_\alpha$. Let f be $\ge \nu$-normal witnessing that $p \in \mathbb{P}^\nu_\kappa(\theta)$. If there is a β with $\alpha < \beta < \nu$ and $f(\gamma_\beta) \ge \gamma_{\beta+1}$, then let $q = p \cup \{(\gamma_\beta, f(\gamma_\beta))\}$. Then $q \le p$. But for all $r \in E_\beta$, $q \perp r$, contradicting that $p \le \bigvee E_\beta$. Thus, for all β with $\alpha < \beta < \nu$, $f(\gamma_\beta) < \gamma_{\beta+1}$. Hence, $f(\lambda) = \lambda$.

We will find a $q \le p$ such that $q \perp r$ for all $r \in E_{\alpha+1}$. Define $g : \kappa^+ \to \kappa^+$ as follows. For all $\zeta \le \gamma_\alpha$, let $g(\zeta) = f(\zeta)$. Let $g(\gamma_\alpha + 1) = \gamma_{\alpha+2}$. If $\gamma_\alpha < \zeta < \lambda$, let $g(\zeta + 1) = g(\zeta) + 1$. For limit ζ with $\gamma_\alpha < \zeta < \lambda$, let $g(\zeta) = \sup_{\xi<\zeta} g(\xi)$. For all $\zeta \ge \lambda$, let $g(\zeta) = f(\zeta)$. g is continuous at λ, since λ is indecomposable. So g is $\ge \nu$-normal. Let $q = p \cup \{(\gamma_{\alpha+1}, g(\gamma_{\alpha+1}))\}$. Then $g(\gamma_{\alpha+1}) > \gamma_{\alpha+2}$ implies that $r \perp q$ for all $r \in E_{\alpha+1}$, contradicting $p \le \bigvee E_{\alpha+1}$. Therefore, I wins $\mathscr{G}^\nu_\kappa(\kappa^+)$.

Let $C = \{\alpha < \kappa^+ : \alpha = \omega^\alpha\}$. C is club. Let $C^\nu = \{\alpha < \kappa^+ : \alpha = \omega^\alpha$ and $\mathrm{cf}(\alpha) \ge \nu\}$. C^ν is $\ge \nu$-club.

Claim 2. If $p \in \mathbb{P}^\nu_\kappa(\theta)$, $\alpha \in C^\nu$, and $p \subseteq \alpha \times \alpha$, then $p \cup \{(\alpha, \alpha)\} \in \mathbb{P}^\nu_\kappa(\theta)$.

Let f be a $\ge \nu$-normal function witnessing that $p \in \mathbb{P}^\nu_\kappa(\theta)$. Let $\beta = \sup\{\zeta + 1 : \zeta \in \mathrm{dom}(p)\}$. Then $\beta < \alpha$ and $\mathrm{cf}(\beta) < \nu$. For $\zeta < \beta$, let $g(\zeta) = f(\zeta)$. Let $g(\beta) = \sup_{\gamma<\beta} f(\gamma)$. (Note that $g(\beta) < \alpha$.) Let $g(\zeta + 1) = g(\zeta) + 1$ for all $\beta < \zeta < \alpha$. Set $g(\zeta) = \sup_{\xi<\zeta} g(\xi)$ for limit $\beta < \zeta < \alpha$. Finally, let $g(\zeta) = \zeta$ for all $\zeta \ge \alpha$. We claim that g is $\ge \nu$-normal. It suffices to show that $\forall \zeta < \alpha$, $g(\zeta) < \alpha$, since g is strictly increasing. But this is trivial, since α is indecomposable. Thus, $p \cup \{(\alpha,\alpha)\} \in \mathbb{P}^{\nu}_{\kappa}(\theta)$.

For a maximal antichain W in $\mathbb{P}^{\nu}_{\kappa}(\theta)$, let

$$C(W) = \{\lambda \in C^{\nu} : \forall\alpha < \lambda\,\exists\beta < \lambda\,\forall p \subseteq \alpha\times\alpha\,\exists q \subseteq \beta\times\beta\,(q \in W \text{ and } q\|p)\}. \tag{6.7}$$

$C(W)$ is $\geq \nu$-club (since $\kappa^{<\theta} = \kappa$).

Claim 3. II wins $\Gamma^{\kappa}_{\kappa}(\infty)$.

Let I fix $p_0 \in \mathbb{P}^{\nu}_{\kappa}(\theta)$. On round 0, when I plays a partition W_0 of p_0, let II choose some $\lambda_0 \in C(W_0)$ such that $p_0 \subseteq \lambda_0 \times \lambda_0$ and play $B^0_0 = \{p \in W_0 : p \subseteq \lambda_0 \times \lambda_0\}$. $|B^0_0| \leq \kappa$, since $\kappa^{<\theta} = \kappa$. On round $\alpha < \kappa$, when I plays a partition W_α of p_0, let II choose some $\lambda_\alpha > \sup_{\beta<\alpha}\lambda_\beta$ such that $\lambda_\alpha \in \bigcap_{\beta\leq\alpha} C(W_\beta)$ and play for all $\gamma \leq \alpha$, $B^{\alpha}_{\gamma} = \{p \in W_\gamma : p \subseteq \lambda_\alpha \times \lambda_\alpha\}$. Also, we require that II choose $\lambda_{\alpha+1} > \lambda_\alpha \cdot 2$. Let $\lambda = \sup_{\alpha<\kappa}\lambda_\alpha$. $\lambda \in \bigcap_{\alpha<\kappa} C(W_\alpha)$, since these sets are all $\geq \nu$-club. Let $q = p_0 \cup \{(\lambda,\lambda)\}$. $q \in \mathbb{P}^{\nu}_{\kappa}(\theta)$ by Claim 2, since $\lambda \in C^{\nu}$.

We claim that $\forall r \leq q$, $\forall\alpha < \kappa$, $\exists s \in \bigcup_{\alpha\leq\gamma<\kappa} B^{\gamma}_{\alpha}$ such that $s\|r$. Let $r \leq q$ and $\alpha < \kappa$. Let $r' = r \cap (\lambda \times \lambda)$. Since $|r| < \nu$, $\exists\beta < \kappa$ such that $r' \subseteq \gamma \times \gamma$ for some $\gamma < \lambda_\beta$. Take such a $\beta > \alpha$. $\lambda_\beta \in C(W_\alpha)$ implies $\exists s \subseteq \lambda_\beta \times \lambda_\beta$ such that $s \in W_\alpha$ and $s\|r'$. Such an s must be in B^{β}_{α}. Let f witness that $r' \cup s \in \mathbb{P}^{\nu}_{\kappa}(\theta)$ and h witness that $r \in \mathbb{P}^{\nu}_{\kappa}(\theta)$. Note that $r' \cup s \subseteq \lambda_\beta \times \lambda_\beta$.

Let $\eta = \sup\{\zeta + 1 : \zeta \in \operatorname{dom}(s \cup r')\}$. We now have two cases. Suppose $h(\lambda) = \lambda$. Then let $g(\zeta) = f(\zeta)$, $\forall\zeta < \eta$. $g(\eta) = \sup_{\zeta<\eta} f(\zeta)$. $g(\zeta+1) = g(\zeta)+1$, $\forall\eta \leq \zeta < \lambda$. $g(\zeta) = \sup_{\xi<\zeta} g(\xi)$, for all limit $\eta < \zeta < \lambda$. $g(\lambda) = \lambda$. $g(\zeta) = h(\zeta)$ $\forall\lambda < \zeta < \kappa^+$. g witnesses that $r \cup s \in \mathbb{P}^{\nu}_{\kappa}(\theta)$. Otherwise, $h(\lambda) > \lambda$. Then let $\eta \leq \xi < \lambda$ be least such that $g(\xi) < h(\xi)$. Define $g'(\zeta) = g(\zeta)$ for all $\zeta \leq \xi$, and $g'(\zeta) = h(\zeta)$ for all $\zeta > \xi$. Then g' witnesses that $r \cup s \in \mathbb{P}^{\nu}_{\kappa}(\theta)$. Therefore, II wins $\Gamma^{\kappa}_{\kappa}(\infty)$. ⊣

Open Problems 62.
1. Is $\mathbb{P}^{\nu}_{\kappa}(\theta)$ $(<\kappa,\kappa)$-distributive; or more generally, does the forcing $\mathbb{P}^{\nu}_{\kappa}(\theta)$ preserve all κ-stationary subsets of $[\lambda]^{\leq\kappa}$?
2. Assuming $\kappa^{<\kappa} = \kappa$, does preservation of all κ-stationary subsets of $[\lambda]^{\leq\kappa}$ $(\forall\lambda > \kappa)$ plus $(<\kappa,\kappa)$-distributivity imply II has a winning strategy for $\Gamma^{\kappa}_{\kappa}(\lambda)$ $(\Gamma^{\kappa}_{\kappa}(\infty))$?

§7. Adding κ-club subsets of $[\lambda]^{\leq\kappa}$ without adding new κ-sequences.

Recall that by Theorem 17, if κ is regular and $\kappa \leq \lambda$, every κ-stationary $X \subseteq [\lambda]^{\leq\kappa}$ can be decomposed into λ many disjoint κ-stationary subsets.

The following generalizes a proposition of Kamburelis in [14] to regular uncountable κ. As a consequence, if $\kappa^{<\kappa} = \kappa$, we can always add a κ-club set through a κ-stationary subset of κ^+. This contrasts with the necessity of a stationary subset of λ being fat in order to add a club set through it, (see [1]). (Huuskonen, Hyttinen, and Rautila found the precise characterization of subsets of λ through which it is possible to shoot a ν-club subset of λ [8].)

Theorem 63. *Suppose $\lambda > \kappa = \kappa^{<\kappa}$ and $S \subseteq [\lambda]^{\leq\kappa}$ is κ-stationary. Then there is a (κ,∞)-distributive, λ^+-c.c. forcing $\mathbb{P}_S$ which adds a new κ-club set through S.*

PROOF. Let $\mathbb{P}_S$ be the set of all one-to-one functions f such that $\mathrm{dom}(f)$ is an ordinal less than κ^+, $\mathrm{ran}(f) \subseteq \lambda$, and for each ordinal $\zeta \leq \mathrm{dom}(f)$ with cofinality κ, $f[\zeta] \in S$. Let $g \leq f \leftrightarrow g \supseteq f$. $\mathbb{P}_S$ is separative and atomless.

Claim. $\mathbb{P}_S$ is (κ, ∞)-distributive.

Let $\{D_\alpha : \alpha < \kappa\}$ be a family of open dense subsets of $\mathbb{P}_S$. Let $g \in \mathbb{P}_S$. Let $(\lambda)^{<\kappa}$ denote the set of sequences of elements of λ of length less than κ. There is a family $\{g_s : s \in (\lambda)^{<\kappa}\} \subseteq \mathbb{P}_S$ such that

1. $g_{\langle\,\rangle} \leq g$
2. $s \supseteq t \rightarrow g_s \leq g_t$
3. $\mathrm{dom}(s) = \alpha \rightarrow g_s \in D_\alpha$
4. $\mathrm{ran}(s) \subseteq \mathrm{ran}(g_s)$.

We proceed inductively on the length of $s \in (\lambda)^{<\kappa}$. Choose one $g_{\langle\,\rangle} \in D_0$ such that $g_{\langle\,\rangle} \leq g$. Suppose $\alpha = \beta + 1 < \kappa$. Let $s \in (\lambda)^\alpha$ and $t = s \restriction \beta$. If $\mathrm{ran}(s) \subseteq \mathrm{ran}(g_t)$, let $f = g_t$. Otherwise, $\{s(\beta)\} = \mathrm{ran}(s) \setminus \mathrm{ran}(g_t)$, so let $f = g_t \cup \{\langle \mathrm{dom}(g_t), s(\beta)\rangle\}$. Choose some $g_s \in D_\alpha$ such that $g_s \leq f$.

Now suppose $\alpha < \kappa$ is a limit ordinal. Let $s \in (\lambda)^\alpha$, $f = \bigcup_{\beta<\alpha} g_{s\restriction\beta}$, and $\delta = \mathrm{dom}(f)$. $\mathrm{ran}(s) \subseteq \mathrm{ran}(f)$. If $\mathrm{cf}(\delta) \neq \kappa$, then $f \in \mathbb{P}_S$. If $\mathrm{cf}(\delta) = \kappa$, then $\mathrm{cf}(\alpha) < \kappa$ implies there is some $\beta < \alpha$ such that for all $\beta \leq \beta' < \alpha$, $g_{s\restriction\beta'} = g_{s\restriction\beta}$. So $f = g_{s\restriction\beta} \in \mathbb{P}_S$. Take some $g_s \leq f$ with $g_s \in D_\alpha$.

Let $C = \{x \in [\lambda]^{\leq\kappa} : \forall s \in (x)^{<\kappa}\ \mathrm{ran}(g_s) \subseteq x\}$. C is κ-club, assuming $\kappa^{<\kappa} = \kappa$. Let $x \in C \cap S$ and fix an enumeration $\langle \zeta_\alpha : \alpha < \kappa\rangle$ of x. For each $\alpha < \kappa$, let $s_\alpha = \langle \zeta_\beta : \beta < \alpha\rangle$. Let $f = \bigcup_{\alpha<\kappa} g_{s_\alpha}$. Note that $x = \bigcup_{\alpha<\kappa} \mathrm{ran}(s_\alpha) \subseteq \bigcup_{\alpha<\kappa} \mathrm{ran}(g_{s_\alpha}) \subseteq x$, since $x \in C$. Therefore, $\mathrm{ran}(f) = x$. $x \in S$ implies $f \in \mathbb{P}_S$. Finally, $f \in D_\alpha$ for all $\alpha < \kappa$.

Claim. $\mathbb{P}_S$ adds a κ-club set through S.

Let G be $\mathbb{P}_S$-generic, $\pi = \bigcup G$, and $C^* = \{\pi[\alpha] : \alpha < \kappa^+,\ \mathrm{cf}(\alpha) = \kappa\}$. If $\alpha < \kappa^+$ and $\mathrm{cf}(\alpha) = \kappa$, then $\pi[\alpha] \in S$. Let $x \in [\check{\lambda}]^{\leq\kappa}$. Since π is onto $\check{\lambda}$, there is an $\alpha < \kappa^+$ such that $x \subseteq \pi[\alpha] \in C^*$. So C^* is unbounded in $[\check{\lambda}]^{\leq\kappa}$. Let $\langle x_\alpha : \alpha < \kappa\rangle$ be a strictly increasing sequence in C^*. For each $\alpha < \kappa$, let ζ_α denote the ordinal such that $x_\alpha = \pi[\zeta_\alpha]$. Then $\langle \zeta_\alpha : \alpha < \kappa\rangle$ is also strictly increasing, so $\mathrm{cf}(\sup_{\alpha<\kappa} \zeta_\alpha) = \kappa$. Therefore, $\bigcup_{\alpha<\kappa} x_\alpha = \pi[\sup_{\alpha<\kappa} \zeta_\alpha] \in C^*$. Hence, C^* is κ-closed. ⊣

We now have a natural extension of an example of Jech [11]: the forcing of [2] which shoots a club through a stationary-co-stationary subset of ω_1 with countable conditions yields a (ω, ∞)-distributive Boolean algebra in which $\mathscr{G}^\omega_\omega(\infty)$ is undetermined.

EXAMPLE 64. If $\kappa^{<\kappa} = \kappa$ and $\lambda^{<\kappa} = \lambda$, then there is a Boolean algebra in which the games $\Gamma^\kappa_\kappa(\theta)$ for all $\theta \geq \lambda$, and $\mathscr{G}^\kappa_{<\mu}(\theta)$ for all $\mu \leq \kappa^+$ and $\theta \geq \lambda$ are undetermined.

PROOF. Let $S \subseteq [\lambda]^{\leq\kappa}$ be κ-stationary such that $[\lambda]^{\leq\kappa} \setminus S$ is also κ-stationary. Let $\mathbb{B} = \mathrm{r.o.}(\mathbb{P}_S)$. II does not have a winning strategy for $\Gamma^\kappa_\kappa(\lambda)$, by Proposition 55. On the other hand, (κ, ∞)-distributivity of $\mathbb{B}$ implies I does not have a winning strategy in any game of length $\leq \kappa$, by Theorem 1. ⊣

This improves on the consistency result for undetermined games in [4] (recall Theorem 6), in the sense that it uses assumptions far weaker than $\diamondsuit_{\kappa^+}(S)$ for all stationary sets $S \subseteq \kappa^+$ and $\square_\kappa$. However, whereas that result is consistent for all cardinals $\mu \leq \lambda$ and κ, our above example requires $\mu \leq \kappa^+ \leq \lambda$ and κ is regular.

OPEN PROBLEM 65. Assuming $\omega < \mu \leq \kappa^{<\kappa} < \lambda$, find a Boolean algebra which is $(< \kappa, \kappa)$-distributive and $(\kappa, \lambda, < \mu)$-distributive and forces $[\check{\lambda}]^{\leq\kappa} \setminus V$ to be κ-stationary.

By Theorems 63 and 37 solving Problem 65 would solve Problem 43. We note that if $\mathbb{B}$ is $(\omega, \lambda^{\omega_1})$-distributive, then by a theorem of Magidor [20] there must be an ω_1-Erdős cardinal in K in order for it to even be possible for $\mathbb{B}$ to force $[\check{\lambda}]^{\leq\omega_1} \setminus V$ to be ω_1-stationary.

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