Ramsey theory of the universal homogeneous triangle-free graph

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Ramsey Theory in Logic, Combinatorics and Complexity

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Ramsey Theory, Small and Big

Property	Examples
$\mathcal{K}$ has Ramsey Property $\forall A \leq B \in \mathcal{K}\ \forall k$, $\mathbf{K} \rightarrow (B)^A_k$	finite: linear orders, Boolean algebras finite ordered: graphs, hypergraphs, graphs omitting k-cliques
$\mathcal{K}$ has Small Ramsey Degrees $\forall A\ \exists t_{\mathcal{K}}(A)\ \forall B\ \forall k$, $\mathbf{K} \rightarrow (B)^A_{k, t_{\mathcal{K}}(A)}$	finite: graphs, hypergraphs graphs omitting k-cliques hypergraphs omitting irreducibles
$\mathcal{K}$ has Big Ramsey Degrees $\forall A\ \exists T_{\mathcal{K}}(A)\ \forall k$, $\mathbf{K} \rightarrow (\mathbf{K})^A_{k, T_{\mathcal{K}}(A)}$	the rationals, Rado graph, dense local order $\mathbf{S}(2)$, tournament $\mathbf{S}(3)$ $\mathbb{Q}_n$, $n \geq 2$.

$\mathcal{K}$ a Fraïssé class.

$\mathbf{K}$ = Fraïssé limit of $\mathcal{K}$, called a Fraïssé structure.

Missing Pieces: Forbidden Configurations

A large collection of Fraïssé classes have been shown to have the Ramsey Property or Small Ramsey Degrees, starting in the 1970's and more in recent years motivated by the Kechris-Pestov-Todorcevic correspondence.

However,

Few Fraïssé structures have been shown to have Big Ramsey Degrees,

and

No Fraïssé structure with forbidden configurations had a complete analysis of its Big Ramsey Degrees.

The Problem: Lack of tools for representing such Fraïssé structures and lack of a viable Ramsey theory for such (non-existent) representations.

We address this lack of representations and techniques, starting with my submitted paper, *The Ramsey theory of the universal homogeneous triangle-free graph*, 48 pp.

The beginning of a more general theory

The techniques developed for the triangle-free case are very broad and likely to extend to a large class of Fraïssé structures with forbidden configurations.

I am currently working to extend this research to big Ramsey degrees of

- Universal homogeneous k-clique free graphs (nearing completion)
- Homogeneous partial order (in progress)
- Homogeneous hypergraphs with forbidden configurations (in progress)
- Homogeneous bowtie-free graph (in progress, with Hubička)

Connections with Topological Dynamics

Ramsey Theory	corr.	Topological Dynamics
Ramsey Property $\mathcal{K}$ an order class with RP	KPT $\longleftrightarrow$	Extreme Amenability G is EA
Small Ramsey Degrees precpct expansion $\mathcal{K}^*$ has RP	KPT/NVT $\longleftrightarrow$	Computation of UMF of G X^* is UMF of G
Big Ramsey Degrees $\mathbf{K}$ admits a big R-structure	Zucker $\longleftrightarrow$	Universal Completion Flow Big Ramsey Flow = UCF of G

$\mathcal{K}$ a Fraïssé class
$\mathbf{K} = \mathrm{Flim}(\mathcal{K})$
$G = \mathrm{Aut}(\mathbf{K})$

A Brief (incomplete) Intro to Graph Colorings

Example: Ordered graph A embeds into ordered graph B.

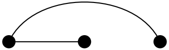


Figure: Ordered Graph A

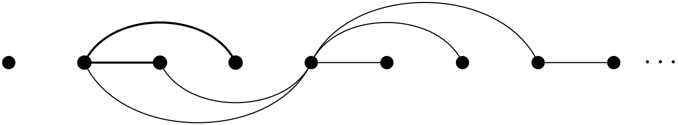
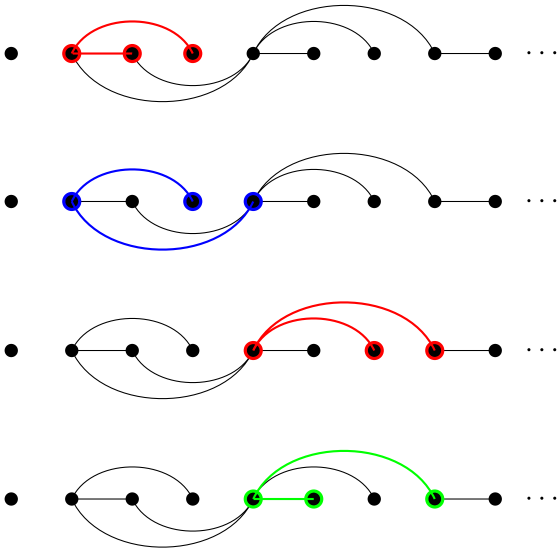


Figure: Ordered Graph B

Some copies of A in B



Thm. (Nešetřil/Rödl 1977 and Abramson/Harrington 1978)
The class of finite ordered graphs has the Ramsey property.

It follows that the class of finite graphs has small Ramsey degrees.

Case Study: Rado Graph

The random graph, also called the Rado graph and denoted $\mathcal{R}$, is the graph on infinitely many nodes such that for each pair of nodes, there is a 50-50 chance that there is an edge between them.

$\mathcal{R}$ is the Fraïssé limit of the class of finite graphs. It is universal for countable graphs: Every countable graph embeds into $\mathcal{R}$.

• Vertices have big Ramsey degree 1. (Folklore)

• Edges have big Ramsey degree 2. (Pouzet/Sauer 1996).

• All finite graphs have finite big Ramsey degree. (Sauer 2006) In this paper is also the set-up for

• Actual degrees were found structurally in (LSV 2006) and computed in (J. Larson 2008).

Other Structures known to have big Ramsey degrees

• the natural numbers (Ramsey 1929)

• the rationals (Galvin, Laver, Devlin 1979)

• the Rado graph and similar binary relational structures (Sauer 2006)

• the countable ultrametric Urysohn space (Nguyen Van Thé 2008)

• the dense local order, circular tournament, $\mathbb{Q}_n$ (Laflamme, NVT, Sauer 2010).

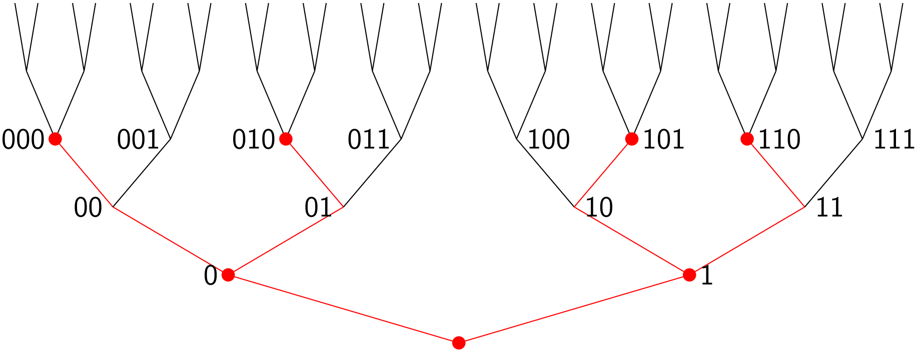
The crux of all but two of these proofs is a theorem of Milliken (or variant).

(The Urysohn space result uses Ramsey's Theorem.)

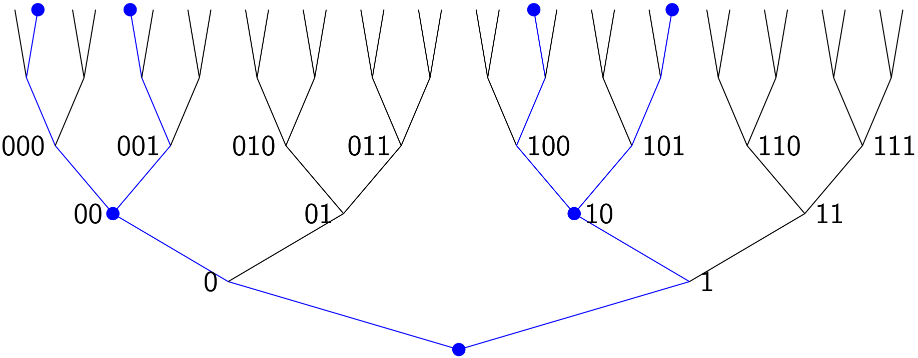
Strong Trees and Milliken's Theorem

A tree $T \subseteq 2^{<\omega}$ is a strong tree iff it is (strongly) isomorphic either to $2^{<\omega}$ or to $2^{\leq k}$ for some finite k.

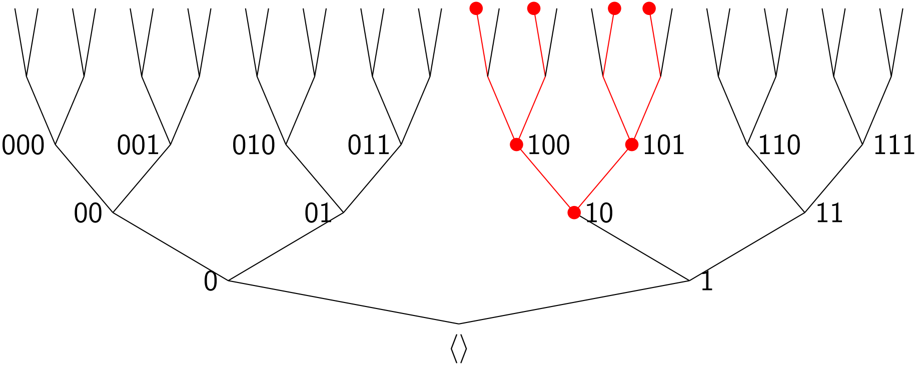
Strong Subtree $\cong 2^{\leq 2}$, Ex. 1



Strong Subtree $\cong 2^{\leq 2}$, Ex. 2



Strong Subtree $\cong 2^{\leq 2}$, Ex. 3



A Ramsey Theorem for Strong Trees

Thm. (Milliken 1979) Let $k \geq 0$, $l \geq 2$, and a coloring of all the subtrees of $2^{<\omega}$ which are isomorphic to $2^{\leq k}$ into l colors. Then there is an infinite strong subtree $S \subseteq 2^{<\omega}$ such that all copies of $2^{\leq k}$ in S have the same color.

Milliken's Theorem builds on the Halpern-Läuchli Theorem.

Thm. (Halpern-Läuchli 1966) Let $d \geq 1$, $l \geq 2$, and $T_i = 2^{<\omega}$ for $i < d$. Given a coloring of the product of level sets of the T_i into l colors,

$$f : \bigcup_{n<\omega} \prod_{i<d} T_i(n) \to l,$$

there are infinite strong trees $S_i \leq T_i$ and an infinite sets of levels $M \subseteq \omega$ where the splitting in S_i occurs, such that f is constant on $\bigcup_{m \in M} \prod_{i<d} S_i(m)$.

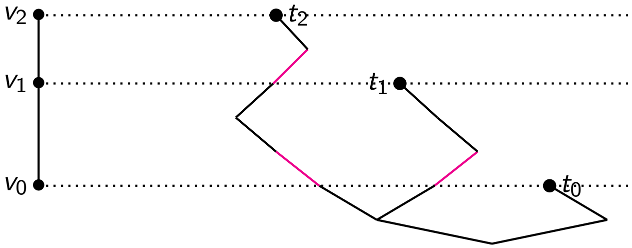
Nodes in Trees can Code Graphs

Let A be a graph. Enumerate the vertices of A as $\langle v_n : n < N \rangle$.

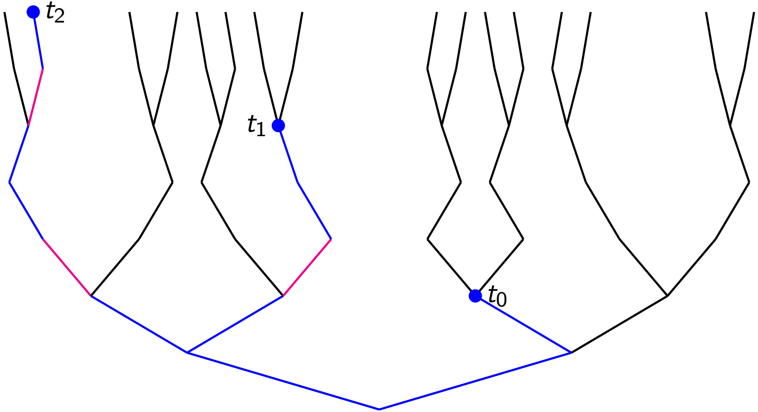
A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes A if and only if for each pair $m < n < N$,

$$v_n \; E \; v_m \Leftrightarrow t_n(|t_m|) = 1.$$

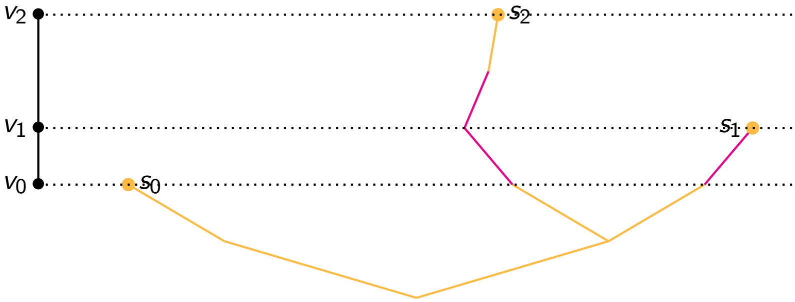
The number $t_n(|t_m|)$ is called the passing number of t_n at t_m.



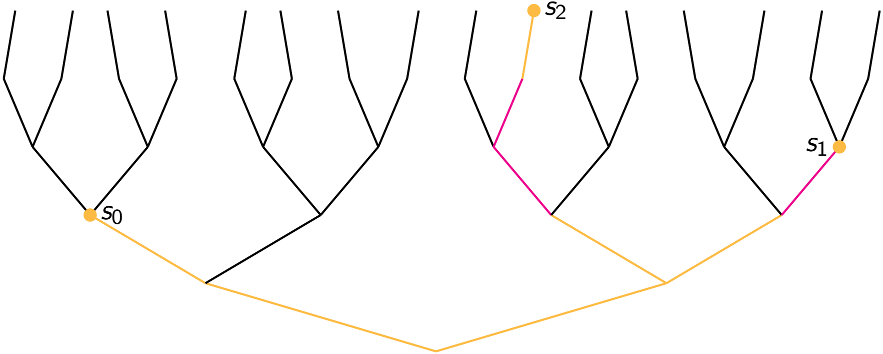
A Strong Tree Envelope



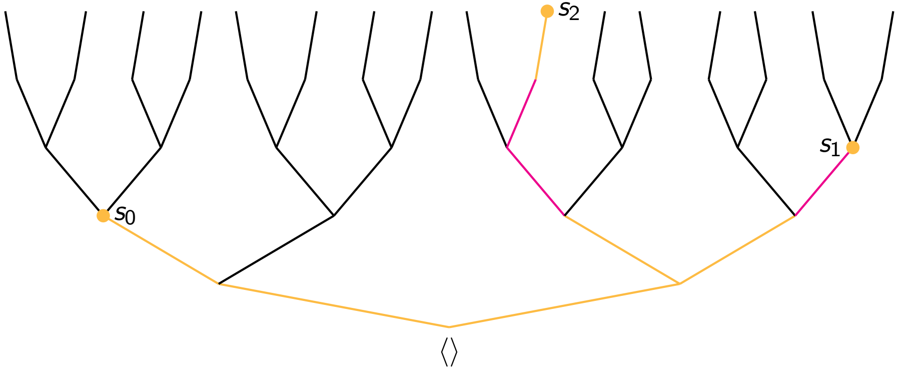
A Different Antichain Coding a Path of Length 2



A Strong Tree Envelope



A different strong tree envelope



Outline of Sauer's Proof: $\mathcal{R}$ has finite big Ramsey degrees

1. The Rado graph is bi-embeddable with the graph coded by all nodes in the tree $2^{<\omega}$.
2. Each finite graph can be coded by finitely many strong similarity types of (diagonal) antichains.
3. Each strongly diagonal antichain can be enveloped into finitely many strong trees.
4. Apply Milliken's Theorem finitely many times to obtain one color for each (strong similarity) type.
5. Choose a strongly diagonal antichain coding the Rado graph.
6. Show that each type persists in each subgraph which is random to obtain exact numbers.

$\mathcal{H}_3$: History of Results

The universal homogeneous triangle-free graph $\mathcal{H}_3$ is the Fraïssé limit of the class of finite triangle-free graphs, $\mathcal{K}_3$.

• Henson constructed $\mathcal{H}_3$ and proved it is weakly indivisible in 1971.

• The Fraïssé class of finite ordered triangle-free graphs $\mathcal{K}_3^<$ has the Ramsey property. (Nešetřil-Rödl 1977/83)

• $\mathcal{H}_3$ is indivisible: Vertex colorings of $\mathcal{H}_3$ have big Ramsey degree 1. (Komjáth/Rödl 1986)

• $\mathcal{H}_3$ has big Ramsey degree 2 for edges. (Sauer 1998)

There progress halted.

Main Obstacles to Big Ramsey Degrees of $\mathcal{H}_3$

"A proof of the big Ramsey degrees for $\mathcal{H}_3$ would need new Halpern-Läuchli and Milliken Theorems, and nobody knows what those should be." (Todorcevic, 2012)

Said the same thing, plus, "There is no simply representable triangle-free graph which is bi-embeddable with $\mathcal{H}_3$." (Sauer, 2013)

"So far, the lack of tools to represent ultrahomogeneous structures is the major obstacle towards a better understanding of their infinite partition properties." (Nguyen Van Thé, Habilitation 2013)

Main Theorem: $\mathcal{H}_3$ has Finite Big Ramsey Degrees

Theorem. (D.) For each finite triangle-free graph A, there is a positive integer $T_{\mathcal{K}_3}(\mathrm{A})$ such that for any coloring of all copies of A in $\mathcal{H}_3$ into finitely many colors, there is a subgraph $\mathcal{H} \le \mathcal{H}_3$, again universal triangle-free, such that all copies of A in $\mathcal{H}$ take no more than $T_{\mathcal{K}_3}(\mathrm{A})$ colors.

Thanks to the following, in addition to Sauer and Todorcevic:
2011 Laver outlined Harrington's 'forcing proof' of Halpern-Läuchli.
2012 First attempts cheered on by Bartošová and Larson.(After 7 months I gave up.)
2015 Cambridge Semseter, Bartošová reminds me about problem. Work ensues.
2016 Prague RT DocCourse - Nešetřil, Rödl, Hubička and company very kind about glitch I found. A few days later, I found the fix.

Structure of Proof: Three Main Parts

I Develop new notion of strong coding tree to represent $\mathcal{H}_3$.

II Prove a Ramsey Theorem for strictly similar finite antichains.

III Apply Ramsey Theorem for strictly similar antichains finitely many times. Then take an antichain of coding nodes coding $\mathcal{H}_3$.

Part I: Strong Coding Trees

Idea: Want correct analogue of strong trees for setting of $\mathcal{H}_3$.

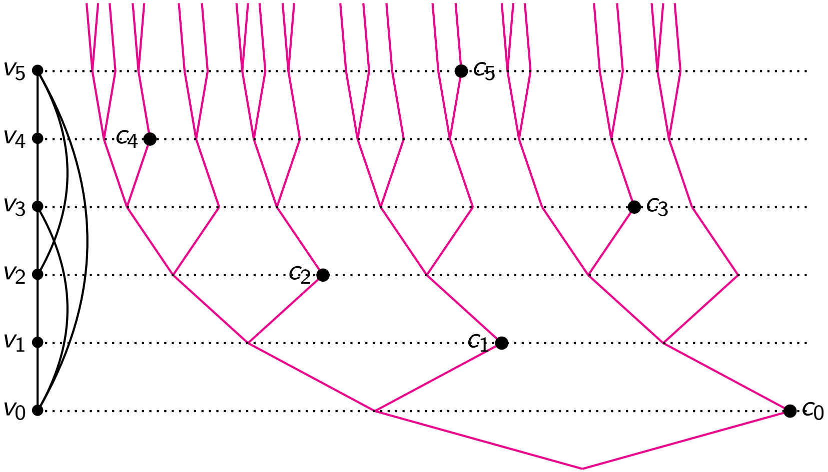
Problem: How to make sure triangles are never encoded but branching is as thick as possible?

First Approach: Strong Triangle-Free Trees

• Use a unary predicate for distinguishing certain nodes to code vertices of a given graph (called coding nodes).

• Make a Branching Criterion so that a node s splits iff all its extensions will never code a triangle with coding nodes at or below the level of s.

Strong triangle-free tree $\mathbb{S}$

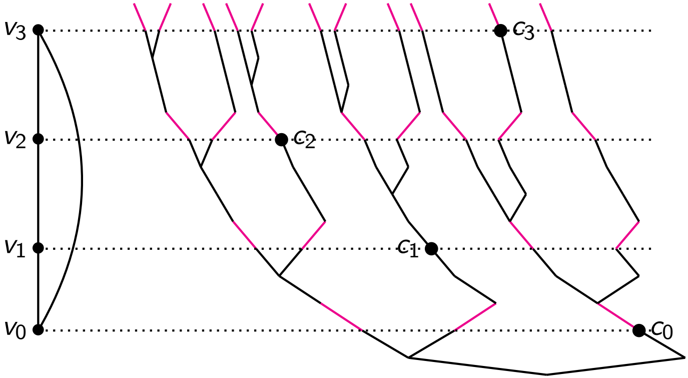


Almost sufficient

One can develop almost all the Ramsey theory one needs on strong triangle-free trees

except for vertex colorings: there is a bad coloring of coding nodes.

Refined Approach: Strong coding tree $\mathbb{T}$



Skew the levels of interest.

The Space of Strong Coding Trees: $\mathcal{T}_3$

$\mathcal{T}_3$ is the collection of all subtrees of $\mathbb{T}$ which are strongly similar to $\mathbb{T}$.

A finite subtree A of a strong coding tree $T \in \mathcal{T}_3$ can be extended to a strong coding subtree of T whenever A is strongly similar to an initial segment of $\mathbb{T}$ and all entanglements of A are witnessed - no types are lost.

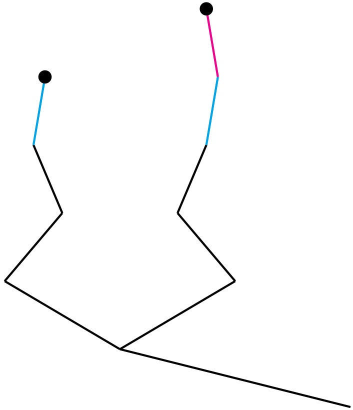
The criteria guaranteeing this are

1. Parallel 1's Criterion: All new sets of parallel 1's in A are witnessed by a coding node in A 'nearby'.
2. A has no pre-determined new parallel 1's in T.

The space $\mathcal{T}_3$ of strong coding trees is very near a topological Ramsey space.

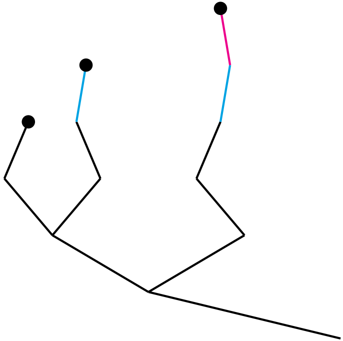
A subtree of $\mathbb{T}$ in which P1C fails

It has parallel 1's not witnessed by a coding node (P1C fails).



A subtree of $\mathbb{T}$ in which P1C holds

Its parallel 1's are witnessed by a coding node.



This gives the basic idea of P1C, though more subtleties are involved.

Part II: A Ramsey Theorem for Strictly Similar Finite Antichains.

Idea: Strict similarity takes into account tree isomorphism and placements of coding nodes and new sets of parallel 1's.

It persists upon taking subtrees in $\mathcal{T}_3$.

Ramsey Theorem for Strong Coding Trees

Theorem. (D.) Let A be a finite subtree of a strong coding tree T, and let c be a coloring of all strictly similar copies of A in T.

Then there is a strong coding tree $S \leq T$ in which all strictly similar copies of A in S have the same color.

This is an analogue of Milliken's Theorem for strong coding trees.

Strict similarity is a strong version of isomorphism, and forms an equivalence relation.

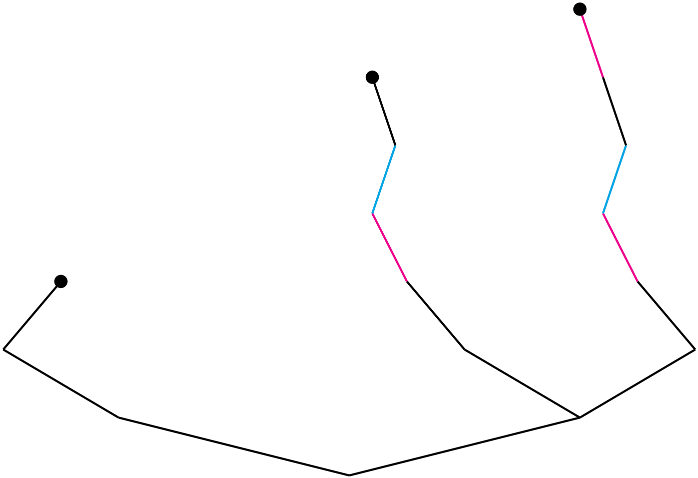
Some Examples of Strict Similarity Types

Let G be the graph with three vertices and no edges.

We show some distinct strict similarity types of trees coding G.

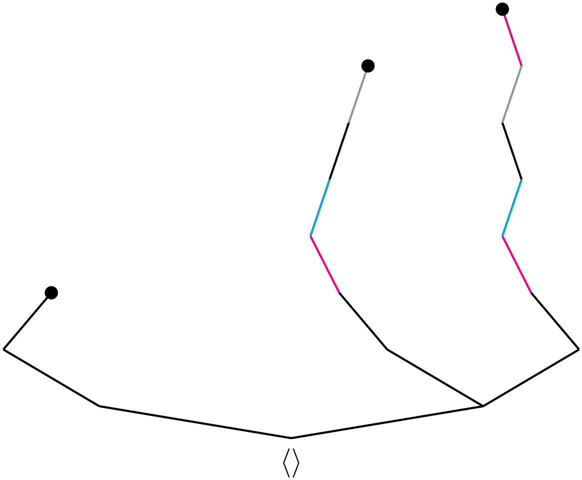
G a graph with three vertices and no edges

A tree A coding G - not P1C but still a valid strict similarity type



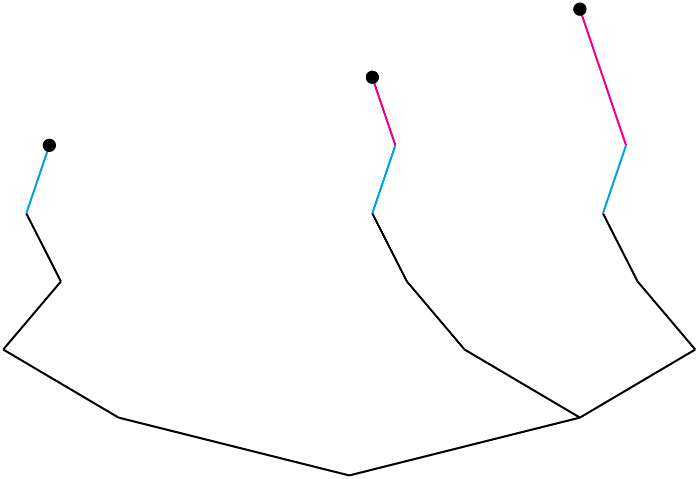
G a graph with three vertices and no edges

B codes G and is strictly similar to A.



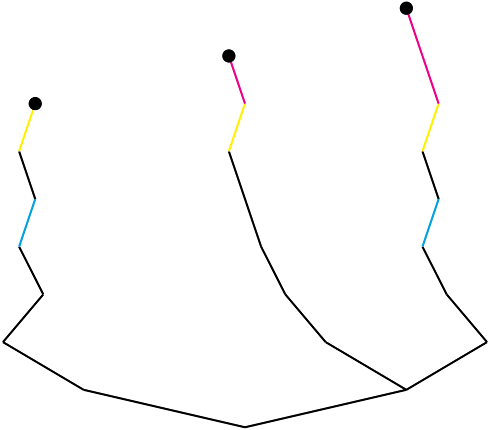
The tree C codes G

C is not strictly similar to A.

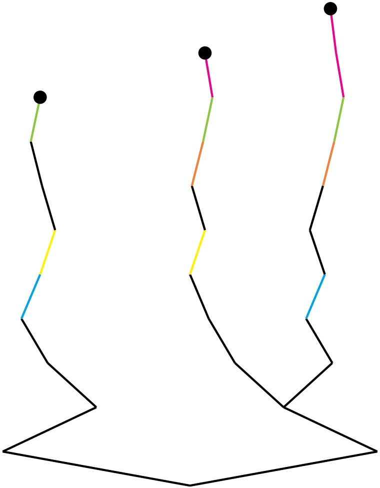


The tree D codes G

D is not strictly similar to either A or C.

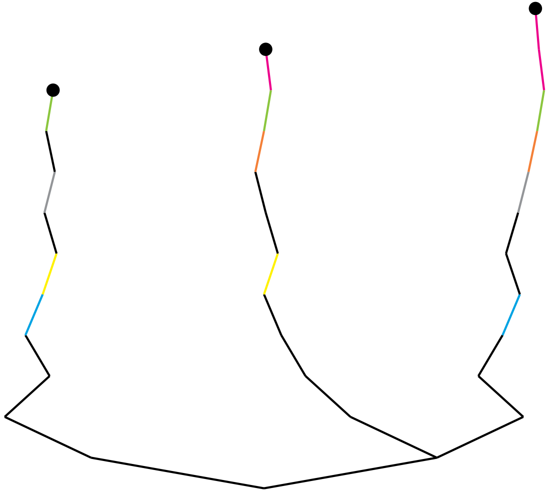


The tree E codes G and is not strictly similar to A - D



E is incremental. More on that later.

The tree F codes G and is strictly similar to E



F is also incremental.

Part III: Apply the Ramsey Theorem to Strictly Similarity Types of Antichains to obtain the Main Theorem.

Bounds for $T_{\mathcal{K}_3}(G)$

1. Let G be a finite triangle-free graph, and let f color the copies of G in $\mathcal{H}_3$ into finitely many colors.
2. Define f' on antichains in $\mathbb{T}$: For an antichain A of coding nodes in $\mathbb{T}$ coding a copy, G_A, of G, define $f'(A) = f(G_A)$.
3. List the strict similarity types of antichains of coding nodes in $\mathbb{T}$ coding G. There are finitely many.
4. Apply the Ramsey Theorem from Part II, once for each strict similarity type, to obtain a strong coding tree $S \leq \mathbb{T}$ in which f' has one color per type.
5. Take an antichain of coding nodes, $\mathbb{A}$ in S, which codes $\mathcal{H}_3$. Let $\mathcal{H}'$ be the subgraph of $\mathcal{H}_3$ coded by $\mathbb{A}$.
6. Then f has no more colors on the copies of G in $\mathcal{H}'$ than the number of (incremental) strict similarity types of antichains coding G.

Reducing the Upper Bounds

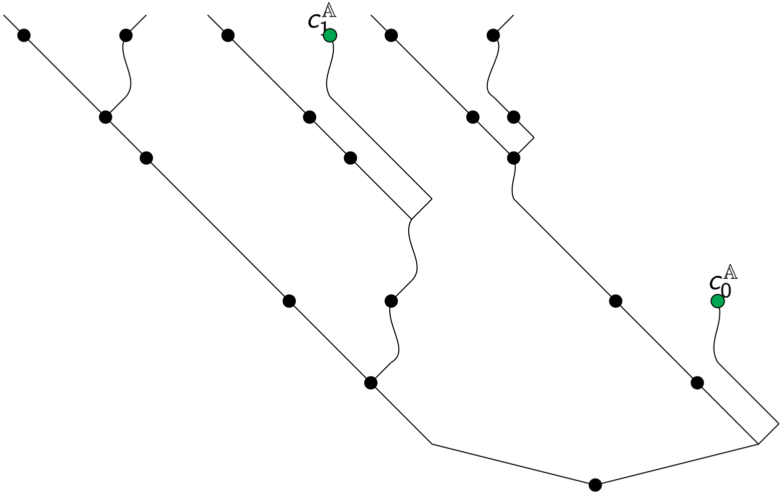
A strong tree U with coding nodes is incremental if whenever a new set of parallel 1's appears in U, all of its subsets appear as parallel 1's at a lower level.

The trees A, B, E, and F are incremental.

The trees C and D are not incremental.

We can take S in the previous slide to be an incremental strong coding tree.

An antichain $\mathbb{A}$ of coding nodes of S coding $\mathcal{H}_3$



The tree minus the antichain of $c_n^{\mathbb{A}}$'s is isomorphic to $\mathbb{T}$.

Part II Expanded: Ideas behind the proof of the Ramsey Theorem for Strictly Similar Finite Trees

(a) Prove new Halpern-Läuchli styleTheorems for strong coding trees.
 – Three new forcings are needed, but the proofs take place in ZFC.

(b) Prove a new Ramsey Theorem for finite trees satisfying Strict P1C.
 – An analogue of Milliken's Theorem.

(c) New notion of envelope.
 – Turns an antichain into a tree satisfying Strict P1C.

(a) Halpern-Läuchli-style Theorem

Thm. (D.) Given a strong coding tree T and

1. B a finite, valid strong coding subtree of T;
2. A a finite subtree of B with $\max(A) \subseteq \max(B)$; and
3. X a level set extending A into T with $A \cup X$ satisfying the P1C and valid in T.

Color all end-extensions Y of A in T for which $A \cup Y$ is strictly similar to $A \cup X$ into finitely many colors.

Then there is a strong coding tree $S \leq T$ end-extending B such that all level sets Y in S with $A \cup Y$ strictly similar to $A \cup X$ have the same color.

Remark. The proof uses three different forcings and Harrington-style ideas. The forcings are best thought of as conducting unbounded searches for finite objects in ZFC.

(b) Ramsey Theorem for Finite Trees satisfying the SP1C

Thm. (D.) Let T be a strong coding tree, and let A be a finite valid subtree of T satisfying the Strict P1C. Suppose all the strictly similar copies of A in T are colored in finitely many colors.

Then there is a strong coding subtree $S \leq T$ such that all strictly similar copies of A in S have the same color.

A tree A satisfies the strict P1C if each new set of parallel 1's is witnessed by a coding node before anything else happens (other occurrences of new parallel 1's, splits, or coding nodes).

(c) Envelopes and Witnessing Coding Nodes

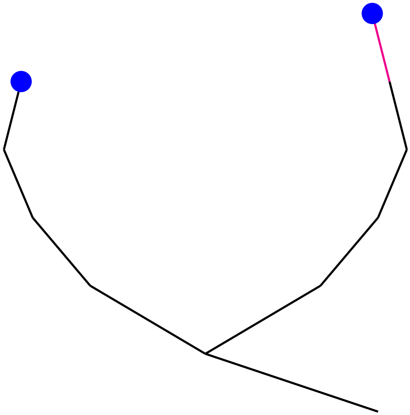
Envelopes add some neutral coding nodes to a finite tree to make it satisfy the strict Parallel 1's Criterion.

Envelopes for an antichain A in a strong coding tree T do not always exist in T.

Instead, given T where the Ramsey theorem has been applied to the strict similarity type of a prototype envelope of A, we take $S \leq T$ and a set of witnessing coding nodes $W \subseteq T$ so that each antichain in S has an envelope in T, using coding nodes from W.

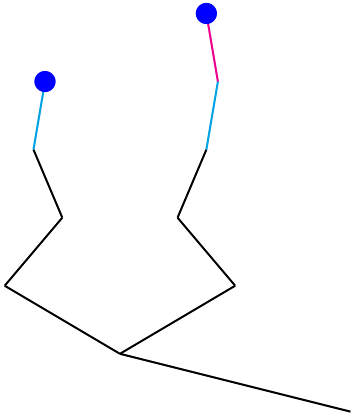
We now give some examples of envelopes.

H codes a non-edge



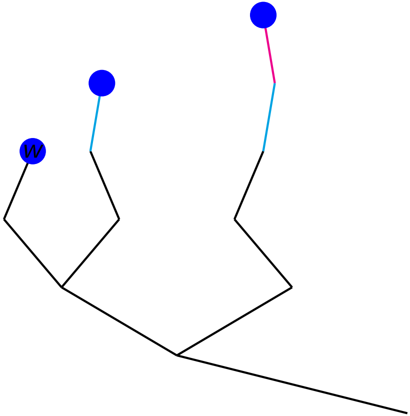
This satisfies the Strict Parallel 1's Criterion, so H is its own envelope.

I codes a non-edge



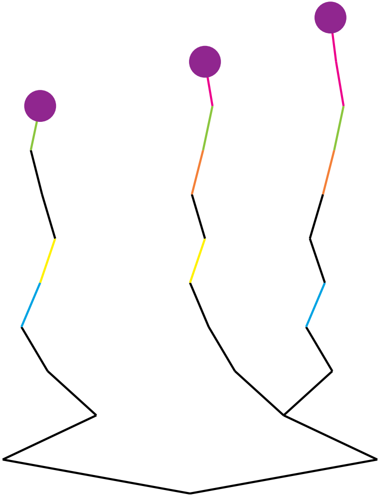
I does not satisfy the Parallel 1's Criterion.

An Envelope $\mathbf{E}(I)$

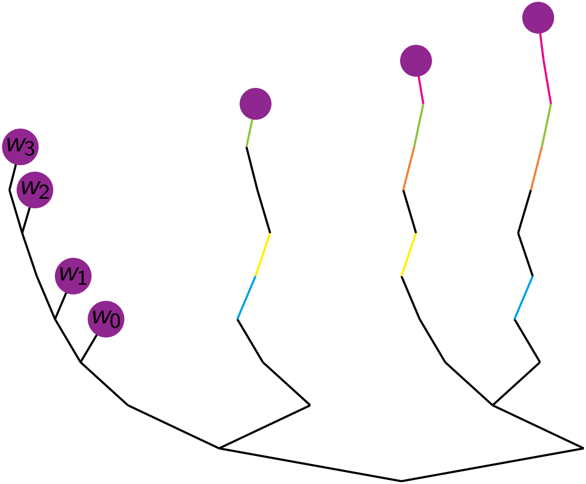


The witnessing coding node w is added to make an envelope.

The incremental tree E from before

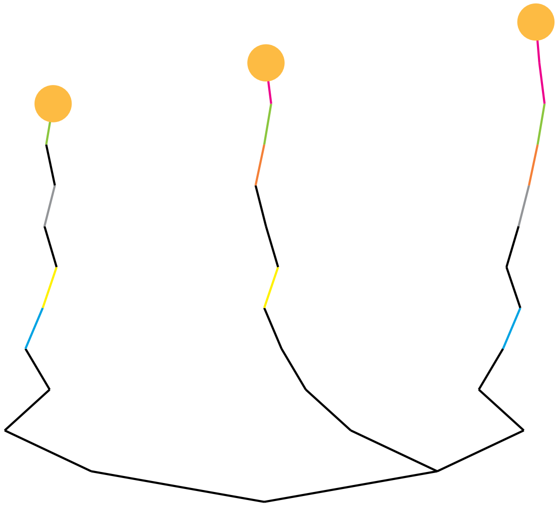


An envelope $\mathbf{E}(E)$

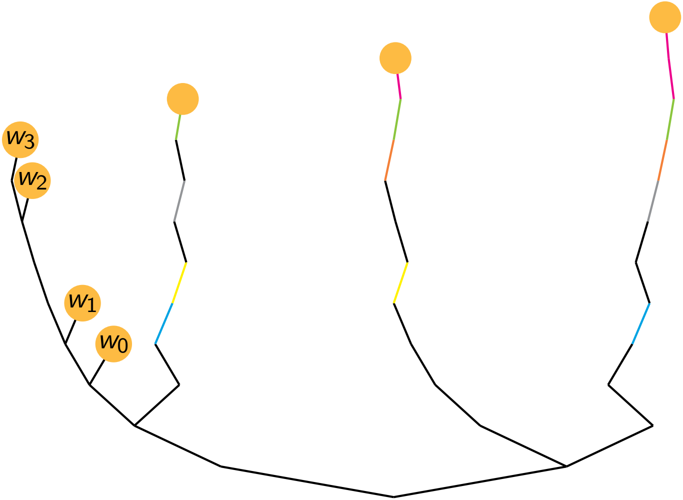


The witnessing coding nodes $w_1, \ldots, w_3$ make an envelope of E.

The tree F from before is strictly similar to E



$\mathbf{E}(F)$ is strictly similar to $\mathbf{E}(E)$



The witnessing coding nodes $w_0, \ldots, w_3$ make an envelope of F.

The Ramsey Thm for Strictly Similar Antichains follows

1. Let A be a finite antichain A of coding nodes inducing an incremental tree; let $E(A)$ be an envelope.

2. A coloring g of all antichains in $\mathbb{T}$ strictly similar to A induces a coloring g' on all strictly similar copies of $E(A)$ in $\mathbb{T}$.

3. Apply the Ramsey Theorem for Trees with the SP1C for g' on $\mathbb{T}$ to obtain $T \leq \mathbb{T}$ in which all copies of $E(A)$ have the same color.

4. Build an incremental strong coding tree $S \leq T$ and a set of witnessing coding nodes $W \subseteq T$ having no parallel 1's with any coding node in S.

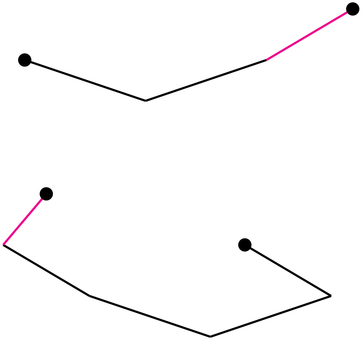
5. Then each copy of A in S has an envelop in T, by adding in some nodes from W.

6. Thus, each copy of A in S has the same color.

Proving the lower bounds in general for big Ramsey degrees of $\mathcal{H}_3$ is a work in progress.

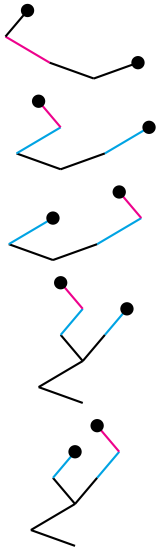
Big Ramsey degrees for edges and non-edges have been computed.

Edges have big Ramsey degree 2 in $\mathcal{H}_3$



These are their own envelopes.
$T_{\mathcal{H}_3}(Edge) = 2$ was obtained in (Sauer 1998) by different methods.

Non-edges have 5 Strict Similarity Types (D.)



Remarks

I am almost finished extending these methods to the universal k-clique-free graphs $\mathcal{H}_k$, for all $k \geq 4$.

To adapt these methods to other structures $\mathcal{S}$ with forbidden configurations, one needs to find the correct Branching Criteria, Extension Criteria guaranteeing a finite subtree can be extended inside a tree coding $\mathcal{S}$, and Ramsey theorems for relevant structures.

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II(a) HL - Case (i): level set X contains a splitting node

List the immediate successors of $\max(A)$ as $s_0, \dots, s_d$, where s_d denotes the node which the splitting node in X extends.

Let $T_i = \{t \in T : t \supseteq s_i\}$, for each $i \le d$.

Fix κ large enough so that $\kappa \to (\aleph_1)^{2d}_{\aleph_0}$ holds.

Such a κ is guaranteed in ZFC by a theorem of Erdős and Rado.

The forcing for Case (i)

$\mathbb{P}$ is the set of conditions p such that p is a function of the form

$$p : \{d\} \cup (d \times \vec{\delta}_p) \to T \restriction l_p,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $l_p \in L$, such that

(i) $p(d)$ is *the* splitting node extending s_d at level l_p;

(ii) For each $i < d$, $\{p(i,\delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \restriction l_p$.

(iii) $\mathrm{ran}(p)$ has no pre-determined new parallel 1's in T.

$q \le p$ if and only if $\vec{\delta}_q \supseteq \vec{\delta}_p$, $l_q \ge l_p$, and

(i) $q(d) \supset p(d)$, and $q(i,\delta) \supset p(i,\delta)$ for each $\delta \in \vec{\delta}_p$ and $i < d$; and

(ii) The set $\{q(i,\delta) : (i,\delta) \in d \times \vec{\delta}_p\} \cup \{q(d)\}$ has no new sets of parallel 1's above $\{p(i,\delta) : (i,\delta) \in d \times \vec{\delta}_p\} \cup \{p(d)\}$.

Case (i)

The forcing is used to find a good set of starting nodes where it is possible to extend them to homogeneous levels.

We alternate between building the subtree by hand and using the forcing to find the next level where homogeneity is guaranteed.

Remarks. (1) No generic extension is actually used.

(2) These forcings are not simply Cohen forcings; the partial orderings are stronger in order to guarantee that the new levels we obtain by forcing are extendible inside T to another strong coding tree.

(3) The assumption that $A \cup X$ satisfies the Parallel 1's Criterion is necessary.

The rest of II

II(a) Case (ii): level set X contains a coding node.

This case is more complex and requires preliminary forcings to obtain cone-homogeneity, an induction proof to construct a cone-homogeneous strong coding tree, and another forcing to obtain the Halpern-Läuchli style theorem.

II(b): Ramsey Theorem for Finite Trees with Strict P1C.

This is obtained by induction using II(a).

II(c) was elaborated on already.