The universal triangle-free graph has finite big Ramsey degrees

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Subtitle

Forcing techniques are useful for solving problems on graphs, in ZFC.

This talk will highlight some of the main concepts in my paper, *The universal triangle-free graph has finite big Ramsey degrees*, 48 pp, submitted.

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Ramsey Property, and Degrees Small and Big

Thm. (Nešetřil/Rödl 1977 and Abramson/Harrington 1978)
The class of finite ordered graphs has the Ramsey property:

Given finite ordered graphs A and B such that A embeds into B, and given a number $l \geq 2$, there is a finite ordered graph C containing a copy of B such that for any coloring of the copies of A in C into l colors, there is some copy B' of B in C such that all copies of A in B' have the same color.

Finite Ordered Graphs

Example: Ordered graph A embeds into ordered graph B.

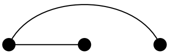


Figure: Ordered Graph A

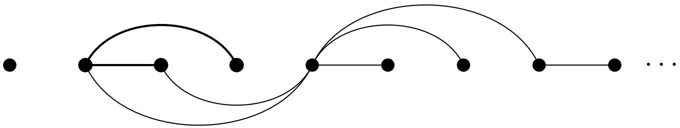
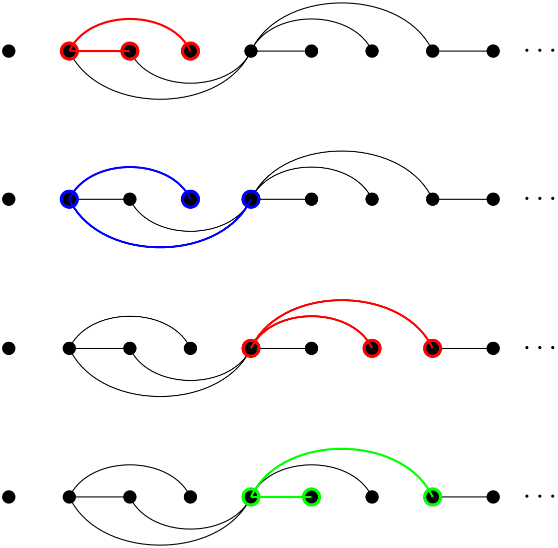


Figure: Ordered Graph B

Some copies of A in B



Finite Unordered Graphs have finite Small Ramsey Degrees

Thm. The class of all finite unordered graphs has finite small Ramsey degrees:

For each finite graph A there is a number bound $t(\mathrm{A})$ such that for each finite graph B into which A embeds, and for each $l \geq 2$, there is a finite graph C such that for any coloring of the copies of A in C into l colors, there is a copy B' of B in C such that the copies of A in B' take on no more than $t(\mathrm{A})$ colors.

Infinite Ramsey Theory on Graphs and Big Ramsey Degrees

The random graph, also called the Rado graph and denoted $\mathcal{R}$, is the graph on infinitely many nodes such that for each pair of nodes, there is a 50-50 chance that there is an edge between them.

The random graph is equivalently

1. universal for countable graphs: Every countable graph embeds into $\mathcal{R}$.
2. homogeneous: Every isomorphism between two finite subgraphs in $\mathcal{R}$ is extendible to an automorphism of $\mathcal{R}$.

Vertex Colorings in the Rado graph $\mathcal{R}$

Thm. (Folklore) Given any coloring of vertices in $\mathcal{R}$ into finitely many colors, there is a subgraph $\mathcal{R}'$, which is again a random graph, such that the vertices in $\mathcal{R}'$ all have the same color.

We say that the Rado graph is indivisible, or has big Ramsey degree 1.

Edge Colorings in $\mathcal{R}$

Thm. (Pouzet/Sauer) Given any coloring of the edges in $\mathcal{R}$ into finitely many colors, there is a subgraph $\mathcal{R}'$, again a random graph, such that the edges in $\mathcal{R}'$ take no more than two colors.

Can we get down to one color?

No! but...

Colorings of Copies of Any Finite Graph in $\mathcal{R}$

Thm. (Sauer) Given any finite graph A, there is a finite number $T(\mathrm{A})$ such that for any $l \geq 2$ and any coloring of all the copies of A in $\mathcal{R}$ into l colors, there is a subgraph $\mathcal{R}'$, again a random graph, such that the set of copies of A in $\mathcal{R}'$ take on no more than $T(\mathrm{A})$ colors.

In the jargon, we say that the big Ramsey degrees for $\mathcal{R}$ are finite.

$T(\text{edge}) = 2$.

$T(\text{triangle}) = 16$.

Lower bounds were structurally found by Sauer and computed by J. Larson.

More generally,

Fraïssé classes of finite structures with the Ramsey property: Boolean algebras, vector spaces over a finite field, ordered graphs, ordered hypergraphs, ordered graphs omitting k-cliques, ordered metric spaces, and many others.

Fraïssé classes of finite structures with finite small Ramsey degrees: Graphs, hypergraphs, graphs omitting k-cliques, and many more.

Def. (Kechris, Pestov, Todorcevic 2005) An infinite relational structure $\mathcal{S}$ has finite big Ramsey degrees if for each finite substructure A of $\mathcal{S}$, there is a finite number $T(A)$ such that for any coloring of the copies of A in $\mathcal{S}$ into finitely many colors, there is a substructure $\mathcal{S}'$ of $\mathcal{S}$, isomorphic to $\mathcal{S}$, in which the copies of A take no more than $T(A)$ colors.

Infinite Structures known to have finite big Ramsey degrees

- the natural numbers (Ramsey 1929)
- the rationals (Devlin 1979)
- the Rado graph (Sauer 2006)
- the countable ultrametric Urysohn space (Nguyen Van Thé 2008)
- the dense local order (Laflamme, NVT, Sauer 2010).
- A couple of others.

The crux of all but two of these proofs is a theorem of Milliken.

(The Urysohn space result uses Ramsey's Theorem.)

Connections with Topological Dynamics

Thm. (Kechris/Pestov/Todorcevic 2005) Aut(Flim $\mathcal{K}$) is extremely amenable (has the fixed point on compacta property) if and only if $\mathcal{K}$ has the Ramsey property and consists of rigid elements.

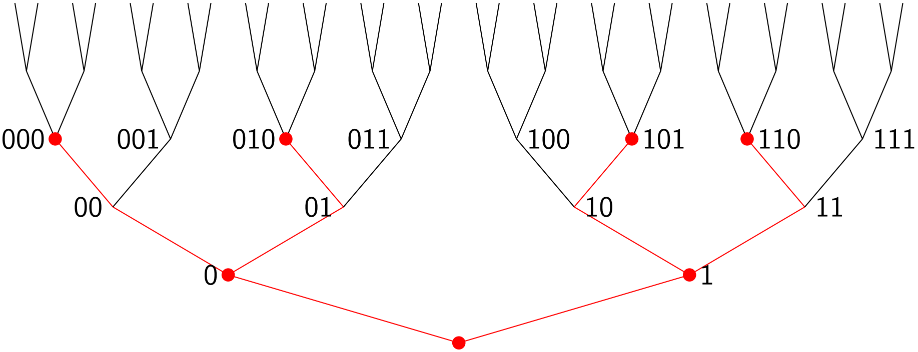
Thm. (Nguyen Van Thé 2013) Extended this to connect Fraïssé classes that have small Ramsey degrees with universal minimal flows.

Thm. (Zucker 2017) Characterized universal completion flows of Aut(Flim $\mathcal{K}$) in terms of big Ramsey degrees.

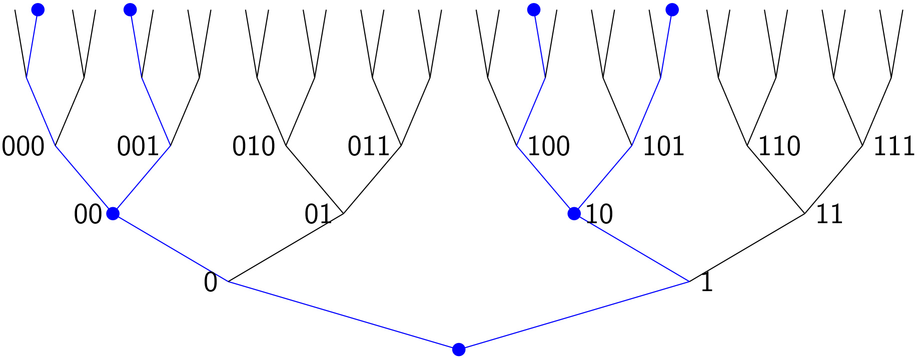
Strong Trees and Milliken's Theorem

A tree $T \subseteq 2^{<\omega}$ is a strong tree iff it is either isomorphic to $2^{<\omega}$ or to $2^{\leq k}$ for some finite k.

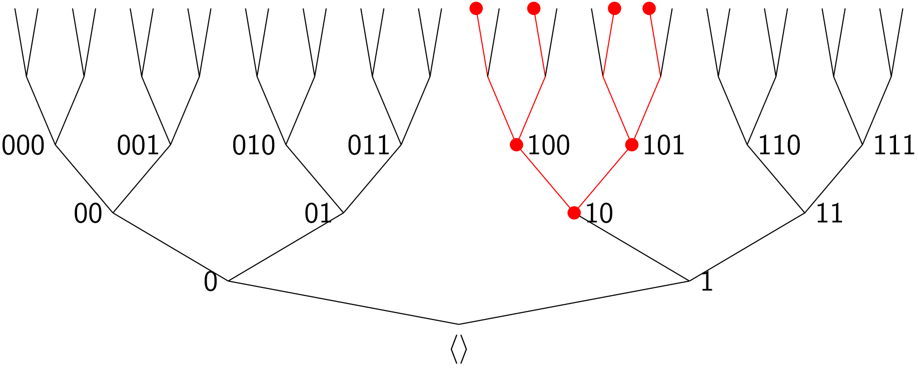
Strong Subtree $\cong 2^{\leq 2}$, Ex. 1



Strong Subtree $\cong 2^{\leq 2}$, Ex. 2



Strong Subtree $\cong 2^{\leq 2}$, Ex. 3



A Ramsey Theorem for Strong Trees

Thm. (Milliken 1979) Let $k \geq 0$, $l \geq 2$, and a coloring of all the subtrees of $2^{<\omega}$ which are isomorphic to $2^{\leq k}$ into l colors. Then there is an infinite strong subtree $S \subseteq 2^{<\omega}$ such that all copies of $2^{\leq k}$ in S have the same color.

Milliken's Theorem builds on the Halpern-Läuchli Theorem.

Thm. (Halpern-Läuchli 1966) Let $d \geq 1$, $l \geq 2$, and $T_i = 2^{<\omega}$ for $i < d$. Given a coloring of the product of level sets of the T_i into l colors,

$$f : \bigcup_{n<\omega} \prod_{i<d} T_i(n) \to l,$$

there are infinite strong trees $S_i \leq T_i$ and an infinite sets of levels $M \subseteq \omega$ where the splitting in S_i occurs, such that f is constant on $\bigcup_{m \in M} \prod_{i<d} S_i(m)$.

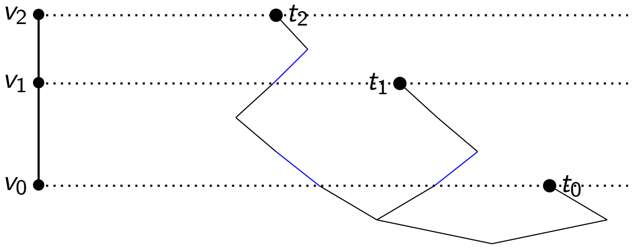
Nodes in Trees can Code Graphs

Let A be a graph. Enumerate the vertices of A as $\langle v_n : n < N \rangle$.

A set of nodes $\{t_n : n < N\}$ in $2^{<\omega}$ codes A if and only if for each pair $m < n < N$,

$$v_n \; E \; v_m \Leftrightarrow t_n(|t_m|) = 1.$$

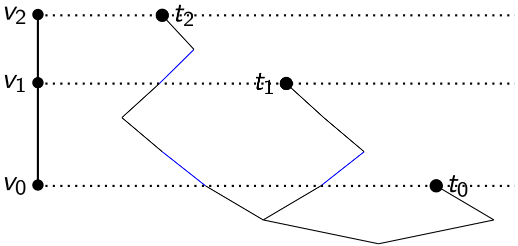
The number $t_n(|t_m|)$ is called the passing number of t_n at t_m.



Diagonal Trees Code Graphs

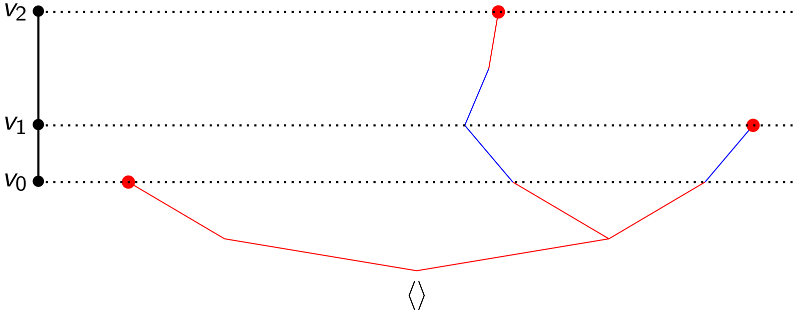
A tree T is diagonal if there is at most one meet or terminal node per level.

T is strongly diagonal if passing numbers at splitting levels are all 0 (except for the right extension of the splitting node).

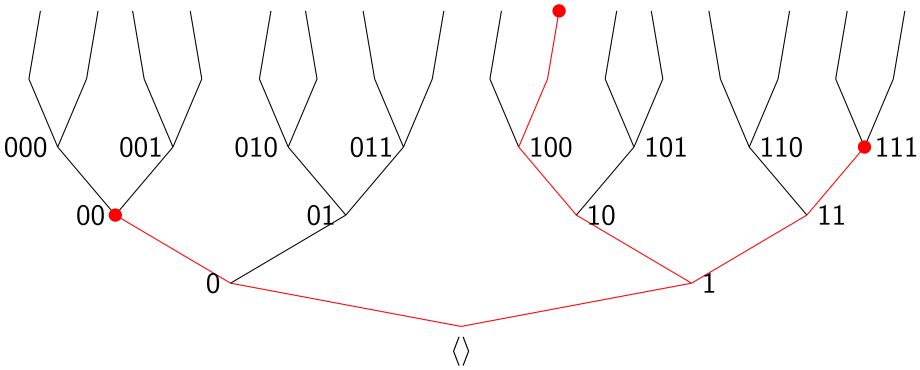


Every graph can be coded by the terminal nodes of a diagonal tree. Moreover, there is a strongly diagonal tree which codes $\mathcal{R}$.

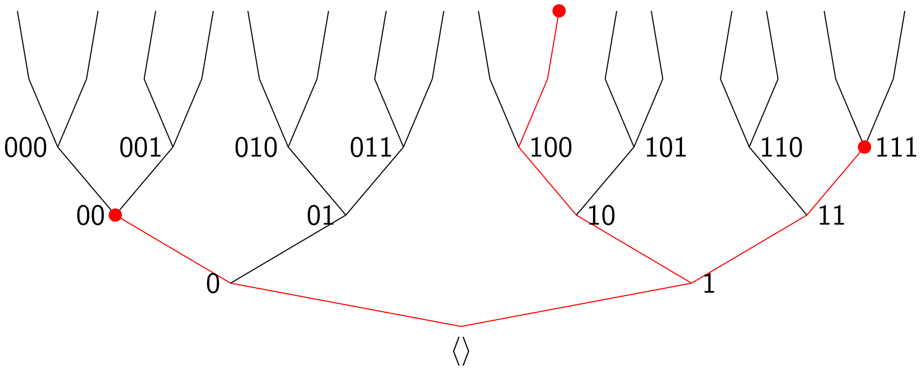
A Different Strongly Diagonal Tree Coding a Path



Strongly diagonal trees can be enveloped into strong trees



Another strong tree envelope



Outline of Sauer's Proof: $\mathcal{R}$ has finite big Ramsey degrees

1. The Rado graph is bi-embeddable with the graph coded by all nodes in the tree $2^{<\omega}$.
2. Each finite graph can be coded by finitely many strong similarity types of strongly diagonal trees.
3. Each strongly diagonal tree can be enveloped into a finite strong tree.
4. Apply Milliken's Theorem finitely many times to obtain one color for each type.
5. Choose a strongly diagonal subtree coding the Rado graph, and show that each type persists in each subgraph which is again random.

The Universal Homogeneous Triangle-Free Graph $\mathcal{H}_3$

The universal triangle-free graph $\mathcal{H}_3$ is the triangle-free graph on infinitely many vertices into which every countable triangle-free graph embeds.

Equivalently, $\mathcal{H}_3$ is homogeneous: Any isomorphism between two finite subgraphs of $\mathcal{H}_3$ extends to an automorphism of $\mathcal{H}_3$.

$\mathcal{H}_3$ is the Fraïssé limit of the Fraïssé class of finite triangle-free graphs, $\mathcal{K}_3$.

$\mathcal{H}_3$ was constructed by Henson in 1971. Henson also constructed universal k-clique-free graphs for each $k \geq 3$.

History of Results

Theorem. (Henson 1971) $\mathcal{H}_3$ is weakly indivisible.

Theorem. (Nešetřil-Rödl 1977/83) The Fraïssé class of finite ordered triangle-free graphs $\mathcal{K}_3^<$ has the Ramsey property. This implies finite small Ramsey degrees for $\mathcal{K}_3$.

Theorem. (Komjáth/Rödl 1986) $\mathcal{H}_3$ is indivisible: Vertex colorings of $\mathcal{H}_3$ have big Ramsey degree 1.

Theorem. (Sauer 1998) $\mathcal{H}_3$ has big Ramsey degree 2 for edges.

What about big Ramsey degrees in $\mathcal{H}_3$ for other finite triangle-free graphs?

Main Obstacles

"A proof of the big Ramsey degrees for $\mathcal{H}_3$ would need new Halpern-Läuchli and Milliken Theorems, and nobody knows what those should be." (Todorcevic, 2012)

Said the same thing, plus, "There is no simply representable triangle-free graph which is bi-embeddable with $\mathcal{H}_3$." (Sauer, 2013)

"So far, the lack of tools to represent ultrahomogeneous structures is the major obstacle towards a better understanding of their infinite partition properties." (Nguyen Van Thé, 2013 Habilitation)

Main Theorem: $\mathcal{H}_3$ has Finite Big Ramsey Degrees

Theorem. (D.) For each finite triangle-free graph A, there is a positive integer $T(\mathrm{A}, \mathcal{K}_3)$ such that for any coloring of all copies of A in $\mathcal{H}_3$ into finitely many colors, there is a subgraph $\mathcal{H} \leq \mathcal{H}_3$, again universal triangle-free, such that all copies of A in $\mathcal{H}$ take no more than $T(\mathrm{A}, \mathcal{K}_3)$ colors.

This is the first result on big Ramsey degrees of a homogeneous structure omitting a non-trivial substructure.

Structure of Proof: Three Main Parts

I Develop new notion of strong coding tree to represent $\mathcal{H}_3$.

II Prove a Ramsey Theorem for strictly similar finite antichains.

(a) Prove new Halpern-Läuchli Theorems for strong coding trees.
 – Three new forcings are needed, but the proofs take place in ZFC.

(b) Prove a new Ramsey Theorem for finite trees satisfying Strict P1C.
 – correct analogue of Milliken's Theorem.

(c) New notion of envelope.
 – Involves new notions of incremental strong coding tree and sets of witnessing coding nodes.

III Construct a strongly diagonal subset of coding nodes coding $\mathcal{H}_3$ and apply the Ramsey Theorem for strictly similar antichains.

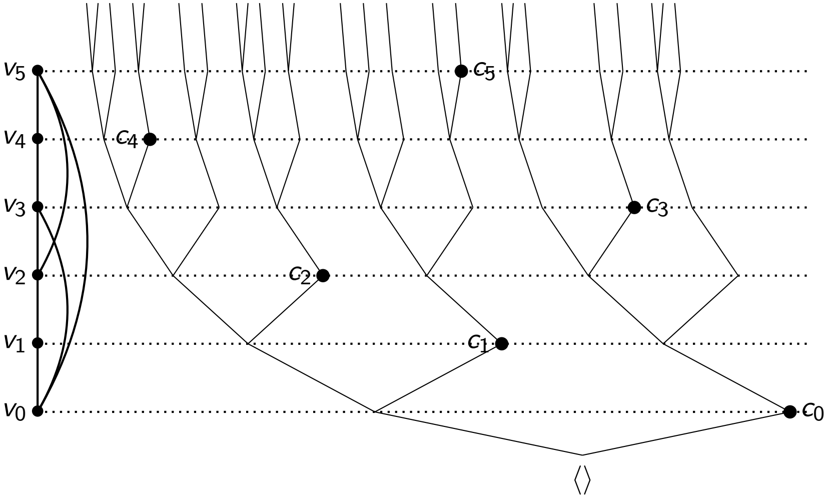
Part I: Strong Coding Trees

Strong Triangle-Free Trees

Trees with a unary predicate for distinguishing certain nodes to code vertices of a given graph, (called coding nodes), and which branch as much as possible, subject to coding no triangles.

The only forbidden structures are sets of coding nodes c_i, c_j, c_k, with lengths $|c_i| < |c_j| < |c_k|$, such that $c_j(|c_i|) = c_k(|c_i|) = c_k(|c_j|) = 1$ as this codes a triangle.

Strong triangle-free tree $\mathbb{S}$



Almost sufficient

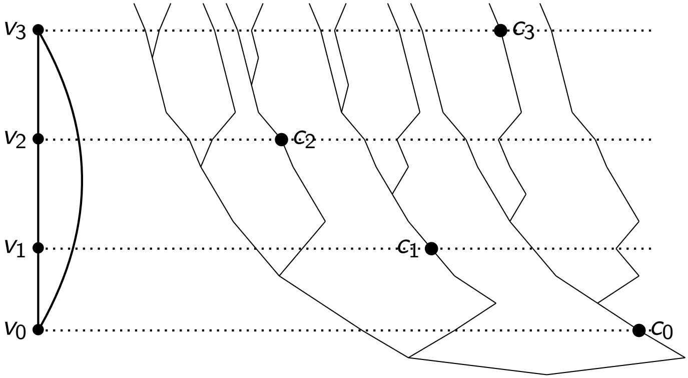
One can develop almost all the Ramsey theory one needs on strong triangle-free trees

except for the base case, vertex colorings via colorings of coding nodes: there is a bad coloring for these.

To get around this, we stretch and skew the trees so that at most one coding or one splitting node occurs at each level.

These skewed trees densely coding $\mathcal{H}_3$ are called strong coding trees.

Strong coding tree $\mathbb{T}$



A subtree $T \subseteq \mathbb{T}$ is a strong coding tree if T is strongly similar to $\mathbb{T}$.

A Space of Strong Coding Trees

$\mathcal{T}(\mathbb{T})$ is the collection of all subtrees of $\mathbb{T}$ which are strongly similar to $\mathbb{T}$.

A finite subtree A of a strong coding tree $T \in \mathcal{T}(\mathbb{T})$ can be extended to a strong coding subtree of T if A satisfies the following:

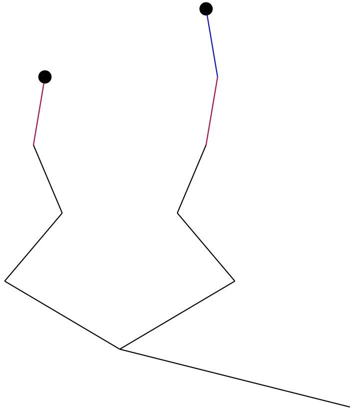
Parallel 1's Criterion: Ensures that no types are lost.
"All new sets of parallel 1's are witnessed by a coding node."

No pre-determined new parallel 1's.

The space $\mathcal{T}(\mathbb{T})$ of strong coding trees is very near a topological Ramsey space.

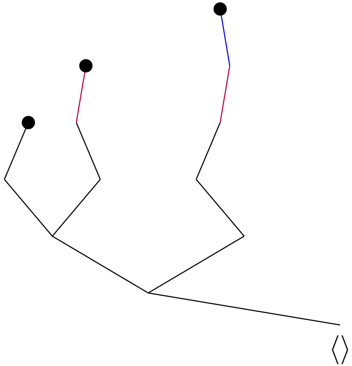
A subtree of $\mathbb{T}$ which cannot be extended to a s.c.t.

It has parallel 1's not witnessed by a coding node (P1C fails).



A subtree of $\mathbb{T}$ extendible to a strong coding tree

This tree satisfies the Parallel 1's Criterion: All its parallel 1's are witnessed by some coding node.



Part II: A Ramsey Theorem for Strictly Similar Finite Antichains.

Ramsey Theorem for Strong Coding Trees

Theorem. (D.) Let A be a finite subtree of a strong coding tree T, and let c be a coloring of all copies of A in T.

Then there is a strong coding tree $S \leq T$ in which all strictly similar copies of A in S have the same color.

This is an analogue of Milliken's Theorem for strong coding trees.

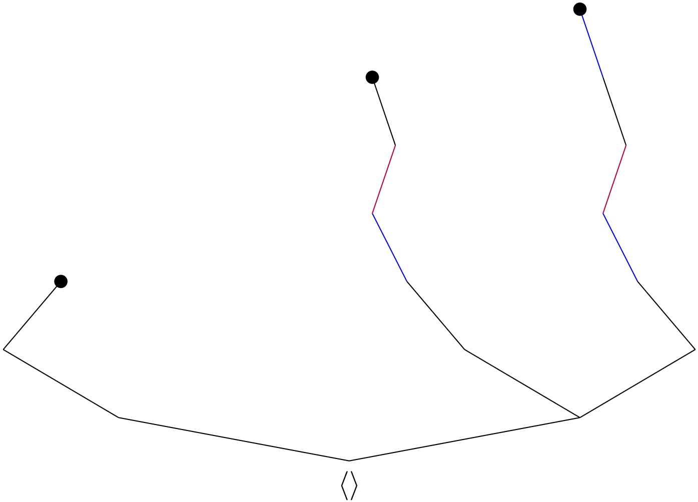
Strict similarity takes into account tree isomorphism, placement of coding nodes, and placement of new sets of parallel 1's. Strict similarity is an equivalence relation.

Let G be the graph with three vertices and no edges.

We show some distinct strict similarity types of trees coding G.

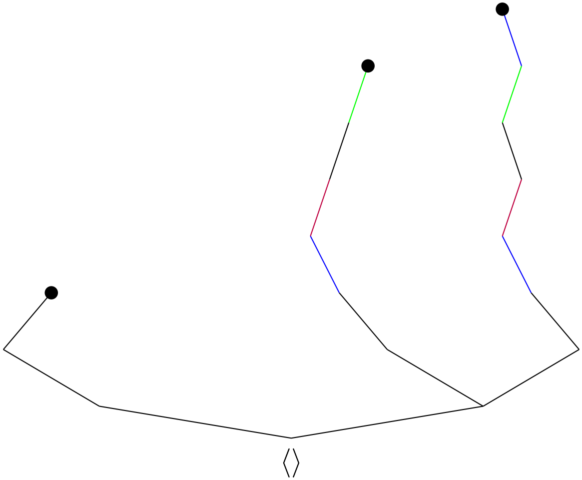
G a graph with three vertices and no edges

A tree A coding G



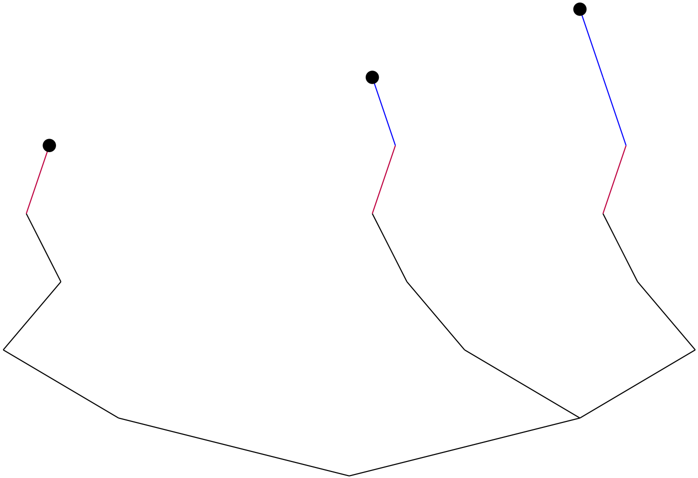
G a graph with three vertices and no edges

B codes G and is strictly similar to A.



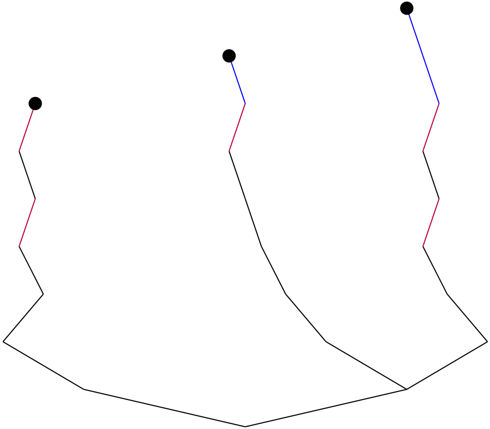
The tree C codes G

C is not strictly similar to A.

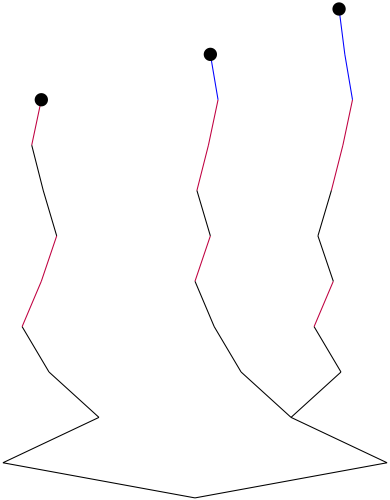


The tree D codes G

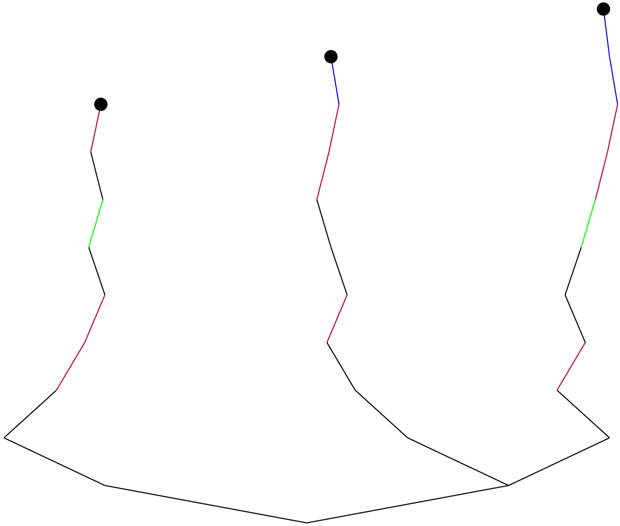
D is not strictly similar to either A or C.



The tree E codes G and is not strictly similar to A - D



The tree F codes G and is strictly similar to E



Reducing the Upper Bounds

A strong tree U with coding nodes is incremental if whenever a new set of parallel 1's appears in U, all of its subsets appear as parallel 1's at a lower level.

The trees A, B, E, and F are incremental.

The trees C and D are not incremental.

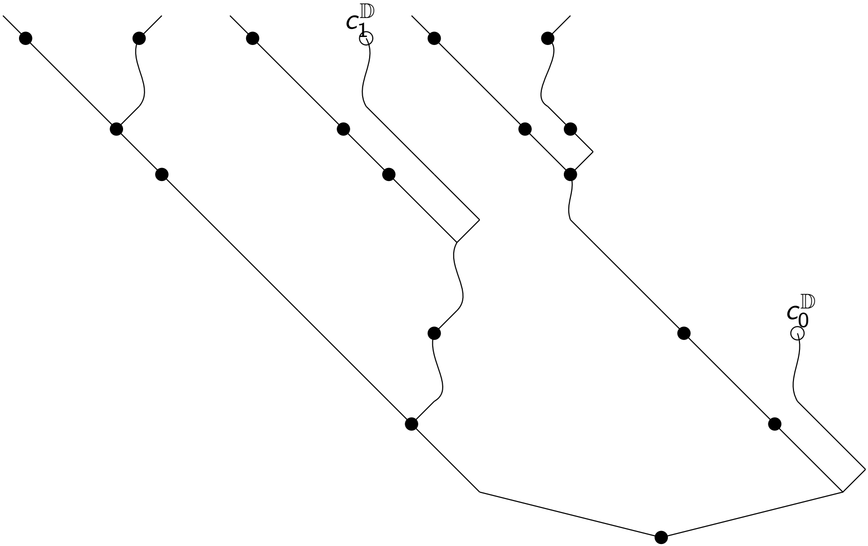
The notion of incremental aids in the proof of the next theorem, while simultaneously reducing the upper bounds on the big Ramsey degrees for finite triangle-free graphs in $\mathcal{H}_3$.

Part III: Apply the Ramsey Theorem for Strictly Similar Antichains to obtain the Main Theorem.

Bounds for $T(G, \mathcal{K}_3)$

1. Let G be a finite triangle-free graph, and let f color the copies of G in $\mathcal{H}_3$ into finitely many colors.
2. The strong coding tree $\mathbb{T}$ codes $\mathcal{H}_3$. For each antichain A of coding nodes in $\mathbb{T}$ coding a copy G_A of G, define $f'(A) = f(G_A)$.
3. List the finitely many strict similarity types of antichains of coding nodes in $\mathbb{T}$ coding G.
4. Apply the Ramsey Theorem for Strict Similarity Types once for each type coding G to obtain a strong coding tree $S \leq \mathbb{T}$ in which f' has one color per type.
5. Take a strongly diagonal subtree $\mathbb{D}$ in S which codes $\mathcal{H}_3$, and let $\mathcal{H}'$ be the subgraph of $\mathcal{H}_3$ coded by $\mathbb{D}$.
6. Then f has no more colors on the copies of G in $\mathcal{H}'$ than the number of incremental strict similarity types of antichains coding G.

Constructing a diagonal set $\mathbb{D}$ of coding nodes coding $\mathcal{H}_3$



Remarks

1. Proving the actual lower bounds is in progress.
2. The methods are currently being generalized to the universal k-clique-free graphs $\mathcal{H}_k$, for all $k \geq 4$.

Part II Expanded:

Ideas behind the proof of
the Ramsey Theorem for Strictly Similar Subtrees

(a) Halpern-Läuchli-style Theorem

Thm. (D.) Given a strong coding tree T and

1. B a finite valid strong coding subtree of T;
2. A a finite subtree of B with $\max(A) \subseteq \max(B)$; and
3. X a level set extending A into T with $A \cup X$ satisfying the P1C and valid in T.

Color all end-extensions Y of A in T for which $A \cup Y$ is strictly similar to $A \cup X$ into finitely many colors.

Then there is a strong coding tree $S \leq T$ end-extending B such that all level sets Y in S with $A \cup Y$ strictly similar to $A \cup X$ have the same color.

Remark. The proof uses three different forcings. The forcings are best thought of as conducting unbounded searches for finite objects in ZFC.

Case (i): X contains a splitting node

List the immediate successors of $\max(A)$ as $s_0, \dots, s_d$, where s_d denotes the node which the splitting node in X extends.

Let $T_i = \{t \in T : t \supseteq s_i\}$, for each $i \le d$.

Fix κ large enough so that $\kappa \rightarrow (\aleph_1)^{2d}_{\aleph_0}$ holds.

Such a κ is guaranteed in ZFC by a theorem of Erdős and Rado.

The forcing for Case (i)

$\mathbb{P}$ is the set of conditions p such that p is a function of the form

$$p : \{d\} \cup (d \times \vec{\delta}_p) \to T \restriction l_p,$$

where $\vec{\delta}_p \in [\kappa]^{<\omega}$ and $l_p \in L$, such that

(i) $p(d)$ is *the* splitting node extending s_d at level l_p;

(ii) For each $i < d$, $\{p(i, \delta) : \delta \in \vec{\delta}_p\} \subseteq T_i \restriction l_p$.

$q \le p$ if and only if $\vec{\delta}_q \supseteq \vec{\delta}_p$, $l_q \ge l_p$, and

(i) $q(d) \supset p(d)$, and $q(i, \delta) \supset p(i, \delta)$ for each $\delta \in \vec{\delta}_p$ and $i < d$; and

(ii) The set $\{q(i, \delta) : (i, \delta) \in d \times \vec{\delta}_p\} \cup \{q(d)\}$ has no new sets of parallel 1's above $\{p(i, \delta) : (i, \delta) \in d \times \vec{\delta}_p\} \cup \{p(d)\}$.

Case (i): X contains a splitting node

The forcing is used to find a good set of starting nodes where it is possible to extend them to homogeneous levels.

We alternate building the subtree by hand with using the forcing to find the next level where homogeneity is guaranteed.

Remarks. (1) No generic extension is actually used.

(2) These forcings are not simply Cohen forcings; the partial orderings are stronger in order to guarantee that the new levels we obtain by forcing are extendible inside T to another strong coding tree.

(3) The assumption that $A \cup X$ satisfies the Parallel 1's Criterion is necessary.

(b) Ramsey Theorem for Finite Trees satisfying the SP1C

Case (ii), when X contains a coding node requires a different forcing. A third forcing and induction are required to obtain

Thm. (D.) Let T be a strong coding tree, and let A be a finite valid subtree of T satisfying the Strict P1C. Suppose all the strictly similar copies of A in T are colored in finitely many colors.

Then there is a strong coding subtree $S \leq T$ such that all strictly similar copies of A in S have the same color.

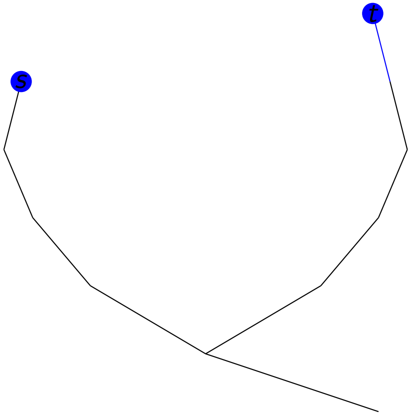
A tree A satisfies the strict P1C if each new set of parallel 1's is witnessed by a coding node before anything else happens (other occurrences of new parallel 1's, splits, or coding nodes).

(c) Envelopes and Witnessing Coding Nodes

Envelopes add some neutral coding nodes to a given finite tree to make it satisfy the strict Parallel 1's Criterion.

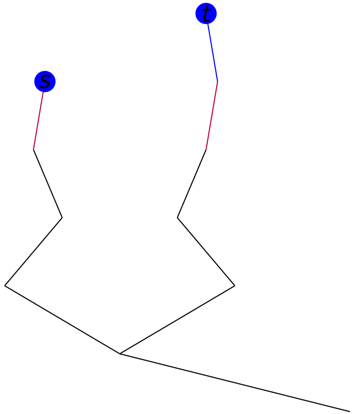
The additional coding nodes are to witness the places where new sets of parallel 1's occur.

A codes a non-edge



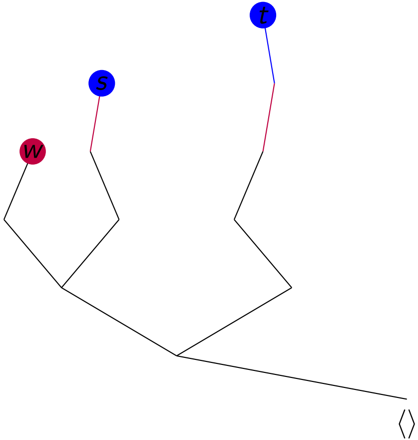
This satisfies the Parallel 1's Criterion, so A is its own envelope.

B codes a non-edge



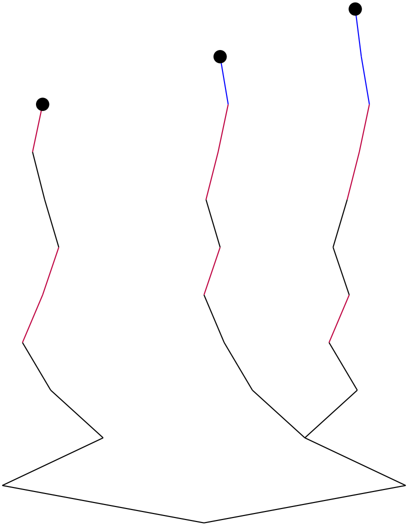
B does not satisfy the Parallel 1's Criterion.

An Envelope E(B)

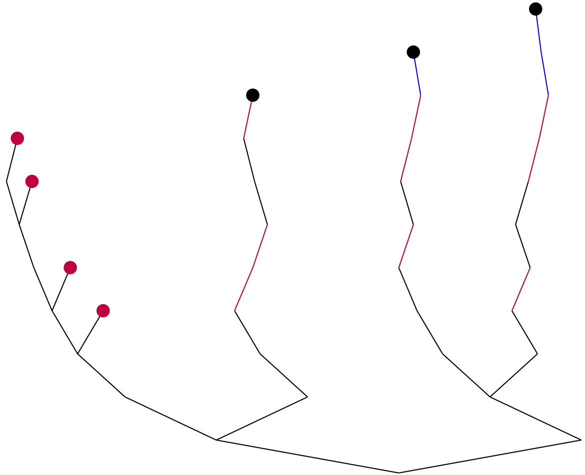


The envelope $E(B)$ satisfies the Parallel 1's Criterion.

An incremental tree C coding three vertices with no edges



An envelope of the incremental tree C

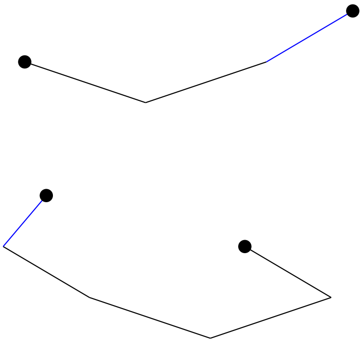


The Ramsey Thm for Strictly Similar Antichains follows

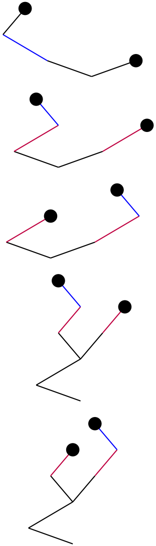
1. Let A be a finite antichain A of coding nodes inducing an incremental tree; let $E(A)$ be an envelope.
2. A coloring f of all antichains in $\mathbb{T}$ strictly similar to A induces a coloring f' on all strictly similar copies of $E(A)$ in $\mathbb{T}$.
3. Apply the Ramsey Theorem for Trees with the SP1C for f' on $\mathbb{T}$ to obtain $T \leq \mathbb{T}$ in which all copies of $E(A)$ have the same color.
4. Build an incremental strong coding tree $S \leq T$ and a set of witnessing coding nodes $W \subseteq T$ having no parallel 1's with any coding node in S.
5. Then each copy of A in S has an envelop in T, by adding in some nodes from W.
6. Thus, each copy of A in S has the same color.

To finish: the big Ramsey degrees of edges and non-edges

Edges have 2 Strict Similarity Types (Sauer 1998)



Non-edges have 5 Strict Similarity Types (D.)



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