Perfect tree forcings for singular cardinals

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Joint work with Daniel Hathaway and Karel Prikry

Motivation: Distributive Laws in Boolean Algebras, 1960's

A forcing $\mathbb{P}$ is (λ, κ)-distributive if $\mathbb{P}$ adds no new functions from λ into κ.

Motivating Question (Solovay 1960's): Which cardinals κ can be first failures of (ω, κ)-distributivity in some forcing?

Work on this and other distributivity problems appeared in

[Prikry 67] *On models constructed using perfect sets.* (unpublished)

[Namba 71] *Independence proof of (ω, ω_α)-distributive laws in complete Boolean algebras.*

[Namba 72] $(\omega_1, 2)$*-distributive law and perfect sets in generalized Baire space.*

[Bukovský 76] *Changing cofinality of $\aleph_2$.* (69 unpublished)

[D-Hathaway-Prikry] includes [Prikry 67] and proves some further properties about his tree forcings.

Related Question (Vopěnka 1966): Can one change the cofinality of $\aleph_2$ to $\aleph_0$ without collapsing $\aleph_1$?

Results on either of the two questions have implications for the other.

Other related work includes

[Prikry 68] *Changing measurable into accessible cardinals.*

[Bukovský-Copláková 90] *Minimal collapsing extensions of models of ZFC.*

Full Answer to Solovay's Question

Theorem. Assume $(\forall \mu < \kappa)\ \mu^\omega < \kappa$, and either κ is regular or $\mathrm{cf}(\kappa) = \omega$. Then there is a forcing $\mathbb{P}$ which adds a new ω sequence of ordinals in κ, but no new bounded ω-sequences in κ.

$\mathrm{cf}(\kappa) = \omega$ due to Prikry (1967), and κ regular due to Namba (1971).

The cardinal arithmetic assumption is necessary.

My interest stemmed from co-stationarity of the ground model:

Thm. (D. 08) Forcing a new ω-sequence into κ over L makes $(\mathcal{P}_\mu(\lambda))^{L[G]} \setminus L$ stationary in $(\mathcal{P}_\mu(\lambda))^{L[G]}$, for all cardinals $\mu < \lambda$ in $L[G]$ with $\lambda \geq \kappa$ and μ regular.

The perfect tree forcing of Prikry

Fix an increasing sequence $\langle \kappa_n : n < \omega \rangle$ of regular cardinals and define

$$\mathcal{X} = \prod_{n<\omega} \kappa_n$$

Give $\mathcal{X}$ the product topology.
(A space of Stone mentioned by Motto Ros yesterday.)

Let $\kappa = \sup_{n<\omega} \kappa_n$. A subset $P \subseteq \mathcal{X}$ is perfect if it is closed, and given any point $f \in P$, every neighborhood of f in P has size κ^ω.

A perfect tree is the tree induced by some perfect set.

$\mathbb{P}$ is the set of all perfect subtrees of $\widehat{\mathcal{X}}$, partially ordered by inclusion.

Strong Splitting Normal Form

A perfect tree $T \in \mathbb{P}$ is in strong splitting normal form if there is a strictly increasing sequence $\langle l_n : n < \omega \rangle$ of levels such that all nodes in T of length l_n have κ_n immediate successors, and all nodes of other lengths do not split.

The set of all perfect trees in strong splitting normal form is dense in $\mathbb{P}$.

This and other results use singularity and a Ramsey-style lemma on Laver-like trees on

$$\widehat{\mathcal{X}} = \bigcup_{m<\omega} \prod_{n \leq m} \kappa_n.$$

3-Parameter Distributivity - stratified covering properties

A forcing $\mathbb{Q}$ is $(\lambda, \kappa, < \mu)$-distributive if for each $g : \lambda \to \kappa$ in $V^{\mathbb{Q}}$, there is a function $f : \lambda \to [\kappa]^{<\mu}$ in V such that $(\forall \alpha < \lambda)\ g(\alpha) \in f(\alpha)$.

Thm. (DHP) In the perfect tree forcing of Prikry,

(1) $(\omega, \kappa, < \mu)$-distributivity fails for all $\mu < \kappa$; but

(2) $(\omega, \infty, < \kappa)$-distributivity holds; and

(3) $(\mathfrak{d}, \infty, < \kappa)$-distributivity fails.

(1) is straightforward.

(2) follows from a Sacks-like property of $\mathbb{P}$.

(3) uses dominated-by families.

Connections with $\mathcal{P}(\omega)/\mathit{fin}$

The distributivity number $\mathfrak{h}$ is the smallest cardinal ν for which $\mathcal{P}(\omega)/\mathit{fin}$ adds a new subset of ν; equivalently, (ν, ∞)-distributivity fails.

$\aleph_1 \le \mathfrak{h} \le \mathfrak{d} \le \mathfrak{c}$.

Thm. (DHP)
(1) $\mathcal{P}(\omega)/\mathit{fin}$ completely embeds into $\mathbb{P}$. Hence,
(2) $\mathbb{P}$ is not $(\mathfrak{h}, 2)$-distributive.
(3) κ^ω to $\mathfrak{h}$.

(1) uses the base tree matrix method of Balcar-Pelant-Simon (1980), similarly to work of Bukovský-Copláková (1990) for regular cardinals κ.

(3) uses an antichain in $\mathbb{P}$ of size κ^ω.

Minimal degrees of constructibility

A well-studied notion, going back to Sacks forcing.

Thm. (DHP) If all κ_n are measurable then $\mathbb{P}$ adds a minimal degree of constructibility for new ω-sequences:

Given $T \in \mathbb{P}$ and $\dot{A}$ such that

$$T \Vdash \dot{A} : \omega \to \check{V} \text{ and } \dot{A} \notin \check{V},$$

then $T \Vdash \dot{G} \in \check{V}[\dot{A}]$.

Some of the many Open Problems

Question 1. What is the optimal requirement on the κ_n for to have a minimal degree of constructibility for new ω-sequences?
(see [Brown-Groszek 06])

Question 2. What about analogues for singular cardinals of uncountable cofinality?

Thank you for your kind attention!