Applications of High Dimensional Ellentuck spaces

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Boolean algebras
Lattices
Alegbraic logic, universal Algebra
Set theory
Topology - general, point-free, set-theoretic

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Ellentuck Space

The Ellentuck space is the space $[\omega]^\omega$ with topology generated by basic open sets

$$[s, A] = \{X \in [\omega]^\omega : s \sqsubset X \subseteq A\},$$

where $s \in [\omega]^{<\omega}$ and $A \in [\omega]^\omega$.

Thm. (Ellentuck) The Ellentuck space is a topological Ramsey space: Given $\mathcal{X} \subseteq [\omega]^\omega$ with the property of Baire, for any basic open set $[s, A]$, there is a member $B \in [s, A]$ such that

$$\text{either } [s, B] \subseteq \mathcal{X} \text{ or else } [s, B] \cap \mathcal{X} = \emptyset.$$

Forcing with members of the Ellentuck space partially ordered by $\subseteq^*$ adds a Ramsey ultrafilter.

Forcing Ultrafilters

$([\omega]^\omega, \subseteq^*)$ is forcing equivalent to $\mathcal{P}(\omega)/\mathrm{Fin}$.

A natural extension of this Boolean algebra: $\mathcal{P}(\omega \times \omega)/\mathrm{Fin} \otimes \mathrm{Fin}$.

$X \in \mathrm{Fin} \otimes \mathrm{Fin}$ iff $X \subseteq \omega \times \omega$ and $\forall^\infty n \in \omega$, $\{i \in \omega : (n, i) \in X\} \in \mathrm{Fin}$.

$\mathcal{P}(\omega^2)/\mathrm{Fin}^{\otimes 2}$ adds an ultrafilter $\mathcal{U}_2$, the next best thing to a p-point:

$$\mathcal{U}_2 \to (\mathcal{U}_2)^2_{r,4}.$$

The projection to the first coordinates, $\pi_1(\mathcal{U}_2)$, is a Ramsey ultrafilter, generic for $\pi_1(\mathcal{P}(\omega^2)/\mathrm{Fin}^{\otimes 2}) \cong \mathcal{P}(\omega)/\mathrm{Fin}$.

Extending $\mathrm{Fin}^{\otimes 2}$ to all uniform barriers

Recursively construct ideals on ω^{k+1}: $\mathrm{Fin}^{\otimes k+1} = \mathrm{Fin} \otimes \mathrm{Fin}^{\otimes k}$.

$\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$ forces an ultrafilter $\mathcal{U}_k$: for each $j < k$, $\pi_j(\mathcal{U}_k) \cong \mathcal{U}_j$.

Replace ω^k by $[\omega]^k$; $\mathrm{Fin}^{\otimes k}$ by the ideal I_k on $[\omega]^k$ determined by $\mathrm{Fin}^{\otimes k}$.

$\mathcal{P}([\omega]^k)/I_k \cong \mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$.

$[\omega]^k$ is a uniform barrier on ω of rank k.

This construction of I_k can be extended to all uniform barriers on ω.

Hierarchy of Boolean Algebras and forced ultrafilters

Example. Schreier barrier: $S = \{s \in [\omega]^{<\omega} : |s| = \min s + 1\}$.

For $X \subseteq S$, $X_n = \{s \in X : \min s = n\}$.

$I_S = \{X \subseteq S : \forall^\infty n\, (X_n \in I_{S_n})\}$.

For any uniform barrier B on ω, $\mathcal{P}(B)/I_B$ forces an ultrafilter $\mathcal{U}_B$ on countable base set B.

Fact. If B projects to C, then $\mathcal{P}(C)/I_C$ embeds as a complete subalgebra of $\mathcal{P}(B)/I_B$, and $\mathcal{U}_C$ is isomorphic to a projection of $\mathcal{U}_B$.

Initial motivation for h.d. Ellentuck spaces: Cofinal Types

A function $f : \mathcal{U} \to \mathcal{V}$ between ultrafilters is cofinal if f maps each filter base for $\mathcal{U}$ to a filter base for $\mathcal{V}$.

$\mathcal{U}$ is Tukey reducible to $\mathcal{U} \geq_T \mathcal{V}$ iff there is a cofinal map from $\mathcal{U}$ into $\mathcal{V}$.

The equivalence relation defined by $\mathcal{U} \equiv_T \mathcal{V}$ iff $\mathcal{U} \leq_T \mathcal{V}$ and $\mathcal{V} \leq_T \mathcal{U}$ is a coarsening of the Rudin-Keisler equivalence relation of isomorphism.

Thm.

1. (Folklore) The ultrafilter $\mathcal{U}_2$ forced by $\mathcal{P}(\omega^2)/\mathrm{Fin}^{\otimes 2}$ is Rudin-Keisler minimal above the Ramsey ultrafilter $\pi_1(\mathcal{U}_2)$.
2. (Blass, D., Raghavan) $\mathcal{U}_2 \geq_T \pi_1(\mathcal{U}_2)$ and $\mathcal{U}_2$ is not Tukey maximal.

Initial Tukey Structures

So what exactly is Tukey below $\mathcal{U}_2$?

Thm. [D1]

1. $\mathcal{U}_2$ is Tukey minimal above its projected Ramsey ultrafilter $\pi_1(\mathcal{U}_2)$.
2. For each $k \geq 2$, The ultrafilter $\mathcal{U}_k$ forced by $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$ has initial Tukey structure exactly a chain of length k. Likewise for its initial Rudin-Keisler structure.
3. [D2 and unpublished] For each uniform barrier B of infinite rank, $\mathcal{U}_B$ has initial Tukey and RK structures which are chains of length 2^ω, and they form a hierarchy via projection to barriers of smaller rank.

Remark. These results rely on new topological Ramsey spaces and canonization theorems for equivalence relations.

The 2-dimensional Ellentuck space $\mathcal{E}_2$

Goal: Construct a topological Ramsey space dense in $(\mathrm{Fin} \otimes \mathrm{Fin})^+$.

Q. Which subsets of $(\mathrm{Fin} \otimes \mathrm{Fin})^+$ should we allow?

A. Fix a particular order $\prec$ of the members of non-decreasing sequences of natural numbers of length 2 in order type ω so that each infinite set is the limit of its finite approximations.

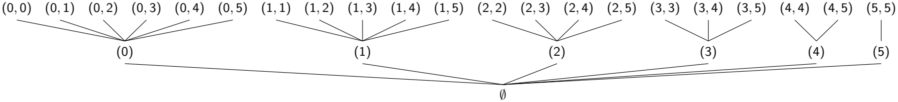
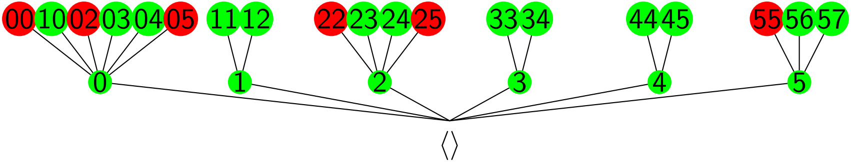
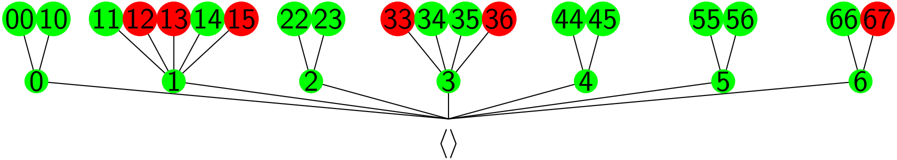


Figure: $\mathbb{W}_2$

$\mathcal{E}_2$ consists of all subsets of $\mathbb{W}_2$ for which the $\prec$-preserving bijection is also a tree-isomorphism.



A member of $\mathcal{E}_2$



A member of $\mathcal{E}_2$

The collection of $X \subseteq \mathbb{W}_2$ for which the $\prec$-preserving bijection from $\mathbb{W}_2$ to X preserves the tree structure induces the finite approximations.
The basic open sets of $\mathcal{E}_2$ are of the form

$$[s, A] = \{X \in \mathcal{E}_2 : s \sqsubset A \subseteq X\}.$$

Thm. [D1] $\mathcal{E}_2$ satisfies the 4 axioms of Todorcevic, and hence is a topological Ramsey space: Every subset with the property of Baire is Ramsey.

That $\mathcal{E}_2$ is a topological Ramsey space was heavily utilized when proving the canonization theorem for equivalence relations on barriers on $\mathcal{E}_2$.

This was applied to show that the generic ultrafilter forced by $\mathcal{P}(\omega^2)/\mathrm{Fin}^{\otimes 2}$ has, up to cofinal equivalence, exactly one Tukey type below it, namely that of its projected Ramsey ultrafilter.

The 3-dimensional Ellentuck space $\mathcal{E}_3$

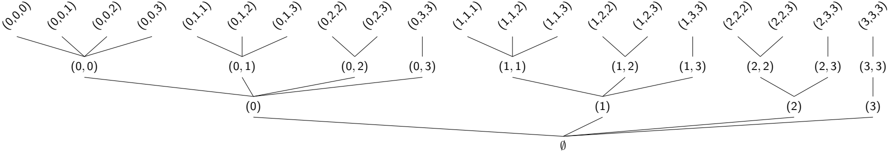
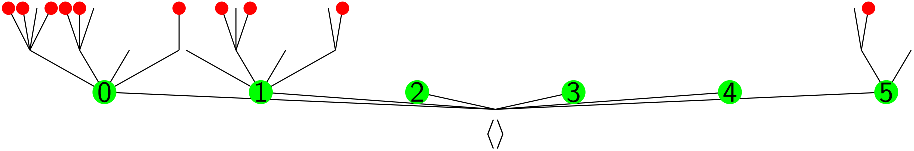


Figure: $\mathbb{W}_3$



A member of $\mathcal{E}_3$

$\mathcal{E}_S$ for S the Schreier barrier

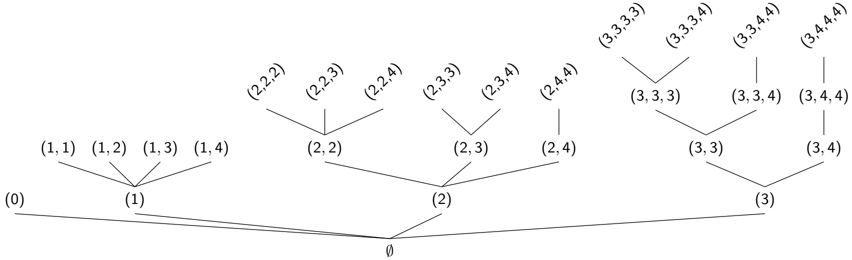


Figure: $\mathbb{W}_{\mathcal{S}}$

$X \in \mathcal{E}_S$ only if $X \subseteq \mathbb{W}_S$, for each n for which X has non-empty intersection with the subtree above (n), that restriction of X is in $\mathcal{E}_n$, and more structural requirements which are defined recursively from the structural requirements for th $\mathcal{E}_k$.

The Infinite Dimensional Ellentuck Spaces

Thm. [D2] For each uniform barrier B, there is a topological Ramsey space $\mathcal{E}_B$ which is dense in I_B^+.

Hence, $(\mathcal{E}_B, \subseteq^{I_B})$ is forcing equivalent to $\mathcal{P}(B)/I_B$.

Thus, the restriction of $\mathcal{P}(B)$ to $\mathcal{E}_B$ produces infinitary Ramsey theory, for those partitions into sets satisfying the property of Baire in the Ellentuck topology.

This was a necessary, though not sufficient, step in proving the initial Tukey structures below the ultrafilters $\mathcal{U}_B$.

Other Applications of Extended Ellentuck Spaces

A hierarchy of new Banach spaces

In [Arias, D., Girón, Mijares], we constructed a new Banach spaces using the Tsirelson norm construction over fronts of finite rank on the $\mathcal{E}_k$ spaces.

This forms a hierarchy of spaces over ℓ_p, with spaces formed from $\mathcal{E}_k$ projecting (in many different ways) to spaces from $\mathcal{E}_j$, for $j < k$.

My motivation for this project was to shed new light on distortion problems. Much work still needs to be done in this direction.

Preservation of ultrafilters by Product Sacks Forcing

Thm. [Y.Y. Zheng] The ultrafilters forced by $\mathcal{P}(\omega^k)/\mathrm{Fin}^{\otimes k}$ are preserved by products of Sacks forcing with countable support.

She first proved a *Moderately-Abstract Parametrized Ellentuck Theorem* for $\mathcal{R} \times \mathbb{R}^\omega$, for a large class of topological Ramsey spaces.

She then showed that $\mathcal{E}_k$ spaces satisfy the premises of this parametrization theorem, which is applied to obtain the theorem above.

A Barren Extension

Thm. [Henle, Mathias, Woodin] Let M be a transitive model of ZF + $\omega \to (\omega)^\omega$ and N its Hausdorff extension, that is the extension $M[\mathcal{U}]$ where $\mathcal{U}$ is the Ramsey ultrafilter forced by $\mathcal{P}(\omega)/\mathrm{Fin}$.

Then M and N have the same sets of ordinals; moreover, every sequence in N of elements of M lies in M.

In particular, this theorem holds when M is the Solovay model $L(\mathbb{R})$.

A Hierarchy of Barren Extensions

Thm. [D., Hathaway] Fix a uniform barrier B. Let M be a transitive model of ZF + every subset of $\mathcal{E}_B$ is Ramsey, and let $N = M[\mathcal{U}_B]$ be the generic extension obtained by forcing with $(\mathcal{E}_B, \subseteq^{I_B})$.
Then M and N have the same sets of ordinals; moreover, every sequence in N of elements of M lies in M.

Thus, there is a hierarchy of models $L(\mathbb{R})[\mathcal{U}_B]$ with stronger and stronger fragments of choice, in the form of containing an ultrafilter $\mathcal{U}_B$ and all $\mathcal{U}_C$ where C is a uniform barrier obtained by a projection of B, all of which are barren extenions of $L(\mathbb{R})$.

References

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