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GLOBAL CO-STATIONARITY OF THE GROUND MODEL FROM A NEW COUNTABLE LENGTH SEQUENCE

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ABSTRACT. Suppose $V \subseteq W$ are models of ZFC with the same ordinals, and that for all regular cardinals κ in W, V satisfies $\square_\kappa$. If $W \setminus V$ contains a sequence $r : \omega \to \gamma$ for some ordinal γ, then for all cardinals $\kappa < \lambda$ in W with κ regular in W and $\lambda \geq \gamma$, $(\mathscr{P}_\kappa(\lambda))^W \setminus V$ is stationary in $(\mathscr{P}_\kappa(\lambda))^W$. That is, a new ω-sequence achieves global co-stationarity of the ground model.

1. Introduction

Suppose $V \subseteq W$ are models of ZFC with the same ordinals, κ is regular and uncountable in W, and $\lambda \geq (\kappa^+)^W$. We say that *the ground model is co-stationary in* W if $(\mathscr{P}_\kappa(\lambda))^W \setminus (\mathscr{P}_\kappa(\lambda))^V$ is stationary in $(\mathscr{P}_\kappa(\lambda))^W$. Note that $(\mathscr{P}_\kappa(\lambda))^V = (\mathscr{P}_\kappa(\lambda))^W \cap V$; hence, $(\mathscr{P}_\kappa(\lambda))^W \setminus (\mathscr{P}_\kappa(\lambda))^V = (\mathscr{P}_\kappa(\lambda))^W \setminus V$. We shall subsequently drop the superscript for the larger model W with the convention that $\mathscr{P}_\kappa(\lambda)$ denotes $(\mathscr{P}_\kappa(\lambda))^W$.

Abraham in [1] showed that if $\mathbb{P}$ is a c.c.c. forcing which adds a new real, then $\mathscr{P}_{\aleph_1}(\lambda) \setminus V$ is stationary in $V^{\mathbb{P}}$ for all $\lambda \geq \aleph_2$. Answering a question of Abraham, Gitik showed in [5] that a new real in the larger model is enough to obtain co-stationarity of the ground model. In fact, he showed more.

Theorem 1.1 (Gitik [5])**.** *Let $V \subseteq W$ be models of ZFC with the same ordinals, κ a regular uncountable cardinal in W, and $\lambda \geq (\kappa^+)^W$. Suppose that there is a real in $W \setminus V$. Then $\mathscr{P}_\kappa(\lambda) \setminus V$ is stationary in W.*

We shall say that the ground model is *globally co-stationary* if for some κ_0, $\mathscr{P}_\kappa(\lambda) \setminus V$ is stationary for all cardinals $\lambda > \kappa \geq \kappa_0$ in W, with κ regular in W. Theorem 1.1 shows that any new real in the larger model achieves global co-stationarity of the ground model.

In this paper, we are interested in how the ground model is affected when the larger model contains a new sequence of countable length.

Question 1.2. Is a new sequence of length ω enough to ensure global co-stationarity of the ground model?

An upper bound for this was obtained by Dobrinen and Friedman in [4].

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Theorem 1.3 (Dobrinen:Friedman [4])**.** *Suppose $\nu \geq \omega_1$ and there is a proper class of ν-Erdős cardinals in V'. Then there is a class generic extension V of V' in which the following holds: Suppose $\kappa' > \nu$ is regular, and $\mathbb{P}$ adds a new function $r : \omega_1 \to \nu$ and is κ'-c.c. (or just satisfies the $(\rho^+, \rho^+, < \rho)$-distributive law for all successor cardinals $\rho \geq \kappa'$, and is θ-c.c. for the least regular limit cardinal $\theta \geq \kappa'$). Then $\mathscr{P}_\kappa(\lambda) \setminus V$ is stationary in $V^{\mathbb{P}}$ for each regular $\kappa \geq \kappa'$ and all $\lambda \geq \kappa^+$ in $V^{\mathbb{P}}$.*

As shown in [4], when $\nu = \omega_1$, the proper class of ν-Erdős cardinals is necessary, as in that case, a covering theorem of Magidor from [8] applies. In fact, it was proved in [4] that global co-stationarity of the ground model in a forcing extension obtained by an $\aleph_2$-c.c. forcing which adds a new subset of $\aleph_1$ is equiconsistent with a proper class of ω_1-Erdős cardinals. However, Theorem 1.3 left open the following questions.

Question 1.4. Suppose $\mathbb{P}$ adds no new reals but does add a new ω-sequence. Let ν be the least cardinal such that $\mathbb{P}$ adds a new function $r : \omega \to \nu$. Does it follow that $(\mathscr{P}_\kappa(\lambda))^V$ is co-stationary in $V^{\mathbb{P}}$ for all cardinals $\aleph_1 < \kappa < \lambda$ in $V^{\mathbb{P}}$ with κ regular in $V^{\mathbb{P}}$ and $\lambda \geq \nu$? (This is Question 1.3 from [4].)

Question 1.5. If $\nu > \omega_1$, is a proper class of ν-Erdős cardinals necessary to achieve global co-stationarity of the ground model from a new sequence $r : \omega_1 \to \nu$?

Question 1.6. Are ν-Erdős cardinals necessary to achieve global co-stationarity of the ground model if there is a new sequence $r : \omega \to \nu$?

The Main Theorem gives a negative answer to Question 1.6 and a positive answer to Question 1.4 in L. Moreover, the larger model need not be a forcing extension of the ground model.

Main Theorem. *Let $V \subseteq W$ be models of ZFC with the same ordinals. Suppose ν is regular in V and is the least cardinal in V such that there is a new sequence $r : \omega \to \nu$ in $W \setminus V$. Suppose that for each regular $\kappa \geq \nu$ in W there is a non-reflecting stationary subset of $\{\alpha < \kappa^+ : cf(\alpha) = \nu\}$ in V. Then for all cardinals $\aleph_1 \leq \kappa < \lambda$ in W with κ regular in W and $\lambda \geq \nu$, $\mathscr{P}_\kappa(\lambda) \setminus V$ is stationary in W.*

The reader is referred to [4] and [3] for more results on co-stationarity of the ground model for forcings which add new subsets of uncountable cardinals. Question 1.5 is still open.

2. Definitions and basic facts

Throughout this paper, standard set-theoretic notation is used. $\alpha, \beta, \gamma, \delta, \varepsilon$ are used to denote ordinals, while $\kappa, \lambda, \mu, \nu, \rho, \theta$ are used to denote cardinals. $\mathscr{P}_\kappa(X) = \{x \subseteq X : |x| < \kappa\}$. Usually we use $[X]^{<\omega}$ instead of $\mathscr{P}_\omega(X)$ to denote the collection of finite subsets of X. $(X)^{<\omega}$ denotes the tree of finite sequences of elements of X ordered by end-extension. Given an ordinal δ and some $X \subseteq \delta$, $\lim(X)$ denotes the set of limit points of X.

Scott and Solovay asked for which cardinals $\nu \geq \omega$ is there a complete Boolean algebra which adds a new sequence $r : \omega \to \nu$ without adding any sequences $g : \omega \to \theta$ for any $\theta < \nu$. Namba showed in [10] that for such a ν, the following two conditions must hold.

1. Either $\mathrm{cf}(\nu) = \omega$ or ν is regular.
2. For all $\theta < \nu$, $|\theta^\omega| < \nu$.

These properties apply to all extension universes with the same ordinals, not just those obtained by forcing.

Fact 2.1. Suppose $V \subseteq W$ are models of ZFC with the same ordinals, ν is the least ordinal such that there is a new sequence $r : \omega \to \nu$ in $W \setminus V$.

(1) If $(\mathrm{cf}(\nu))^V > \omega$, then ν is a regular cardinal in V.
(2) For all $\theta < \nu$, $|\theta^\omega| < \nu$ in V.

Proof. (1) Since ν is the least ordinal such that there is a new sequence $r : \omega \to \nu$ in $W \setminus V$, ν must be a cardinal in V and r must be cofinal in ν. Suppose $\omega < (\mathrm{cf}(\nu))^V < \nu$. Let $\langle \nu_\alpha : \alpha < \mathrm{cf}(\nu)\rangle$ be a cofinal sequence in ν. Define $g : \omega \to \mathrm{cf}(\nu)$ by $g(n) =$ the least α such that $\nu_\alpha > r(n)$. g is unbounded in $cf(\nu)$, since r is cofinal in ν. But this implies $g \in W \setminus V$, since $(\mathrm{cf}(\nu))^V > \omega$, which contradicts ν being the least range of a new ω-sequence.

(2) Suppose not. Then there is some $\theta < \nu$ such that $|\theta^\omega| \geq \nu$ in V. Let $h : \nu \to \theta^\omega$ be a 1-1 function in V. Define $g : \omega \times \omega \to \theta$ by $g(i, j) = (h(r(i)))(j)$. Then $g \in V$. For each $i < \omega$, define $f_i : \omega \to \theta$ by $f_i(j) = g(i, j)$. Then the sequence $\langle f_i : i < \omega\rangle$ is in V. But $r(i) = h^{-1}(h(r(i))) = h^{-1}(f_i)$, a contradiction to $r \notin V$. □

Remark 2.2. (2) implies that if ν is the least cardinal such that there is a new sequence $r : \omega \to \nu$ in $W \setminus V$, then either $\nu \geq \aleph_2$ or $\nu = 2$.

Next, we review some facts about co-stationarity of the ground model. Co-stationarity of the ground model in $\mathscr{P}_\kappa(\lambda)$ implies co-stationarity of the ground model for certain other cardinals.

Theorem 2.3 (Menas [9])**.** *Let $A \subseteq B$ with $|A| \geq \kappa$. For $Y \subseteq \mathscr{P}_\kappa(B)$, let $Y \restriction A = \{y \cap A : y \in Y\}$. If $C \subseteq \mathscr{P}_\kappa(B)$ is club, then $C \restriction A$ contains a club set in $\mathscr{P}_\kappa(A)$.*

The next fact follows easily from Theorem 2.3.

Fact 2.4 ([4])**.** Let $V \subseteq W$ be models of ZFC with the same ordinals and κ be regular and $\lambda > \kappa$ in W. If $\mathscr{P}_\kappa(\lambda) \setminus V$ is stationary in W, then for all $\mu \geq \lambda$, $\mathscr{P}_\kappa(\mu) \setminus V$ is also stationary in W.

Fact 2.5 ([4])**.** Let $V \subseteq W$ be models of ZFC with the same ordinals. If κ is a cardinal in W and $\nu > \kappa$ is the least cardinal in V such that $W \setminus V$ has a new function from κ into ν, then $\forall \lambda \geq \nu$, $\mathscr{P}_{\kappa^+}(\lambda) \setminus V$ contains a cone. Moreover, for all cardinals ρ, λ in W with ρ regular in W, $\kappa < \rho \leq \nu \leq \lambda$, and $\mathrm{cf}(\nu) \geq \rho$ in V, then $\mathscr{P}_\rho(\lambda) \setminus V$ contains a cone.

Note, however, that this tells us nothing about whether $\mathscr{P}_\rho(\lambda) \setminus V$ is stationary in W for $\lambda > \rho > \nu$.

Finally, we give the necessary definitions concerning non-reflecting stationary sets.

Definition 2.6 ([6])**.** A stationary set $S \subseteq \kappa$ is *non-reflecting* if for all limit ordinals $\alpha < \kappa$, $S \cap \alpha$ is not stationary in α.

Definition 2.7 ([2])**.** Let κ be a cardinal and $S \subseteq \kappa^+$. $\square_\kappa(S)$ holds if there is a sequence $\langle C_\alpha : \alpha \in \lim(\kappa^+)\rangle$ such that for each $\alpha \in \lim(\kappa^+)$,

(1) $C_\alpha \subseteq \alpha$ is club in α;

(2) $\mathrm{cf}(\alpha) < \kappa \to \mathrm{o.t.}(C_\alpha) < \kappa$; and
(3) if $\beta < \alpha$ and $\beta \in \lim(C_\alpha)$, then $C_\beta = C_\alpha \cap \beta$ and $\beta \notin S$.

$\square_\kappa$ denotes $\square_\kappa(\emptyset)$.

It is well known that the existence of non-reflecting stationary subsets of κ^+ follows from $\square_\kappa$ and hence holds in L. In fact, more is true.

Fact 2.8. Let $\omega \leq \nu < \kappa$ with ν regular and assume $\square_\kappa$ holds. Then there exists a non-reflecting stationary $S \subseteq \{\alpha < \kappa^+ : \mathrm{cf}(\alpha) = \nu\}$ for which $\square_\kappa(S)$ holds.

3. Main theorem

Let $V \subseteq W$ be models of ZFC with the same ordinals. When $r : \omega \to \nu$ is in $W \setminus V$, we say that r *achieves global co-stationarity of the ground model* if for all cardinals $\omega < \kappa < \lambda$ in W with κ regular in W and $\lambda \geq \nu$, $\mathscr{P}_\kappa(\lambda) \setminus V$ is stationary in W. In this section we show, assuming certain combinatorial principals, that if ν is regular and is the least ordinal such that there exists a new sequence $r : \omega \to \nu$, then r achieves global co-stationarity of the ground model.

Our proof of the Main Theorem will use certain trees constructed by Gitik to prove Theorem 1.1. For each regular $\kappa > \omega$, Gitik constructs a certain subtree $T \subseteq (\kappa^+)^{<\omega}$ and constructs three branches through T to code the new real. In Theorem 3.3, given an $r : \omega \to \nu$ in $W \setminus V$, we will construct two branches through Gitik's tree which, along with a particular function f, suffice to code a new cofinal sequence from ω into ν, which thus cannot be in V. The next lemma gives conditions under which the function we use exists and can be naturally extracted from work of Todorčević in [12]. We thank Justin Moore for pointing it out and include his proof. Given a pairset a, we let $a^0 = \min(a)$ and $a^1 = \max(a)$.

Lemma 3.1. *Suppose $\aleph_0 \leq \nu < \kappa$ are regular cardinals and that there exists a non-reflecting stationary subset of $\{\alpha < \kappa : cf(\alpha) = \nu\}$. Then there is a function $f : [\kappa]^2 \to \nu$ satisfying the following:*

$$(*f) \qquad \begin{array}{l} \textit{Suppose } A, B \subseteq [\kappa]^2,\ |A| = |B| = \kappa, \\ \forall a_0, a_1 \in A\ a_0 \cap a_1 = \emptyset,\ \textit{and } \forall b_0, b_1 \in B\ b_0 \cap b_1 = \emptyset. \\ \textit{Then } \forall \varepsilon < \nu\ \exists a \in A\ \exists b \in B\ \forall i, j < 2\ f(a(i), b(j)) > \varepsilon. \end{array}$$

Proof. Let S be a non-reflecting stationary subset of $\{\alpha < \kappa : \mathrm{cf}(\alpha) = \nu\}$. For each limit ordinal $\alpha < \kappa$, let $C_\alpha \subseteq \alpha$ be a club in α such that $C_\alpha \cap S = \emptyset$. Let $C_0 = \emptyset$, and for each ordinal $0 < \alpha < \kappa$, let $C_{\alpha+1} = \{\alpha\}$. For $\alpha < \beta < \kappa$, the *trace from α to β* is

$$\mathrm{tr}(\alpha, \beta) = \{\beta_i : \beta_0 = \beta, \beta_{i+1} = \min(C_{\beta_i} \setminus \alpha) \text{ if } \beta_i > \alpha\}. \tag{3.1}$$

Define $f : [\kappa]^2 \to \nu$ by

$$f(\alpha, \beta) = \max\{\mathrm{o.t.}(C_\xi \cap \alpha) : \xi \in \mathrm{tr}(\alpha, \beta) \cap S\}. \tag{3.2}$$

Let A, B be as in $(*f)$, and let $\varepsilon < \nu$. Choose a sequence $\langle a_\alpha : \alpha < \kappa\rangle$ of elements of A such that $\alpha < \beta < \kappa \to a^1_\alpha < a^0_\beta$. Let $A_1 = \{a^1_\alpha : \alpha < \kappa\}$. $\mathrm{Lim}(A_1)$ is club in κ, so let $\delta \in \lim(A_1) \cap S$. There is a subsequence $\langle a_{\alpha_i} : i < \nu\rangle$ such that $\sup_{i<\nu} a^1_{\alpha_i} = \delta$, and for each $i < \nu$, $\mathrm{o.t.}(C_\delta \cap a^0_{\alpha_i}) > \varepsilon$.

Fix $b \in B$ such that $b^0 > \delta$. Fix $k < 2$ and let $\beta = b^k$. Note that for $\zeta \in \mathrm{tr}(\delta, \beta)$, $\delta \in C_\zeta$ iff $\zeta = \delta + 1$. Let $\delta_0 = \sup\{\sup(C_\zeta \cap \delta) : \zeta \in \mathrm{tr}(\delta, \beta) \text{ and } \zeta > \delta\}$. Then $\delta_0 < \delta$, since for all $\zeta \in \mathrm{tr}(\delta, \beta)$ such that $\zeta > \delta$, $C_\zeta \cap \delta$ is bounded in δ.

Claim: For each ξ such that $\delta_0 < \xi < \delta$, $\delta \in \mathrm{tr}(\xi, \beta)$. Let $\beta = \beta_0 > \beta_1 > \cdots > \beta_n = \delta$ enumerate $\mathrm{tr}(\delta, \beta)$. Let ξ be such that $\delta_0 < \xi < \delta$. Calculate $\mathrm{tr}(\xi, \beta)$: Let $\zeta_0 = \beta = \beta_0$. Assume $l < n$ and $\zeta_l = \beta_l$. Let $\zeta_{l+1} = \min(C_{\zeta_l} \setminus \xi)$. $\beta_l \in \mathrm{tr}(\delta, \beta)$ and $\beta_l > \delta$, so $\sup(C_{\zeta_l} \cap \delta) = \sup(C_{\beta_l} \cap \delta) < \delta_0$. Hence, $C_{\zeta_l} \cap \delta = C_{\zeta_l} \cap \xi$, which implies $C_{\zeta_l} \setminus \xi = C_{\zeta_l} \setminus \delta$. Therefore, $\zeta_{l+1} = \min(C_{\zeta_l} \setminus \xi) = \min(C_{\beta_l} \setminus \delta) = \beta_{l+1}$. By induction, $\zeta_n = \beta_n = \delta$.

Now choose $i < \nu$ for which $\delta_0 < a^0_{\alpha_i}$. $\delta \in \mathrm{tr}(a^m_{\alpha_i}, \beta)$ for each $m < 2$, so $f(a^m_{\alpha_i}, \beta) \geq \mathrm{o.t.}(C_\delta \cap a^m_{\alpha_i}) > \varepsilon$. $\square$

Remark 3.2. J. Moore has pointed out that the "2" in Lemma 3.1 can be replaced by "finite".

The next theorem contains most of the work towards proving the Main Theorem.

Theorem 3.3. *Let $V \subseteq W$ be models of ZFC with the same ordinals. Suppose $\nu \geq \aleph_2$ is regular in V and is the least cardinal in V such that there is a new sequence $r : \omega \to \nu$ in $W \setminus V$. Suppose $\kappa \geq \nu$ and κ is regular in W. If in V there is a non-reflecting stationary subset of $\{\alpha < \kappa^+ : cf(\alpha) = \nu\}$, then $\mathscr{P}_\kappa(\kappa^+) \setminus V$ is stationary in W.*

Proof. Let $\nu \geq \aleph_2$ be a regular cardinal in W, and assume that ν is least such that there is a sequence $r : \omega \to \nu$ in $W \setminus V$. In W, let $\kappa > \nu$ be a regular cardinal, and let $C \subseteq \mathscr{P}_\kappa(\kappa^+)$ be club. By a theorem due to Kueker in [7], there is a function $g : [\kappa^+]^{<\omega} \to \mathscr{P}_\kappa(\kappa^+)$ such that C_g is a club subset of C, where $C_g = \{x \in \mathscr{P}_\kappa(\kappa^+) : (\forall y \in [x]^{<\omega})\ g(y) \subseteq x\}$. Define $\mathrm{cl}^0(x) = x$, $\mathrm{cl}^1(x) = \bigcup\{g(y) : y \in [x]^{<\omega}\} \cup x$, and $\mathrm{cl}^{n+1}(x) = \mathrm{cl}^1(\mathrm{cl}^n(x))$. Define $\mathrm{cl}_g(x) = \bigcup_{n<\omega} \mathrm{cl}^n(x)$. Note: For each $x \in \mathscr{P}_\kappa(\kappa^+)$, $\mathrm{cl}_g(x) \in C_g$.

The following paragraph is extracted from Gitik's proof of Theorem 1.1 in [5]. We take the liberty of revising a bit of the notation.

Let $\langle T_0, \prec \rangle$ be the tree of all finite increasing sequences from $\{\alpha < \kappa^+ : \mathrm{cf}(\alpha) = \kappa\}$. For every $\langle \gamma \rangle$ from $\mathrm{Lev}_{T_0}(1)$, the first level of T_0, and for every $\bar{\eta} \succ \langle \gamma \rangle$ in T_0, define the ordinal $\alpha(\bar{\eta})$ to be the supremum of $\gamma \cap \mathrm{cl}_g(\bar{\eta})$. Then $\alpha(\bar{\eta}) < \gamma$ since $\mathrm{cf}(\gamma) = \kappa$. By XI Lemma 3.7 of Shelah in [11], we can shrink the tree above $\langle \gamma \rangle$ to a tree $T_0^{\langle \gamma \rangle}$ such that every splitting in $T_0^{\langle \gamma \rangle}$ is still a stationary subset of κ^+, and there exists a $\gamma_* < \gamma$ such that for every $\bar{\eta} \in T_0^{\langle \gamma \rangle}$, $\alpha(\bar{\eta}) < \gamma_*$. Let $T_1 = \bigcup\{T_0^{\langle \gamma \rangle} : \langle \gamma \rangle \in \mathrm{Lev}_{T_0}(1)\}$. Repeat the process for each node in $\mathrm{Lev}_{T_1}(2)$, and so forth. In this manner, we obtain a tree T' such that $\forall \bar{\eta} = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle \in T'$,

(i) $\mathrm{Succ}_{T'}(\bar{\eta})$ is a stationary subset of $\{\alpha < \kappa^+ : \mathrm{cf}(\alpha) = \kappa\}$;

(ii) $\exists \alpha(\bar{\eta}) < \alpha_{n-1}$ such that $\forall \bar{\zeta} >_T \bar{\eta}$ in T', $\alpha_{n-2} \leq \bigcup(\alpha_{n-1} \cap \mathrm{cl}_g(\bar{\zeta})) < \alpha(\bar{\eta})$.

Using (i), shrink T' level by level to obtain a T which satisfies, in addition to (i) and (ii), also

(iii) $\exists \beta(\bar{\eta})$ such that $\forall \gamma \in \mathrm{Succ}_T(\bar{\eta})$, $\beta(\bar{\eta}) = \alpha(\bar{\eta}^\frown \langle \gamma \rangle) < \gamma$.

Note: If $\bar{\eta} = \langle \alpha_0, \ldots, \alpha_{n-1} \rangle$, then $\forall \alpha_n \in \mathrm{Succ}_T(\bar{\eta})$,

$$\alpha_{n-1} \leq \bigcup(\alpha_n \cap \mathrm{cl}_g(\bar{\eta}^\frown \langle \alpha_n \rangle)) < \alpha(\bar{\eta}^\frown \langle \alpha_n \rangle) = \beta(\bar{\eta}) < \alpha_n. \tag{3.3}$$

We point out the following useful property of the tree T and the chosen $\beta(\bar{\eta})$'s.

Fact 3.4. If b is a branch through T and $x = \mathrm{cl}_g(b)$, then for each $k < \omega$, $b(k) = \min(x \setminus \beta(b \upharpoonright k))$.

Firstly, $b(k) \in b \subseteq x$. Secondly, for each $k+1 < n < \omega$, $\bigcup(b(k) \cap \mathrm{cl}_g(b \restriction n)) < \beta(b \restriction k) < b(k)$, by (ii) and (iii). Therefore, $[\beta(b \restriction k), b(k)) \cap x = \emptyset$.

By Lemma 3.1, let $f : [\kappa^+]^2 \to \nu$ satisfy $(*f)$ in V. We construct two branches b, c through T which we will close under g. The resulting two elements of C will code an ω-sequence unbounded in ν, hence not in V.

Construct a strictly increasing sequence $\langle b^\delta(0), c^\delta(0) : \delta < \kappa^+\rangle$ such that

(a) $\forall \delta < \kappa^+$, $b^\delta(0), c^\delta(0) \in \mathrm{Succ}_T(\langle\rangle)$;
(b) $\forall \delta < \varepsilon < \kappa^+$, $b^\delta(0) < c^\delta(0) < b^\varepsilon(0) < c^\varepsilon(0) < \kappa^+$;
(c) $\forall \delta < \kappa^+$, $b^\delta(0) < \beta(\langle b^\delta(0)\rangle) < c^\delta(0)$.

Let $A(0) = \{\{b^\delta(0), c^\delta(0)\} : \delta < \kappa^+\}$. Break up $A(0)$ into two disjoint sets each of cardinality κ^+. Since $(*f)$ holds, there exist $\delta_0 < \varepsilon_0 < \kappa^+$ such that $f(b^{\delta_0}(0), c^{\varepsilon_0}(0)) > r(0)$. Let $b(0) = b^{\delta_0}(0)$ and $c(0) = c^{\varepsilon_0}(0)$. For $0 < n < \omega$, choose $b(n)$ and $c(n)$ as follows: Given $b \restriction n$ and $c \restriction n$, construct a strictly increasing sequence $\langle b^\delta(n), c^\delta(n) : \delta < \kappa^+\rangle$ such that

(a) $\beta(c \restriction n) < b^0(n)$;
(b) $\forall \delta < \kappa^+$, $b^\delta(n) \in \mathrm{Succ}_T(b \restriction n)$, $c^\delta(n) \in \mathrm{Succ}_T(c \restriction n)$;
(c) $\forall \delta < \varepsilon < \kappa^+$, $b^\delta(n) < c^\delta(n) < b^\varepsilon(n) < c^\varepsilon(n) < \kappa^+$;
(d) $\forall \delta < \kappa^+$, $b^\delta(n) < \beta((b \restriction n)^\frown\langle b^\delta(n)\rangle) < c^\delta(n)$.

Let $A(n) = \{\{b^\delta(n), c^\delta(n)\} : \delta < \kappa^+\}$. Split $A(n)$ into two disjoint pieces each of cardinality κ^+ and apply $(*f)$ to obtain $\delta_n < \varepsilon_n < \kappa^+$ such that $f(b^{\delta_n}(n), c^{\varepsilon_n}(n)) > r(n)$. Let $b(n) = b^{\delta_n}(n)$ and $c(n) = c^{\varepsilon_n}(n)$.

Now we have branches b, c through T such that for each $n < \omega$, $f(b(n), c(n)) > r(n)$. Therefore, b, c code an unbounded function from ω into ν. Since $(\mathrm{cf}(\nu))^V > \omega$, this function must be in $W \setminus V$. Since $f \in V$, at least one of b, c must be in $W \setminus V$. Let $x = \mathrm{cl}_g(b)$ and $y = \mathrm{cl}_g(c)$.

Claim. b and c can be recovered from V, using x and y as oracles.

Proof. $b(0) = \min(x \setminus \beta(\langle\rangle))$ and $c(0) = \min(y \setminus \beta(\langle\rangle))$, by Fact 3.4. Suppose we have $b(n)$ and $c(n)$. $b(n+1) = \min(x \setminus \beta(b \restriction (n+1)))$, by Fact 3.4. The β function is not necessarily in V, since it is constructed using g. However, we know that $\beta(b \restriction (n+1)) = \beta((b \restriction n)^\frown\langle b^{\delta_n}(n)\rangle) < c^{\delta_n}(n) < c^{\varepsilon_n}(n) = c(n) < b^0(n+1) \leq b(n+1)$. Therefore, $b(n+1) = \min(x \setminus c(n))$. $c(n+1) = \min(y \setminus \beta(c \restriction (n+1))$, by Fact 3.4. $\beta(c \restriction (n+1)) < b^0(n+1) \leq b(n+1) < c(n+1)$. Therefore, $c(n+1) = \min(y \setminus b(n+1))$. □

Hence, at least one of x, y is not in V. Therefore, $C \cap (\mathscr{P}_\kappa(\kappa^+) \setminus V) \neq \emptyset$. □

Remark 3.5. In the proof of Theorem 3.3, it is not necessary for the function f to satisfy $(*f)$, but rather the following weak version: $f : [\kappa^+]^2 \to \nu$ is such that for each $A \subseteq [\kappa^+]^2$ satisfying ($|A| = \kappa^+$ and $\forall a, b \in A\ a \cap b = \emptyset$), for each $\varepsilon < \nu$ there are $a, b \in A$ such that $f(a^0, b^1) > \varepsilon$. It is open whether this is strictly weaker.

Main Theorem. *Let $V \subseteq W$ be models of ZFC with the same ordinals. Suppose ν is regular in V and is the least cardinal in V such that there is a new ω-sequence $r : \omega \to \nu$ in $W \setminus V$. Suppose that for each regular $\kappa \geq \nu$ in W there is a non-reflecting stationary subset of $\{\alpha < \kappa^+ : cf(\alpha) = \nu\}$ in V. Then for all cardinals $\aleph_1 \leq \kappa < \lambda$ in W with κ regular in W and $\lambda \geq \nu$, $\mathscr{P}_\kappa(\lambda) \setminus V$ is stationary in W.*

Proof. If $\nu < \aleph_2$, then $\nu = 2$ by Remark 2.2. Gitik's Theorem 1.1 then gives the result. So assume $\nu \geq \aleph_2$. By Theorem 3.3 and Fact 2.4, for each $\nu \leq \kappa < \lambda$ in W

with κ regular in W, $\mathscr{P}_\kappa(\lambda)\setminus V$ is stationary in W. For $\aleph_1 \leq \kappa < \nu \leq \lambda$, $\mathscr{P}_\kappa(\lambda)\setminus V$ is stationary in W, by Fact 2.5 along with the fact that ν is regular in V. $\square$

Corollary 3.6. *Suppose $V \models$ ZFC and satisfies $\square_\kappa$ whenever κ is a regular cardinal in V. Let $W \supseteq V$ be any model of ZFC with the same ordinals as V. Suppose ν is regular in V and is the least cardinal in V such that there is a sequence $r : \omega \to \nu$ in $W \setminus V$. Then r achieves global co-stationarity of V in W.*

Example 3.7 (Namba forcing over L)**.** Let $\nu \geq \aleph_2$ be regular, and let $\mathbb{P}_\nu$ denote Namba forcing on ν. $\mathbb{P}_\nu$ is the collection of subtrees $p \subseteq (\nu)^{<\omega}$ such that for each node $t \in p$ above the stem of p, t has ν-many immediate successors. $q \leq p$ iff $q \supseteq p$. Namba proved that if $\nu^\omega = \nu$, then $\mathbb{P}_\nu$ adds a new sequence $r : \omega \to \nu$, but for all $\theta < \nu$, $\mathbb{P}_\nu$ adds no new sequences from ω into θ [10].

In L, $\square_\kappa$ holds for *all* cardinals κ. Moreover, $\nu^\omega = \nu$, since $\nu \geq \aleph_2$ is regular and L satisfies GCH. Hence, by the Main Theorem, for all cardinals $\aleph_1 \leq \kappa < \lambda$ in $L^{\mathbb{P}_\nu}$ with κ regular in $L^{\mathbb{P}_\nu}$ and $\lambda \geq \nu$, $\mathscr{P}_\kappa(\lambda) \setminus L$ is stationary in $L^{\mathbb{P}_\nu}$.

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