



Proof theory for lattice-ordered groups



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ABSTRACT

Proof-theoretic methods are developed and exploited to establish properties of the variety of lattice-ordered groups. In particular, a hypersequent calculus with a cut rule is used to provide an alternative syntactic proof of the generation of the variety by the lattice-ordered group of automorphisms of the real number chain. Completeness is also established for an analytic (cut-free) hypersequent calculus using cut elimination and it is proved that the equational theory of the variety is co-NP complete.

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1. Introduction

A *lattice-ordered group* (ℓ -group) is an algebraic structure

$$(L, \wedge, \vee, \cdot, ^{-1}, 1)$$

such that (L, \wedge, \vee) is a lattice, $(L, \cdot, ^{-1}, 1)$ is a group, and \cdot preserves the order in both arguments; i.e., $a \leq b$ implies $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$ for all $a, b, c \in L$. It follows also from this definition that the lattice (L, \wedge, \vee) is distributive and that $1 \leq a \vee a^{-1}$ for all $a \in L$. We refer to [1] for proofs and further standard facts about this class of structures.

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Commutative ℓ -groups include the real, rational, and integer numbers with the standard total order and addition. For non-commutative examples, consider a chain (totally-ordered set) Ω and denote by $\text{Aut}(\Omega)$ the set of all order-preserving bijections on Ω . Then $\text{Aut}(\Omega)$ constitutes an ℓ -group $\mathbf{Aut}(\Omega)$ under coordinate-wise lattice operations, functional composition, and functional inverse. It was proved by Holland in [10] that the variety of ℓ -groups \mathcal{LG} is generated by $\mathbf{Aut}(\mathbb{R})$, where \mathbb{R} is the real number chain, or indeed by any $\mathbf{Aut}(\Omega)$, where Ω is an n -transitive chain for all n (i.e., for any two n -tuples of elements of Ω there is a bijection that maps the first tuple to the second). The standard proof relies on Holland’s embedding theorem, which states, analogously to Cayley’s theorem for groups, that every ℓ -group embeds into an ℓ -group $\mathbf{Aut}(\Omega)$ for some chain Ω [9]. Although not every ℓ -group embeds into $\mathbf{Aut}(\mathbb{R})$, each identity that fails in some ℓ -group fails, by the embedding theorem, in some automorphism ℓ -group, and a simple argument then shows that the identity must also fail in $\mathbf{Aut}(\mathbb{R})$. This generation result for \mathcal{LG} was subsequently exploited by Holland and McCleary to provide an algorithm for checking if an identity is valid in all ℓ -groups [11].

The first main contribution of this paper is a new syntactic (and first axiom of choice free) proof that $\mathbf{Aut}(\mathbb{R})$ generates the variety \mathcal{LG} of ℓ -groups. A proof system is defined in a one-sided hypersequent framework such that derivability of a hypersequent (interpreted as a disjunction of group terms) implies the validity of a corresponding identity in all ℓ -groups. A rule is then added to the system and it is shown, following closely the Holland–McCleary algorithm of [11], that this augmented system derives all identities (rewritten in a certain form) that are valid in $\mathbf{Aut}(\mathbb{R})$. Finally, it is proved syntactically that applications of this rule can be eliminated from derivations. Hence an identity is valid in $\mathbf{Aut}(\mathbb{R})$ if and only if it is valid in all ℓ -groups, and so, by Birkhoff’s variety theorem, $\mathbf{Aut}(\mathbb{R})$ generates \mathcal{LG} . This proof illustrates the usefulness of proof-theoretic methods for tackling algebraic problems, and is similar to proofs of generation of varieties by dense chains via density elimination (see [4,14]) or of properties such as interpolation and amalgamation via cut elimination (see, e.g., [7,18]).

The second main contribution is the introduction of a first analytic (cut-free) proof calculus for ℓ -groups. In contrast to the well-developed proof theory for well-behaved families of varieties of residuated lattices (which provide algebraic semantics for substructural logics, see [2,3,7,14,17,18]), there has been relatively little success in obtaining cut-free systems for algebraic structures related to ℓ -groups. Hypersequent calculi have been defined for abelian ℓ -groups and related varieties in [13,15–17], but a calculus for the general non-commutative case has until now been lacking. The virtue of such a calculus is illustrated by the fact that we obtain not only the known decidability result for the equational theory of ℓ -groups, but also, via cut elimination, a (first) procedure for obtaining proofs of valid ℓ -group identities in equational logic (i.e., using only defining identities of \mathcal{LG}). More generally, the analytic hypersequent calculus presented here provides a crucial first step towards developing a uniform proof theory for the wide range of algebras and logics related in some way to ℓ -groups: in particular, MV-algebras and GMV-algebras (which may be viewed as intervals in abelian ℓ -groups [20] and ℓ -groups [6,8], respectively) and commutative cancellative residuated lattices (which may be viewed as ℓ -groups with a co-nucleus [19]).

The final contribution of the paper is a first proof that the equational theory of ℓ -groups is co-NP complete, matching the complexity of the equational theories of both abelian ℓ -groups [21] and distributive lattices [12].

2. Preliminaries

Let us call a variable x and its inverse x^{-1} *literals*. Using De Morgan identities valid in all ℓ -groups, we consider only *normalized* ℓ -group terms s, t built from literals and the operation symbols $1, \wedge, \vee$, and \cdot , with an inductively defined inverse:

$$\begin{array}{ll} \bar{1} & = 1 & \overline{(s \cdot t)} & = \bar{t} \cdot \bar{s} \\ \bar{x} & = x^{-1} & \overline{(s \wedge t)} & = \bar{s} \vee \bar{t} \\ \overline{x^{-1}} & = x & \overline{(s \vee t)} & = \bar{s} \wedge \bar{t}. \end{array}$$

As usual we write st for $s \cdot t$, omit brackets in group terms (built using literals, \cdot , and 1), and define $t^0 = 1$ and $t^{n+1} = t \cdot t^n$ for $n \in \mathbb{N}$. We also write $s \leq t$ as an abbreviation for the identity $s \wedge t \approx s$.

Given a class of ℓ -groups \mathcal{K} , we let $\mathcal{K} \models s \approx t$ denote that the ℓ -group identity $s \approx t$ is valid in all members of \mathcal{K} . Using standard distributivity laws in ℓ -groups, every ℓ -group term t is equivalent either to 1 or to a meet of joins of group terms (see [1] for basic facts about ℓ -groups). That is, for some index sets I and $J_i \neq \emptyset$ ($i \in I$) and group terms t_{ij} , assuming also $\bigwedge \emptyset = 1$,

$$\mathcal{L}\mathcal{G} \models t \approx \bigwedge_{i \in I} \bigvee_{j \in J_i} t_{ij}.$$

Moreover, for any ℓ -group terms s, t ,

$$\mathcal{L}\mathcal{G} \models s \approx t \quad \Leftrightarrow \quad \mathcal{L}\mathcal{G} \models 1 \leq (\overline{st}) \wedge (\overline{ts}),$$

and for any finite family of ℓ -group terms $(t_i)_{i \in I}$,

$$\mathcal{L}\mathcal{G} \models 1 \leq \bigwedge_{i \in I} t_i \quad \Leftrightarrow \quad \mathcal{L}\mathcal{G} \models 1 \leq t_i \text{ for each } i \in I.$$

Hence checking the validity of ℓ -group identities amounts to checking the validity of inequations of the form $1 \leq t$ where t is a join of group terms. Following proof-theoretic treatments of related classes of algebras (see [2,3,13–18]), we present such terms and identities here in the framework of sequents and hypersequents.

An ℓ -*sequent*, denoted by Γ, Δ, Π , or Σ , is a finite (possibly empty) sequence of ℓ -group terms, written

$$(t_1, \dots, t_n),$$

and interpreted as an ℓ -group term by

$$\mathcal{I}() = 1 \quad \text{and} \quad \mathcal{I}(t_1, \dots, t_n) = t_1 \cdot \dots \cdot t_n.$$

The inverse of an ℓ -sequent is defined as

$$\overline{(t_1, \dots, t_n)} = (\overline{t_n}, \dots, \overline{t_1}).$$

For ℓ -sequents Γ and Δ , we denote their concatenation by (Γ, Δ) or simply by Γ, Δ . An ℓ -sequent will be called *basic* if it is a sequence of literals, and a basic ℓ -sequent Γ will be called *group valid* if $1 \approx \mathcal{I}(\Gamma)$ is valid in all groups.

An ℓ -*hypersequent*, denoted by \mathcal{G} or \mathcal{H} , is a finite (possibly empty) multiset of ℓ -sequents, written

$$\Gamma_1 \mid \dots \mid \Gamma_n,$$

and interpreted, when $n \geq 1$, as an ℓ -group term by

$$\mathcal{I}(\Gamma_1 \mid \dots \mid \Gamma_n) = \mathcal{I}(\Gamma_1) \vee \dots \vee \mathcal{I}(\Gamma_n).$$

An ℓ -hypersequent \mathcal{G} will be called *basic* if it contains only basic ℓ -sequents. It will be called *valid* in an ℓ -group \mathbf{L} if $\mathbf{L} \models 1 \leq \mathcal{I}(\mathcal{G})$, and *ℓ -valid* if it is valid in all ℓ -groups.

An ℓ -*hypersequent rule* is the set of its *instances*, each instance consisting of a finite set of ℓ -hypersequents called the *premises* and an ℓ -hypersequent called the *conclusion* of the instance. Typically, such rules are written schematically using s, t to denote arbitrary ℓ -terms and $\Gamma, \Pi, \Sigma, \Delta$ and \mathcal{G}, \mathcal{H} to denote arbitrary

$\frac{}{\mathcal{G} \mid \Gamma} \text{ (GV)}$ <p style="text-align: center;">Γ group valid</p>	$\frac{}{\mathcal{G} \mid \Delta \mid \bar{\Delta}} \text{ (EM)}$
$\frac{\mathcal{G} \mid \Gamma, \Delta \quad \mathcal{G} \mid \bar{\Delta}, \Sigma}{\mathcal{G} \mid \Gamma, \Sigma} \text{ (CUT)}$	$\frac{\mathcal{G}}{\mathcal{G} \mid \mathcal{H}} \text{ (EW)} \quad \frac{\mathcal{G} \mid \mathcal{H} \mid \mathcal{H}}{\mathcal{G} \mid \mathcal{H}} \text{ (EC)}$

Fig. 1. The basic ℓ -hypersequent calculus $\mathcal{G}\ell$.

ℓ -sequents and ℓ -hypersequents, respectively. An ℓ -hypersequent calculus $\mathcal{G}\mathcal{L}$ is a set of ℓ -hypersequent rules. A $\mathcal{G}\mathcal{L}$ -derivation of an ℓ -hypersequent \mathcal{G} from a set of ℓ -hypersequents S is a finite tree of ℓ -hypersequents with root \mathcal{G} such that each node is either in S or the node and its parents form an instance of a rule of $\mathcal{G}\mathcal{L}$. In this case, we write $S \vdash_{\mathcal{G}\mathcal{L}} \mathcal{G}$ and say that \mathcal{G} is $\mathcal{G}\mathcal{L}$ -derivable from S .

A rule is said to be $\mathcal{G}\mathcal{L}$ -derivable if for each of its instances, the conclusion is $\mathcal{G}\mathcal{L}$ -derivable from the premises, and $\mathcal{G}\mathcal{L}$ -admissible if for each of its instances, whenever the premises are $\mathcal{G}\mathcal{L}$ -derivable, the conclusion is $\mathcal{G}\mathcal{L}$ -derivable. A rule will also be called ℓ -sound if for each of its instances, whenever the premises are ℓ -valid, the conclusion is ℓ -valid, and ℓ -invertible if for each of its instances, whenever the conclusion is ℓ -valid, the premises are ℓ -valid.

3. A basic calculus

Fig. 1 presents a proof system $\mathcal{G}\ell$ for basic ℓ -hypersequents, where (GV) stands for *group valid*, (EM) for *excluded middle*, (EW) for *external weakening*, and (EC) for *external contraction*. Occurrences of the sequents Γ in (GV), Δ and $\bar{\Delta}$ in (EM), (Γ, Δ) , $(\bar{\Delta}, \Sigma)$, and (Γ, Σ) in (CUT), and the sequents in \mathcal{H} in (EW) and (EC) will be called *active* in applications of these rules in $\mathcal{G}\ell$ -derivations.

We remark that (EW) is useful for constructing $\mathcal{G}\ell$ -derivations but is not strictly necessary: that is, (EW) is admissible in the calculus $\mathcal{G}\ell$ without (EW). This is because instances of (EW) can be permuted upwards over instances of any other rule in a derivation, while its effect on the axioms is subsumed by the form of the axioms themselves. Note also that we could avoid using (EC) by redefining ℓ -hypersequents as finite sets of ℓ -sequents.

To better understand $\mathcal{G}\ell$, let us consider some useful $\mathcal{G}\ell$ -derivable rules.

Lemma 3.1. *The following rules are $\mathcal{G}\ell$ -derivable:*

$$\frac{\mathcal{G} \mid \Gamma \quad \mathcal{G} \mid \Sigma}{\mathcal{G} \mid \Gamma, \Sigma} \text{ (MIX)} \quad \frac{\mathcal{G} \mid \Gamma, \Delta}{\mathcal{G} \mid \Gamma \mid \Delta} \text{ (SPLIT)}$$

$$\frac{\mathcal{G} \mid \Gamma, \Sigma}{\mathcal{G} \mid \Gamma, \Delta, \bar{\Delta}, \Sigma} \text{ (SIMP)} \quad \frac{\mathcal{G} \mid \Gamma, \Sigma \quad \mathcal{G} \mid \Pi, \Delta}{\mathcal{G} \mid \Gamma, \Delta \mid \Pi, \Sigma} \text{ (COM)}$$

Proof. Observe first that (MIX) consists of instances of (CUT) where $\Delta = ()$. For (SPLIT) and (SIMP), we make use of $\mathcal{G}\ell$ -derivations:

$$\frac{\frac{\mathcal{G} \mid \Gamma, \Delta}{\mathcal{G} \mid \Gamma, \Delta \mid \Delta} \text{ (EW)} \quad \frac{}{\mathcal{G} \mid \bar{\Delta} \mid \Delta} \text{ (EM)}}{\mathcal{G} \mid \Gamma \mid \Delta} \text{ (CUT)} \quad \frac{\mathcal{G} \mid \Gamma, \Sigma \quad \frac{}{\mathcal{G} \mid \bar{\Sigma}, \Delta, \bar{\Delta}, \Sigma} \text{ (GV)}}{\mathcal{G} \mid \Gamma, \Delta, \bar{\Delta}, \Sigma} \text{ (CUT)}$$

The following $\mathcal{G}\ell$ -derivation takes care of (COM):

$$\frac{\frac{\mathcal{G} \mid \Pi, \Delta}{\mathcal{G} \mid \Gamma, \Delta \mid \Pi, \Delta} \text{ (EW)} \quad \frac{\frac{\mathcal{G} \mid \Gamma, \Sigma}{\mathcal{G} \mid \Gamma, \Delta, \bar{\Delta}, \Sigma} \text{ (SIMP)}}{\mathcal{G} \mid \Gamma, \Delta \mid \bar{\Delta}, \Sigma} \text{ (SPLIT)}}{\mathcal{G} \mid \Gamma, \Delta \mid \Pi, \Sigma} \text{ (CUT)} \quad \square$$

Example 3.2. We illustrate Gl and some Gl -derivable rules in the following derivation of an ℓ -hypersequent corresponding to the inequation $1 \leq xx \vee yy \vee \overline{xy}$:

$$\frac{\frac{\frac{\overline{xx, x, \overline{x} \mid y, y}}{\overline{xx, x, \overline{x} \mid y, y \mid \overline{x}}} \text{ (SPLIT)} \quad \frac{\overline{xx \mid y, y, \overline{y}, \overline{y}}}{\overline{xx \mid y, y, \overline{y} \mid \overline{y}}} \text{ (SPLIT)}}{\overline{xx \mid y, y \mid \overline{x} \mid \overline{x}}} \text{ (SPLIT)} \quad \frac{\overline{xx \mid y, y \mid \overline{y} \mid \overline{y}}}{\overline{xx \mid y, y \mid \overline{y}}} \text{ (SPLIT)}}{\overline{xx \mid y, y \mid \overline{x}}} \text{ (EC)} \quad \frac{\overline{xx \mid y, y \mid \overline{y}}}{\overline{xx \mid y, y \mid \overline{y}}} \text{ (EC)}}{\overline{xx \mid y, y \mid \overline{x}, \overline{y}}} \text{ (MIX)}$$

We show now that Gl derives only ℓ -valid basic ℓ -hypersequents.

Lemma 3.3. *The following (quasi-)identities are valid in all ℓ -groups:*

- (i) $xy \wedge 1 \leq x \vee y$
- (ii) $(1 \leq xy \vee z) \Rightarrow (1 \leq x \vee y \vee z)$
- (iii) $(1 \leq y \vee x) \Rightarrow 1 \leq y \vee x^2$
- (iv) $(1 \leq x \vee y) \ \& \ (1 \leq x \vee z) \Rightarrow 1 \leq x \vee yz$.

Proof. We show that each of these (quasi-)identities holds in any ℓ -group \mathbf{L} with arbitrary elements $a, b, c \in L$:

(i) Note first that $1 \leq b(ab)^{-1} \vee a1$ because $1 \leq a^{-1} \vee a$. So also $1 \leq (a \vee b)(ab)^{-1} \vee (a \vee b)1$. By distributivity, $1 \leq (a \vee b)((ab)^{-1} \vee 1)$. Hence $ab \wedge 1 = ((ab)^{-1} \vee 1)^{-1} \leq a \vee b$.

(ii) From (i), $(ab \wedge 1) \vee c \leq a \vee b \vee c$. If $1 \leq ab \vee c$, then by distributivity also $1 \leq (ab \wedge 1) \vee c$, so $1 \leq a \vee b \vee c$ as required.

(iii) If $1 \leq a \vee b$, then also $1 \leq a \vee b(a \vee b)$. I.e., $1 \leq a \vee ba \vee b^2$ and hence $1 \leq a \vee (ba \wedge 1) \vee b^2$, by distributivity. Note that $(ba \wedge 1)^{-1}(a \vee b^2) = (1 \vee (a^{-1}b^{-1}))(a \vee b^2) = a \vee b^2 \vee a^{-1}b^{-1}a \vee a^{-1}b$. Using $1 \leq 1 \vee b^2 = (a(a^{-1}b^{-1}a)a^{-1}b) \vee b^2$, (ii) yields $1 \leq (ba \wedge 1)^{-1}(a \vee b^2)$. I.e., $ba \wedge 1 \leq a \vee b^2$. So $1 \leq a \vee b^2$.

(iv) If $1 \leq a \vee b$ and $1 \leq a \vee c$, then also $1 \leq a \vee (b \wedge c)$. But then by (iii), $1 \leq a \vee (b \wedge c)^2$ and from $(b \wedge c)^2 \leq bc$ it follows that $1 \leq a \vee bc$ as required. \square

Lemma 3.4. *If a basic ℓ -hypersequent \mathcal{G} is Gl -derivable, then it is ℓ -valid.*

Proof. By induction on the height of a Gl -derivation of \mathcal{G} . The base case follows because in all ℓ -groups, $1 \leq 1 \vee b$ and $1 \leq a \vee b \vee b^{-1}$. For the induction step, if the last application of a rule is (EW) or (EC), then we are done because $1 \leq a$ implies $1 \leq a \vee b$ and $1 \leq a \vee a$ implies $1 \leq a$ in all ℓ -groups. If the last application of a rule is (CUT), then we use the fact that if $1 \leq a \vee bc$ and $1 \leq a \vee c^{-1}d$, then by Lemma 3.3 (iv), $1 \leq a \vee bcc^{-1}d = a \vee bd$. \square

4. An augmented basic calculus

We now define Gl^* as the extension of Gl with the following rule for basic ℓ -hypersequents:

$$\frac{\mathcal{G} \mid \Delta \quad \mathcal{G} \mid \overline{\Delta}}{\mathcal{G}} \quad (*)$$

where Δ is not group valid.

The condition that Δ is not group valid is clearly necessary; otherwise, Δ and $\overline{\Delta}$ are both group valid and the premises are derivable using (GV) for any basic ℓ -hypersequent \mathcal{G} . Moreover, it is not at all obvious

that the rule is ℓ -sound. In particular, it is not true that in any ℓ -group, if $1 \leq a \vee b$ and $1 \leq a \vee b^{-1}$ with $b \neq 1$, then $1 \leq a$. Consider, e.g., the direct product of the additive ℓ -group of the integers $\mathbf{Z} \times \mathbf{Z}$ where $(0, 0) \leq (0, -1) \vee (1, 0)$ and $(0, 0) \leq (0, -1) \vee (-1, 0)$, but $(0, 0) \not\leq (0, -1)$. Observe, however, that for soundness, we require something weaker: namely, that for any basic ℓ -hypersequent \mathcal{G} and basic ℓ -sequent Δ that is not valid in all groups, if $\mathcal{G} \mid \Delta$ and $\mathcal{G} \mid \overline{\Delta}$ are both valid in *all* ℓ -groups, then \mathcal{G} is valid in all ℓ -groups. Equivalently, for any n and for any elements a_1, \dots, a_n, b of the ω -generated free group interpreted in the corresponding free ℓ -group, whenever $1 \leq a_1 \vee \dots \vee a_n \vee b$ and $1 \leq a_1 \vee \dots \vee a_n \vee b^{-1}$, either $1 \leq a_1 \vee \dots \vee a_n$ or $1 = b$.

First, we establish the completeness of the augmented calculus with respect to $\mathbf{Aut}(\mathbb{R})$: that is, we show that any basic ℓ -hypersequent that is valid in $\mathbf{Aut}(\mathbb{R})$ is \mathbf{Gl}^* -derivable. Let us say that a set S of inequations of the form $x > y$ where x, y are variables from some set X is *chain-consistent* if S is satisfiable as a set of first-order formulas in the structure consisting of \mathbb{R} with the standard order; otherwise S is said to be *chain-inconsistent*. It follows by an easy induction on the number of distinct variables occurring in S that S is chain-inconsistent if and only if there exists a sequence z_1, z_2, \dots, z_n of variables from X such that $(z_i > z_{i+1}) \in S$ for $1 \leq i \leq n - 1$ and $(z_n > z_1) \in S$. The base case is trivial. For the induction step, eliminate a variable x (adding $y > z$ whenever $y > x$ and $x > z$ are in S , and then remove all inequations containing an occurrence of x) to obtain a set of inequations S' that is chain-consistent if and only if S is chain-consistent; the desired result then follows using the induction hypothesis applied to S' .

Observe now that $\Gamma \sim \Delta$ if and only if $(\Gamma, \overline{\Delta})$ is group valid defines an equivalence relation (namely, equivalence in groups) on the set of basic ℓ -sequents. Let $[\Gamma] = \{\Delta : (\Gamma, \overline{\Delta}) \text{ is group valid}\}$ denote the corresponding equivalence classes. For each such equivalence class $[\Gamma]$, we define a fresh variable $a_{[\Gamma]}$. We will say that a basic ℓ -hypersequent \mathcal{G} is *chain-inconsistent* if there exist $(\Gamma_i, \overline{\Delta_i}) \in \mathcal{G}$ for $1 \leq i \leq n$ such that $\{a_{[\Delta_i]} > a_{[\Gamma_i]} : 1 \leq i \leq n\}$ is a chain-inconsistent set of inequations; otherwise \mathcal{G} is said to be *chain-consistent*.

Lemma 4.1. *If \mathcal{G} is chain-inconsistent, then \mathcal{G} is \mathbf{Gl}^* -derivable.*

Proof. Suppose that $(\Gamma_i, \overline{\Delta_i}) \in \mathcal{G}$ for $1 \leq i \leq n$ where $\{a_{[\Delta_i]} > a_{[\Gamma_i]} : 1 \leq i \leq n\}$ is a chain-inconsistent set of inequations. We prove the lemma by induction on n .

For the base case $n = 1$, we have $a_{[\Gamma_1]} = a_{[\Delta_1]}$, which means also that $[\Gamma_1] = [\Delta_1]$. So $\Gamma_1, \overline{\Delta_1}$ is group valid and \mathcal{G} is \mathbf{Gl}^* -derivable using (GV).

For the induction step, $\{a_{[\Delta_i]} > a_{[\Gamma_i]} : 1 \leq i \leq n\}$ is chain-inconsistent and hence we may assume without loss of generality that $a_{[\Gamma_1]} = a_{[\Delta_2]}$. So $\Gamma_1, \overline{\Delta_2}$ is group valid. Then the required \mathbf{Gl}^* -derivation ends with

$$\frac{\frac{\frac{\overline{\mathcal{G} \mid \Gamma_1, \overline{\Delta_2}} \quad (\text{GV}) \quad \frac{\vdots}{\overline{\mathcal{G} \mid \Gamma_2, \overline{\Delta_1}}} \quad (\text{COM})}{\mathcal{G} \mid \Gamma_1, \overline{\Delta_1} \mid \Gamma_2, \overline{\Delta_2}} \quad (\text{EC})}{\mathcal{G} \mid \Gamma_1, \overline{\Delta_1}} \quad (\text{EC})}{\mathcal{G}} \quad (\text{EC})$$

where the right premise is \mathbf{Gl}^* -derivable using the fact that $\{a_{[\Delta_1]} > a_{[\Gamma_2]}\} \cup \{a_{[\Delta_i]} > a_{[\Gamma_i]} : 3 \leq i \leq n\}$ is a chain-inconsistent set of inequations and applying the induction hypothesis. \square

Observe now that an ℓ -hypersequent \mathcal{G} containing variables x_1, \dots, x_n is valid in $\mathbf{Aut}(\mathbb{R})$ if and only if for every n -tuple \vec{f} of functions in $\mathbf{Aut}(\mathbb{R})$,

$$id \leq \bigvee \{\mathcal{I}(\Gamma)^{\mathbf{Aut}(\mathbb{R})}(\vec{f}) : \Gamma \in \mathcal{G}\},$$

where id is the identity function on \mathbb{R} . Moreover, this inequation holds in $\mathbf{Aut}(\mathbb{R})$ if and only if for every point $p \in \mathbb{R}$, the value of at least one $\mathcal{I}(\Gamma)^{\mathbf{Aut}(\mathbb{R})}(\vec{f})$ at p is greater than or equal to p . Equivalently, there is no point $p \in \mathbb{R}$ such that the value of every $\mathcal{I}(\Gamma)^{\mathbf{Aut}(\mathbb{R})}(\vec{f})$ at p is strictly less than p .

Lemma 4.2. *If \mathcal{G} is valid in $\mathbf{Aut}(\mathbb{R})$, then \mathcal{G} is \mathbf{Gl}^* -derivable.*

Proof. Suppose that $\mathbf{Aut}(\mathbb{R}) \models \mathcal{G}$, and consider the set of equivalence classes of initial subsequences of sequents in \mathcal{G} :

$$is(\mathcal{G}) = \{[\Gamma] : (\Gamma, \Delta) \in \mathcal{G}\}.$$

We apply the rule $(*)$ backwards finitely many times to \mathcal{G} using a representative sequent $\Gamma, \overline{\Delta}$ of $[\Gamma, \overline{\Delta}]$ for all distinct $[\Gamma], [\Delta] \in is(\mathcal{G})$. We will show that every leaf $\mathcal{G} \mid \mathcal{H}$ resulting from this process is chain-inconsistent and therefore \mathbf{Gl}^* -derivable. It follows that \mathcal{G} is also \mathbf{Gl}^* -derivable.

Let us suppose for a contradiction that $\mathcal{G} \mid \mathcal{H}$ is chain-consistent. Then for each $[\Gamma] \in is(\mathcal{G})$ we can assign $a_{[\Gamma]}$ to a real number $r_{[\Gamma]}$ such that for any distinct $[\Gamma], [\Delta] \in is(\mathcal{G})$, either $(\Gamma, \overline{\Delta}) \in \mathcal{H}$ and $r_{[\Delta]} > r_{[\Gamma]}$ or $(\Delta, \overline{\Gamma}) \in \mathcal{H}$ and $r_{[\Gamma]} > r_{[\Delta]}$. Moreover, we may assume that $r_{\square} > r_{[\Gamma]}$ for each $\Gamma \in \mathcal{G}$.

Now for each variable x we define a map \hat{x} that sends $r_{[\Gamma]}$ to $r_{[\Gamma, x]}$ when $[\Gamma], [\Gamma, x] \in is(\mathcal{G})$ and $r_{[\Delta, \overline{x}]}$ to $r_{[\Delta]}$ when $[\Delta, \overline{x}], [\Delta] \in is(\mathcal{G})$.

Let us check that \hat{x} does not violate the order-preserving condition. It suffices, by Lemma 4.1, to show that if \hat{x} violates the condition, then $\mathcal{G} \mid \mathcal{H}$ is \mathbf{Gl}^* -derivable. First suppose that \hat{x} maps $r_{[\Gamma]}$ to $r_{[\Gamma, x]}$ and $r_{[\Delta]}$ to $r_{[\Delta, x]}$, but $r_{[\Delta]} > r_{[\Gamma]}$ and $r_{[\Gamma, x]} > r_{[\Delta, x]}$. Then \mathcal{H} contains sequents in $[\Gamma, \overline{\Delta}]$ and $[\Delta, x, \overline{x}, \overline{\Gamma}]$, and is \mathbf{Gl}^* -derivable using (SPLIT) and (GV). Alternatively, suppose that \hat{x} maps $r_{[\Gamma]}$ to $r_{[\Gamma, x]}$ and $r_{[\Delta, \overline{x}]}$ to $r_{[\Delta]}$, but $r_{[\Delta, \overline{x}]} > r_{[\Gamma]}$ and $r_{[\Gamma, x]} > r_{[\Delta]}$. Then \mathcal{H} contains sequents in $[\Gamma, x, \overline{\Delta}]$ and $[\Delta, \overline{x}, \overline{\Gamma}]$, and is \mathbf{Gl}^* -derivable using (SPLIT) and (GV). Other cases are very similar.

Finally, we extend each \hat{x} to a function in $Aut(\mathbb{R})$ (for example, linearly between the given points). Consider now each $\Gamma \in \mathcal{G}$ and its evaluation as a function $\mathcal{I}(\hat{\Gamma})$ in $Aut(\mathbb{R})$. This function maps $r_{\square} \in \mathbb{R}$ to $r_{[\Gamma]} \in \mathbb{R}$ where $r_{\square} > r_{[\Gamma]}$. But this contradicts the assumption that \mathcal{G} is valid in $\mathbf{Aut}(\mathbb{R})$, so we are done. \square

The proof of the previous lemma follows quite closely the argument for the adequacy of the Holland–McCleary algorithm given in [11]. For readers familiar with this algorithm, recall that a basic ℓ -hypersequent \mathcal{G} corresponds to a disjunction of group terms that, for simplicity, we may assume are already in reduced form. We consider, in both cases, the initial subterms of these group terms and all the ways that they can be totally ordered. The Holland–McCleary algorithm searches for a failure or inconsistency, and hypersequents are interpreted accordingly as conjunctions. Likewise, a sequent Γ in a hypersequent is not interpreted as the inequality $1 \leq \Gamma$, but rather as the strict inequality $1 > \Gamma$. In essence, hypersequents are viewed as sets/conjunctions of strict inequalities. The Holland–McCleary algorithm automatically closes each such set under transitivity and, in the case where the set is chain-consistent, this provides a total ordering of the group terms involved. Such a set is called a *diagram* if it is further equipped with labeled *arrows*: for example, we have an arrow with label x for each pair of group terms $(\Gamma, \Gamma x)$ in the diagram. Moreover, in each step of the algorithm, new elements of $is(\mathcal{G})$ are inserted in the possible relative positions of the diagram, creating new inequalities. A backward application of the rule $(*)$ has the same effect, but without closing under transitivity at each step; however, exhaustive applications of $(*)$ to $is(\mathcal{G})$, as described in the above proof, subsume transitivity closure. Finally, checking whether \hat{x} violates the order-preserving condition corresponds to the Holland–McCleary algorithm checking whether arrows with label x *cross*. Here, branches involving inconsistency in the order give rise to derivable hypersequents.

5. A generation theorem

We have shown that the calculus $\mathsf{G}\ell$ derives only ℓ -valid basic ℓ -hypersequents (Lemma 3.4) and that the augmented calculus $\mathsf{G}\ell^*$ derives all basic ℓ -hypersequents valid in $\mathbf{Aut}(\mathbb{R})$ (Lemma 4.2). To complete the circle and show that derivability in these calculi, ℓ -validity, and validity in $\mathbf{Aut}(\mathbb{R})$ all coincide, it suffices to show that $(*)$ is $\mathsf{G}\ell$ -admissible. This will also provide a syntactic proof that the variety of ℓ -groups is generated by $\mathbf{Aut}(\mathbb{R})$.

We begin with some useful terminology. Given a $\mathsf{G}\ell$ -derivation d of $\mathcal{G} \mid \mathcal{H}$, we inductively label sequents occurring in d as \mathcal{G} -sequents as follows. To be precise, we label occurrences of sequents, thus allowing for some occurrences of a sequent in a hypersequent to be \mathcal{G} -sequents and some not to be; we trust the reader to interpret correctly which are the corresponding occurrences of a sequent in the premises and the conclusion of a rule. Each sequent in \mathcal{G} at the root is a \mathcal{G} -sequent. For the induction step, we consider an instance of a rule in d and the following three cases:

- (i) The instance has premises $\mathcal{G}' \mid \Gamma, \Delta$ and $\mathcal{G}' \mid \overline{\Delta}, \Sigma$ and conclusion $\mathcal{G}' \mid \Gamma, \Sigma$: each \mathcal{G} -sequent in \mathcal{G}' in the conclusion is a \mathcal{G} -sequent in \mathcal{G}' in the premises, and if Γ, Σ is a \mathcal{G} -sequent in the conclusion, then Γ, Δ and $\overline{\Delta}, \Sigma$ are \mathcal{G} -sequents in the premises.
- (ii) The instance has premise \mathcal{G}' and conclusion $\mathcal{G}' \mid \mathcal{H}'$: each \mathcal{G} -sequent in \mathcal{G}' in the conclusion is a \mathcal{G} -sequent in \mathcal{G}' in the premise.
- (iii) The instance has premise $\mathcal{G}' \mid \mathcal{H}' \mid \mathcal{H}'$ and conclusion $\mathcal{G}' \mid \mathcal{H}'$: each \mathcal{G} -sequent in $\mathcal{G}' \mid \mathcal{H}'$ in the conclusion is a \mathcal{G} -sequent in $\mathcal{G}' \mid \mathcal{H}' \mid \mathcal{H}'$ (twice if it occurs in \mathcal{H}') in the premise.

It follows that each (occurrence of a) sequent occurring in d is strictly either a \mathcal{G} -sequent or an \mathcal{H} -sequent.

A $\mathsf{G}\ell$ -derivation of $\mathcal{G} \mid \mathcal{H}$ will be called \mathcal{H} -cut-free if it contains no application of (CUT) with an active \mathcal{H} -sequent (an \mathcal{H} -cut). Consider an \mathcal{H} -cut in a $\mathsf{G}\ell$ -derivation of $\mathcal{G} \mid \mathcal{H}$

$$\frac{\mathcal{G}' \mid \mathcal{H}' \mid \Gamma, \Delta \quad \mathcal{G}' \mid \mathcal{H}' \mid \overline{\Delta}, \Sigma}{\mathcal{G}' \mid \mathcal{H}' \mid \Gamma, \Sigma}$$

where the \mathcal{G} -sequents are in \mathcal{G}' and the \mathcal{H} -sequents are in $\mathcal{H}' \mid \Gamma, \Sigma$ in the conclusion and in $\mathcal{H}' \mid \Gamma, \Delta$ and $\mathcal{H}' \mid \overline{\Delta}, \Sigma$ in the premises. We call this \mathcal{H} -cut *significant* if it is not the case that both $\mathcal{H}' \mid \Gamma, \Delta$ and $\mathcal{H}' \mid \overline{\Delta}, \Sigma$ are $\mathsf{G}\ell$ -derivable.

We obtain the following partial cut-elimination lemma.

Lemma 5.1. *If $\vdash_{\mathsf{G}\ell} \mathcal{G} \mid \mathcal{H}$, then either $\vdash_{\mathsf{G}\ell} \mathcal{H}$ or there is an \mathcal{H} -cut-free $\mathsf{G}\ell$ -derivation of $\mathcal{G} \mid \mathcal{H}$.*

Proof. We show that an uppermost significant \mathcal{H} -cut can be eliminated from a $\mathsf{G}\ell$ -derivation d of $\mathcal{G} \mid \mathcal{H}$. It then follows that all significant \mathcal{H} -cuts can be eliminated inductively from d to obtain a significant- \mathcal{H} -cut-free $\mathsf{G}\ell$ -derivation d' of $\mathcal{G} \mid \mathcal{H}$. If d' contains no applications of \mathcal{H} -cuts, then there is an \mathcal{H} -cut-free $\mathsf{G}\ell$ -derivation of $\mathcal{G} \mid \mathcal{H}$. Otherwise, consider a lowest (non-significant) \mathcal{H} -cut in d' with conclusion $\mathcal{G}' \mid \mathcal{H}' \mid \Gamma, \Sigma$ where the \mathcal{G} -sequents are in \mathcal{G}' and all other sequents are \mathcal{H} -sequents. Note that, by assumption, $\vdash_{\mathsf{G}\ell} \mathcal{H}' \mid \Gamma, \Sigma$. Below this application in d' , the only rule applications with active \mathcal{H} -sequents are (EW) and (EC). So \mathcal{H} is derivable from $\mathcal{H}' \mid \Gamma, \Sigma$ using (EW) and (EC). But also $\mathcal{H}' \mid \Gamma, \Sigma$ is $\mathsf{G}\ell$ -derivable, so $\vdash_{\mathsf{G}\ell} \mathcal{H}$.

To show that an uppermost significant \mathcal{H} -cut can be eliminated from a $\mathsf{G}\ell$ -derivation of $\mathcal{G} \mid \mathcal{H}$, we prove that if both $\mathcal{G}_1 \mid \mathcal{H}_1 \mid \Gamma_1, \Sigma \mid \dots \mid \Gamma_1, \Sigma$ and $\mathcal{G}_2 \mid \mathcal{H}_2 \mid \overline{\Sigma}, \Gamma_2 \mid \dots \mid \overline{\Sigma}, \Gamma_2$ are significant- \mathcal{H} -cut-free $\mathsf{G}\ell$ -derivable where the \mathcal{G} -sequents are in \mathcal{G}_1 and \mathcal{G}_2 and all other sequents are \mathcal{H} -sequents, then $\mathcal{G}_1 \mid \mathcal{G}_2 \mid \mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1, \Gamma_2$ is significant- \mathcal{H} -cut-free $\mathsf{G}\ell$ -derivable, proceeding by induction on the combined heights of these derivations.

We consider the last steps of both derivations. If either step is an application of (EW) or (EC), then the claim follows immediately using the induction hypothesis and possibly an additional application of (EW) or (EC).

Suppose that both last steps are (non-significant) \mathcal{H} -cuts or instances of (GV) or (EM) whose active sequents are \mathcal{H} -sequents. It follows that $\mathcal{H}_1 \mid \Gamma_1, \Sigma \mid \dots \mid \Gamma_1, \Sigma$ and $\mathcal{H}_2 \mid \overline{\Sigma}, \Gamma_2 \mid \dots \mid \overline{\Sigma}, \Gamma_2$ are $\mathcal{G}\ell$ -derivable. So, using (CUT) and (EW), $\mathcal{G}_1 \mid \mathcal{G}_2 \mid \mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1, \Gamma_2$ is significant- \mathcal{H} -cut-free $\mathcal{G}\ell$ -derivable as required.

If either a group valid sequent or $\Delta \mid \overline{\Delta}$ occurs in $\mathcal{G}_1 \mid \mathcal{G}_2 \mid \mathcal{H}_1 \mid \mathcal{H}_2$, then $\mathcal{G}_1 \mid \mathcal{G}_2 \mid \mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1, \Gamma_2$ is clearly significant- \mathcal{H} -cut-free $\mathcal{G}\ell$ -derivable. Also, if both Γ_1, Σ and $\overline{\Sigma}, \Gamma_2$ are group valid, then Γ_1, Γ_2 is group valid and $\mathcal{G}_1 \mid \mathcal{G}_2 \mid \mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1, \Gamma_2$ is derivable by (GV).

If $\mathcal{G}_2 = \mathcal{G}'_2 \mid \overline{\Gamma_2}, \Sigma$, then we have a significant- \mathcal{H} -cut-free $\mathcal{G}\ell$ -derivation

$$\frac{\frac{\frac{\frac{\vdots}{\mathcal{G}_1 \mid \mathcal{H}_1 \mid \Gamma_1, \Sigma \mid \dots \mid \Gamma_1, \Sigma} \text{ (EC)}}{\mathcal{G}_1 \mid \mathcal{G}'_2 \mid \overline{\Gamma_2}, \overline{\Gamma_1} \mid \mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1, \Gamma_2} \text{ (EM)}}{\mathcal{G}_1 \mid \mathcal{G}'_2 \mid \overline{\Gamma_2}, \Sigma \mid \mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1, \Gamma_2} \text{ (CUT)}}{\frac{\frac{\frac{\vdots}{\mathcal{G}_1 \mid \mathcal{H}_1 \mid \Gamma_1, \Sigma} \text{ (EC)}}{\mathcal{G}_1 \mid \mathcal{G}'_2 \mid \Gamma_1, \Sigma \mid \mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1, \Gamma_2} \text{ (EW)}}{\mathcal{G}_1 \mid \mathcal{G}'_2 \mid \overline{\Gamma_2}, \Sigma \mid \mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1, \Gamma_2} \text{ (CUT)}} \text{ (CUT)}$$

The case where $\mathcal{G}_1 = \mathcal{G}'_1 \mid \overline{\Sigma}, \overline{\Gamma_1}$ is very similar.

Suppose now that one of the derivations ends with a \mathcal{G} -cut; e.g., $\mathcal{G}_2 = \mathcal{G}'_2 \mid \Delta_1, \Delta_2$ and we have

$$\frac{\frac{\frac{\vdots}{\mathcal{G}'_2 \mid \Delta_1, \Pi \mid \mathcal{H}_2 \mid \overline{\Sigma}, \Gamma_2 \mid \dots \mid \overline{\Sigma}, \Gamma_2} \text{ (CUT)}}{\mathcal{G}'_2 \mid \Delta_1, \Delta_2 \mid \mathcal{H}_2 \mid \overline{\Sigma}, \Gamma_2 \mid \dots \mid \overline{\Sigma}, \Gamma_2} \text{ (CUT)}}{\mathcal{G}'_2 \mid \overline{\Pi}, \Delta_2 \mid \mathcal{H}_2 \mid \overline{\Sigma}, \Gamma_2 \mid \dots \mid \overline{\Sigma}, \Gamma_2} \text{ (CUT)}$$

We apply the induction hypothesis to the derivations of $\mathcal{G}_1 \mid \mathcal{H}_1 \mid \Gamma_1, \Sigma \mid \dots \mid \Gamma_1, \Sigma$ and $\mathcal{G}'_2 \mid \Delta_1, \Pi \mid \mathcal{H}_1 \mid \mathcal{H}_2 \mid \overline{\Sigma}, \Gamma_2 \mid \dots \mid \overline{\Sigma}, \Gamma_2$ to obtain an \mathcal{H} -significant-cut-free $\mathcal{G}\ell$ -derivation of $\mathcal{G}_1 \mid \mathcal{G}'_2 \mid \Delta_1, \Pi \mid \mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1, \Gamma_2$. Similarly, we obtain an \mathcal{H} -significant-cut-free $\mathcal{G}\ell$ -derivation of $\mathcal{G}_1 \mid \mathcal{G}'_2 \mid \overline{\Pi}, \Delta_2 \mid \mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1, \Gamma_2$. So we obtain an \mathcal{H} -significant-cut-free $\mathcal{G}\ell$ -derivation of $\mathcal{G}_1 \mid \mathcal{G}_2 \mid \mathcal{H}_1 \mid \mathcal{H}_2 \mid \Gamma_1, \Gamma_2$ using (CUT). \square

Lemma 5.2. *(*) is $\mathcal{G}\ell$ -admissible.*

Proof. It suffices, using the rule (EC), to prove that whenever $\vdash_{\mathcal{G}\ell} \mathcal{G} \mid \Gamma \mid \dots \mid \Gamma$ and $\vdash_{\mathcal{G}\ell} \mathcal{H} \mid \overline{\Gamma}$ where Γ is not group valid, then $\vdash_{\mathcal{G}\ell} \mathcal{G} \mid \mathcal{H}$. If Γ is not group valid, then $\not\vdash_{\mathcal{G}\ell} \Gamma$. So by Lemma 5.1, we may prove the claim by induction on the height of a $\Gamma \mid \dots \mid \Gamma$ -cut-free $\mathcal{G}\ell$ -derivation of $\mathcal{G} \mid \Gamma \mid \dots \mid \Gamma$. For the base case, there are two possibilities. If there is a group valid sequent or occurrence of $\Delta \mid \overline{\Delta}$ in \mathcal{G} , then clearly $\vdash_{\mathcal{G}\ell} \mathcal{G} \mid \mathcal{H}$. Otherwise, $\mathcal{G} = (\mathcal{G}' \mid \overline{\Gamma})$ and, because $\vdash_{\mathcal{G}\ell} \mathcal{H} \mid \overline{\Gamma}$, also $\vdash_{\mathcal{G}\ell} \mathcal{G} \mid \mathcal{H}$ using (EW). For the induction step, suppose that $\mathcal{G} \mid \Gamma \mid \dots \mid \Gamma$ is the conclusion of an instance of (CUT) with premises $\mathcal{G}' \mid \Gamma_1, \Sigma \mid \Gamma \mid \dots \mid \Gamma$ and $\mathcal{G}' \mid \overline{\Sigma}, \Gamma_2 \mid \Gamma \mid \dots \mid \Gamma$ where $\mathcal{G} = (\mathcal{G}' \mid \Gamma_1, \Gamma_2)$. By the induction hypothesis twice, $\vdash_{\mathcal{G}\ell} \mathcal{G}' \mid \Gamma_1, \Sigma \mid \mathcal{H}$ and $\vdash_{\mathcal{G}\ell} \mathcal{G}' \mid \overline{\Sigma}, \Gamma_2 \mid \mathcal{H}$. Hence, using (CUT), $\vdash_{\mathcal{G}\ell} \mathcal{G}' \mid \Gamma_1, \Gamma_2 \mid \mathcal{H}$ as required. The cases of (EW) and (EC) are straightforward. \square

Putting together Lemmas 3.4, 4.2, and 5.2, we obtain:

Theorem 5.3. *For any basic ℓ -hypersequent \mathcal{G} , the following are equivalent:*

- (1) \mathcal{G} is $\mathcal{G}\ell$ -derivable.
- (2) \mathcal{G} is ℓ -valid.

$\frac{}{\mathcal{G} \mid \Gamma}$ (GV)	$\frac{\mathcal{G}}{\mathcal{G} \mid \mathcal{H}}$ (EW)	$\frac{\mathcal{G} \mid \mathcal{H} \mid \mathcal{H}}{\mathcal{G} \mid \mathcal{H}}$ (EC)
Γ group valid		
$\frac{\mathcal{G} \mid \Gamma \quad \mathcal{G} \mid \Delta}{\mathcal{G} \mid \Gamma, \Delta}$ (MIX)		$\frac{\mathcal{G} \mid \Gamma, \Sigma \quad \mathcal{G} \mid \Pi, \Delta}{\mathcal{G} \mid \Gamma, \Delta \mid \Pi, \Sigma}$ (COM)

Fig. 2. The basic ℓ -hypersequent calculus \mathbf{GLG}_b .

- (3) \mathcal{G} is valid in $\mathbf{Aut}(\mathbb{R})$.
 (4) \mathcal{G} is \mathbf{GL}^* -derivable.

It follows that an identity is valid in all ℓ -groups if and only if it is valid in $\mathbf{Aut}(\mathbb{R})$. Hence, as the variety generated by a class of algebras \mathcal{K} is, by Birkhoff's variety theorem, the smallest class satisfying the same identities as \mathcal{K} , we obtain:

Theorem 5.4. (See [10].) *The variety of ℓ -groups is generated by $\mathbf{Aut}(\mathbb{R})$.*

Note that Holland's original proof of this generation theorem in [10] is based on his embedding theorem for ℓ -groups [9] and, unlike the proof presented here, makes use of the axiom of choice. Let us remark also that although the Holland–McCleary algorithm decides the ℓ -validity of ℓ -group identities, it does not produce corresponding derivations in equational logic (i.e., derivations from defining ℓ -group identities). However, by applying the constructive $(*)$ -elimination procedure described here to the derivation of an ℓ -group identity in \mathbf{GL}^* , we obtain a \mathbf{GL} -derivation of the identity. This derivation may then easily be translated into an equational logic derivation.

6. An analytic calculus

Although \mathbf{GL} and \mathbf{GL}^* provide bases for decision procedures for ℓ -groups, these calculi are not analytic. They contain rules, (CUT) and $(*)$, where terms occurring in the premises may not occur as subterms in the conclusion. Observe that, following the proofs in Sections 4 and 5, a basic ℓ -hypersequent \mathcal{G} is valid in $\mathbf{Aut}(\mathbb{R})$ if and only if it is derivable in the analytic calculus consisting of (GV), (EM), (EW), (EC), and the restricted $(*)$ rule

$$\frac{\mathcal{G} \mid \Gamma, \bar{\Pi} \quad \mathcal{G} \mid \Pi, \bar{\Gamma}}{\mathcal{G} \mid \Gamma, \Delta \mid \Pi, \Sigma}$$

where $\Gamma, \bar{\Pi}$ is not group valid.

However, this restricted rule makes use of a side-condition and cannot be interpreted as a quasiequation valid in all ℓ -groups. Also the completeness proof for the calculus is heavily dependent on the particular class of algebraic structures and cannot be expected to generalize easily to other classes.

In this section we establish soundness and completeness for an analytic basic ℓ -hypersequent calculus \mathbf{GLG}_b , presented in Fig. 2, that replaces (CUT) and (EM) in \mathbf{GL} with the analytic rules (MIX) and (COM). The key step in the proof is to show that (CUT) is \mathbf{GLG}_b -admissible, obtained as the culmination of a series of lemmas establishing the \mathbf{GLG}_b -admissibility of related rules.

Let us write $\mathcal{G}[\Gamma]$ to denote a basic ℓ -hypersequent \mathcal{G} with specific occurrences (perhaps none) of a sequence of literals Γ ; a subsequent reference to $\mathcal{G}[\Delta]$ then denotes the ℓ -hypersequent obtained by replacing these occurrences with Δ .

Lemma 6.1. *The following rule is \mathcal{GLG}_b -admissible:*

$$\frac{\Delta \quad \mathcal{G}[]}{\mathcal{G}[\Delta]}$$

Proof. We proceed by induction on the height of a \mathcal{GLG}_b -derivation of $\mathcal{G}[]$. For the base case, it suffices to observe that if $\Gamma[]$ is group valid, then, because Δ is group valid, also $\Gamma[\Delta]$ is group valid. For the induction step, suppose that the last step in the derivation of $\mathcal{G}[]$ is an application of (COM) of the form

$$\frac{\mathcal{H}[] \mid \Gamma_1[], \Pi_2[] \quad \mathcal{H}[] \mid \Pi_1[], \Gamma_2[]}{\mathcal{H}[] \mid \Gamma_1[], \Gamma_2[] \mid \Pi_1[], \Pi_2[]} \text{ (COM)}$$

Then by the induction hypothesis twice,

$$\vdash_{\mathcal{GLG}_b} \mathcal{H}[\Delta] \mid \Gamma_1[\Delta], \Pi_2[\Delta] \quad \text{and} \quad \vdash_{\mathcal{GLG}_b} \mathcal{H}[\Delta] \mid \Pi_1[\Delta], \Gamma_2[\Delta].$$

So by an application of (COM) as required,

$$\vdash_{\mathcal{GLG}_b} \mathcal{H}[\Delta] \mid \Gamma_1[\Delta], \Gamma_2[\Delta] \mid \Pi_1[\Delta], \Pi_2[\Delta].$$

The cases where the last step in the derivation of $\mathcal{G}[]$ is an application of (EW), (EC), or (MIX) are very similar. \square

Lemma 6.2. *The following rule is \mathcal{GLG}_b -admissible:*

$$\frac{\mathcal{G} \mid \Gamma, t, \Delta \quad \mathcal{G} \mid \Pi, \bar{t}, \Sigma}{\mathcal{G} \mid \Gamma, \Sigma \mid \Pi, \Delta}$$

Proof. Let us define $(\Gamma_1, t, \Gamma_2, t, \dots, \Gamma_n, t, \Gamma_{n+1})@(\Pi, \bar{t}, \Sigma)$, where the t and \bar{t} are distinguished (but perhaps not all) occurrences in the sequents, as

$$(\Gamma_1, \Sigma \mid \Pi, \Gamma_2, \Sigma \mid \dots \mid \Pi, \Gamma_n, \Sigma \mid \Pi, \Gamma_{n+1}),$$

noting that $(\Gamma)@(\Pi, \bar{t}, \Sigma)$ is just Γ .

Then it suffices (using (EC)) to prove that the following rule is \mathcal{GLG}_b -admissible:

$$\frac{\Delta_1[t] \mid \dots \mid \Delta_n[t] \quad \mathcal{G} \mid \Pi, \bar{t}, \Sigma}{\mathcal{G} \mid (\Delta_1[t])@(\Pi, \bar{t}, \Sigma) \mid \dots \mid (\Delta_n[t])@(\Pi, \bar{t}, \Sigma)}$$

We proceed by induction on the height of a \mathcal{GLG}_b -derivation of $\Delta_1[t] \mid \dots \mid \Delta_n[t]$.

For the base case, we may assume without loss of generality that $\Delta_1[t]$ is group valid. We prove that $\vdash_{\mathcal{GLG}_b} \mathcal{G} \mid (\Delta_1[t])@(\Pi, \bar{t}, \Sigma)$ by (a new) induction on the number of terms in $\Delta_1[t]$. Clearly, if $\Delta_1[t]$ contains no distinguished occurrences of t , then $(\Delta_1[t])@(\Pi, \bar{t}, \Sigma) = \Delta_1$, and $\mathcal{G} \mid (\Delta_1[t])@(\Pi, \bar{t}, \Sigma)$ is an instance of (GV). Otherwise, there are three possibilities. First, $\Delta_1[t]$ can have the form

$$\Gamma_1, t, \Gamma_2, t, \dots, \Gamma_k, t, \bar{t}, \Gamma_{k+1}, t, \dots, \Gamma_n, t, \Gamma_{n+1}.$$

Applying the induction hypothesis to $\Gamma_1, t, \Gamma_2, t, \dots, \Gamma_k, \Gamma_{k+1}, t, \dots, \Gamma_n, t, \Gamma_{n+1}$,

$$\vdash_{\mathcal{GLG}_b} \mathcal{G} \mid \Gamma_1, \Sigma \mid \Pi, \Gamma_2, \Sigma \mid \dots \mid \Pi, \Gamma_k, \Gamma_{k+1}, \Sigma \mid \dots \mid \Pi, \Gamma_n, \Sigma \mid \Pi, \Gamma_{n+1}.$$

But also $\vdash_{\mathcal{G}\mathcal{L}\mathcal{G}_b} \mathcal{G} \mid \Pi, \bar{t}, \Sigma$, so using (COM) and (EW),

$$\vdash_{\mathcal{G}\mathcal{L}\mathcal{G}_b} \mathcal{G} \mid \Gamma_1, \Sigma \mid \Pi, \Gamma_2, \Sigma \mid \dots \mid \Pi, \Gamma_k, \Sigma \mid \Pi, \bar{t}, \Gamma_{k+1}, \Sigma \mid \dots \mid \Pi, \Gamma_n, \Sigma \mid \Pi, \Gamma_{n+1}.$$

Now suppose that $\Delta_1[t]$ has the form

$$\Gamma_1, t, \Gamma_2, t, \dots, \Gamma_k, \bar{t}, t, \Gamma_{k+1}, t, \dots, \Gamma_n, t, \Gamma_{n+1}.$$

Applying the induction hypothesis to $\Gamma_1, t, \Gamma_2, t, \dots, \Gamma_k, \Gamma_{k+1}, t, \dots, \Gamma_n, t, \Gamma_{n+1}$,

$$\vdash_{\mathcal{G}\mathcal{L}\mathcal{G}_b} \mathcal{G} \mid \Gamma_1, \Sigma \mid \Pi, \Gamma_2, \Sigma \mid \dots \mid \Pi, \Gamma_k, \Gamma_{k+1}, \Sigma \mid \dots \mid \Pi, \Gamma_n, \Sigma \mid \Pi, \Gamma_{n+1}.$$

But also $\vdash_{\mathcal{G}\mathcal{L}\mathcal{G}_b} \mathcal{H} \mid \Pi, \bar{t}, \Sigma$, so by (COM) and (EW) as required:

$$\vdash_{\mathcal{G}\mathcal{L}\mathcal{G}_b} \mathcal{G} \mid \Gamma_1, \Sigma \mid \Pi, \Gamma_2, \Sigma \mid \dots \mid \Pi, \Gamma_k, \bar{t}, \Sigma \mid \Pi, \Gamma_{k+1}, \Sigma \mid \dots \mid \Pi, \Gamma_n, \Sigma \mid \Pi, \Gamma_{n+1}.$$

Finally, $\Delta_1[t]$ might have the form

$$\Gamma_1, t, \Gamma_2, t, \dots, \Gamma_k^1, s, \bar{s}, \Gamma_k^2, t, \dots, \Gamma_n, t, \Gamma_{n+1}.$$

Applying the induction hypothesis to $\Gamma_1, t, \Gamma_2, t, \dots, \Gamma_k^1, \Gamma_k^2, t, \dots, \Gamma_n, t, \Gamma_{n+1}$,

$$\vdash_{\mathcal{G}\mathcal{L}\mathcal{G}_b} \mathcal{G} \mid \Gamma_1, \Sigma \mid \Pi, \Gamma_2, \Sigma \mid \dots \mid \Pi, \Gamma_k^1, \Gamma_k^2, \Sigma \mid \dots \mid \Pi, \Gamma_n, \Sigma \mid \Pi, \Gamma_{n+1}.$$

But then by [Lemma 6.1](#), we obtain as required

$$\vdash_{\mathcal{G}\mathcal{L}\mathcal{G}_b} \mathcal{G} \mid \Gamma_1, \Sigma \mid \Pi, \Gamma_2, \Sigma \mid \dots \mid \Pi, \Gamma_k^1, s, \bar{s}, \Gamma_k^2, \Sigma \mid \dots \mid \Pi, \Gamma_n, \Sigma \mid \Pi, \Gamma_{n+1}.$$

Now let us return to the inductive step of the main induction proof. Note that the cases of (EW) and (EC) follow immediately using the induction hypothesis. Suppose that the derivation ends with an application of (COM) of the form

$$\frac{\mathcal{H} \mid \Gamma_1, t, \dots, \Gamma_k^1, \Theta_r^2, t, \dots, \Theta_m, t, \Theta_{m+1} \quad \mathcal{H} \mid \Theta_1, t, \dots, \Theta_r^1, \Gamma_k^2, t, \dots, \Gamma_p, t, \Gamma_{p+1}}{\mathcal{H} \mid \Gamma_1, t, \dots, \Gamma_k^1, \Gamma_k^2, t, \dots, \Gamma_p, t, \Gamma_{p+1} \mid \Theta_1, t, \dots, \Theta_r^1, \Theta_r^2, t, \dots, \Theta_m, t, \Theta_{m+1}}$$

where $\mathcal{H} = \Delta_3[t] \mid \dots \mid \Delta_n[t]$. By the induction hypothesis twice,

$$\begin{aligned} & \vdash_{\mathcal{G}\mathcal{L}\mathcal{G}_b} \mathcal{H}' \mid \Gamma_1, \Sigma \mid \Pi, \Gamma_2, \Sigma \mid \dots \mid \Pi, \Gamma_k^1, \Theta_r^2, \Sigma \mid \dots \mid \Pi, \Theta_m, \Sigma \mid \Pi, \Theta_{m+1} \\ \text{and} \quad & \vdash_{\mathcal{G}\mathcal{L}\mathcal{G}_b} \mathcal{H}' \mid \Theta_1, \Sigma \mid \Pi, \Theta_2, \Sigma \mid \dots \mid \Pi, \Theta_r^1, \Gamma_k^2, \Sigma \mid \dots \mid \Pi, \Gamma_p, \Sigma \mid \Pi, \Gamma_{p+1} \end{aligned}$$

where $\mathcal{H}' = \mathcal{G} \mid (\Delta_3[t]) @ (\Pi, \bar{t}, \Sigma) \mid \dots \mid (\Delta_n[t]) @ (\Pi, \bar{t}, \Sigma)$. Hence, using (EW) and (COM),

$$\begin{aligned} & \vdash_{\mathcal{G}\mathcal{L}\mathcal{G}_b} \mathcal{H}' \mid \Gamma_1, \Sigma \mid \dots \mid \Pi, \Gamma_k^1, \Gamma_k^2, \Sigma \mid \dots \mid \Pi, \Gamma_p, \Sigma \mid \Pi, \Gamma_{p+1} \mid \\ & \Theta_1, \Sigma \mid \dots \mid \Pi, \Theta_r^1, \Theta_r^2, \Sigma \mid \dots \mid \Pi, \Theta_m, \Sigma \mid \Pi, \Theta_{m+1}. \end{aligned}$$

Now suppose that the last step in the derivation is an application of (MIX),

$$\frac{\mathcal{H} \mid \Gamma_1, t, \dots, \Gamma_k^1 \quad \mathcal{H} \mid \Gamma_k^2, t, \dots, \Gamma_p, t, \Gamma_{p+1}}{\mathcal{H} \mid \Gamma_1, t, \dots, \Gamma_k^1, \Gamma_k^2, t, \dots, \Gamma_p, t, \Gamma_{p+1}}$$

where $\mathcal{H} = \Delta_2[t] \mid \dots \mid \Delta_n[t]$. By the induction hypothesis twice,

$$\begin{aligned} & \vdash_{\mathcal{GLG}_b} \mathcal{H}' \mid \Gamma_1, \Sigma \mid \Pi, \Gamma_2, \Sigma \mid \dots \mid \Pi, \Gamma_k^1 \\ \text{and } & \vdash_{\mathcal{GLG}_b} \mathcal{H}' \mid \Gamma_k^2, \Sigma \mid \dots \mid \Pi, \Gamma_p, \Sigma \mid \Pi, \Gamma_{p+1} \end{aligned}$$

where $\mathcal{H}' = \mathcal{G} \mid (\Delta_2[t])@(\Pi, \bar{t}, \Sigma) \mid \dots \mid (\Delta_n[t])@(\Pi, \bar{t}, \Sigma)$. Hence, using (EW) and (MIX),

$$\vdash_{\mathcal{GLG}_b} \mathcal{H}' \mid \Gamma_1, \Sigma \mid \dots \mid \Pi, \Gamma_k^1, \Gamma_k^2, \Sigma \mid \dots \mid \Pi, \Gamma_p, \Sigma \mid \Pi, \Gamma_{p+1}. \quad \square$$

Lemma 6.3. *The following rule is \mathcal{GLG}_b -admissible:*

$$\frac{\mathcal{G} \mid \Gamma, t \quad \mathcal{G} \mid \bar{t}, \Delta}{\mathcal{G} \mid \Gamma, \Delta}$$

Proof. It suffices, using (EC), to prove the \mathcal{GLG}_b -admissibility of

$$\frac{\mathcal{G} \mid \Gamma_1, t \mid \dots \mid \Gamma_n, t \quad \mathcal{H} \mid \bar{t}, \Delta}{\mathcal{G} \mid \mathcal{H} \mid \Gamma_1, \Delta \mid \dots \mid \Gamma_n, \Delta}$$

proceeding by induction on the height of a \mathcal{GLG}_b -derivation of $\mathcal{G} \mid \Gamma_1, t \mid \dots \mid \Gamma_n, t$.

For the base case, we have two possibilities. If a sequent in \mathcal{G} is group valid, then clearly $\vdash_{\mathcal{GLG}_b} \mathcal{G} \mid \mathcal{H} \mid \Gamma_1, \Delta \mid \dots \mid \Gamma_n, \Delta$. Otherwise, some Γ_i, t is group valid and therefore of the form $\Gamma_i^1, \bar{t}, \Gamma_i^2, t$ where Γ_i^1 and Γ_i^2 are both group valid. Hence by Lemma 6.1, $\vdash_{\mathcal{GLG}_b} \mathcal{H} \mid \Gamma_i^1, \bar{t}, \Gamma_i^2, \Delta$ and the result follows using (EW).

For the induction step, the cases of (EW) and (EC) follow immediately using the induction hypothesis. Now suppose that the last step in the derivation is an application of (COM) of the form

$$\frac{\mathcal{G} \mid \Gamma_1^1, \Gamma_2^2, t \mid \dots \mid \Gamma_n, t \quad \mathcal{G} \mid \Gamma_2^1, \Gamma_1^2, t \mid \dots \mid \Gamma_n, t}{\mathcal{G} \mid \Gamma_1^1, \Gamma_1^2, t \mid \Gamma_2^1, \Gamma_2^2, t \mid \dots \mid \Gamma_n, t}$$

By the induction hypothesis twice,

$$\vdash_{\mathcal{GLG}_b} \mathcal{G} \mid \mathcal{H} \mid \Gamma_1^1, \Gamma_2^2, \Delta \mid \dots \mid \Gamma_n, \Delta \quad \text{and} \quad \vdash_{\mathcal{GLG}_b} \mathcal{G} \mid \mathcal{H} \mid \Gamma_2^1, \Gamma_1^2, \Delta \mid \dots \mid \Gamma_n, \Delta.$$

Hence, using (COM),

$$\vdash_{\mathcal{GLG}_b} \mathcal{G} \mid \mathcal{H} \mid \Gamma_1^1, \Gamma_1^2, \Delta \mid \Gamma_2^1, \Gamma_2^2, \Delta \mid \dots \mid \Gamma_n, \Delta.$$

Now suppose that the application of (COM) is of the form

$$\frac{\mathcal{G} \mid \Gamma_1^1 \mid \dots \mid \Gamma_n, t \quad \mathcal{G} \mid \Gamma_2, t, \Gamma_1^2, t \mid \dots \mid \Gamma_n, t}{\mathcal{G} \mid \Gamma_1^1, \Gamma_1^2, t \mid \Gamma_2, t \mid \dots \mid \Gamma_n, t}$$

By the induction hypothesis twice,

$$\vdash_{\mathcal{GLG}_b} \mathcal{G} \mid \mathcal{H} \mid \Gamma_1^1 \mid \dots \mid \Gamma_n, \Delta \quad \text{and} \quad \vdash_{\mathcal{GLG}_b} \mathcal{G} \mid \mathcal{H} \mid \Gamma_2, t, \Gamma_1^2, \Delta \mid \dots \mid \Gamma_n, \Delta.$$

But $\mathcal{H} \mid \bar{t}, \Delta$ is \mathcal{GLG}_b -derivable, so by Lemma 6.2,

$$\vdash_{\mathcal{GLG}_b} \mathcal{G} \mid \mathcal{H} \mid \Gamma_2, \Delta \mid \Gamma_1^2, \Delta \mid \dots \mid \Gamma_n, \Delta.$$

Hence, using (MIX),

$$\vdash_{\mathbf{GLG}_b} \mathcal{G} \mid \mathcal{H} \mid \Gamma_1^1, \Gamma_1^2, \Delta \mid \Gamma_2, \Delta \mid \dots \mid \Gamma_n, \Delta.$$

Now suppose that the last step in the derivation is an application of (MIX):

$$\frac{\mathcal{G} \mid \Gamma_1^1 \mid \dots \mid \Gamma_n, t \quad \mathcal{G} \mid \Gamma_1^2, t \mid \dots \mid \Gamma_n, t}{\mathcal{G} \mid \Gamma_1^1, \Gamma_1^2, t \mid \dots \mid \Gamma_n, t}$$

By the induction hypothesis twice,

$$\vdash_{\mathbf{GLG}_b} \mathcal{G} \mid \mathcal{H} \mid \Gamma_1^1 \mid \dots \mid \Gamma_n, \Delta \quad \text{and} \quad \vdash_{\mathbf{GLG}_b} \mathcal{G} \mid \mathcal{H} \mid \Gamma_1^2, \Delta \mid \dots \mid \Gamma_n, \Delta$$

and the desired result follows by an application of (MIX). \square

Lemma 6.4. *The following rule is \mathbf{GLG}_b -admissible:*

$$\frac{\mathcal{G}[t, \bar{t}]}{\mathcal{G}[]}$$

Proof. We proceed by induction on the height of a \mathbf{GLG}_b -derivation of $\mathcal{G}[t, \bar{t}]$. For the base case, we simply observe that if a basic ℓ -sequent $\Gamma[t, \bar{t}]$ is group valid, then also $\Gamma[]$ is group valid.

For the induction step, the cases of (EW) and (EC) follow immediately using the induction hypothesis. Suppose now that the last step in the derivation is an application of (COM). The only tricky case has the form

$$\frac{\mathcal{H}[t, \bar{t}] \mid \Gamma_1[t, \bar{t}], t, \Pi_2[t, \bar{t}] \quad \mathcal{H}[t, \bar{t}] \mid \Pi_1[t, \bar{t}], \bar{t}, \Gamma_2[t, \bar{t}]}{\mathcal{H}[t, \bar{t}] \mid \Gamma_1[t, \bar{t}], t, \bar{t}, \Gamma_2[t, \bar{t}] \mid \Pi_1[t, \bar{t}], \Pi_2[t, \bar{t}]} \quad (\text{COM})$$

By the induction hypothesis twice,

$$\vdash_{\mathbf{GLG}_b} \mathcal{H}[] \mid \Gamma_1[], t, \Pi_2[] \quad \text{and} \quad \vdash_{\mathbf{GLG}_b} \mathcal{H}[] \mid \Pi_1[], \bar{t}, \Gamma_2[].$$

But then, by Lemma 6.2, $\vdash_{\mathbf{GLG}_b} \mathcal{H}[] \mid \Gamma_1[], \Gamma_2[] \mid \Pi_1[], \Pi_2[]$ as required.

Now suppose that the last step in the derivation is an application of (MIX). The only tricky case has the form

$$\frac{\mathcal{H}[t, \bar{t}] \mid \Gamma_1[t, \bar{t}], t \quad \mathcal{H}[t, \bar{t}] \mid \bar{t}, \Gamma_2[t, \bar{t}]}{\mathcal{H}[t, \bar{t}] \mid \Gamma_1[t, \bar{t}], t, \bar{t}, \Gamma_2[t, \bar{t}]}$$

By the induction hypothesis twice,

$$\vdash_{\mathbf{GLG}_b} \mathcal{H}[] \mid \Gamma_1[], t \quad \text{and} \quad \vdash_{\mathbf{GLG}_b} \mathcal{H}[] \mid \bar{t}, \Gamma_2[].$$

But then, by Lemma 6.3, $\vdash_{\mathbf{GLG}_b} \mathcal{H}[] \mid \Gamma_1[], \Gamma_2[]$ as required. \square

Lemma 6.5. (CUT) *is \mathbf{GLG}_b -admissible.*

Proof. Suppose that $\vdash_{\mathbf{GLG}_b} \mathcal{G} \mid \Gamma, \Delta$ and $\vdash_{\mathbf{GLG}_b} \mathcal{G} \mid \bar{\Delta}, \Sigma$. Then, using (MIX), also $\vdash_{\mathbf{GLG}_b} \mathcal{G} \mid \Gamma, \Delta, \bar{\Delta}, \Sigma$. So, using Lemma 6.4 repeatedly, $\vdash_{\mathbf{GLG}_b} \mathcal{G} \mid \Gamma, \Sigma$. \square

Structural Rules		
$\frac{}{\Delta, \bar{\Delta}} \text{ (ID)}$	$\frac{\Delta, \Gamma}{\Gamma, \Delta} \text{ (CYCLE)}$	$\frac{\mathcal{G}}{\mathcal{G} \mid \mathcal{H}} \text{ (EW)}$ $\frac{\mathcal{G} \mid \mathcal{H} \mid \mathcal{H}}{\mathcal{G} \mid \mathcal{H}} \text{ (EC)}$
$\frac{\mathcal{G} \mid \Gamma \quad \mathcal{G} \mid \Delta}{\mathcal{G} \mid \Gamma, \Delta} \text{ (MIX)}$ $\frac{\mathcal{G} \mid \Gamma, \Sigma \quad \mathcal{G} \mid \Pi, \Delta}{\mathcal{G} \mid \Gamma, \Delta \mid \Pi, \Sigma} \text{ (COM)}$		
Operational Rules		
$\frac{\mathcal{G} \mid \Gamma, t, s, \Delta}{\mathcal{G} \mid \Gamma, t \cdot s, \Delta} \text{ } (\cdot)$ $\frac{\mathcal{G} \mid \Gamma, \Delta}{\mathcal{G} \mid \Gamma, 1, \Delta} \text{ (1)}$		
$\frac{\mathcal{G} \mid \Gamma, t, \Delta \quad \mathcal{G} \mid \Gamma, s, \Delta}{\mathcal{G} \mid \Gamma, t \wedge s, \Delta} \text{ } (\wedge)$ $\frac{\mathcal{G} \mid \Gamma, t, \Delta}{\mathcal{G} \mid \Gamma, t \vee s, \Delta} \text{ } (\vee_1)$ $\frac{\mathcal{G} \mid \Gamma, s, \Delta}{\mathcal{G} \mid \Gamma, t \vee s, \Delta} \text{ } (\vee_2)$		

Fig. 3. The ℓ -hypersequent calculus GLG .

All the rules of GLG_b are Gl -derivable and, conversely, (EM) is GLG_b -derivable and (CUT) is GLG_b -admissible. Hence, we obtain:

Theorem 6.6. *A basic ℓ -hypersequent \mathcal{G} is GLG_b -derivable if and only \mathcal{G} is ℓ -valid.*

7. A full calculus

In this section, we establish soundness and completeness for an analytic calculus GLG , presented in Fig. 3, that derives all (not just basic) valid ℓ -hypersequents and uses simple initial sequents rather than all group valid sequents.

Example 7.1. The inequation $(x \cdot y) \wedge 1 \leq x \vee y$ can be transformed into an inequation $1 \leq ((\bar{y} \cdot \bar{x}) \vee 1) \cdot (x \vee y)$, which is derived in GLG as follows:

$$\begin{array}{c}
 \frac{\frac{\bar{y}, y}{\bar{y}, y} \text{ (ID)} \quad \frac{\bar{x}, x}{\bar{x}, x} \text{ (ID)}}{\bar{y}, \bar{x}, x \mid y} \text{ (COM)} \\
 \frac{\bar{y}, \bar{x}, x \mid y}{\bar{y} \cdot \bar{x}, x \mid y} \text{ } (\cdot) \\
 \frac{\bar{y} \cdot \bar{x}, x \mid y}{\bar{y} \cdot \bar{x}, x \mid 1, y} \text{ (1)} \\
 \frac{\bar{y} \cdot \bar{x}, x \mid 1, y}{\bar{y} \cdot \bar{x}, x \mid (\bar{y} \cdot \bar{x}) \vee 1, y} \text{ } (\vee_2) \\
 \frac{\bar{y} \cdot \bar{x}, x \mid (\bar{y} \cdot \bar{x}) \vee 1, y}{(\bar{y} \cdot \bar{x}) \vee 1, x \mid (\bar{y} \cdot \bar{x}) \vee 1, y} \text{ } (\vee_1) \\
 \frac{(\bar{y} \cdot \bar{x}) \vee 1, x \mid (\bar{y} \cdot \bar{x}) \vee 1, x \vee y}{(\bar{y} \cdot \bar{x}) \vee 1, x \vee y \mid (\bar{y} \cdot \bar{x}) \vee 1, x \vee y} \text{ } (\vee_2) \\
 \frac{(\bar{y} \cdot \bar{x}) \vee 1, x \vee y \mid (\bar{y} \cdot \bar{x}) \vee 1, x \vee y}{(\bar{y} \cdot \bar{x}) \vee 1, x \vee y} \text{ (EC)} \\
 \frac{(\bar{y} \cdot \bar{x}) \vee 1, x \vee y}{((\bar{y} \cdot \bar{x}) \vee 1) \cdot (x \vee y)} \text{ } (\cdot)
 \end{array}$$

We observe first that the ℓ -sequent rules of GLG together with (MIX) provide a simple calculus for groups.

Lemma 7.2. *The following are equivalent for any basic ℓ -sequent Γ :*

- (1) Γ is group valid.
- (2) Γ is derivable using (ID), (CYCLE), and (MIX).

Proof. (1) \Rightarrow (2) follows by an induction on the length of a group valid basic ℓ -sequent Γ . For the base case, Γ is empty and derivable using (ID). For the induction step, Γ must be of the form $\Gamma_1, t, \bar{t}, \Gamma_2$. But

then Γ_1, Γ_2 is group valid and, by the induction hypothesis, derivable using (ID), (CYCLE), and (MIX). So we obtain a derivation using these rules that ends with

$$\frac{\frac{\frac{\frac{\vdots}{\Gamma_1, \Gamma_2}}{\Gamma_2, \Gamma_1} \text{ (CYCLE)}}{t, \bar{t}} \text{ (ID)}}{t, \bar{t}, \Gamma_2, \Gamma_1} \text{ (MIX)}}{\Gamma_1, t, \bar{t}, \Gamma_2} \text{ (CYCLE)}$$

(2) \Rightarrow (1) follows from the ℓ -soundness of (ID), (CYCLE), and (MIX). \square

Theorem 7.3. *An ℓ -hypersequent \mathcal{G} is ℓ -valid if and only if \mathcal{G} is GLG -derivable.*

Proof. The right-to-left direction follows by a straightforward induction on the height of a GLG -derivation and the ℓ -soundness of the rules. For the left-to-right direction, we first observe that (\cdot) , (1), (\wedge) , and the following GLG -derivable (using (\vee_1) , (\vee_2) , and (EC)) rule

$$\frac{\mathcal{G} \mid \Gamma, t, \Delta \mid \Gamma, s, \Delta}{\mathcal{G} \mid \Gamma, t \vee s, \Delta} \text{ (}\vee\text{)}$$

are all ℓ -invertible. Denote by $\text{mc}(\Gamma)$ the multiset of the lengths of all occurrences of terms in an ℓ -sequent Γ , and by $\text{mc}(\mathcal{H})$, the multiset containing $\text{mc}(\Gamma)$ for each occurrence of a Γ in \mathcal{H} . These multisets of multisets of positive integers are well-ordered by the standard multiset well-ordering of [5]. Moreover, for any premise \mathcal{H}' of an instance of (\cdot) , (1), (\wedge) , and (\vee) with conclusion \mathcal{H} , clearly $\text{mc}(\mathcal{H}')$ is strictly smaller than $\text{mc}(\mathcal{H})$. Hence if \mathcal{G} is ℓ -valid, then, by a straightforward induction on $\text{mc}(\mathcal{G})$, it is GLG -derivable from a finite set of ℓ -valid basic ℓ -hypersequents. By Theorem 6.6, each of these basic ℓ -hypersequents is GLG_b -derivable. But GLG contains the rules (COM), (MIX), (EW), and (EC), and (GV) is GLG -derivable, so these basic ℓ -hypersequents are also GLG -derivable. Hence \mathcal{G} is GLG -derivable. \square

Note also that by adding to GLG an “exchange” rule

$$\frac{\mathcal{G} \mid \Pi, \Delta, \Gamma, \Sigma}{\mathcal{G} \mid \Pi, \Gamma, \Delta, \Sigma} \text{ (EX)}$$

or, alternatively, by reinterpreting ℓ -sequents as multisets of terms, we obtain a one-sided version of the calculus for abelian ℓ -groups introduced in [16].

8. Co-NP completeness

In this last section, we provide a first proof of co-NP completeness for the equational theory of ℓ -groups or, equivalently, the word problem for free ℓ -groups. Hardness is already guaranteed by the fact that the equational theory of distributive lattices is co-NP complete [12]. For inclusion, it suffices (by the reasoning in Section 2) to prove that checking the ℓ -validity of ℓ -hypersequents is in co-NP. Roughly, the idea is to apply the ℓ -sound and ℓ -invertible operational rules (\cdot) , (1), (\wedge) , and (\vee) backwards to reach basic ℓ -hypersequents, and then use the procedure (based on the Holland–McCleary algorithm [11]) described in Section 4 to check ℓ -validity non-deterministically. Note that the derived operational rule (\vee) may lead to an exponential increase in the size of the ℓ -hypersequents considered, but we can avoid this problem by introducing new variables.

Lemma 8.1. *The following rule is ℓ -sound and ℓ -invertible:*

$$\frac{\mathcal{G} \mid \Gamma, x \mid \bar{x}, \Delta}{\mathcal{G} \mid \Gamma, \Delta}$$

where x does not occur in $\mathcal{G} \mid \Gamma, \Delta$.

Proof. The ℓ -invertibility of the rule follows using the ℓ -soundness of (SPLIT) and (SIMP). To establish ℓ -soundness, note first that it suffices to show that the rule is ℓ -sound when Δ is empty. Suppose that this is the case and consider the general situation where the ℓ -hypersequent $\mathcal{G} \mid \Gamma, x \mid \bar{x}, \Delta$ is ℓ -valid and x does not occur in $\mathcal{G} \mid \Gamma, \Delta$. Replacing x with Δ, y where y does not occur in $\mathcal{G} \mid \Gamma, \Delta$, it follows easily that $\mathcal{G} \mid \Gamma, \Delta, y \mid \bar{y}$ is ℓ -valid. But then, by assumption, also $\mathcal{G} \mid \Gamma, \Delta$ is ℓ -valid.

It suffices now to prove that if $\mathcal{G}[x] \mid \bar{x}$ is ℓ -valid, then $\mathcal{G}[1]$ is ℓ -valid, proceeding by induction on the number of operation symbols occurring in $\mathcal{G}[x]$. The induction step is straightforward using the ℓ -sound and ℓ -invertible rules (\wedge), (\vee), (\cdot), and (1). For the base case, $\mathcal{G}[x] \mid \bar{x}$ is an ℓ -valid basic ℓ -hypersequent. By Lemma 5.1, there is an \bar{x} -cut-free $\text{GL}\mathcal{L}\mathcal{G}_b$ -derivation of $\mathcal{G}[x] \mid \bar{x}$. We prove that $\mathcal{G}[1]$ is ℓ -valid by a new induction on the height of such an \bar{x} -cut-free $\text{GL}\mathcal{L}\mathcal{G}_b$ -derivation. For the base case, if $\mathcal{G}[x]$ is ℓ -valid, then so also is $\mathcal{G}[1]$. Otherwise, $\mathcal{G}[x] \mid \bar{x}$ has the form $\mathcal{G}'[x] \mid x \mid \bar{x}$, and clearly $\mathcal{G}'[1] \mid 1$ is ℓ -valid. The induction step is straightforward. \square

Corollary 8.2. *The following rule is ℓ -sound and ℓ -invertible:*

$$\frac{\mathcal{G} \mid \Gamma, x \mid \bar{x}, t, y \mid \bar{x}, s, y \mid \bar{y}, \Delta}{\mathcal{G} \mid \Gamma, t \vee s, \Delta} \quad (\vee')$$

where x and y do not occur in $\mathcal{G} \mid \Gamma, t \vee s, \Delta$.

Theorem 8.3. *The equational theory of $\mathcal{L}\mathcal{G}$ is co-NP complete.*

Proof. As mentioned above, it suffices to prove that checking the ℓ -validity of an ℓ -hypersequent \mathcal{G} is in co-NP. Note first that because the rules (\wedge), (\vee'), (\cdot), and (1) are both ℓ -sound and ℓ -invertible, \mathcal{G} fails to be ℓ -valid if and only if any backwards proof search using these rules leads to a set of basic ℓ -hypersequents, at least one of which is not ℓ -valid. Moreover, the depth of the proof search tree is bounded by the number of occurrences of operational symbols in \mathcal{G} and the sizes of the basic ℓ -hypersequents are linear in the size of \mathcal{G} . We choose non-deterministically one of these basic ℓ -hypersequents \mathcal{G}' .

Now consider the process described in Lemma 4.2 applied to \mathcal{G}' using (*). \mathcal{G}' fails to be ℓ -valid if and only if at least one of the resulting basic ℓ -hypersequents obtained in this way is not ℓ -valid. Moreover, both the depth of the proof search tree and the sizes of the resulting basic ℓ -hypersequents are polynomial in the size of \mathcal{G}' (the rule (*) is applied using sequents of the form $\Gamma, \bar{\Delta}$ where Γ and Δ are initial subsequences of sequents occurring in \mathcal{G}'). We choose non-deterministically one of these basic ℓ -hypersequents \mathcal{H} .

Finally, \mathcal{H} fails to be ℓ -valid if and only if it is chain-consistent and this can be checked in polynomial time in the size of \mathcal{H} ; we may consider it as the problem of checking the satisfiability of an ordering of variables over the real number chain where the number of variables is linear in the size of \mathcal{H} . \square

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