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Distributive residuated frames and generalized bunched implication algebras

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Abstract. We show that all extensions of the (non-associative) Gentzen system for distributive full Lambek calculus by simple structural rules have the cut elimination property. Also, extensions by such rules that do not increase complexity have the finite model property, hence many subvarieties of the variety of distributive residuated lattices have decidable equational theories. For some other extensions, we prove the finite embeddability property, which implies the decidability of the universal theory, and we show that our results also apply to generalized bunched implication algebras. Our analysis is conducted in the general setting of residuated frames.

1. Introduction

Motivation and history. Residuated lattices form algebraic semantics for substructural logics and have been of growing interest in recent years, both because of the interconnections between order-algebra and proof-theory, which their study provides, but also because they are related to areas such as classical algebra, logic, theoretical computer science, philosophy, and mathematical linguistics, to mention a few. In particular, examples of residuated lattices include the ideals of a ring (under the lattice structure, but also including the usual multiplication and division of ideals), lattice-ordered groups, Boolean and Heyting algebras, MV-algebras and relation algebras. On the other hand, substructural logics include, apart from classical logic, intuitionistic, relevance, linear, many-valued, Hajek's basic logic and the logic of bunched implications. An account of residuated lattices and substructural logics can be found in [8]. Among such examples, numerous ones have a distributive lattice base; this paper is concerned with the distributive case.

Distributive residuated lattices appear naturally and also have a simpler representation [6] than general ones. However, some useful methods and techniques already developed do not apply to the distributive case. In particular, relation semantics, known as residuated frames and introduced in [7], have

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turned out to be a very useful tool and provide a very natural setting for the investigation of both algebraic and logical properties in the area [6, 9]. We develop such frames in the distributive case and use them to obtain various results in logic and in algebra.

The study of residuated frames inspired the substructural hierarchy developed in [3, 4, 5], where the third level involving hyper-sequent calculi is also developed. We do not pursue this direction here, but anticipate that distributive frames can serve as a basis of an alternative hierarchy (for distributive varieties only) and that a similar development of distributive hyper-sequent calculi is possible. The benefits are that some axiomatizations that are beyond the third level of the usual hierarchy are now within the first three levels of the distributive hierarchy, so they become amenable to distributive versions of the above results, based on the tools of this paper.

Outline. After defining (distributive) residuated lattices and generalized bunched implication logics, which correspond to variants of full Lambek calculus, we introduce in Section 2 relational semantics for these algebras and logics, which we call distributive residuated frames and which are the main tool of the paper. These are in some sense analogous to Kripke frames for intuitionistic logic, which in turn are based on the result that the underlying lattice is distributive and in the finite case is captured by the poset of join-irreducibles; in this capacity the downsets of Kripke frames yield a class of algebras that generate the variety. However, distributive residuated frames follow the lines of relational semantics for substructural logic, which need not satisfy distributivity and thus are necessarily two sorted (finite lattices can be captured by a polarity relation between the sets of join- and meet-irreducible elements); the resulting lattice-based *Galois* algebra is constructed by this polarity in a way similar to the Dedekind-MacNeille completion of a poset. These frames introduce some redundancy (many different frames can represent the same lattice) and in the distributive case they can be 'folded' into a one-sorted Kripke-like frame; this relationship is discussed in Section 7.

The benefit of the two sorted approach and of the associated redundancies is that it allows us to connect frames directly with proof-theoretic sequent-calculus systems, and via this bridge, import methods of proof theory to the study of relational semantics (for example, as in the proof of the finite model property). At the same time, it allows for transferring algebraic ideas to establish proof-theoretic results via residuated frames (as for example, in the proof of cut elimination). They also yield algebraic constructions, as they come equipped with an algebraic embedding (for example, in the proof of the finite embeddability property). The paper draws much from [7], where the theory of residuated frames is developed, and considers the distributive case. This raises the need for more complicated syntactic terms, but once the correct setting has been established, many of the proofs are analogous to ones in [7] (and are omitted here).

We consider *simple* equations, namely equations in the fragment over $\{\wedge, \vee, \cdot, 1, \top\}$ (in other words, they do not involve the connectives $\backslash, /, \to$) and corresponding conditions on residuated frames, called *simple* conditions. The latter end up being universal strict Horn sentences in the two-sorted language of frames, and we prove that a frame satisfies a simple condition if and only if its dual algebra satisfies the corresponding simple equation. This allows us to identify constructions that produce algebras in a given variety. We show that almost all of our results persist when we consider extensions with such simple equations.

Having defined distributive residuated frames in Section 2, in Section 3 we introduce distributive Gentzen frames, which are expansions with a partial algebra and which satisfy conditions which have a natural algebraic and natural proof-theoretic meaning. We prove that this partial algebra is (quasi)embeddable into the Galois algebra of the frame.

In Section 4 we consider a sequent calculus and define a distributive Gentzen frame from it, where the associated algebra is the free algebra of terms/propositional formulas. The associated map from that algebra to the dual algebra of the frame can be used to show that the cut-rule of the system is redundant, a result that is usually proved syntactically via complicated triple induction. Cut elimination is a very desirable property in proof theory and we prove that it holds also in the presence of *simple* structural rules as they correspond to simple conditions on the frame.

Cut elimination is usually the first step toward decidability (of the equational theory). In Section 5 we show the finite model property (namely the corresponding variety of algebras is generated by its finite members) by considering a modification of the above frame, used for cut elimination. In effect, given an invalid equation/sequent, a counter-model is provided by the Galois algebra; the definition of the frame makes use of all of the unsuccessful attempts (proof-figures) of that sequent that one can construct using the rules of the calculus. The main complication in proving finiteness of the Galois algebra is that there are infinitely many such proof attempts and infinitely many sequents involved in them, due to the presence of the external contraction rule in our system (corresponding to one of the inequalities of idempotency of meet). We undertake a careful investigation of the possible proof-figures and establish permutability results, where one proof can be transformed into another such that some applications of the contraction rule are performed higher up in the proof. This leads to a contraction-controlled proof and finally to only a *finite* number of possible proof-figures that one needs to consider in order to check the validity of a sequent, implying the finiteness of the counter-model. From this, one can also extract a decidability algorithm. The results are again valid for simple extensions with rules that do not increase an appropriate measure of complexity.

For extensions with the equation/rule of integrality $x \leq 1$, we prove a stronger result, the finite embeddability property, in Section 6, which leads to the decidability of the universal theory of our varieties. Given an algebra in our variety and a finite subset of it, we construct a frame whose Galois algebra is still in the variety, it still contains a copy of the finite subset where all the operations inside the subset are computed as before, and further, the Galois algebra is finite. Integrality plays an important role in the proof of finiteness, but once it is present, the addition of further simple equations does not affect the validity of the result. The residuated frame bears some similarities to the one in the proof of the finite model property, but this time it is based on algebraic (as opposed to proof-theoretic) data.

Finally, as mentioned above, in Section 7, we analyze the relationship between the two-sorted (residuated) and the one-sorted (Kripke-like) frames that one may consider, and which form relational semantics for the logics/varieties under investigation.

2. Residuated structures and distributive residuated frames

We start by recalling the definitions of the structures that we study and by developing the main tool of the paper, distributive residuated frames.

Residuated structures. A *residuated lattice* is an algebra of the form $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ where $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid and the following *residuation* property holds for all $x, y, z \in A$

$$xy \leq z \quad \text{iff} \quad x \leq z/y \quad \text{iff} \quad y \leq x\backslash z. \tag{res}$$

An *FL-algebra* is of the form $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ where $(A, \wedge, \vee, \cdot, \backslash, /, 1)$ is a residuated lattice and 0 is an arbitrary element of A. We denote the variety of FL-algebras by FL. The variety of *distributive* FL-algebras, where the lattice reduct is distributive, is denoted by DFL.

A *Brouwerian algebra* is a residuated lattice where multiplication coincides with meet, while a *Heyting algebra* is an FL-algebra with the same property together with the stipulation that 0 is the least element. In such algebras, it turns out that for all elements a, b, we have $a\backslash b = b/a$ and we denote the common value by $a \to b$. Furthermore, it turns out that they have a top element and that this element coincides with 1.

We consider algebras that have two residuated-lattice structures on them, one of them assumed to be of the Brouwerian/Heyting algebra nature. In particular, a *generalized bunched implication algebra*, or *GBI-algebra* for short, is an algebra of the form $(A, \wedge, \vee, \cdot, \backslash, /, \to, 1, \top)$ such that $(A, \wedge, \vee, \top)$ is a lattice with top element $\top$, $(A, \cdot, 1)$ is a monoid, and for all $x, y, z \in A$, we have

$$x \wedge y \leq z \iff y \leq x \to z,$$
$$x \cdot y \leq z \iff y \leq x\backslash z \iff x \leq z/y.$$

Such an algebra is said to be *bounded*, or a *bGBI-algebra*, if the lattice reduct is bounded and the signature is expanded with a constant operation $\bot$ that denotes the least element of the lattice. It is *commutative* if the monoid is commutative. A *BI-algebra* is defined to be a commutative bGBI-algebra. Since the meet operation is residuated by the Heyting arrow $\to$, it follows that meet distributes over all existing joins, hence the lattice is distributive.

BI-algebras, or *bunched implication algebras* are the algebraic models of *bunched implication logic* [13]. This logic is part of separation logic and has received considerable attention in the past two decades in computer science since it is well suited to reasoning about concurrent resources in parallel programs [14]. Our results apply to the commutative as well as the non-commutative version, with or without bottom element. Also, our results apply to non-associative versions of residuated lattices and GBI-algebras.

We will also make use of the following definitions of residuated structures that either lack associativity, unit, or the lattice operations. A *po-groupoid* is a structure $\mathbf{G} = (G, \leq, \cdot)$ where $\leq$ is a partial order on G and the binary operation $\cdot$ is order preserving. A *residuated po-groupoid*, or *rpo-groupoid*, is an expansion $\mathbf{G} = (G, \leq, \cdot, \backslash, /)$ of a po-groupoid, where $\leq$ is a partial order on G and the residuation property (res) holds. If $\leq$ is a lattice order, then $(G, \wedge, \vee, \cdot, \backslash, /)$ is said to be a *$r\ell$-groupoid*, and if this algebra is extended with a constant 1 that is a multiplicative unit, or with an arbitrary constant 0, then it is said to be a *$r\ell u$-groupoid* or a *$r\ell z$-groupoid*, respectively. Note that a residuated lattice is an associative $r\ell u$-groupoid, and an FL-algebra is an associative $r\ell uz$-groupoid.

We will refer to distributive $r\ell u$-groupoids as nDRL-algebras and their expansions with the residual $\to$ of $\wedge$ as nGBI-algebras. (Here 'n' stands for "not necessarily associative".) The variety of all nDRL-algebras (nGBI-algebras) is denoted by nDRL (respectively, nGBI) and the associative subvarieties are denoted by DRL and GBI.

Distributive residuated frames. Given a binary relation $N \subseteq W \times W'$ between two sets, we define

$$X^{\rhd} = \{z \in W' : x \; N \; z \text{ for all } x \in X\} \text{ and}$$
$$Z^{\lhd} = \{x \in W : x \; N \; z \text{ for all } z \in Z\}.$$

It is well known and easy to see that the map γ_N on the powerset $\mathcal{P}(W)$, where $\gamma_N(X) = X^{\rhd\lhd}$, is a closure operator (expansive, monotone, and idempotent), and that every closure operator on a powerset $\mathcal{P}(W)$ is of the form γ_N for some $N \subseteq W \times W'$. Also, the image of γ_N forms a complete lattice, under the operations given by $X \wedge Y = X \cap Y$ and $X \vee_{\gamma_N} Y = \gamma_N(X \cup Y)$, and all complete lattices are of this form (up to isomorphism).

In [7], a similar characterization is given for complete residuated lattices. For the image of γ_N to be a residuated lattice, it is enough for the set W to support an associative ternary relation $\circ$ and a unary relation E that is the unit of $\circ$, and for N to be a *nuclear* relation, namely for every $x, y \in W, z \in W'$, there exist subsets $x \backslash\!\backslash z$ and $z /\!/ y$ of W' such that

$$x \circ y \, N \, z \quad \text{iff} \quad y \, N \, x \backslash\!\backslash z \quad \text{iff} \quad x \, N \, z /\!/ y. \qquad [\text{nuc}\circ]$$

The corresponding condition for γ_N is that it is a nucleus. In general, a map γ on a po-groupoid $\mathbf{G}$ is called a *nucleus* if it is a closure operator such that $\gamma(x) \cdot \gamma(y) \leq \gamma(x \cdot y)$ for all $x, y \in G$.

For a ternary relation $\circ$, we write $X \circ Y$ for

$$\{w \in W : (x, y, w) \in \circ, \text{ for some } x \in X, \, y \in Y\}$$

and $x \circ y$ for $\{x\} \circ \{y\}$. The relation is said to be *associative* if it satisfies $(x \circ y) \circ z = x \circ (y \circ z)$, i.e., if it satisfies the following equivalence

$$\exists u[(x, y, u) \in \circ \text{ and } (u, z, w) \in \circ] \quad \text{iff} \quad \exists v[(x, v, w) \in \circ \text{ and } (y, z, v) \in \circ],$$

and to *have unit* $E \subseteq W$ if $x \circ E = \{x\} = E \circ x$, i.e., if

$$\exists e \in E[(x, e, y) \in \circ] \quad \text{iff} \quad x = y \quad \text{iff} \quad \exists e \in E[(e, x, y) \in \circ].$$

The additional operations on the image of γ_N that provide the residuated-lattice structure are

$$X \circ_{\gamma_N} Y = \gamma_N(X \circ Y), X/Y = \{z : \{z\} \circ Y \subseteq X\}, Y \backslash X = \{z : Y \circ \{z\} \subseteq X\},$$

and $1 = \gamma_N(E)$. Also, every complete residuated lattice is (isomorphic to one) of this form; see [7] for details. We proceed to present a similar characterization for the distributive case.

Given a lattice expansion $\mathbf{L} = (L, \wedge, \vee, \curlywedge)$, a nucleus γ on L (with respect to $\curlywedge$) is called *distributive* if it satisfies $\gamma(x \curlywedge y) = \gamma(x) \wedge \gamma(y)$.

Lemma 2.1. *Let $\mathbf{L} = (L, \wedge, \vee, \curlywedge)$ be a lattice expansion and γ a distributive $\curlywedge$-nucleus on $\mathbf{L}$. Then $\curlywedge_\gamma = \wedge$ on the image L_γ of γ. If, furthermore, $\curlywedge$ is a residuated operation on $\mathbf{L}$, then $\mathbf{L}_\gamma$ is distributive.*

Proof. As γ is a $\curlywedge$-nucleus on $\mathbf{L}$, we have $\gamma(\gamma(x) \curlywedge \gamma(y)) = \gamma(x \curlywedge y)$ for all $x, y \in L$. So, $\gamma(x) \curlywedge_\gamma \gamma(y) = \gamma(x) \wedge \gamma(y)$ since $\curlywedge$ is a distributive nucleus. Thus, for $x, y \in L_\gamma$, $x \curlywedge_\gamma y = x \wedge y$, namely $\curlywedge_\gamma = \wedge$. Moreover, since $\curlywedge_\gamma$ is a residuated operation on $\mathbf{L}_\gamma$, the latter is distributive. □

Corollary 2.2. *Let $\curlywedge$ be a ternary relation on a set W and γ a distributive $\curlywedge$-nucleus on $\mathcal{P}(W)$. Then $\mathcal{P}(W)_\gamma$ is distributive and it satisfies $\curlywedge_\gamma = \cap$.*

Proof. Clearly, $\mathcal{P}(W)$ is a complete lattice and $\curlywedge$ distributes over arbitrary unions, so $\curlywedge$ is residuated on $\mathcal{P}(W)$. □

For a ternary relational structure $\curlywedge$ on a set W, a relation $N \subseteq W \times W'$ is called *distributively nuclear* if it is nuclear with respect to $\curlywedge$, i.e., for all $x, y \in W, z \in W'$, there exist subsets denoted $x \curlyeqsucc z, z \curlyeqprec y$ of W' such that

$$x \curlywedge y \, N \, z \quad \text{iff} \quad y \, N \, x \curlyeqsucc z \quad \text{iff} \quad x \, N \, z \curlyeqprec y, \qquad [\text{nuc}\curlywedge]$$

and it satisfies the following conditions of associativity, exchange, integrality, and contraction (to be read as downward implications, with the first one being a bi-implication):

$$\frac{x \curlywedge (y \curlywedge w)\ N\ z}{(x \curlywedge y) \curlywedge w\ N\ z}\ [\curlywedge a] \qquad \frac{x \curlywedge y\ N\ z}{y \curlywedge x\ N\ z}\ [\curlywedge e]$$

$$\frac{x\ N\ z}{x \curlywedge y\ N\ z}\ [\curlywedge i] \qquad \frac{x \curlywedge x\ N\ z}{x\ N\ z}\ [\curlywedge c]$$

Note that $[\curlywedge e]$ can be replaced by

$$\frac{y\ N\ z}{x \curlywedge y\ N\ z}\ [\curlywedge i_\ell]$$

in view of the following two derivations:

$$\cfrac{\cfrac{y\ N\ z}{y \curlywedge x\ N\ z}\ [\curlywedge i]}{x \curlywedge y\ N\ z}\ [\curlywedge e] \qquad \cfrac{\cfrac{x \curlywedge y\ N\ z}{(y \curlywedge x) \curlywedge (y \curlywedge x)\ N\ z}\ [\curlywedge i], [\curlywedge i_\ell], [\mathrm{nuc}\curlywedge]}{y \curlywedge x\ N\ z}\ [\curlywedge c]$$

Lemma 2.3. *Given a set W, a ternary relational structure $\curlywedge$ on W and $N \subseteq W \times W'$, we have that γ_N is a distributive nucleus on $\mathcal{P}(W, \curlywedge)$ iff N is a distributively nuclear relation.*

Proof. Given the correspondence between nuclei and nuclear relations, it is enough to show that the distributivity conditions correspond. For brevity, we write γ_N simply as γ. The distributivity condition $\gamma(X \curlywedge Y) = \gamma(X) \cap \gamma(Y)$ for γ is equivalent to the inequalities $\gamma(X) \cap \gamma(Y) \subseteq \gamma(X \curlywedge Y)$, $\gamma(X \curlywedge Y) \subseteq \gamma(X)$, and $\gamma(X \curlywedge Y) \subseteq \gamma(Y)$.

By basic properties of $^{\lhd}$ and $^{\rhd}$, we can see that the inclusion $\gamma(X \curlywedge Y) \subseteq \gamma(X)$ is equivalent to $X^{\rhd} \subseteq (X \curlywedge Y)^{\rhd}$, namely to the condition that for all $z \in W'$, $X\ N\ z$ implies $(X \curlywedge Y)\ N\ z$. Specializing this to singletons yields $(\curlywedge i)$. Conversely, for all $z \in W'$, if $X\ N\ z$, then $x\ N\ z$ for all $x \in X$, hence $x \curlywedge y\ N\ z$ for all $x \in X$ and $y \in Y$, by $(\curlywedge i)$; so $(X \curlywedge Y)\ N\ z$.

Note that $\gamma(X) \cap \gamma(Y) \subseteq \gamma(X \curlywedge Y)$ is equivalent to $\gamma(X) \subseteq \gamma(X \curlywedge X)$. The forward direction follows by choosing $Y = X$. For the converse direction, using twice the fact that γ is a $\curlywedge$-nucleus, we have

$$\begin{aligned}\gamma(X) \cap \gamma(Y) &\subseteq \gamma(\gamma(X) \cap \gamma(Y)) \subseteq \gamma([\gamma(X) \cap \gamma(Y)] \curlywedge [\gamma(X) \cap \gamma(Y)]) \\ &\subseteq \gamma(\gamma(X) \curlywedge \gamma(Y)) = \gamma(X \curlywedge Y).\end{aligned}$$

Now, $\gamma(X) \subseteq \gamma(X \curlywedge X)$ is equivalent to $(X \curlywedge X)^{\rhd} \subseteq X^{\rhd}$, namely to the condition that for all $z \in W'$, $(X \curlywedge X)\ N\ z$ implies $X\ N\ z$. Specializing this to singletons yields $(\curlywedge c)$. Conversely, for all $z \in W'$, if $(X \curlywedge X)\ N\ z$, then $x \curlywedge x\ N\ z$ for all $x \in X$, hence $x\ N\ z$ for all $x \in X$; so $X\ N\ z$. ☐

A (*distributive*) *residuated frame* is a structure of the form

$$\mathbf{W} = (W, W', N, \circ, \backslash\!\backslash, /\!/, \curlywedge, \leftthreetimes, \rightthreetimes),$$

where $\circ$ and $\curlywedge$ are ternary relations on W, $N \subseteq W \times W'$ is $\circ$-nuclear with respect to $\backslash\!\backslash$ and $/\!/$, and is distributively $\curlywedge$-nuclear with respect to $\leftthreetimes$ and $\rightthreetimes$.

It follows that the image $\mathbf{W}^+ = (\gamma_N[\mathcal{P}(W)], \cap, \cup_{\gamma_N}, \circ_{\gamma_N}, \backslash, /)$ of γ_N is a distributive $r\ell$-groupoid, called the *Galois algebra of* $\mathbf{W}$, and denoted by $\mathbf{W}^+$. The (*bunched*) *Galois algebra* $\mathbf{W}^{+\!\!\!\!+}$ is the expansion of this (non-associative) residuated lattice with two operations: $X \to Y = \{z : X \curlywedge z \subseteq Y\}$ and $\top = W$, and it is an nGBI-algebra. Our results will hold for both constructions, so we will use the first notation for both of them in most of the paper.

An *associative* frame is such that $\circ_\gamma$ is associative, a *unital* distributive residuated frame is an expansion of a distributive residuated frame with a set $E \subseteq W$ such that $1 := \gamma_N(E)$ is a unit for $\circ_\gamma$, and a distributive residuated *zero* frame is an expansion with a distinguished subset $D \subseteq W$ as interpretation for the constant 0. The first two conditions are respectively equivalent to

- $[(x \circ y) \circ z]^\rhd = [x \circ (y \circ z)]^\rhd$; (associativity)
- $(x \circ E)^\rhd = \{x\}^\rhd = (E \circ x)^\rhd$, for all $x \in W$. (unit)

Below, we refer to these various frames simply as (distributive) residuated frames and often suppress the adjective 'bunched' before 'Galois algebra'.

To provide an example, given a GBI-algebra $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, \to, 1, \top)$, define the residuated frame $\mathbf{W_A} = (A, A, \leq, \cdot, \backslash, /, \wedge, \to, \leftarrow, \{1\}, \{\top\})$, where $\cdot, \backslash, /, \wedge, \to$ are considered as ternary relations (e.g., $\cdot(x, y, z) \iff x \cdot y = z$), and $\leftarrow(x, y, z) \iff y \to x = z$). It is easy to see that $\mathbf{W}_\mathbf{A}^{+\!\!\!\!+}$ is based on the Dedekind-MacNeille completion of the lattice reduct of A, hence if A is complete (e.g., finite), then the Galois algebra is isomorphic to $\mathbf{A}$. Moreover, it follows from the next section that $\mathbf{A}$ embeds into $\mathbf{W}_\mathbf{A}^{+\!\!\!\!+}$ as a GBI-algebra.

As a second example, if $\mathbf{A} = (A, \wedge, \vee, \cdot, \backslash, /, 1)$ is a distributive residuated lattice, then we define the distributive residuated frame $\mathbf{W_A}$ by setting $x \to y$ equal to $\{z : x \wedge z \leq y\} = y \leftarrow x$. Then the Galois algebra is still a completion of $\mathbf{A}$, but it may not be the Dedekind-MacNeille completion of $\mathbf{A}$ since the Galois algebra contains all residuals for the meet operation, while the Dedekind-MacNeille completion adds joins and meets only for subsets that do not have one in $\mathbf{A}$. So if binary meet does not distribute over some existing infinite join, then the Galois algebra will contain an extra element, but the Dedekind-MacNeille completion will not.

Alternatively, given a distributive residuated lattice $\mathbf{A}$, we can define $\mathbf{W}'_\mathbf{A} = (A, S_A \times A, N, \cdot, \backslash\!\backslash, /\!/, \wedge, \leftthreetimes, \rightthreetimes, \{1\})$ where S_A is the set of all polynomials of $(A, \cdot, \wedge)$ that have a single variable and which appears only once, usually denoted by $u = u(\_)$. Also, the relation N is defined by $x\ N\ (u, b)$ iff $u(x) \leq b$, and where $x \backslash\!\backslash (u, b) = (u(x \cdot \_), b)$, $(u, b) /\!/ y = (u(\_ \cdot y), b)$, $x \leftthreetimes (u, b) = (u(x \wedge \_), b)$, and $(u, b) \rightthreetimes y = (u(\_ \wedge y), b)$. It will follow that $\mathbf{A}$ embeds into $\mathbf{W}_\mathbf{A}^+$ as a distributive residuated lattice. Such embeddings are the main focus of Section 3.

Simple conditions and equations. Let $t_0, t_1, \ldots, t_n$ be elements of the free bi-unital bigroupoid in the signature $\{\circ, \varepsilon, \curlywedge, \delta\}$ over a countable set of variables, with t_0 a linear term (every variable appears once), and let $\mathbf{W}$ be a distributive frame. Here, similar to the definition of a term function, $t_i^{\mathbf{W}}$ denotes the function from W to $\mathcal{P}(W)$ induced by t_i. Also, in the following if X is a set, then $X\ N\ y$ means that $x\ N\ y$ for all $x \in X$.

A *simple* condition is an implication (the assumptions are read conjunctively) of the form

$$\frac{t_1\ N\ q \quad \cdots \quad t_n\ N\ q}{t_0\ N\ q}\ [\mathrm{r}]$$

where q is a variable not occurring in $t_0, t_1, \ldots, t_n$. For example, $\curlywedge$-exchange $[\curlywedge e]$, $\curlywedge$-contraction $[\curlywedge c]$, $\curlywedge$-integrality $[\curlywedge i]$, $\curlywedge$-associativity $[\curlywedge a]$, and $\circ$-associativity $[\circ a]$ are simple structural conditions, and so is

$$\frac{x \circ (y_1 \curlywedge y_2)\ N\ z}{(x \circ y_1) \curlywedge (x \circ y_1)\ N\ z}\ [\mathrm{mdm}]$$

where [mdm] stands for "multiplication distributes over meet".

We say that $\mathbf{W}$ satisfies condition [r] if for all $z \in W'$, and for all sequences $\bar{x}$ of elements of W matching the variables involved in $t_0, t_1, \ldots, t_n$, the conjunction of the conditions $t_i^{\mathbf{W}}(\bar{x})\ N\ z$, for $i \in \{1, \ldots, n\}$, implies $t_0^{\mathbf{W}}(\bar{x})\ N\ z$.

Note that in the multi-sorted first-order language of $\mathbf{W}$, the only predicate symbol is the relation N, terms in the first sort are elements of the above free bi-unital bigroupoid, and terms in the second sort are repeated applications of terms of the first sort as denominators in $\backslash\!\backslash$ and $/\!/$, as well as in $\curlyvee$ and $\curlywedge$, on (eventually) variables of the second sort, for example $t_1 \backslash\!\backslash (t_2 \curlyvee (q /\!/ t_3))$. However, given the nuclear property of N, the most general atomic formulas are of the form $t\ N\ q$, where t is a biunital bigroupoid term and q is a variable (or ε or δ, if we assume that 0 or $\bot$ are in the type); for example, $t_4\ N\ t_1 \backslash\!\backslash (t_2 \curlyvee (q /\!/ t_3))$ is equivalent to $(t_2 \curlywedge (t_1 \circ t_4)) \circ t_3\ N\ q$. It is then clear that simple conditions are exactly the strict universal Horn formulas in this language, with the restriction of linearity of t_0. The latter restriction is not essential and any strict universal Horn formula can be converted into such a linearized one, as essentially follows from the analysis below.

Note that the condition [r] and the inequality $\varepsilon = (t_0 \leq t_1 \vee \cdots \vee t_n)$ are interdefinable. We denote by $\varepsilon(\mathrm{r})$ the inequality corresponding to the above condition and by $\mathrm{R}(\varepsilon)$ the condition corresponding to the above inequality. Such inequalities are called *simple*. For example, the inequality corresponding to [mdm] is $xy_1 \wedge xy_2 \leq x(y_1 \wedge y_2)$.

In nGBI and nDRL, every equation ε over $\{\wedge, \vee, \cdot, 1, \top\}$ is equivalent to a conjunction of inequalities of the form above. To show this, we distribute all products and meets over all joins to reach a form $s_1 \vee \cdots \vee s_m = t_1 \vee \cdots \vee t_n$, where s_i, t_j are unital bi-groupoid terms. Such an equation is in turn equivalent to the conjunction of the two inequalities $s_1 \vee \cdots \vee s_m \leq t_1 \vee \cdots \vee t_n$ and $t_1 \vee \cdots \vee t_n \leq s_1 \vee \cdots \vee s_m$. Finally, the first one is equivalent to the conjunctions of the inequalities $s_j \leq t_1 \vee \cdots \vee t_n$. Likewise, the second inequality is written as a conjunction, as well.

We now rewrite each of the conjuncts, say $s \leq t_1 \vee \cdots \vee t_n$, in a form for which s is a linear term. For each variable x that appears $k > 1$ times in s, we replace each occurrence of x in the equation by $x_1 \vee x_2 \vee \cdots \vee x_k$, where $x_1, \ldots, x_k$ are variables that do not occur in $s \leq t_1 \vee \cdots \vee t_n$. As multiplication and meet distribute over join, the new equation can be written in the form $s'_1 \vee \cdots \vee s'_p \leq t'_1 \vee \cdots \vee t'_q$, where all the terms are obtained from variables by taking products and meets. Let s'_l be one of the $k!$-many linear terms among $s'_1, \ldots, s'_p$. The last equation clearly implies the equation $s'_l \leq t'_1 \vee \cdots \vee t'_q$, but it is actually equivalent to it, as the latter implies $s \leq t_1 \vee \cdots \vee t_n$ by setting all duplicate copies of each variable equal to each other. For example, if the equation to be linearized is $x^2 \wedge y \leq (x \wedge y) \vee yx$, then we get, successively:

$$(x_1 \vee x_2)^2 \wedge y \leq [(x_1 \vee x_2) \wedge y] \vee y(x_1 \vee x_2),$$
$$(x_1^2 \wedge y) \vee (x_1x_2 \wedge y) \vee (x_2x_1 \wedge y) \vee (x_2^2 \wedge y) \leq (x_1 \wedge y) \vee (x_2 \wedge y) \vee yx_1 \vee yx_2,$$
$$x_1x_2 \wedge y \leq (x_1 \wedge y) \vee (x_2 \wedge y) \vee yx_1 \vee yx_2,$$
$$x_1 \wedge y \leq v \;\;\&\;\; x_2 \wedge y \leq v \;\;\&\;\; yx_1 \leq v \;\;\&\;\; yx_2 \leq v \Longrightarrow x_1x_2 \wedge y \leq v,$$

and the simple condition that corresponds to it is:

$$\frac{x_1 \curlywedge y \; N \; z \quad x_2 \curlywedge y \; N \; z \quad y \circ x_1 \; N \; z \quad y \circ x_2 \; N \; z}{(x_1 \circ x_2) \curlywedge y \; N \; z} \; R(\varepsilon)$$

Given an equation ε, let $\mathrm{R}(\varepsilon)$ denote the set of conditions associated with each of these conjuncts (inequalities) obtained from ε in the way described above.

En route to transforming simple conditions to equations over $\{\wedge, \vee, \cdot, 1\}$ and vice versa, we established the following theorem, whose proof is an easy adaptation of the corresponding proof in [7].

Theorem 2.4.

(1) *Every equation over* $\{\wedge, \vee, \cdot, 1, \top\}$ *is equivalent to a conjunction of simple equations.*

(2) *Every equation* ε *over* $\{\wedge, \vee, \cdot, 1, \top\}$ *is equivalent to* $\mathrm{R}(\varepsilon)$*, relative to* nGBI*. More precisely, for every* $\mathbf{G}$ *in* nGBI*,* $\mathbf{G}$ *satisfies* ε *iff* $\mathbf{W_G}$ *satisfies* $\mathrm{R}(\varepsilon)$*.*

(3) *Let* $\mathbf{W}$ *be a distributive residuated frame and let* ε *be an equation over* $\{\wedge, \vee, \cdot, 1, \top\}$*. Then* $\mathbf{W}$ *satisfies* $\mathrm{R}(\varepsilon)$ *iff* $\mathbf{W}^+$ *satisfies* ε *iff* $\mathbf{W}^+$ *satisfies* $\mathrm{R}(\varepsilon)$*.*

We say that a set R of conditions is *preserved by* $(\_)^+$ if for every distributive residuated frame $\mathbf{W}$, if $\mathbf{W}$ satisfies R, then $\mathbf{W}^+$ satisfies R. The next corollary follows directly from Theorem 2.4.

Corollary 2.5. *All simple conditions are preserved by* $(\_)^+$*.*

For example, the conditions of $[\curlywedge e]$, $[\circ a]$ and [mdm] are preserved by $(\_)^+$.

3. Proof theory as inspiration for Gentzen frames

In this section, we develop a theory parallel to that of [7], where we draw inspiration from proof theory and consider expansions of a distributive residuated frame with a (partial) algebra and provide conditions under which there is a natural embedding (or some more general map) from that (partial) algebra into the Galois algebra of the residuated frame.

The sequent calculi GBI and DRL. We write Fm for the algebra of terms (over some fixed countable set of variables) in the language of residuated lattices. These terms also serve as propositional formulas in the associated substructural logic. Let $(Fm^\circ, \circ, \curlywedge, \varepsilon)$ be the free unital bi-groupoid generated by the set Fm, namely ε is a unit for $\circ$; often we expand this to a bi-unital bi-groupoid by adding a constant δ which serves as a unit for $\curlywedge$, and in this case we take Fm to be all formulas over GBI-algebras. We will be lax about this and use $\mathbf{Fm}^\circ$ to denote either one of these structures.

S_{Fm° denotes the set of unary linear polynomials of Fm°, namely unary polynomials obtained from terms where the variable occurs exactly once. We write $u(x)$ for the value of the polynomial u at x, and we also write $u(\_)$ for u itself; for example, we write $\_ \circ y$ for the polynomial u defined by $u(x) = x \circ y$. The basic object of the forthcoming logical system is a *sequent*, namely a pair $(x, b) \in Fm^\circ \times Fm$, traditionally written $x \Rightarrow b$. A *sequent rule* is a pair $(\{s_1, \dots, s_n\}, s_0)$ where $s_0, \dots, s_n$ are sequents and is presented in the form

$$\frac{s_1 \quad s_2 \quad \cdots \quad s_n}{s_0} \qquad \text{or} \qquad \frac{}{s_0}$$

with rules of the latter form referred to as *axioms* for $n = 0$; we call $s_1, \dots, s_n$ the *assumptions* or *premises* of the rule and s_0 its *conclusion*. Finally, a *Gentzen system* is a set of sequent rules.

We will consider the Gentzen system **nGBI** for non-commutative, non-associative bunched implication logic, given by the rules (or rule schemes) in Figure 1 and all their uniform substitution instances (i.e., a, b, c range over Fm, x, y range over Fm° and u ranges over S_{Fm°). A double horizontal line indicates that the rule can be applied in both directions. The name of a particular sequent rule is listed after the rule in parentheses. We also consider its associative

$$\frac{u(x \circ (y \circ z)) \Rightarrow c}{\overline{u((x \circ y) \circ z) \Rightarrow c}} \ (\circ a)$$

version **GBI**, as well as the fragment **DRL** of **GBI** that does not contain $\to$ and $\top$. Systems that are lower-bounded contain the additional rules

$$\frac{}{u(\bot) \Rightarrow a} \ (\bot \mathrm{L}) \qquad \frac{x \Rightarrow \delta}{x \Rightarrow \bot} \ (\bot \mathrm{R})$$

We will let $\mathbf{L}$ denote any one of those systems, since our results apply to all of them, as well as numerous extensions and extensions of fragments.

$$\frac{x \Rightarrow a \quad u(a) \Rightarrow c}{u(x) \Rightarrow c}\ (\text{CUT}) \qquad \frac{}{a \Rightarrow a}\ (\text{Id}) \qquad \frac{u(x \curlywedge (y \curlywedge z)) \Rightarrow c}{u((x \curlywedge y) \curlywedge z) \Rightarrow c}\ (\curlywedge a)$$

$$\frac{u(x \curlywedge y) \Rightarrow c}{u(y \curlywedge x) \Rightarrow c}\ (\curlywedge e) \qquad \frac{u(x) \Rightarrow c}{u(x \curlywedge y) \Rightarrow c}\ (\curlywedge i) \qquad \frac{u(x \curlywedge x) \Rightarrow c}{u(x) \Rightarrow c}\ (\curlywedge c)$$

$$\frac{x \Rightarrow a \quad u(b) \Rightarrow c}{u(x \circ (a\backslash b)) \Rightarrow c}\ (\backslash\text{L}) \qquad \frac{a \circ x \Rightarrow b}{x \Rightarrow a\backslash b}\ (\backslash\text{R}) \qquad \frac{x \Rightarrow a \quad u(b) \Rightarrow c}{u((b/a) \circ x) \Rightarrow c}\ (/\text{L}) \qquad \frac{x \circ a \Rightarrow b}{x \Rightarrow b/a}\ (/\text{R})$$

$$\frac{u(a \circ b) \Rightarrow c}{u(a \cdot b) \Rightarrow c}\ (\cdot\text{L}) \qquad \frac{x \Rightarrow a \quad y \Rightarrow b}{x \circ y \Rightarrow a \cdot b}\ (\cdot\text{R}) \qquad \frac{u(\varepsilon) \Rightarrow a}{u(1) \Rightarrow a}\ (1\text{L}) \qquad \frac{}{\varepsilon \Rightarrow 1}\ (1\text{R})$$

$$\frac{u(a) \Rightarrow c \quad u(b) \Rightarrow c}{u(a \vee b) \Rightarrow c}\ (\vee\text{L}) \qquad \frac{x \Rightarrow a}{x \Rightarrow a \vee b}\ (\vee\text{R}\ell) \qquad \frac{x \Rightarrow b}{x \Rightarrow a \vee b}\ (\vee\text{R}r)$$

$$\frac{u(a \curlywedge b) \Rightarrow c}{u(a \wedge b) \Rightarrow c}\ (\wedge\text{L}) \qquad \frac{x \Rightarrow a \quad x \Rightarrow b}{x \Rightarrow a \wedge b}\ (\wedge\text{R})$$

$$\frac{x \Rightarrow a \quad u(b) \Rightarrow c}{u(x \curlywedge (a \to b)) \Rightarrow c}\ (\to\text{L}) \qquad \frac{x \curlywedge a \Rightarrow b}{x \Rightarrow a \to b}\ (\to\text{R}) \qquad \frac{u(\delta) \Rightarrow c}{u(\top) \Rightarrow c}\ (\top\text{L}) \qquad \frac{}{x \Rightarrow \top}\ (\top\text{R})$$

Figure 1. The systems **nGBI**, **GBI**, **nDRL** and **DRL**.

A *proof* in **L** is defined inductively as an (upward growing) tree in the usual way, where the proved sequent is at the bottom. If there is a proof of a sequent s in **L** from assumptions S, then we write $S \vdash_{\mathbf{L}} s$ and say that s is *provable in* **L** *from* S. If S is empty we simply write $\vdash_{\mathbf{L}} s$ and say that s is provable in **L**.

Note that the rules

$$\frac{u(a) \Rightarrow c}{u(a \wedge b) \Rightarrow c}\ (\wedge\text{L}\ell) \qquad \frac{u(b) \Rightarrow c}{u(a \wedge b) \Rightarrow c}\ (\wedge\text{L}r)$$

are derivable in **L**. Indeed,

$$\cfrac{\cfrac{u(a) \Rightarrow c}{u(a \curlywedge b) \Rightarrow c}\ (\curlywedge i)}{u(a \wedge b) \Rightarrow c}\ (\wedge\text{L}) \qquad \cfrac{\cfrac{u(b) \Rightarrow c}{u(a \curlywedge b) \Rightarrow c}\ (\curlywedge i_\ell)}{u(a \wedge b) \Rightarrow c}\ (\wedge\text{L})$$

We take $W = Fm^\circ$ and $W' = S_W \times Fm$, where S_W is the set of all unary linear polynomials in W, and define the relation N by

$$x\ N\ (u, a) \quad \text{iff} \quad \vdash_{\mathbf{L}} (u(x) \Rightarrow a).$$

Then

$$x \circ y\ N\ (u, a) \text{ iff } \vdash_{\mathbf{L}} u(x \circ y) \Rightarrow a \text{ iff } x\ N\ (u(\_ \circ y), a) \text{ iff } y\ N\ (u(x \circ \_), a),$$
$$x \curlywedge y\ N\ (u, a) \text{ iff } \vdash_{\mathbf{L}} u(x \curlywedge y) \Rightarrow a \text{ iff } x\ N\ (u(\_ \curlywedge y), a) \text{ iff } y\ N\ (u(x \curlywedge \_), a).$$

Hence, N is a nuclear relation with respect to both $\circ$ and $\curlywedge$, where the appropriate subsets of W' are given by

$$(u,a) /\!\!/ x = \{(u(\_ \circ x), a)\} \ \text{ and } \ x \backslash\!\!\backslash (u,a) = \{(u(x \circ \_), a)\},$$
$$(u,a) \leftthreetimes x = \{(u(\_ \curlywedge x), a)\} \ \text{ and } \ x \rightthreetimes (u,a) = \{(u(x \curlywedge \_), a)\}.$$

We denote the resulting distributive residuated frame by $\mathbf{W_L}$.

We say that an nGBI-algebra $\mathbf{G}$ *satisfies* the sequent $x \Rightarrow a$, or that the sequent *holds* or is *valid* in $\mathbf{G}$, if for every homomorphism $f\colon \mathbf{Fm} \to \mathbf{G}$, $f(x^{\mathbf{Fm}}) \leq f(a)$. Here $x^{\mathbf{Fm}}$ denotes the formula obtained from x by replacing $\circ$ with $\cdot$, ε with 1, and $\curlywedge$ with $\wedge$. It is easy to see that **nGBI** is sound with respect to the variety of nGBI-algebras. The proof proceeds by induction on the rules (and axioms) of **nGBI**. For (\L) and (/L) we use the monotonicity of $\cdot$ and $\wedge$, while for ($\vee$L) we use the distributivity of $\wedge$ over $\vee$. We will show that the converse is also true, i.e., nGBI-algebras provide a complete semantics.

Gentzen frames. A *distributive Gentzen ru-frame of type* $\mathcal{L}$ for $\{\cdot, 1, \wedge\} \subseteq \mathcal{L}$ is a pair $(\mathbf{W}, \mathbf{B})$ where

(i) $\mathbf{W} = (W, W', N, \circ, \backslash\!\!\backslash, /\!\!/, \{\varepsilon\}, \curlywedge, \rightthreetimes, \leftthreetimes)$ is a distributive *ru*-frame, where $\circ$ and $\curlywedge$ are binary *operations*,

(ii) $\mathbf{B}$ is a partial $\mathcal{L}$-algebra,

(iii) $(W, \circ, \varepsilon, \curlywedge)$ is a bi-groupoid with unit for $\circ$ generated by $B \subseteq W$,

(iv) there is an injection of B into W' (under which we will identify B with a subset of W') and

(v) N satisfies the $\mathcal{L}$-conditions of **nGBIN** (Figure 2) for all $a, b \in B$, $x, y \in W$, and $z \in W'$.

Note that the names of Gentzen frame conditions are enclosed in *square brackets* to distinguish them from the corresponding sequent rule names (in parentheses). A condition is understood to hold only in case all the expressions in it make sense. For example, [$\wedge$L] is read as: if $a, b, a \wedge b \in B$, $z \in W'$, and $a \curlywedge b \; N \; z$, then $a \wedge b \; N \; z$.

We note that condition [\L] is, by [nuc$\curlywedge$], equivalent to

$$\frac{x \; N \; a \quad b \; N \; z}{x \circ (a \backslash b) \; N \; z}$$

A *distributive Gentzen ruz-frame* is a distributive Gentzen *ru*-frame extended with the set $\{\varepsilon\}^{\lhd}$, and (iv),(v) are modified as follows:

(iv′) there is an injection of $B \cup \{\varepsilon\}$ into W' (under which we will identify $B \cup \{\varepsilon\}$ with a subset of W') and

(v′) N satisfies the conditions of **nGBIN** (Figure 2) for all $a, b \in B$, $x, y \in W$, and $z \in W'$, as well as the two conditions

$$\frac{x \; N \; \varepsilon}{x \; N \; 0} \, [0\text{R}] \qquad \frac{}{0 \; N \; \varepsilon} \, [0\text{L}]$$

$$\frac{x\ N\ a \quad a\ N\ z}{x\ N\ z}\ [\mathrm{CUT}] \qquad \frac{}{a\ N\ a}\ [\mathrm{Id}] \qquad \frac{x \curlywedge (y \curlywedge w)\ N\ z}{(x \curlywedge y) \curlywedge w\ N\ z}\ [\curlywedge a]$$

$$\frac{x \curlywedge y\ N\ z}{y \curlywedge x\ N\ z}\ [\curlywedge e] \qquad \frac{x\ N\ z}{x \curlywedge y\ N\ z}\ [\curlywedge i] \qquad \frac{x \curlywedge x\ N\ z}{x\ N\ z}\ [\curlywedge c]$$

$$\frac{x\ N\ a \quad b\ N\ z}{a\backslash b\ N\ x\backslash\!\!\backslash z}\ [\backslash \mathrm{L}] \qquad \frac{x\ N\ a\backslash\!\!\backslash b}{x\ N\ a\backslash b}\ [\backslash \mathrm{R}] \qquad \frac{x\ N\ a \quad b\ N\ z}{b/a\ N\ z/\!\!/x}\ [/\mathrm{L}] \qquad \frac{x\ N\ b/\!\!/a}{x\ N\ b/a}\ [/\mathrm{R}]$$

$$\frac{a \circ b\ N\ z}{a \cdot b\ N\ z}\ [\cdot \mathrm{L}] \qquad \frac{x\ N\ a \quad y\ N\ b}{x \circ y\ N\ a \cdot b}\ [\cdot \mathrm{R}] \qquad \frac{\varepsilon\ N\ z}{1\ N\ z}\ [1\mathrm{L}] \qquad \frac{}{\varepsilon\ N\ 1}\ [1\mathrm{R}]$$

$$\frac{a\ N\ z \quad b\ N\ z}{a \vee b\ N\ z}\ [\vee \mathrm{L}] \qquad \frac{x\ N\ a}{x\ N\ a \vee b}\ [\vee \mathrm{R}\ell] \qquad \frac{x\ N\ b}{x\ N\ a \vee b}\ [\vee \mathrm{R}r]$$

$$\frac{a \curlywedge b\ N\ z}{a \wedge b\ N\ z}\ [\wedge \mathrm{L}] \qquad \frac{x\ N\ a \quad x\ N\ b}{x\ N\ a \wedge b}\ [\wedge \mathrm{R}]$$

$$\frac{x\ N\ a \quad b\ N\ z}{a \to b\ N\ x \leftthreetimes z}\ [\to \mathrm{L}] \qquad \frac{x\ N\ a \leftthreetimes b}{x\ N\ a \to b}\ [\to \mathrm{R}] \qquad \frac{\delta\ N\ z}{\top\ N\ z}\ [\top \mathrm{L}] \qquad \frac{}{x\ N\ \top}\ [\top \mathrm{R}]$$

FIGURE 2. The theory **nGBIN**.

We also consider extensions with the conditions

$$\frac{}{\bot\ N\ z}\ [\bot \mathrm{L}] \qquad \frac{x\ N\ \delta}{x\ N\ \bot}\ [\bot \mathrm{R}]$$

It is possible to relax the condition that B is a common subset of W and W' by considering maps from B to W and W', but we will not make use of such a generalization here.

A *cut-free distributive Gentzen frame* is defined in the same way, but it is not stipulated to satisfy the [CUT] condition. It is easy to see that $(\mathbf{W_{nDRL}}, \mathbf{Fm})$ is a distributive Gentzen frame. Also, given a GBI-algebra $\mathbf{A}$, the pair $(\mathbf{W_A}, \mathbf{A})$ is a distributive Gentzen frame. We will see more examples of distributive Gentzen frames in the following sections.

For readers familiar with display logic, we mention that the system **nGBI** does not enjoy the display property, however it satisfies the conditions **nGBIN**, which do enjoy the nuclear property (an analogue of the display property). In this sense, (distributive) residuated frames could be seen as a framework that is more general than display logic, or as a non-syntactic version of display logic, in that the display-logic rendering of **nGBI**, our version of **nGBI**, as well as other 'algebraic' situations, give rise to residuated frames, all of which satisfy the nuclear/display property.

Residuated frames and *Gentzen frames* are defined in [7] in the same way as their distributive versions, but with no mention of (and no requirements associated with) the operation $\curlywedge$; the conditions $(\wedge \mathrm{L}\ell)$ and $(\wedge \mathrm{L}r)$ are used instead of the condition $(\wedge \mathrm{L})$. As these conditions are derivable from $(\wedge \mathrm{L})$, every distributive Gentzen frame is also a Gentzen frame, so the results of [7] apply. In particular we state the following results of [7, Thm 2.6, Cor 2.7] for distributive Gentzen frames; the cases for the (possibly) additional connectives $\curlywedge, \curlyvee, \curlywedgedownarrow, \top, \bot$ are similar to the cases of the other connectives handled in [7].

Theorem 3.1. *Let* $(\mathbf{W}, \mathbf{B})$ *be a cut-free distributive Gentzen frame of type* $\mathcal{L}$. *For all* $a, b \in B$, *for all* $X, Y \in \mathbf{W}^+$, *and for every connective* $\bullet \in \mathcal{L}$, *if* $a \bullet^{\mathbf{B}} b$ *is defined, then*

(i) $1^{\mathbf{B}} \in \gamma_N(\varepsilon) \subseteq \{1_{\mathbf{B}}\}^{\lhd}$, $0^{\mathbf{B}} \in \{\varepsilon\}^{\lhd} \subseteq \{0^{\mathbf{B}}\}^{\lhd}$.
(ii) $\top^{\mathbf{B}} \in \gamma_N(\delta) \subseteq \{\top_{\mathbf{B}}\}^{\lhd}$, $\bot^{\mathbf{B}} \in \{\delta\}^{\lhd} \subseteq \{\bot^{\mathbf{B}}\}^{\lhd}$.
(iii) *If* $a \in X \subseteq \{a\}^{\lhd}$ *and* $b \in Y \subseteq \{b\}^{\lhd}$, *then* $a \bullet^{\mathbf{B}} b \in X \bullet^{\mathbf{W}^+} Y \subseteq \{a \bullet^{\mathbf{B}} b\}^{\lhd}$.
(iv) *In particular,* $a \bullet^{\mathbf{B}} b \in \{a\}^{\lhd} \bullet^{\mathbf{W}^+} \{b\}^{\lhd} \subseteq \{a \bullet^{\mathbf{B}} b\}^{\lhd}$.
(v) *If, additionally,* N *satisfies* [CUT], *then* $\{a\}^{\lhd} \bullet^{\mathbf{W}^+} \{b\}^{\lhd} = \{a \bullet^{\mathbf{B}} b\}^{\lhd}$.

Corollary 3.2. *If* $(\mathbf{W}, \mathbf{B})$ *is a distributive Gentzen frame of type* $\mathcal{L}$, *the map* $x \mapsto \{x\}^{\lhd}$ *from* $\mathbf{B}$ *to* $\mathbf{W}^+$ *is an* $\mathcal{L}$*-homomorphism from the partial algebra* $\mathbf{B}$ *into* $\mathbf{W}^+$*; it is injective if the restriction of* N *to* $B \times B$ *is antisymmetric.*

4. Cut elimination

Let $(\mathbf{W}, \mathbf{B})$ be a cut-free Gentzen frame. For the rest of the section, we assume that $\mathbf{B}$ is a total $\mathcal{L}$-algebra. For every homomorphism $f\colon \mathbf{Fm} \to \mathbf{B}$, we let $\bar{f}\colon \mathbf{Fm} \to \mathbf{W}^+$ be the $\mathcal{L}$-homomorphism that extends the assignment $p \mapsto \{f(p)\}^{\lhd}$, for all variables p of $\mathbf{Fm}$. (More generally, we may define the assignment by $p \mapsto Q_p$, where Q_p is any set with $\{f(p)\}^{\rhd\lhd} \subseteq Q_p \subseteq \{f(p)\}^{\lhd}$.)

Lemma 4.1. [7] *If* $(\mathbf{W}, \mathbf{B})$ *is a cut-free distributive Gentzen frame and* $\mathbf{B}$ *is a total algebra, then for every homomorphism* $f\colon \mathbf{Fm} \to \mathbf{B}$, *we have that* $f(a) \in \bar{f}(a) \subseteq \{f(a)\}^{\lhd}$, *for all* $a \in Fm$. *If* $(\mathbf{W}, \mathbf{B})$ *is a distributive Gentzen frame, then* $\bar{f}(a) = \{f(a)\}^{\lhd}$, *for all* $a \in Fm$.

Let $(\mathbf{W}, \mathbf{B})$ be a cut-free distributive Gentzen frame. Note that every map $f\colon Fm \to B$ extends inductively to a map $f^\circ\colon Fm^\circ \to W$ by $f^\circ(x \circ^{\mathbf{Fm}^\circ} y) = f^\circ(x) \circ^W f^\circ(y)$ and $f^\circ(x \curlywedge^{\mathbf{Fm}^\circ} y) = f^\circ(x) \curlywedge^W f^\circ(y)$. Likewise, every homomorphism $f\colon \mathbf{Fm} \to \mathbf{G}$ into an $\mathcal{L}$-algebra $\mathbf{G}$ extends to a homomorphism $f^\circ\colon \mathbf{Fm}^\circ \to \mathbf{G}$. A sequent $x \Rightarrow a$ is said to be *valid* in $(\mathbf{W}, \mathbf{B})$ if for every homomorphism $f\colon \mathbf{Fm} \to \mathbf{B}$, we have $f^\circ(x)\ N\ f(a)$. Note that a sequent $x \Rightarrow a$ is valid in an nGBI-algebra $\mathbf{G}$ iff it is valid in the Gentzen frame $(\mathbf{W}_{\mathbf{G}}, \mathbf{G})$, namely if for all homomorphisms $f\colon \mathbf{Fm} \to \mathbf{G}$, we have $f^\circ(x) \le f(a)$.

Theorem 4.2. *If* $(\mathbf{W}, \mathbf{B})$ *is a cut free distributive Gentzen frame of type* $\mathcal{L}$ *and* $\mathbf{B}$ *is a total algebra, then every sequent that is valid in* $\mathbf{W}^+$ *is also valid in* $(\mathbf{W}, \mathbf{B})$.

The adaptation of the result in [7] to the distributive case uses the fact that $\curlywedge$ is defined element-wise in $\mathcal{P}(W)$ and that γ_N is a $\curlywedge$-nucleus. The following corollaries have proofs analogous to results in [7]. Also, they hold for both **nGBI** and **nDRL**.

Corollary 4.3. (1) *If a sequent is valid in* **nGBI**, *then it is valid in all cut-free distributive Gentzen frames* $(\mathbf{W}, \mathbf{B})$ *where* $\mathbf{B}$ *is a total algebra.*
(2) *A sequent is provable in* **nGBI** *iff it is valid in* nGBI.
(3) *The free algebra in* nGBI *is embeddable in* $\mathbf{W}^+_{\mathbf{nGBI}}$.
(4) *The system* **nGBI** *enjoys the cut elimination property.*

For a given set R of conditions, a *distributive residuated* R*-frame* is simply a distributive residuated frame that satisfies R. We denote by $\mathsf{nGBI}_{\mathrm{R}}$ the subvariety of nGBI axiomatized by $\varepsilon(\mathrm{R}) = \{\varepsilon(r) : r \in \mathrm{R}\}$. By Theorem 4.2, we have the following.

Corollary 4.4. *If a sequent is valid in* $\mathsf{nGBI}_{\mathrm{R}}$, *then it is valid in all distributive residuated* R*-frames.*

We can also prove cut elimination for extensions of the systems we have considered by simple structural rules.

Corollary 4.5. (1) *The system* $\mathbf{nGBI}_{\mathrm{R}}$ *enjoys the cut elimination property, for every set* R *of rules that are preserved by* $(\_)^+$, *and in particular for the set* $\mathrm{R} = \mathrm{R}(\varepsilon)$ *with simple rules for an equation* ε *over* $\{\wedge, \vee, \cdot, 1, \top\}$.
(2) *The basic systems* $\mathbf{nGBI}_{\mathrm{R}}$, *where* R *is a subset of* $\{[\circ a], [\circ e], [\circ c], [\circ i]\}$, *have the cut elimination property.*
(3) *Moreover, every variety of distributive residuated lattices axiomatized by equations over* $\{\wedge, \vee, \cdot, 1, \top\}$ *has a corresponding cut-free Gentzen system.*

5. Finite model property

The finite model property for **DRL** was established in [12] and for **BI** it was proved in [10]. We extend these results by proving the finite model property (FMP) for many simple extensions of **DRL** and of **GBI**, actually for many simple extensions nGBI, namely axiomatized by certain equations/sequents that do not involve divisions and implication, but otherwise can have any combination of the other connectives. Given a sequent/equation, the decision procedure that follows from any FMP result about finitely axiomatized theories is to run a model-checker for finite models of the theory to find a possible counterexample to the sequent/equation and also a theorem prover to identify a possible proof of it.

Although not stated explicitly in [12], it can be inferred from the proof that it is possible to use only the model-checker, since an upper-bound for a countermodel (if it exists) can be estimated. The proof of the FMP of **DRL** given there is based on a proof search for the given sequent, but because of the rule $(\curlywedge c)$, the naive exhaustive proof search is not finite; the FMP is established in [12] without showing or claiming that a finite proof search is possible. Our first result in this section is to show that a finite proof search is possible, and from there we easily deduce the FMP, for all the extensions mentioned above, including extensions of **GBI**. Also, we give a canonical form to which any proof can be rewritten, and which has very limited applications of ($\curlywedge c$); this reduces even further the number of potential proofs in our first result that need to be examined during the proof search.

Free algebras. Recall that $\mathbf{Fm}^\circ = (Fm^\circ, \circ, \curlywedge, \varepsilon, \delta)$ denotes the bi-unital bi-groupoid over the set Fm of **GBI** formulas. We call the elements of Fm° *structures* and we will be considering their structure trees; this is in direct analogy to the formula tree, and following usual practice, we assume that the root is at the top and the leaves at the bottom of the tree. Proofs, however, will still be thought of as trees where the root is at the bottom. We will also consider the free algebras $\mathbf{Fm}^\circ/a$, $\mathbf{Fm}^\circ/ae$, and $\mathbf{Fm}^\circ/aec$, which are obtained by taking the quotient by the equivalence relation that renders $\curlywedge$ associative, or associative and commutative, or associative commutative and idempotent. As usual, each element of these sets is an equivalence class of structures from $\mathbf{Fm}^\circ$. However, we can also represent each element of $\mathbf{Fm}^\circ/a$ by a variation of a structure tree, where $\curlywedge$ denotes an n-ary operation for every n and where in the tree, a $\curlywedge$-node has a finite number of children; we call such terms *flat*. Given such a representation, we can obtain an element of $\mathbf{Fm}^\circ$ by fixing a specific way to insert parentheses; we chose to always associate to the left. Under this convention, $\mathbf{Fm}^\circ/a$ can be identified with a subset of $\mathbf{Fm}^\circ$.

In the flat representation, if the order of the list of subtrees of a $\curlywedge$-node does not matter, namely the child nodes form a multiset, this represents an element of $\mathbf{Fm}^\circ/ae$. Given a fixed total ordering on $\mathbf{Fm}^\circ$ under which children will be listed from left to right, we can identify each element of $\mathbf{Fm}^\circ/ae$ with an element of $\mathbf{Fm}^\circ/a$ (and thus an element of $\mathbf{Fm}^\circ$). We will use the term *commutative flat representation* of an element in $\mathbf{Fm}^\circ$ for that particular element of $\mathbf{Fm}^\circ$ (left associated and all subtrees of $\curlywedge$-nodes ordered by the above convention), as well as for the flattened version of this where parentheses are removed at these $\curlywedge$-nodes and the structure tree is represented with $\curlywedge$-nodes that have arbitrary arity. Finally, $\mathbf{Fm}^\circ/aec$ can be viewed as a subset of $\mathbf{Fm}^\circ/a$, where in that (commutative) flat representation, all child nodes are distinct. So, for example, an element of Fm°/ae can be viewed as an equivalence class $[b \circ (c \curlywedge (a \curlywedge c))]_{ae}$ of an element of Fm°, where a, b, c are formulas, or the object $b \circ (a \curlywedge c \curlywedge c)$, assuming the ordering $a < b < c$, or as the element $b \circ ((a \curlywedge c) \curlywedge c)$ of Fm°. Likewise, the element $[b \circ (c \curlywedge (a \curlywedge c))]_a$ of Fm°/a can be represented as

$$b \circ (c \curlywedge a \curlywedge c) \quad \text{or} \quad b \circ ((c \curlywedge a) \curlywedge c);$$

we could even allow the representation $b \circ (\curlywedge(c, a, c))$.

The composition of the canonical homomorphism $\mathbf{Fm}^\circ \to \mathbf{Fm}^\circ/aec$ with the insertion of $\mathbf{Fm}^\circ/aec$ into $\mathbf{Fm}^\circ$ resulting from the commutative flat representation is denoted by r . For a structure x, we call $r(x)$ its *full reduction* and we move freely between the representations of $r(x)$ as an equivalence class or an element of Fm° in its commutative flat version. It is clear that $r(x) = r(y)$ iff the sequents $x \Rightarrow p$ and $y \Rightarrow p$, where p is a propositional variable, are interderivable using $(\curlywedge a)$, $(\curlywedge e)$, and bidirectional $(\curlywedge c)$. For a sequent $x \Rightarrow a$, we call $r(x) \Rightarrow a$ its *full reduction.*

Reduction and multiplicity. We call a sequent *n-reduced at a $\curlywedge$-node* if in the flat representation, there are at most n duplicate copies of any immediate subtree. We call it *n-reduced* if it is n-reduced at all $\curlywedge$-nodes. A *reduction* of a sequent is a sequent obtained by applications of the rules $(\curlywedge a)$, $(\curlywedge e)$, and $(\curlywedge c)$. We mention that the order in which $(\curlywedge c)$-rules are applied to obtain a reduction, namely to which $\curlywedge$-nodes we apply contraction first, does not matter (see also the discussion in the subsection on contraction-controlled proofs), and the resulting reductions are always inter-derivable using the rules $(\curlywedge a)$ and $(\curlywedge e)$; this also explains the ability to select representatives from equivalence classes in the free algebras above.

An *n-reduction* of a sequent is a reduction which happens to be an n-reduced sequent. Note that the full reduction $r(x) \Rightarrow a$ is a 1-reduction of $x \Rightarrow a$. One can see this by applying contraction at the lowest $\curlywedge$ in the tree, then again taking the commutative flat representation and again contraction at the leaves, etc.

Given a rule, we define its *multiplicity* as the least number n such that if all premises are 1-reduced sequents, then the conclusion is n-reduced. Note that the multiplicity of $(\to\mathrm{L})$ is 3 since $a \to b$ could be part of x and also part of u. To be more specific, x has to have $\curlywedge$ at its root and, in its flat version, have $a \to b$ as one of its children, and also $u(\_)$ has to be of the form $v(y \curlywedge \_)$ and y has to have the same property as x above. So, if the assumptions of the rule are 1-reduced, then the conclusion of the rule is always 1-reduced at all other $\curlywedge$ nodes except for the one where it may be 3-reduced. Likewise, $(\backslash\mathrm{L})$ has multiplicity 1 for instances where x is non-empty, while it has multiplicity 2 for instances where $x = \varepsilon$, $u(\_)$ has the form $v(y \curlywedge \_)$, and y has $\curlywedge$ at its root and, in its flat version, has $a\backslash b$ as one of its children. The same holds for the rule $(/\mathrm{L})$. The remaining rules have multiplicity 1 since they do not allow for the combination of substructures using $\curlywedge$.

A proof is called *n-reduced* if every sequent in it is n-reduced.

Lemma 5.1 (n-reduced proofs for 1-reductions)**.** *If a sequent is provable in a simple extension of* **nGBI** *where every rule has multiplicity at most n, then every* 1*-reduction of it has an n-reduced proof.*

Proof. We prove this by induction on the depth of the proof of the given sequent $y \Rightarrow d$. The base case of an initial sequent is obvious. Assume that the last step of the proof is

$$\frac{y_1 \Rightarrow d_1 \quad y_2 \Rightarrow d_2}{y \Rightarrow d}\ (r)$$

We now apply the rules $(\curlywedge a)$, $(\curlywedge e)$, and $(\curlywedge c)$ to obtain 1-reductions $y_1' \Rightarrow d_1$ and $y_2' \Rightarrow d_2$ of $y_1 \Rightarrow d_1$ and $y_2 \Rightarrow d_2$. Note that the rule (r) is still applicable, and we call $y' \Rightarrow d'$ its conclusion:

$$\frac{y_1' \Rightarrow d_1 \quad y_2' \Rightarrow d_2}{y' \Rightarrow d'}\ (r)$$

For example, for $(\wedge\mathrm{R})$ we have $y_1 = y_2$, so we apply the exact same reductions to both and then $y_1' = y_2'$, so the rule $(\wedge\mathrm{R})$ is still applicable. For the rule $(\backslash\mathrm{L})$, for example, the reductions are done independently on $x \Rightarrow a$ and on $u(b) \Rightarrow c$. Applying to $y \Rightarrow d$ the combination/union of the reductions involved in $y_1 \Rightarrow d_1$ and $y_2 \Rightarrow d_2$ results in a sequent $\bar{y} \Rightarrow \bar{d}$. If there were contractions/reductions applied to parts of $y_1 \Rightarrow d_1$ or $y_2 \Rightarrow d_2$ that involve the principal sequents, we can reinstate these parts by applications of $(\curlywedge i)$ to obtain a reduction $y' \Rightarrow d'$ of $y \Rightarrow d$, staying within the realm of n-reduced sequents. By repeated applications of $(\curlywedge a)$, $(\curlywedge e)$, and $(\curlywedge c)$, we can prove (using only n-reduced sequents) a 1-reduction $y_r \Rightarrow d_r$ of $\bar{y} \Rightarrow \bar{d}$ and hence also of $y \Rightarrow d$.

$$\dfrac{\dfrac{\dfrac{\dfrac{y_1 \Rightarrow d_1}{y_1' \Rightarrow d_1}\ (\curlywedge a,e,c) \quad \dfrac{y_2 \Rightarrow d_2}{y_2' \Rightarrow d_2}\ (\curlywedge a,e,c)}{y' \Rightarrow d'}\ (r)}{\bar{y} \Rightarrow \bar{d}}\ (\curlywedge i)}{y_r \Rightarrow d_r}\ (\curlywedge a,e,c)$$

By the induction hypothesis, $y_1' \Rightarrow d_1$ and $y_2' \Rightarrow d_2$ have n-reduced proofs where all sequents are n-reduced, so we can replace the top lines of the above proof-figure by these n-reduced proofs. Also, because $y_1' \Rightarrow d_1$ and $y_2' \Rightarrow d_2$ are themselves 1-reduced, and the rule (r) has multiplicity at most n, we get that $y' \Rightarrow d'$ is n-reduced. The resulting proof involves only n-reduced sequents. $\square$

The argument above is along the lines of [8, Lemma 4.10]. The applicability of the rule (r) alludes to some permutation of the rule $(\curlywedge c)$ up in the proof and we make this precise in the subsection on contraction-controlled proofs.

Complexity measure for extensions of DRL. Following [12], we define the complexity $m(x)$ for $x \in Fm^\circ$ inductively as follows:

- $m(1) = m(\top) = m(0) = m(\bot) = m(p) = 1$, for every variable p
- $m(a \bullet b) = m(a) + m(b) + 1$, where $\bullet$ is any logical connective
- $m(\varepsilon) = 0$
- $m(x \circ y) = m(x) + m(y)$
- $m(x \curlywedge y) = max\{m(x), m(y)\}$.

We define $m(x \Rightarrow a) = m(x) + m(a)$. This complexity measure can be used to show that **DRL** is decidable even when expanded by certain structural rules.

We say that a rule in a sequent system *does not increase complexity* upward if for each instance of the rule, the complexity of each sequent in the premises is at most as big as the complexity of the conclusion. We can easily see that the rules in the system **DRL** do not increase complexity.

We now show by induction on m that the set of n-reduced sequents of complexity at most m, and constructed from a finite set S of formulas, is finite. Indeed, if the statement is true for all $k < m$, then consider first a structure of complexity m that does not have $\curlywedge$ at the root of its structure tree. Then all of its subtrees (if any) must have complexity less than m, and so these subtrees can be chosen from a finite set of structures by the induction hypothesis, and thus there are finitely many structures of this form of complexity m. If, now, the structure tree has $\curlywedge$ at its root, then consider the set of all of its immediate substructures in the flat representation. These subtrees have complexity at most m, so by the above argument can be taken from a finite set; moreover, they have at most n repetitions since the structure is n-reduced. As there are only finitely many such choices, there are only finitely many structures of this form of complexity m.

The set S will be taken below to be the (finite) set $\mathrm{Sub}(x \Rightarrow a)$ of subformulas of all the formulas that appear in a given sequent $x \Rightarrow a$. Since all rules of **nDRL** (except the cut rule) have the *subformula property*, namely every formula in the premises is a subformula of a formula in the conclusion, all cut-free proofs of $x \Rightarrow a$ involve only sequents over $\mathrm{Sub}(x \Rightarrow a)$. A *proof scheme* is defined in the same way as a proof without the assumption that the leaves are axioms.

Corollary 5.2 (Finite proof search)**.** *Given a sequent s, there are only finitely many proof-schemes that need to be investigated in order to check if the sequent is provable in an extension of* **nDRL** *with finitely many simple rules none of which increases complexity.*

Proof. Using the invertibility of $(\curlywedge c)$, as an instance of $(\curlywedge i)$, we see that a sequent is provable iff its full reduction is provable. Also, by Lemma 5.1, the full reduction is provable iff it has an n-reduced proof, where n is the maximal multiplicity of each rule. All sequents in the proof are constructed from the finite set of formulas in $\mathrm{Sub}(s)$. Also, since each rule does not increase complexity upward, all sequents involved in a proof of a given sequent s have complexity at most $m(s)$. Therefore, the sequents in the proof are selected from a finite set of sequents, and by the argument above, since they are made from a finite set of formulas, they are n-reduced and have bounded complexity. This does not imply that there are only finitely many proofs, as sequents could be repeated. However, we can assume that the proof has no repetitions on any of its branches since we can simply omit the part of the proof between repeated sequents. Therefore, there is a bound on the length of each branch, namely on the height of a proof. Together with the bound on the maximum degree of a rule (the number of its premises), this imposes a bound on the total number of sequents (distinct or not) in the proof. Hence, there are only finitely many proof-schemes to be checked. □

Complexity measure for extensions of nGBI. We are indebted to Revantha Ramanayake for pointing out to us that the complexity measure used for the decidability of **nDFL** does not work for the additional rules of **nGBI**, hence necessitating the more sophisticated argument given here.

Given a sequent $x \Rightarrow a$, we define its tree (growing downward) in the usual way. The symbol $\Rightarrow$ at the top/root has as its right-child node the formula tree of a and as its left-child node the tree of x (where the latter is naturally stratified, with structural connectives appearing above the logical ones). As a result, each node of the tree is either $\Rightarrow$, $\circ$, $\curlywedge$, $\vee$, $\wedge$, $\cdot$, $\to$, $\backslash$, $/$, or a propositional variable (at a leaf). We call *formula-nodes* those nodes that have as label a logical connective or a propositional variable. Each node of the tree also carries a sign in the standard way, guided by order-preservation considerations; we call this the *position sign* of the node. (In detail, we first assign a *polarity sign* to each edge. Both edges below the connectives $\circ, \curlywedge, \vee, \wedge, \cdot$ are given the polarity sign $+$; the same is true for the right edge below $\Rightarrow$, $\to$ and $\backslash$, as well as the left edge below $/$; the other edges of these connectives receive a $-$ polarity sign. Then, as usual, we define the position sign of a node in the tree to be $+$ if there is an even number of $-$ polarity signs in the edges of the branch above it to the root, and a $-$ sign otherwise.)

Now we give a new definition that is not considered in the literature. Given a sequent s, we define a direction on the edges of the sequent graph of s. We understand upward edges as positively directed and downward edges as negatively directed, so the *direction sign* for an edge can be interpreted as its direction. To determine the value of the direction sign of an edge, we take the product of two other signs: the position sign of the node above the edge and the polarity sign of the edge.

For example, a $\circ$ can appear only in a negative position in a sequent and each of the edges below it are positive (as $\circ$ is positive in both coordinates); since negative times positive yields negative, both edges below a $\circ$ point downward. The same holds for $\curlywedge$. Also, the edges below the connectives $\cdot, \wedge, \vee$ are pointing downward if the connective is in negative position, and upward if the connective is in positive position. The same holds for the right-hand-side edge of $\backslash$, $\to$, and $\Rightarrow$, and similarly for the left-hand-side edge of $\backslash$; the remaining edges have the opposite orientation.

To this general rule we add some special rules which result in some edges having two directions. We add an upward edge from a negative $\to$ to a negative $\curlywedge$ or $\wedge$ which is directly above it. Likewise, but in a more delicate way, we add such edges from a negative $\backslash$ to a negative $\circ$ or $\cdot$ that is directly above it, as long as this is the left edge below the $\circ$ or $\cdot$, and the same for $/$, as long as this is the right edge. We also add an edge upward from a negative $\backslash$ to a positive $\backslash$ above it provided this is the left edge. Further, we add an edge from a negative $/$ upward to a $/$ provided this is the right edge. Note that the right edge of $\Rightarrow$ points toward it and the left edge points away from it. For the purposes of the resulting directed graph, we find it helpful to consider these as one edge, since they have a consistent orientation. If the left-hand side of a sequent is empty and the main connective on the right-hand side is one of $\to$, $\backslash$, or $/$, then we follow the same convention of considering the two edges that stem from it as a single directed edge. We also have an edge from a negative $/$, going through $\Rightarrow$ to a positive $\backslash$. Finally, we add an edge from a negative $\backslash$, going through $\Rightarrow$ to a positive $/$. The *multiplicative length* of a sequent is defined by considering all oriented paths in it and counting the maximum number of *polarized multiplicative connectives*; these are defined to be $\circ$ and $\cdot$ in negative position and $\backslash$, $/$ in positive position. Note that not all paths pass through the root $\Rightarrow$. We clarify that paths are allowed to go through the same edge in different directions, provided it has two directions, but the connective that the special edge points to is not allowed to be repeated in the path. In considering what paths can be realized in a tree, we allow for a sequent tree to be read in a way that $\curlywedge$ is considered in its flat version with multiple child-nodes, or in any of the forms obtained by inserting parentheses.

It is easy to see by inspection that the multiplicative length of each premise of each rule of **nGBI** is no bigger than the multiplicative length of the conclusion of that rule. The use of the bidirectional edges is explained by the rules ($\to$L), ($\backslash$L), and ($/$L); the directed paths of the premise $x \Rightarrow a$ are included in the directed paths of the conclusion, because we can move in the additional direction of the edge. We note that, unfortunately, the rule ($\circ a$) can change the multiplicative length of a sequent. To handle **GBI**, we consider a flat version of $\circ$, thus internalizing associativity and also having a smaller number of polarized multiplicative connectives. This does not affect the argument significantly as each rule, including ($\circ a$), is non-increasing under this new form of a sequent tree. As for the associativity of $\curlywedge$, the tree is allowed to be viewed under its non-flattened version for $\circ$ in order for directed paths to be realized.

The $\circ$-tree of a sequent t is the subtree of the sequent tree of t consisting of just the $\circ$ nodes and edges for the paths between them. It then follows that in every sequent t in a proof of s, the $\circ$-tree of t has height no more than the multiplicative length of s.

We now argue that there are only finitely many sequents that could appear in a 3-reduced proof of s and that this number is computable. In particular, we argue that there are only finitely many 3-reduced sequents that are formed by subformulas of s and which have $\circ$-tree of height less than or equal to the multiplicative length of s; we do this by induction on the height h of the $\circ$-tree of a sequent. We make crucial use of the fact that we need to consider only 3-reduced proofs, namely these proofs consist of 3-multisets (multisets where every element appears at most 3 times) of substructures at every $\curlywedge$-level of the sequent trees. Clearly, the number of 3-multisets over a finite set S, namely of functions from S to $\{0, 1, 2, 3\}$, is $4^{|S|}$. We focus on the structure on the left-hand side of sequents, and prove there are finitely many choices; combined with a choice of a subformula of s for the right-hand side, this yield finitely many choices for such sequents.

If $h = 0$, then the structure has no $\circ$, hence it consists of formulas separated by $\curlywedge$. In other words, it is a 3-multiset of subformulas of s and there are only a finite number of these. We assume that the result holds for $h < k$ and prove it for $h = k$. First we prove this result for $\circ$-structures, namely structures where $\circ$ is the main structural connective. Each of the two child nodes of $\circ$ will be a $\curlywedge$-structure with $\circ$-height less than or equal to k, so there are only finitely many such choices, by the induction hypothesis. The result for $\curlywedge$-structures then follows by the fact that they will be 3-multisets of $\circ$-structures of $\circ$-height up to k.

For the associative case, the above argument needs to be modified slightly. Now, $\circ$ may have multiple child nodes. However, we can bound this number, say by the total number of polarized multiplicative connectives of the original sequent s, therefore the finiteness argument still works. Also, further structural rules can be added as long as they also respect the non-increasing nature of the multiplication length. The rule of exchange ($\circ e$) is one such example, but one can consider other examples where the multiplicity of the rule is higher than 3.

Finite models. For a sequent s of some extension **L** of **nGBI** by simple rules, we define $s^{\leftarrow}$ to be the least set of sequents such that $s \in s^{\leftarrow}$ and if $(\{t_1, \ldots, t_n\}, t)$ is an instance of a rule of **L** and $t \in s^{\leftarrow}$, then $t_1, \ldots, t_n \in s^{\leftarrow}$. Clearly $s^{\leftarrow}$ is the set of all sequents involved in an exhaustive proof search for s. By the subformula property, all sequents in $s^{\leftarrow}$ are over the set $\mathrm{Sub}(s)$.

Theorem 5.3. *Any extension of any fragment of* **nGBI** *containing* $\circ, \curlywedge, \varepsilon, \delta$ *by finitely many simple rules that do not increase complexity has the FMP.*

Proof. Let N denote the relation in the frame $\mathbf{W_{nGBI}}$ and let s be a sequent that is not provable in **nGBI**. Let N_s be the relation defined by

$$x \; N_s \; (u, a) \qquad \text{iff} \qquad x \; N \; (u, a) \text{ or } (u(x) \Rightarrow a) \notin s^{\leftarrow}.$$

Following the arguments in [7], it is easy to see that N_s is nuclear and satisfies the conditions **nGBIN**. So, $(\mathbf{W}_s, \mathbf{Fm})$ is a distributive Gentzen frame, where $\mathbf{W}_s$ uses N_s as the nuclear relation.

To show that $\mathbf{W}_s^+$ is finite, we show that there only finitely many basic closed sets, namely sets of the form $\{z\}^{\lhd}$ for $z \in W'$, since every other element of $\mathbf{W}_s^+$ is an intersection of such sets. First note that every equivalence class $[x]_{aec}$ modulo associativity, commutativity, and idempotency of $\curlywedge$ contains the 1-reduced structure $r(x)$. Since there are only finitely many 1-reduced sequents over $\mathrm{Sub}(s)$ of bounded complexity, this means that there are only finitely many such equivalence classes. Also, note that every basic closed set $\{(u, b)\}^{\lhd}$ is a union of such equivalence classes. Indeed, let x and y be equivalent structures. We have that $x \in \{(u, b)\}^{\lhd}$ iff $x \; N_s \; (u, b)$ iff the sequent $u(x) \Rightarrow b$ is either provable or not in $s^{\leftarrow}$. Since x and y are equivalent the sequents $u(x) \Rightarrow b$ and $u(y) \Rightarrow b$ are interderivable using the rules $(\curlywedge a)$, $(\curlywedge e)$, and bidirectional $(\curlywedge c)$, so one is provable iff the other is, and also one is in $s^{\leftarrow}$ iff the other is. Thus, $x \in \{(u,b)\}^{\lhd}$ iff $y \in \{(u,b)\}^{\lhd}$.

Furthermore, s fails in $\mathbf{W}_s^+$. Indeed, let s be the sequent $x \Rightarrow a$ and let $b = x^{\mathbf{Fm}}$ (i.e., b is the term x with every $\circ$ replaced by $\cdot$, every $\curlywedge$ replaced by $\wedge$, every ε replaced by 1, and every δ replaced by $\top$). Note that $x \not\!N_s\, a$ since $x \not\!N\, a$ and $(x \Rightarrow a) = s \in s^{\leftarrow}$. Hence, $x \notin \{a\}^{\lhd_s}$. However, $x \Rightarrow b$ is provable in **nGBI**, so $x \in \{b\}^{\lhd_s}$, and therefore $\{b\}^{\lhd_s} \not\subseteq \{a\}^{\lhd_s}$. Since $(\mathbf{W}_s, \mathbf{Fm})$ is a Gentzen frame, the map ${}^{\lhd_s} : \mathbf{Fm} \to \mathbf{W}_s^+$ is a homomorphism by Corollary 3.2. Consequently, the inequality $b \leq a$ is not valid in $\mathbf{W}_s^+$, so neither is the sequent $x \Rightarrow a$. □

Contraction-controlled proofs. We have shown that the proof search is finite, namely we can focus only on finitely many proof-schemes in order to check the validity of a given sequent; these are all the proof-schemes that involve n-reduced sequents of bounded complexity and have no repetitions on each branch. In this subsection, we undertake a detailed analysis that shows that even fewer proof-schemes are needed and that every proof can be in what we call *contraction-controlled* form. This reveals the structure of these, in some sense canonical, proofs and also can be useful for a practical implementation of the algorithm. Additionally, it illuminates aspects of the proof of Lemma 5.1.

It is easy to see, for example, that the $\curlywedge$-contraction rule $(\curlywedge c)$ can be permuted up above all the right logical rules. If we consider the consequence $x \Rightarrow a\backslash b$ of the rule $(\backslash\mathrm{R})$, we see that we could have applied $(\curlywedge c)$ below it only if x was of the form $u(y \curlywedge y)$, in which case we can rewrite that part of the proof so that $(\curlywedge c)$ is performed above the rule $(\backslash\mathrm{R})$.

$$\dfrac{\dfrac{a \circ u(y \curlywedge y) \Rightarrow b}{u(y \curlywedge y) \Rightarrow a\backslash b}\,(\backslash\mathrm{R})}{u(y) \Rightarrow a\backslash b}\,(\curlywedge c) \quad\rightsquigarrow\quad \dfrac{\dfrac{a \circ u(y \curlywedge y) \Rightarrow b}{a \circ u(y) \Rightarrow b}\,(\curlywedge c)}{u(y) \Rightarrow a\backslash b}\,(\backslash\mathrm{R})$$

The same applies to all right logical rules: $(\curlywedge c)$ can be postponed as we explore the proof upward in favor of a right-logical rule. Even in the rule $(\circ\mathrm{R})$, where the left-hand side $x \circ y$ of the conclusion is separated into two pieces x and y in the assumptions, still any instantiation of $(\curlywedge c)$ in $x \circ y$ has to occur fully in x or fully in y, so it can be performed later above, after the rule has been applied below. In the right rules for the lattice connectives, the situation is even simpler as the left-hand side remains the same, while for the rule (1R), contraction cannot be performed at all immediately below it.

The left-logical rules are not as easy to argue about, but we are actually able to identify $\curlywedge$-contractions that can be permuted up above these rules. For this, we will need to consider the structure tree of a given structure. Given a certain node/subtree x in the structure tree of $u(x)$, we consider the set or *path* of nodes $\uparrow_u x$ that appear above it in the tree; we often identify a node with the subtree it specifies. Given a left logical rule $(\bullet\mathrm{L})$, where $\bullet$ is any logical connective, we can focus on the structure on the left-hand side y of the conclusion $y \Rightarrow c$ of the rule, identify the position of the *active* connective/formula $a \bullet b$ on the structure tree and consider $\uparrow_y(a \bullet b)$, which we call the *path of the rule* ($\bullet$L); we call $a \bullet b$ the *principal level* of the path and the positions of other nodes on the path the *non-principal levels* of the path. If there is an instance of contraction applied to $y \Rightarrow c$ at a node not on the path, then that contraction permutes up above ($\bullet$L) since the contracted part is completely disjoint from the principal formula inside the structure tree. This should be obvious; as a concrete example, we consider ($\backslash$L) and the only two distinct positions off the path in which we could apply contraction to $u(x \circ (a\backslash b)) \Rightarrow c$: inside x, we have

$$\dfrac{\dfrac{v(y \curlywedge y) \Rightarrow a \quad u(b) \Rightarrow c}{u(v(y \curlywedge y) \circ (a\backslash b)) \Rightarrow c}\,(\backslash\mathrm{L})}{u(v(y) \circ (a\backslash b)) \Rightarrow c}\,(\curlywedge c) \quad \rightsquigarrow \quad \dfrac{\dfrac{v(y \curlywedge y) \Rightarrow a}{v(y) \Rightarrow a}\,(\curlywedge c) \quad u(b) \Rightarrow c}{u(v(y) \circ (a\backslash b)) \Rightarrow c}\,(\backslash\mathrm{L})$$

and on a part of u outside x; here $u(y, x)$ denotes as usual a term and two (non-overlapping) occurrences of subterms,

$$\dfrac{\dfrac{x \Rightarrow a \quad u(y \curlywedge y, b) \Rightarrow c}{u(y \curlywedge y, x \circ (a\backslash b)) \Rightarrow c}\,(\backslash\mathrm{L})}{u(y, x \circ (a\backslash b)) \Rightarrow c}\,(\curlywedge c) \quad \rightsquigarrow \quad \dfrac{x \Rightarrow a \quad \dfrac{u(y \curlywedge y, b) \Rightarrow c}{u(y, b) \Rightarrow c}\,(\curlywedge c)}{u(y, x \circ (a\backslash b)) \Rightarrow c}\,(\backslash\mathrm{L})$$

Contractions that are performed at various levels of the path do not permute in general. For example, there is no obvious way to rewrite the following proof scheme so that contraction will be performed above ($\cdot$L):

$$\dfrac{\dfrac{u(v(a \circ b) \curlywedge v(a \cdot b)) \Rightarrow c}{u(v(a \cdot b) \curlywedge v(a \cdot b)) \Rightarrow c}\,(\cdot\mathrm{L})}{u(v(a \cdot b)) \Rightarrow c}\,(\curlywedge c) \quad \rightsquigarrow \quad \dfrac{u(v(a \circ b) \curlywedge v(a \cdot b)) \Rightarrow c}{u(v(a \cdot b)) \Rightarrow c}\,(?)$$

For simple structural rules, the criterion is very similar: contractions permute up as long as they apply to nodes not on the upward path starting at the lowest *structural* connective, namely $\curlywedge$ or $\circ$, that appears explicitly in the conclusion of the rule; the notion of *path* and of principal level are defined, extending the definition for logical rules. For downward ($\curlywedge a$),

$$\dfrac{u(x \curlywedge (y \curlywedge z)) \Rightarrow c}{u((x \curlywedge y) \curlywedge z) \Rightarrow c}\,(\curlywedge a)$$

this external connective is the one between the x and the y in the conclusion of the rule. Contraction off the path can be performed inside/at x or inside/at y or inside/at z or inside u, but outside x, y, and z, still off the path. For the first and last case we have, for example,

$$\dfrac{\dfrac{u(v(x \curlywedge x) \curlywedge (y \curlywedge z)) \Rightarrow c}{u((v(x \curlywedge x) \curlywedge y) \curlywedge z) \Rightarrow c}\,(\curlywedge a)}{u((v(x) \curlywedge y) \curlywedge z) \Rightarrow c}\,(\curlywedge c) \quad \rightsquigarrow \quad \dfrac{\dfrac{u(v(x \curlywedge x) \curlywedge (y \curlywedge z)) \Rightarrow c}{u(v(x) \curlywedge (y \curlywedge z)) \Rightarrow c}\,(\curlywedge c)}{u((v(x) \curlywedge y) \curlywedge z) \Rightarrow c}\,(\curlywedge a)$$

and

$$\dfrac{\dfrac{u(w \curlywedge w, x \curlywedge (y \curlywedge z)) \Rightarrow c}{u(w \curlywedge w, (x \curlywedge y) \curlywedge z) \Rightarrow c}\,(\curlywedge a)}{u(w, (x \curlywedge y) \curlywedge z) \Rightarrow c}\,(\curlywedge c) \quad \rightsquigarrow \quad \dfrac{\dfrac{u(w \curlywedge w, x \curlywedge (y \curlywedge z)) \Rightarrow c}{u(w, (x \curlywedge y) \curlywedge z) \Rightarrow c}\,(\curlywedge c)}{u(w, (x \curlywedge y) \curlywedge z) \Rightarrow c}\,(\curlywedge a)$$

Also, if a $(\curlywedge c)$ is immediately below a $(\curlywedge i)$, in case the $\curlywedge c$ occurs inside/at x, or at a part of u not above $x \curlywedge y$, then $\curlywedge c$ can be easily permuted up. If $\curlywedge c$ happens inside/at y then $\curlywedge c$ is redundant. Finally if $\curlywedge c$ happens at $x \curlywedge y$, namely for $y = x$, then $(\curlywedge i)$ is an application of the inverse of $(\curlywedge c)$ and clearly $(\curlywedge c)$ is redundant.

We say that a $\curlywedge$-contraction is *p-permutable* above another rule (r) in case the above path condition is satisfied, namely it is not applied at any point on the path of (r). We have shown that p-permutability above a rule implies actual permutability above it, for all rules in the system plus all simple rules.

Putting the above together, we see that every rule in the rewritten proof comes with a *cluster* of $(\curlywedge c)$ rules below it. To be precise, a $(\curlywedge c)$ rule is in the cluster of a rule (r) if it is performed at some place below (r) in the proof with no other non-$(\curlywedge c)$ rule between them and further it cannot be p-permuted up above (r).

We now look into these clusters and investigate whether contractions can move within each cluster and/or to higher clusters. In particular, for permuting contractions above other contractions, we note again that if they contract portions that are disjoint in the term tree, for example in $u(x, y)$ one contracts part of x and the other part of y, then these two contractions can be performed in any order. Also, we can see that by parsing the proof from above, it is more general to perform contractions lower in the tree and then further down in the proof perform contractions at higher nodes in the tree, since if done in the other order, we can permute them:

$$\dfrac{\dfrac{u(v(x \curlywedge x) \curlywedge v(x \curlywedge x)) \Rightarrow c}{u(v(x \curlywedge x)) \Rightarrow c}\,(\curlywedge c)}{u(v(x)) \Rightarrow c}\,(\curlywedge c) \quad \rightsquigarrow \quad \dfrac{\dfrac{u(v(x \curlywedge x) \curlywedge v(x \curlywedge x)) \Rightarrow c}{u(v(x) \curlywedge v(x)) \Rightarrow c}\,(\curlywedge c)}{u(v(x)) \Rightarrow c}\,(\curlywedge c)$$

Note that if two contractions $(\curlywedge c)_h$ and $(\curlywedge c)_\ell$ are in the cluster of a rule (r), then $(\curlywedge c)_h$ is performed higher in the proof than $(\curlywedge c)_\ell$, and also, if the lower contraction $(\curlywedge c)_\ell$ cannot be p-permuted up above the higher contraction $(\curlywedge c)_h$, then the lower contraction $(\curlywedge c)_\ell$ cannot be p-permuted up above the rule (r) either. This is simply because $(\curlywedge c)_h$ applies to the upward path of (r) and $(\curlywedge c)_\ell$ applies at a node higher than $(\curlywedge c)_h$. This provides a better understanding of the structure of the clusters, and extends the notion of p-permutability to $(\curlywedge c)$ rules.

We can now formally define a *contraction-controlled* proof as a proof where each cluster of contractions appears below a non-contraction rule, all these contractions are applied on the upward path of the structure tree of the (LHS of the) conclusion of the rule; for the two contractions $(\curlywedge c)_\ell$ and $(\curlywedge c)_h$, we have that $(\curlywedge c)_h$ is performed above $(\curlywedge c)_\ell$ in the proof iff $(\curlywedge c)_h$ operates at a node on the path of the rule that is lower than the node of $(\curlywedge c)_\ell$. We have thus proved the following result.

Theorem 5.4. *Every n-reduced sequent that is provable in a finite simple extension of any reduct of* **nGBI** *has a contraction-controlled proof.*

Since by using $(\curlywedge i)$ we can show that a sequent is provable iff its full reduction (which is 1-reduced) is provable (by a contraction-controlled proof), this provides a more explicit finite proof search decision procedure than Lemma 5.1. Note that all the above results apply also to arbitrary fragments of our calculus, which contain the structural rules for $\curlywedge$.

Fragments containing the structural rules for $\curlywedge$. Making use of the structural rules $(\curlywedge a)$ and $(\curlywedge e)$, we can do even better with respect to contraction-controlled proofs. For this, we will make use of the commutative-flat version of structures, as they incorporate seamlessly the two rules. So we will feel free to work with this data type and take the explicit rules $(\curlywedge a)$ and $(\curlywedge e)$ out of the system.

We say that an application of $(\curlywedge c)$ below a rule *pae-permutes* up above the rule if $(\curlywedge c)$ is applied on the path in the (commutative-flat) structure tree of the conclusion of the rule. We have essentially shown that *pae*-permutability implies actual permutability, but we can do better.

Recall that if the premises of a rule are 1-reduced, then the conclusion in all $\curlywedge$-nodes except one is 1-reduced (at that principal node it is n-reduced, where n is the multiplicity of that rule). We say that a contraction-controlled proof is *ae-reduced* if for each rule (r), with multiplicity n, the cluster of contractions below it is such that at every level strictly above the principal level on the path there is at most one contraction applied, at the principal level there are at most $(n-1)$ contractions applied, and none of the substructures created are repeated in the premises of the rule (r). Therefore, if the lower sequent of a cluster of contractions below a rule (r) is m-reduced, then all the premises of the rule (r) are also m-reduced. Consequently, if an m-reduced sequent has an *ae*-reduced contraction-controlled proof, then all the sequents in the proof are $(n+m)$-reduced, where n is the maximal multiplicity of rules in the system.

Lemma 5.5. *Every sequent that is provable in a simple extension of a fragment of* **GBI** *that contains the structural rules for $\curlywedge$ has an ae-reduced contraction-controlled proof.*

Proof. We need to show that if the contraction is one of at least 2 contractions that are applied to the same $\curlywedge$-level in the structure tree of the conclusion of the rule (r) and that level is not the level of the principal formula, or if it is one of at least n contractions that are applied to the $\curlywedge$-level of the principal formula, then the contraction rule permutes above the rule (r).

This is clear from the fact that if we have an additional copy of the subterm, then its contraction can happen before or after the rule (r) with no difference on the outcome. We give one example using the rule $(\backslash\mathrm{L})$ for the level not at the principal formula (we abbreviate $v'(x \circ a\backslash b)$ as just v', the result of contractions above the path of $a\backslash b$ and up to the node v). The notation $(\curlywedge c)^n$ is used for n applications of contraction and $(\curlywedge c)^*$ denotes some finite number of contractions.

$$\dfrac{\dfrac{\dfrac{x \Rightarrow a \quad u(v' \curlywedge v' \curlywedge v(b)) \Rightarrow c}{u(v' \curlywedge v' \curlywedge v(x \circ a\backslash b)) \Rightarrow c}\,(\backslash\mathrm{L})}{u(v' \curlywedge v' \curlywedge v'(x \circ a\backslash b)) \Rightarrow c}\,(\curlywedge c)^*}{u(v'(x \circ a\backslash b)) \Rightarrow c}\,(\curlywedge c)^2 \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{x \Rightarrow a \quad \dfrac{u(v' \curlywedge v' \curlywedge v(b)) \Rightarrow c}{u(v' \curlywedge v(b)) \Rightarrow c}\,(\curlywedge c)}{u(v' \curlywedge v(x \circ a\backslash b)) \Rightarrow c}\,(\backslash\mathrm{L})}{u(v' \curlywedge v'(x \circ a\backslash b)) \Rightarrow c}\,(\curlywedge c)^*}{u(v'(x \circ a\backslash b)) \Rightarrow c}\,(\curlywedge c)$$

As another example, the rule $(\to\mathrm{L})$ has multiplicity 3, so if we assume that we have 3 contractions at the level of the principal formula, then we show that one of them may be permuted up (here $(a \to b)^{\curlywedge 2}$ stands for $(a \to b) \curlywedge (a \to b)$).

$$\dfrac{\dfrac{a \to b \Rightarrow a \quad u((a \to b)^{\curlywedge 2} \curlywedge b) \Rightarrow c}{u((a\to b)^{\curlywedge 2} \curlywedge (a\to b) \curlywedge (a\to b)) \Rightarrow c}\,(\to\mathrm{L})}{u(a \to b) \Rightarrow c}\,(\curlywedge c)^3 \quad \rightsquigarrow \quad \dfrac{\dfrac{a \to b \Rightarrow a \quad \dfrac{u((a \to b)^{\curlywedge 2} \curlywedge b) \Rightarrow c}{u((a \to b) \curlywedge b) \Rightarrow c}\,(\curlywedge c)}{u((a\to b) \curlywedge (a\to b) \curlywedge (a\to b)) \Rightarrow c}\,(\to\mathrm{L})}{u(a \to b) \Rightarrow c}\,(\curlywedge c)^2$$

□

We have thus obtained a transparent finite proof search decidability process for all simple extensions of fragments of **nGBI** that contain the $\curlywedge$-structural rules. In detail, given an m-reduced sequent, we investigate the ways in which it can serve as the conclusion of a rule; for logical rules, this includes identifying a connective that matches the connective of the rule. This can be done only in finitely many ways, and if we were to apply upward rules other than $(\curlywedge c)$ and investigate all possibilities, the process would terminate as we stay in the setting of $(n+m)$-reduced sequents and no sequent is allowed to appear twice on a branch; here n is the maximum multiplicity of a rule in the system. However, applications of $(\curlywedge c)$ also need to be investigated, but only in a controlled manner. In particular, we first identify the (for simplicity, say logical) rule that will be applied further up after a possible cluster of contractions, by identifying the logical connective to be investigated; assume that it has multiplicity m. We look at the path of the (LHS of the) sequent upward from that connective and we explore (constructing upward the proof) the application of a cluster of contractions performed in successively decreasing positions of the path; we only consider such cases with at most one for each level between $\circ$ nodes and one final application of a sequence of at most $(m-1)$ contractions (just below the application of the logical rule) at the principal level of the path; then the logical rule is applied and we verify that all of its premises are m-reduced.

6. Finite embeddability property

The finite model property for finitely axiomatizable theories implies the decidability of the equational theory. The stronger result of the decidability of the universal theory follows from the stronger condition of the finite embeddability property. A class of algebras $\mathcal{K}$ is said to have the *finite embeddability property* (FEP) if for every algebra $\mathbf{A}$ in $\mathcal{K}$ and every *finite* partial subalgebra $\mathbf{B}$ of $\mathbf{A}$, there exists a finite algebra $\mathbf{D}$ in $\mathcal{K}$ such that $\mathbf{B}$ embeds into $\mathbf{D}$.

For a type $\mathcal{L}$, with $\{\cdot, 1, \wedge\} \subseteq \mathcal{L} \subseteq \{\cdot, 1, \wedge, \vee, \backslash, /, \to, \top, 0, \bot\}$, let $\mathbf{A}$ be an $\mathcal{L}$-algebra that is a meet-semilattice and unital groupoid such that multiplication is compatible with the order; also if $\vee \in \mathcal{L}$, let the lattice reduct be distributive; if one/both divisions are in $\mathcal{L}$, let $\mathbf{A}$ be residuated from the appropriate side; if $\to \in \mathcal{L}$, let it be the residual of $\wedge$, and if $\bot \in \mathcal{L}$, let it be evaluated as the least element (and $\mathbf{A}$ needs to be bounded). We will abbreviate the above by saying that $\mathbf{A}$ is *at least a distributive semilattice unital groupoid*, or just *at least a dsℓu-groupoid*. Assume also that $\mathbf{B}$ is a partial subalgebra of $\mathbf{A}$, i.e., B is any subset of A, and each operation $f^{\mathbf{A}}$ on A induces a partial operation $f^{\mathbf{B}}$ on B defined by $f^{\mathbf{B}}(b_1, \ldots, b_n) := f^{\mathbf{A}}(b_1, \ldots, b_n)$, if this latter value is in B, and undefined otherwise. Define $(W, \cdot, 1, \wedge)$ to be the $\{\cdot, 1, \wedge\}$-subreduct of $\mathbf{A}$ generated by B. We denote by S_W the set of all *sections* (*unary linear polynomials*) of $(W, \cdot, 1, \wedge)$, namely terms in one variable which appears only once in the term. Let $W' = S_W \times B$, and define $x \; N \; (u, b)$ by $u(x) \leq^{\mathbf{A}} b$. We denote by *id* the identity polynomial ($id(x) = x$), and write $u(\_)$ for every section u. Thus, $u' = u(\_ \circ y)$ denotes the section defined by $u'(x) = u(x \circ y)$. Moreover, we define $x \backslash\!\backslash (u, b) = \{(u(x \circ \_), b)\}$, $(u, b) /\!/ y = \{(u(\_ \circ y), b)\}$, $x \curlywedge (u, b) = \{(u(x \curlywedge \_), b)\}$ and $(u, b) \curlywedge y = \{(u(\_ \curlywedge y), b)\}$.

It is easy to see that $\mathbf{W}_{\mathbf{A},\mathbf{B}} = (W, W', N, \cdot, 1, \backslash\!\backslash, /\!/, \wedge, \curlywedge, \curlywedge)$ is a distributive residuated frame. Moreover, $(\mathbf{W}_{\mathbf{A},\mathbf{B}}, \mathbf{B})$ is a distributive Gentzen frame of the same type as $\mathbf{A}$. To see this, observe that if $\vee$ is present in the type, then distributivity of the lattice is needed for the verification of condition (L$\vee$); also residuation guarantees the conditions for the divisions or implication, if the latter are in the type.

By Corollary 3.2 we obtain the following result.

Corollary 6.1. *The map* $\{\_\}^{\lhd} \colon \mathbf{B} \to \mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$ *is an* $\mathcal{L}$*-embedding of the partial subalgebra* $\mathbf{B}$ *of the at least distributive semilattice unital groupoid* $\mathbf{A}$ *into the nGBI-algebra* $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$.

Theorem 6.2. *If an equation over* $\{\wedge, \vee, \cdot, 1, \top\}$ *is valid in an at least distributive semilattice unital groupoid* $\mathbf{A}$*, then it is also valid in* $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$*, for every partial subalgebra* $\mathbf{B}$ *of* $\mathbf{A}$.

Proof. By Theorem 2.4(1), it is enough to consider simple equations ε, i.e., of the form $t_0 \leq t_1 \vee \cdots \vee t_n$, where t_0 is a linear term. Assume that ε is valid in $\mathbf{A}$. By Theorem 2.4, to show that ε is valid in $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$, it is enough to show that the rule $\frac{t_1 N(u,b) \quad \cdots \quad t_n N(u,b)}{t_0 N(u,b)} \mathrm{R}(\varepsilon)$ is valid in the Gentzen frame $(\mathbf{W}, \mathbf{B})$, namely that if $u(t_i) \leq_{\mathbf{A}} b$ for all $i \in \{1, \ldots, n\}$, then $u(t_0) \leq_{\mathbf{A}} b$; here we abused notation slightly by using, for example, b initially as a metavariable and then as an element of B. The latter implication follows directly from the fact that $\mathbf{A}$ satisfies ε. □

Note that the result can be slightly strengthened, in the case where $\vee$ is not in the type, for quasiequations of the form suggested by $\mathrm{R}(\varepsilon)$.

Let $(F, \circ, \varepsilon, \curlywedge)$ be the free unital bigroupoid over $|B|$ generators, where ε is a unit for both $\circ$ and $\curlywedge$. For $x, y \in F$, we write $x \leq^F y$ iff y is obtained from x by deleting some (possibly none) of the generators; also we stipulate $x \leq^F \varepsilon$. For example, $(x \curlywedge (y \circ z)) \curlywedge ((y \circ x) \circ z) \leq^F x \curlywedge (y \circ x)$. When a term is deleted, such as $(y \circ z)$ or the last occurrence of z, then also the operation symbol next to it is deleted. Note that this relation is a partial order on F, as for distinct non-unit x, y, if $x \leq^F y$, then x is a longer string of symbols than y. We denote by $\mathbf{F}$ the resulting ordered algebra.

Moreover, by Kruskal's tree theorem, F has no infinite antichains and no infinite ascending chains (it is dually well-ordered).

The following lemma shows that $\mathbf{F}$ is residuated in a strong sense. For $x \in F$ and $u \in S_F$, $\frac{x}{u}$ is defined by induction on the structure of u as follows:

$$\frac{x}{id} = x, \frac{x}{u \circ y} = \frac{x /\!\!/ y}{u}, \frac{x}{y \circ u} = \frac{y \backslash\!\!\backslash x}{u}, \frac{x}{u \curlywedge y} = \frac{x \curlyvee\!\!\!/ \, y}{u} \text{ and } \frac{x}{y \curlywedge u} = \frac{y \curlyvee\!\!\!\backslash \, x}{u},$$

where $id = \_$ is the identity section and where $\backslash\!\!\backslash, /\!\!/$ are the residuals of $\circ$ and $\curlyvee\!\!\!\backslash, \curlyvee\!\!\!/$ are the residuals of $\curlywedge$ in $\mathbf{F}$ (see following lemma).

Lemma 6.3. (1) *Assume that $x, y, z, w \in F$, $\bullet \in \{\circ, \curlywedge\}$, and $x \bullet y \leq z \bullet w$. Then one of three things must happen: $x \leq z \bullet w$, $y \leq z \bullet w$, or ($x \leq z$ and $y \leq w$).*

(2) *Both $\circ$ and $\curlywedge$ are residuated in $\mathbf{F}$.*

(3) *For all $x \in F$, $u \in S_F$, and $b \in B$, we have $u(x) \leq^F b$ iff $x \leq^F \frac{b}{u}$.*

Proof. We follow the ideas in [1]. For (1), if the displayed $\bullet$ in $x \bullet y$ is not deleted, then it is the same as the displayed $\bullet$ in $z \bullet w$, and clearly $x \leq z$ and $y \leq w$. If it is deleted, then the displayed $\bullet$ in $z \bullet w$ (and therefore also both z and w) appear completely inside x or completely inside y.

For (2), as $\bullet$ is order-preserving on both sides, we only need to show that there is a y (denoted by $x \backslash_\bullet z$) that is a maximum with respect to $x \bullet y \leq z$. Clearly, if $z = \varepsilon$, then $x \backslash_\bullet z = \varepsilon$. If z is a variable, then $x \backslash_\bullet z$ is ε if z occurs in x, and is z otherwise. Next, assume that $z = z_1 \bullet z_2$ (and z_1, z_2 in $z_1 \bullet z_2$ do not contain redundant occurrences of ε). If $x \leq z$, then $x \backslash_\bullet z = \varepsilon$. If not, and $x \leq z_1$, then $x \backslash_\bullet z = z_2$. Indeed, if $x \bullet y \leq z_1 \bullet z_2$, then we obtain $y \leq z_2$ using (1), $x \not\leq z_1 \bullet z_2$, and the fact that $y \leq z_1 \bullet z_2$ implies $y \leq z_2$. Finally, if $x \not\leq z_1 \bullet z_2$ and $x \not\leq z_1$, then $x \backslash_\bullet z = z$. Indeed, if $x \bullet y \leq z_1 \bullet z_2$, then we obtain $y \leq z$, using (1).

Finally, (3) follows by applying (2) repeatedly. □

Theorem 6.4. *If* $\mathbf{A}$ *is at least an integral* ($x \le 1$) *distributive semilattice unital groupoid and* $\mathbf{B}$ *a finite partial subalgebra of* $\mathbf{A}$*, then the distributive* $r\ell u$*-groupoid* $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$ *is finite.*

Proof. We roughly follow the ideas in [1]. Let $h\colon F \to W$ be the (surjective) homomorphism that extends a fixed bijection $x_i \mapsto b_i$ from its generators to B (and replaces $\circ, \curlywedge, \varepsilon$ by $\cdot, \wedge, 1$, respectively). Note that h is order-preserving, where W inherits the order of $\mathbf{A}$. Moreover, h extends naturally to a surjective map from S_F to S_W, which we denote by h, as well.

Consider the new frame $\mathbf{W}^{\mathbf{F}}_{\mathbf{A},\mathbf{B}} = (F, W', h \circ N, \cdot^{\mathbf{F}}, \mathbin{\backslash\!\backslash}_h, \mathbin{/\!/}_h, \curlywedge, \searrow, \swarrow, \{1\})$, where $x\ (h \circ N)\ z$ iff $h(x)\ N\ z$, and $x \mathbin{\backslash\!\backslash}_h z = h(x) \mathbin{\backslash\!\backslash} z$ and $z \mathbin{/\!/}_h y = z \mathbin{/\!/} h(y)$. Using the fact that h is a homomorphism, it is easy to see that $h \circ N$ is nuclear for $\circ$ and distributively nuclear for $\curlywedge$; thus, $\mathbf{W}^{\mathbf{F}}_{\mathbf{A},\mathbf{B}}$ is a distributive residuated frame.

To prove that $\mathbf{W}^{+}_{\mathbf{A},\mathbf{B}}$ is finite, it suffices to prove that it possesses a finite basis of sets $\{z\}^{\lhd} = \{x \in W : x\ N\ z\}$ for $z \in W'$. For this we show that there are only finitely many sets of the form $\{z\}^{\lhd_h} = \{x \in F : x\ (h \circ N)\ z\}$ for $z \in W'$, as $h[\{z\}^{\lhd_h}] = \{z\}^{\lhd}$. Indeed, for all $x \in W$, there is $x' \in F$ with $h(x') = x$ since h is surjective; so, $x = h(x') \in \{(u,b)\}^{\lhd}$ iff $x' \in \{(u,b)\}^{\lhd_h}$, hence $x \in h[\{(u,b)\}^{\lhd_h}]$. Conversely, if $x \in h[\{(u,b)\}^{\lhd_h}]$, then $x = h(x')$ for some $x' \in \{(u,b)\}^{\lhd_h}$, hence $x = h(x') \in \{(u,b)\}^{\lhd}$.

For $x \in F$ and $(u,b) \in W'$, we have $x \in \{(u,b)\}^{\lhd_h}$ iff $u(h(x)) \le b$ iff $h(v(x)) \le b$, for some $v \in S_F$ such that $h(v) = u$, since h is a surjective homomorphism. Equivalently, $v(x) \in h^{-1}({\downarrow_A} b)$ for some $v \in h^{-1}(u)$. Now, since h is order-preserving, $h^{-1}({\downarrow_A} b)$ is a downset in $\mathbf{F}$, and because $\mathbf{F}$ is dually well-ordered, this downset is equal to ${\downarrow} M_b$ for some *finite* $M_b \subseteq F$. So, the above statement is equivalent to $v(x) \le m$, or to $x \le \frac{m}{v}$, for some $v \in h^{-1}(u)$ and some $m \in M_b$. Consequently, $\{(u,b)\}^{\lhd_h} = {\downarrow}\{\frac{m}{v} : m \in M_b, h(v) = u\}$.

Note that the set $\{\frac{m}{v} : m \in M_b, b \in B, h(v) = u, u \in S_W\}$ is finite, being a subset of the finite set ${\uparrow} \bigcup_{b \in B} M_b$, as $m \le \frac{m}{v}$ (or $v(m) \le m$) by integrality. Thus, there are only finitely many choices for $\{(u,b)\}^{\lhd}$. ☐

Corollary 6.5. *Every variety of integral nDRL/nGBI-algebras axiomatized by equations over the signature* $\{\wedge, \vee, \cdot, 1, \top\}$ *has the FEP.*

This result is improved in [2] for many subvarieties of DRL/GBI where multiplication distributes over meet (recall condition [mdm]), and in this case, the assumption of integrality is considerably weakened.

7. Relating distributive residuated frames and Birkhoff frames

We saw in Section 2 that given a GBI-algebra or a distributive residuated lattice, we can construct a distributive residuated frame. However, in either case, there are often much smaller frames that represent the same algebra. A subset J of A is *join-generating* if every element of A is the join of some subset of J, and the notion of *meet-generating* is defined analogously. A lattice is *join-perfect* if every element is the join of completely join-irreducible elements, and *meet-perfect* if every element is the meet of completely meet-irreducibles. A *perfect* lattice is one that has both these properties. In general, it suffices to choose W, W' to be any join-generating and meet-generating subset of A, respectively, and for perfect lattices one can, in particular, choose W, W' to be the set of all completely join-irreducible and all completely meet-irreducible elements, respectively.

For a perfect distributive residuated lattice $\mathbf{A}$, the Galois algebra $(\mathbf{W_A})^+$ is a doubly-algebraic distributive lattice, and such an algebra is completely determined by the poset $J(\mathbf{A})$ of completely join-irreducible elements with the order inherited from $\mathbf{A}$. In particular, the Galois algebra is isomorphic to the set $D(J(\mathbf{A}))$ of downsets of $J(A)$, ordered by inclusion. For finite distributive lattices, this observation is due to Birkhoff, hence we call $(J(\mathbf{A}), \leq, \circ, E)$ the *Birkhoff frame* of $\mathbf{A}$, where the ternary relation $\circ$ is given by $x \circ y = \{z \in J(\mathbf{A}) : z \leq x \cdot y\}$ and $E = \{x \in J(\mathbf{A}) : x \leq 1\}$. Note that since $\cdot$ is order-preserving, $\circ$ is *up-up-down-closed*, i.e., for all $x, x', y, y', z, z' \in J(\mathbf{A})$

$$\circ(x, y, z) \text{ and } x \leq x' \text{ and } y \leq y' \text{ and } z \geq z' \implies \circ(x', y', z')$$

and E is a downset of $J(\mathbf{A})$.

In general, the definition of a Birkhoff frame $(P, \leq, \circ, E)$ is that $(P, \leq)$ is a poset, $\circ$ is an up-up-down-closed ternary relation on P, and $E \in D(P)$. It is *associative* if ${\downarrow}((x \circ y) \circ z) = {\downarrow}(x \circ (y \circ z))$, and *unital* if ${\downarrow}(x \circ E) = {\downarrow}x = {\downarrow}(E \circ x)$ for all $x, y, z \in P$. While Birkhoff frames are considerably simpler than distributive residuated frames, they are not directly related to sequent systems, and they only capture complete perfect DRLs and complete perfect GBI-algebras. Given a Birkhoff frame $\mathbf{P} = (P, \leq, \circ, E)$, one can define a corresponding distributive frame by $F(\mathbf{P}) = (P, P, \not\geq, \circ, \backslash\!\backslash, /\!/, \curlywedge, \curlywedge\!\!\backslash, /\!\!\curlywedge, E)$, where

- $x \backslash\!\backslash z = P - \{y : x \circ y \not\geq z\}, \quad z /\!/ y = P - \{x : x \circ y \not\geq z)\}$,
- $x \curlywedge y = \{z : z \leq x \text{ and } z \leq y\} = {\downarrow}x \cap {\downarrow}y$,
- $x \curlywedge\!\!\backslash\, z = P - \{y : x \curlywedge y \not\geq z\}, \quad z /\!\!\curlywedge\, y = P - \{x : x \curlywedge y \not\geq z)\}$.

Theorem 7.1. *If* $\mathbf{P}$ *is a Birkhoff frame, then* $F(\mathbf{P})$ *is a distributive residuated frame and* $F(\mathbf{P})^+ = D(P)$.

Proof. Let $\mathbf{P} = (P, \leq, \circ, E)$ be a Birkhoff frame. We need to check that $F(\mathbf{P})$ satisfies the nuclear conditions for $\circ$ and $\curlywedge$, and the distributive conditions $[\curlywedge a]$, $[\curlywedge e]$, $[\curlywedge i]$, and $[\curlywedge c]$.

For [nuc$\circ$], we show that $x \circ y \; N \; z \iff y \; N \; x \backslash\!\backslash z$, where N is the relation $\not\geq$. Let $D = \{y : x \circ y \not\geq z\}$ and note that this set is down-closed since $\circ$ is up-closed in the second argument. Hence, $x \backslash\!\backslash z = P - D$ is up-closed, from which it follows that $y \not\geq x \backslash\!\backslash z$ is equivalent to $y \notin x \backslash\!\backslash z$, and this in turn is equivalent to $y \in D$, i.e., $x \circ y \not\geq z$.

The second equivalence for [nuc$\circ$] is similar, and the same reasoning applies to [nuc$\curlywedge$] after observing that $\curlywedge$ is also up-closed in the first and second argument.

The conditions [$\curlywedge a$] and [$\curlywedge e$] follow from the associativity and commutativity of intersection. For [$\curlywedge i$], note that $x \not\geq z$ implies $x' \not\geq z$ for all $x' \leq x$, and for [$\curlywedge c$], if $x \curlywedge x \not\geq z$, then $({\downarrow}x) \not\geq z$, so in particular, $x \not\geq z$.

Finally, note that if $N = \not\geq$, then $x^{\rhd} = \{y : x \not\geq y\} = P - {\downarrow}x$, so $\gamma_N\{x\} = x^{\rhd\lhd} = (P - {\downarrow}x)^{\lhd} = {\downarrow}x$. Hence, $F(\mathbf{P})^+$ has all downsets of $\mathbf{P}$ as elements. $\square$

Furthermore, distributive residuated frames of the form $F(\mathbf{P})$ satisfy the following two conditions from [11]. A Galois relation $N \subseteq W \times W'$ is *separating* if the maps $x \mapsto \gamma_N\{x\}$ and $y \mapsto \gamma'_N\{y\}$ are one-to-one (where $\gamma'_N\{y\} = \{y\}^{\lhd\rhd}$ for $y \in W'$), and N is *reduced* if both

$$\forall x \in W\, \exists y \in W' \text{ s.t. } \neg(x\ N\ y) \text{ and } (\gamma_N\{x\} - \{x\})\ N\ y \text{ and}$$
$$\forall y \in W'\, \exists x \in W \text{ s.t. } \neg(x\ N\ y) \text{ and } (\gamma'_N\{y\} - \{y\})\ N\ z$$

hold. The notion of reduced is easily seen to be equivalent to $\gamma_N\{x\} - \{x\}$ being γ_N-closed and $\gamma'_N\{y\}-\{y\}$ being γ'_N-closed for all $x \in W$ and $y \in W'$. In the Galois algebra, this means that all $\gamma_N\{x\}$ are completely join-irreducible. Conversely, every completely join-irreducible is of the form $\gamma_N\{x\}$ since any γ-closed set is the join of singleton closures. Reduced also implies separating since if N is not separating, then there exist $x_1 \neq x_2 \in W$ such that $\gamma_N\{x_1\} = \gamma_N\{x_2\}$, whence $\gamma_N\{x_1\} - \{x_1\}$ contains x_2, and its closure will add x_1 again.

Now let $\mathbf{W}$ be a reduced distributive residuated frame, and define $G(\mathbf{W}) = (\{\gamma_N\{x\} : x \in W\}, \subseteq, \hat{\circ}, \hat{E})$ where $\hat{\circ} = \{(\gamma_N\{x\}, \gamma_N\{y\}, \gamma_N\{z\}) : \circ(x,y,z)\}$ and $\hat{E} = \{\gamma_N\{x\} : x \in E\}$.

Theorem 7.2. *If* $\mathbf{W}$ *is a reduced distributive residuated frame, then* $G(\mathbf{W})$ *is a Birkhoff frame.*

Proof. In a reduced frame, Galois-closed subsets of the form $\gamma_N\{x\}$ are exactly the completely join-irreducible elements of the Galois algebra. Hence, $G(\mathbf{W}) = J(\mathbf{W}^+)$, and since $\mathbf{W}^+$ with the subset-inclusion order is a distributive lattice, it follows that $G(\mathbf{W})$ is a Birkhoff frame. $\square$

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