Algebra Univers. 78 (2017) 363–376
DOI 10.1007/s00012-017-0466-8
Published online October 17, 2017

Algebra Universalis

The FEP for some varieties of fully distributive knotted residuated lattices

Riquelmi Cardona and Nikolaos Galatos

Abstract. We prove the finite embeddability property for a wide range of varieties of fully distributive residuated lattices and FL-algebras. Part of the axiomatization is assumed to be a knotted inequality and some appropriate generalization of commutativity. The construction is based on distributive residuated frames and extends to subvarieties axiomatized by any division-free equation.

1. Introduction

A class of algebras $\mathcal{K}$ is said to have the *finite embeddability property* (FEP) if for every algebra $\mathbf{A}$ in $\mathcal{K}$ and every *finite* partial subalgebra $\mathbf{B}$ of $\mathbf{A}$, there exists a finite algebra $\mathbf{D}$ in $\mathcal{K}$ such that $\mathbf{B}$ embeds into $\mathbf{D}$. Recall that $\mathbf{B}$ is a finite partial subalgebra of $\mathbf{A}$ if B is a finite subset of A together with partial operations $f^{\mathbf{B}}$ for each n-ary operation $f^{\mathbf{A}}$ on A, where $f^{\mathbf{B}}$ is given by:

$$f^{\mathbf{B}}(b_1,\dots,b_n) = \begin{cases} f^{\mathbf{A}}(b_1,\dots,b_n), & \text{if } f^{\mathbf{A}}(b_1,\dots,b_n) \in \mathbf{B}, \\ \text{undefined}, & \text{if } f^{\mathbf{A}}(b_1,\dots,b_n) \notin \mathbf{B}. \end{cases}$$

The FEP is a strong property, as it yields decidability of the universal first order theory of any finitely axiomatizable class, and generation by finite algebras for (quasi)varieties.

A residuated lattice is an algebra $(A,\wedge,\vee,\cdot,\backslash,/,1)$ where $(A,\wedge,\vee)$ is a lattice, $(A,\cdot,1)$ is a monoid, and for all $a,b,c \in A$, we have $ab \leq c$ iff $a \leq c/b$ iff $b \leq a\backslash c$. As usual, we write $x \leq y$ for $x \vee y = y$. It is not hard to see that the class of residuated lattices is a variety. For more on residuated lattices, see [8] for example.

The FEP was studied for various classes of residuated lattices by W. Blok and C. van Alten in a series of papers [1, 2, 3]. Since residuated lattices form algebraic semantics for substructural logics (see [8]), the FEP for a variety of residuated lattices yields the strong finite model property for the corresponding

Presented by J. Raftery.
Received September 1, 2015; accepted in final form April 4, 2017.
2010 *Mathematics Subject Classification*: Primary: 06F05, Secondary: 08B15, 03B47, 03G10,
Key words and phrases: substructural logic, fully distributive knotted residuated lattice, decidability, finite embeddability property, well quasi order, better quasi order.
The second author acknowledges the support of the grants Simons Foundation 245805 and FWF project START Y544-N23.

Birkhäuser

substructural logic. In that respect, the FEP is a very desirable, but also fairly rare, property.

In [6], among other things, the FEP is established for all subvarieties of integral (satisfying $x \leq 1$) residuated lattices axiomatized by equations over the language of join, multiplication, and 1; the method used is that of *residuated frames.* In that respect, integrality is a strong condition, but already in [17] it is replaced by a weaker condition,

$$x^m \leq x^n, \text{ for any } m \neq n, \text{ with } m \geq 1, n \geq 0,$$

known as a *knotted inequality*; the price to pay for such a generalization is to assume commutativity (of multiplication). The variety of all residuated lattices satisfying a knotted rule does not have the FEP [10] (for more negative results, see [2]); however, in [4] it is shown that the FEP holds for an infinite collection of noncommutative varieties satisfying a knotted rule. (Further subvarieties of these axiomatized by equations of over the language $\{\vee, \cdot, 1\}$ have the FEP.) Each of them is axiomatized by a monoid identity, the simplest of which is $xyx = xxy$, and in general, it is given relative to a vector $a = (a_0, a_1, \ldots, a_r)$ of natural numbers whose sum is $r + 1$ and whose product is 0 (namely an additive, nontrivial, decomposition of the number $r + 1$):

$$xy_1xy_2 \cdots y_rx = x^{a_0}y_1x^{a_1}y_2 \cdots y_rx^{a_r}. \tag{a}$$

In [7], it is shown, by developing a theory for a distributive version of residuated frames, that in the presence of integrality, we can obtain the FEP for all varieties of residuated lattices that are distributive (as lattices) and are axiomatized over the language $\{\wedge, \vee, \cdot, 1\}$. For example, the FEP is established for all integral and *fully distributive* residuated lattices. In all residuated lattices, multiplication distributes over join, but if we further know that both multiplication and join distribute over meet, we call the residuated lattice *fully distributive.* Algebras such as lattice-ordered groups, Heyting algebras, and all semilinear residuated lattices (including MV-algebras and BL-algebras), are fully distributive residuated lattices. Furthermore, fully distributive residuated lattices admit a nice representation theorem [5].

In this paper, we relax the integrality condition with a combination of a knotted inequality and an equation (a), for some decomposition a of a natural number, thus obtaining infinitely many varieties of fully distributive residuated lattices with the FEP, outside the setting of integrality or commutativity.

2. The construction of the algebra D

We consider a variety $\mathcal{D}^n_m(a)$ of fully distributive residuated lattices axiomatized by a knotted rule $x^m \leq x^n$ and an equation of the form (a). (We may also assume that the axiomatization of the variety contains further equations over the language $\{\wedge, \vee, \cdot, 1\}$ as explained in [7].) We will show that $\mathcal{D}^n_m(a)$ has the FEP.

We consider an algebra $\mathbf{A}$ in $\mathcal{D}_m^n(a)$ and a finite partial subalgebra $\mathbf{B}$ of $\mathbf{A}$; let $B = \{b_1, b_2, \ldots, b_k\}$. Define $\mathbf{W} = (W, \circ, \curlywedge, \varepsilon)$ as the $\{\cdot, \wedge, 1\}$-subalgebra of $\mathbf{A}$ generated by B (note that we use different notation for the restriction of the operations of A on W). Observe that polynomials over $(W, \circ, \curlywedge, \varepsilon)$ containing a single variable x must look like $u(x) = (y \circ x \circ z) \curlywedge w$ for $y, z \in W$ and $w \in W \cup \{\top\}$. Here we write $y \circ x \circ z \curlywedge \top$ for $y \circ x \circ z$, in order to have uniform notation, where $\top$ is a new symbol used only for this purpose. Since multiplication distributes over meet, we can even assume that y and z do not have $\curlywedge$ in them. We denote the set of all such polynomials by S_W and we define the set $W' = S_W \times B$, as well as the relation N from W to W', given by

$$x \; N \; (u, b) \text{ iff } u^{\mathbf{A}}(x) \leq^{\mathbf{A}} b.$$

Going forward, we will suppress the upper index in $u^{\mathbf{A}}(x)$ and simply write $u(x)$. For $X \subseteq W$ and $Y \subseteq W'$, we define $X^{\rhd} = \{z \in W' : (\forall x \in X)(x \; N \; z)\}$ and $Y^{\lhd} = \{w \in W : (\forall y \in Y)(w \; N \; y)\}$, and also define the map γ_N on $\mathscr{P}(W)$ by $\gamma_N(X) = X^{\rhd\lhd}$. We denote by $\gamma_N[\mathscr{P}(W)]$ the image of this map and call its members *closed sets*. The algebra

$$\mathbf{W}^+_{\mathbf{A},\mathbf{B}} = (\gamma_N[\mathscr{P}(W)], \cap, \cup_{\gamma_N}, \circ_{\gamma_N}, \backslash, /, \gamma_N(\{\varepsilon\}))$$

is called the *Galois algebra* of $\mathbf{W}_{\mathbf{A},\mathbf{B}}$, where for $X, Y \subseteq W$ we define $X \bullet_N Y = \gamma_N(X \bullet Y)$, for all operations $\bullet \in \{\circ, \cup, \curlywedge\}$.

Lemma 2.1 ([7]). *The structure* $\mathbf{W}_{\mathbf{A},\mathbf{B}} = (W, W', N, \circ, \curlywedge, \varepsilon)$ *supports a distributive residuated frame structure in the sense of* [7]. *Therefore,*

(1) *The algebra* $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ *is a distributive residuated lattice (and* $\curlywedge_N$ *is intersection).*
(2) *The map* $b \mapsto \{(\mathrm{id}, b)\}^{\lhd}$ *is a (partial algebra) embedding of* $\mathbf{B}$ *into* $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$.
(3) $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ *is in* $\mathcal{D}_m^n(a)$.
(4) *Every set in* $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ *is an intersection of sets of the form* $\{(u, b)\}^{\lhd}$ *for* $u \in S_W, b \in B$.
(5) *If* $\mathbf{A}$ *satisfies an equation over the language* $\{\wedge, \vee, \cdot, 1\}$, *then so does* $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$.

We will take $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ to play the role of $\mathbf{D}$ in the definition of the FEP. The above lemma provides the embedding and also membership in $\mathcal{D}_m^n(a)$, so the only thing that remains to be shown is the finiteness of $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$.

By Lemma 2.1(4), it suffices to show that there are only finitely many closed sets of the form $\{(u, b)\}^{\lhd}$ for $u \in S_W, b \in B$. In particular, since B is finite, it suffices to show that for each $b \in B$, the set $C_b = \{\{(u, b)\}^{\lhd} : u \in S_W\}$ is finite. We will show that in C_b ordered under inclusion, all antichains, descending chains, and ascending chains are finite, thus yielding finiteness.

Toward this goal, we will construct a relatively free semilattice monoid $\mathbf{F}$ and a surjective homomorphism $h \colon \mathbf{F} \to \mathbf{W}$. Recall that a *semilattice monoid* is an algebra $(L, \circ, \wedge, 1)$, where $(L, \circ, 1)$ is a monoid, $(L, \wedge)$ is a semilattice and multiplication distributes over meet. Furthermore, it will be important to show that the underlying poset of $\mathbf{F}$ is well partially ordered (it has no infinite descending chains or antichains) or dually well partially ordered (it has no infinite ascending chains or antichains).

To guarantee the existence of such a map h, the auxiliary algebra $\mathbf{F}$ will be chosen to be free over a class of semilattice monoids that includes $\mathbf{W}$ and satisfies the identity (a). $\mathbf{F}$ will satisfy $x^m \leq x^n$ only for selected elements; therefore, it will not be the free algebra in any variety of semilattice monoids containing $\mathbf{W}$, but it will be free enough for our purposes.

We begin by recalling the construction of the free semilattice on a poset and extending it to the construction of a semilattice monoid from a pomonoid.

3. The semilattice construction $\mathscr{M}$.

The construction of a semilattice monoid from a given pomonoid is fairly standard; we include the details of the construction for completeness. Given a poset $\mathbf{P}$, we define $\mathscr{M}(P)$ as the set of all nonempty finitely generated upsets of $\mathbf{P}$. Note that if A is a nonempty finitely generated upset of $\mathbf{P}$, then the set mA of its minimal elements is nonempty and $A = {\uparrow}mA$; in this section only, we will feel free to use the letters A and B to denote sets other than the ones fixed in the definition of the FEP. Also, the union of two nonempty finitely generated upsets A and B is finitely generated by $m(A \cup B) \subseteq mA \cup mB$. Clearly, $\mathscr{M}(P)$ supports a meet semilattice under the operation $A \wedge B = A \cup B$, for $A, B \in \mathscr{M}(P)$. If $\mathbf{P}$ is a pomonoid, then for $A, B \subseteq P$, we can further define

$$A \bullet B = {\uparrow}(AB), \text{ where } AB = \{ab : a \in A, b \in B\}.$$

Lemma 3.1. *If $\mathbf{P}$ is a pomonoid and $A, B \in \mathscr{M}(P)$, then $A \bullet B \in \mathscr{M}(P)$. Specifically,*

$$A \bullet B = {\uparrow}[(mA)(mB)].$$

Proof. Using order preservation, it is easy to see that for all $C, D \subseteq P$, we have $({\uparrow}C)({\uparrow}D) \subseteq {\uparrow}[CD]$. Then for $A, B \in \mathscr{M}(P)$, we have

$$AB = ({\uparrow}mA)({\uparrow}mB) \subseteq {\uparrow}[(mA)(mB)] \subseteq {\uparrow}[AB],$$

so $A \bullet B = {\uparrow}(AB) \subseteq {\uparrow\uparrow}[(mA)(mB)] \subseteq {\uparrow\uparrow}AB$. In particular, we have that $A \bullet B = {\uparrow}[(mA)(mB)]$. □

Given a poset $\mathbf{P}$, we define its semilattice extension $\mathscr{M}(\mathbf{P}) = (\mathscr{M}(P), \wedge)$. Also, given a pomonoid $\mathbf{P}$, we define its semilattice monoid extension as

$$\mathscr{M}(\mathbf{P}) = (\mathscr{M}(P), \wedge, \bullet, {\uparrow}\{1\}).$$

Lemma 3.2. *If $\mathbf{P}$ is a pomonoid, then $\mathscr{M}(\mathbf{P})$ is a semilattice monoid under the operations defined above.*

Proof. It is clear that $\uparrow\{1\}$ is the identity for $\bullet$. Multiplication is associative as shown below. Let $A, B, C \in \mathscr{M}(P)$.

$$\begin{aligned} A \bullet (B \bullet C) &= \uparrow\{ad : a \in A, d \in B \bullet C\} \\ &= \uparrow\{ad : a \in A, d \in \uparrow\{bc : b \in B, c \in C\}\} \\ &= \uparrow\{a(bc) : a \in A, b \in B, c \in C\} \quad \text{(by order preservation)} \\ &= \uparrow\{(ab)c : a \in A, b \in B, c \in C\} \\ &= \uparrow\{dc : d \in \uparrow\{ab : a \in A, b \in B\}, c \in C\} = (A \bullet B) \bullet C. \end{aligned}$$

To see that multiplication distributes over meet, let $A, B, C \in \mathscr{M}(P)$.

$$\begin{aligned} A \bullet (B \wedge C) = A \bullet (B \cup C) &= \uparrow\{ad : a \in A, d \in B \cup C\} \\ &= \bigcup_{\substack{a \in A \\ d \in B \cup C}} \uparrow\{ad\} = \Big(\bigcup_{\substack{a \in A \\ b \in B}} \uparrow\{ab\}\Big) \cup \Big(\bigcup_{\substack{a \in A \\ c \in C}} \uparrow\{ac\}\Big) \\ &= A \bullet B \wedge A \bullet C. \end{aligned}$$

The other equality, $(B \wedge C) \bullet A = B \bullet A \wedge C \bullet A$, can be proven using a symmetric argument. □

The next lemma shows that we can extend pomonoid homomorphisms (i.e., order-preserving monoid homomorphisms) to semilattice homomorphisms on the semilattice extensions created by $\mathscr{M}$, whence $\mathscr{M}$ is a functor from the category of pomonoids to the category of semilattice monoids.

Lemma 3.3. *If* $\mathbf{P}$ *and* $\mathbf{Q}$ *are pomonoids and* $f \colon \mathbf{P} \to \mathbf{Q}$ *is a (surjective) pomonoid homomorphism then* $\mathscr{M}f \colon \mathscr{M}(\mathbf{P}) \to \mathscr{M}(\mathbf{Q})$ *is a (surjective) semilattice monoid homomorphism, where we have* $\mathscr{M}f(A) = \uparrow\{f(a) : a \in mA\}$ *for* $A \in \mathscr{M}(P)$*.*

Proof. We have that $\mathscr{M}f(A)$ is in $\mathscr{M}(Q)$ because it is an upset and the set mA is finite. Since f is order-preserving, we obtain that for any $A, E \subseteq P$ such that $mA \subseteq E \subseteq A$,

$$\uparrow\{f(a) : a \in mA\} = \uparrow\{f(a) : a \in E\} = \uparrow\{f(a) : a \in A\}.$$

To show that $\mathscr{M}f$ is a homomorphism, note that, based on the above observation, we have that for all $A, B \in \mathscr{M}(P)$,

$$\begin{aligned} \mathscr{M}f(A \bullet B) &= \uparrow\{f(d) : d \in m(\uparrow[(mA)(mB)])\} \\ &= \uparrow\{f(d) : d \in m((mA)(mB))\} \\ &= \uparrow\{f(d) : d \in (mA)(mB)\} = \uparrow\{f(a)f(b) : a \in (mA), b \in (mB)\} \\ &= \uparrow\big(\{f(a) : a \in mA\} \cdot \{f(b) : b \in mB\}\big) \\ &= \uparrow\{f(a) : a \in mA\} \bullet \uparrow\{f(b) : b \in mB\} = \mathscr{M}f(A) \bullet \mathscr{M}f(B), \\ \mathscr{M}f(A \wedge B) &= \uparrow\{f(d) : d \in m(A \cup B)\} = \uparrow\{f(d) : d \in mA \cup mB)\} \\ &= \uparrow\{f(d) : d \in mA\} \cup \uparrow\{f(d) : d \in mB\} = \mathscr{M}f(A) \wedge \mathscr{M}f(B). \end{aligned}$$

Let $C \in \mathscr{M}(Q)$ and $mC = \{c_1, \ldots, c_j\}$. If f is surjective, then for every c_i, there exists a $b_i \in P$ such that $f(b_i) = c_i$. Let $B = {\uparrow}\{b_i : 1 \leq i \leq j\}$. It is clear that $B \in \mathscr{M}(P)$ and $\mathscr{M}f(B) = C$. □

Clearly, the above lemma is an extension of the analogous result for posets and semilattices.

Finally, we provide a connection between a semilattice monoid and the free semilattice monoid on its underlying pomonoid. We know that the monoid reduct of a semilattice monoid is actually a pomonoid because $a(b \wedge c) = ab \wedge ac$ implies $b \leq c \Rightarrow ab \leq ac$, and similarly for multiplication on the right.

Given a semilattice $\mathbf{S} = (S, \wedge)$, we denote by $\mathbf{S}_p = (S, \leq)$ its poset reduct. Also, given a semilattice monoid $\mathbf{S} = (S, \wedge, \cdot, 1)$, we denote by $\mathbf{S}_p = (S, \leq, \cdot, 1)$ its corresponding pomonoid reduct.

Lemma 3.4. *For a semilattice monoid* $\mathbf{S} = (S, \wedge, \cdot, 1)$*,* $\psi\colon \mathscr{M}(\mathbf{S}_p) \to \mathbf{S}$*, defined by* $\psi(A) = \bigwedge_{a \in A} a$*, is a surjective homomorphism.*

Proof. For $A, B \in \mathscr{M}(S)$, we have

$$\psi(A \wedge B) = \psi(A \cup B) = \bigwedge_{c \in A \cup B} c = \bigwedge_{c \in A} c \wedge \bigwedge_{c \in B} c = \psi(A) \wedge \psi(B),$$

and

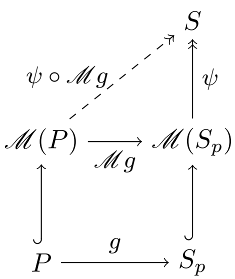
$$\begin{aligned} \psi(A \bullet B) &= \bigwedge({\uparrow}\{ab : a \in A, b \in B\}) = \bigwedge(\{ab : a \in A, b \in B\}) \\ &= \Big(\bigwedge_{a \in A} a\Big) \cdot \Big(\bigwedge_{b \in B} b\Big) = \psi(A) \cdot \psi(B). \end{aligned}$$

For $x \in S$, we have ${\uparrow}\{x\} \in \mathscr{M}(S)$ and $\psi({\uparrow}\{x\}) = x$, so ψ is surjective. □

Clearly, the above lemma works in the absence of a monoid structure in the signature, where $\mathbf{S}$ is a semilattice.

Lemma 3.5. *Let* $\mathbf{P}$ *be a pomonoid,* $\mathbf{S}$ *a semilattice monoid and* $g\colon \mathbf{P} \to \mathbf{S}_p$ *a poset homomorphism. Also assume that there is a* $T \subseteq g[P]$ *such that for all* $s \in S$*, we have* $s = \bigwedge T_s$ *for some finite* $T_s \subseteq T$*. Then* $\psi \circ \mathscr{M}g$ *is surjective.*

Proof. The following diagram describes the situation.



Given $s \in S$, there exists $T_s \subseteq T$ and a finite subset $P_s \subseteq P$ such that $s = \bigwedge T_s$ and $g[P_s] = T_s$. Then

$$(\psi \circ \mathscr{M}g)({\uparrow}P_s) = \bigwedge {\uparrow}g[m{\uparrow}P_s] = \bigwedge {\uparrow}g[P_s] \quad (g \text{ is order-preserving})$$

$$= \bigwedge g[P_s] = \bigwedge T_s = s.$$

Hence, $\psi \circ \mathscr{M} g$ is surjective. □

4. Well partially ordered sets

Recall that a poset is *well partially ordered* (*wpo*) if it contains no infinite descending chains and no infinite antichains. Examples of well partially ordered sets are $(\mathbb{N}, \leq)$ and $(\mathbb{N}^k, \leq)$. If $(P, \leq)$ is a well partially ordered set, then it is known that for each $k \in \mathbb{N}$, P^k is a well partially ordered set under the direct product ordering of P^k. Furthermore, disjoint unions, subposets and homomorphic images of well partially ordered sets are well partially ordered sets [9]. *Well quasi-ordered sets* (abbreviated wqo) are defined in an analogous manner. We will use *dwpo* (*dually well partially ordered*) to refer to posets that are the duals of wpo's.

The following definitions were introduced in [14]. Consider a poset (or quasi-ordered set) $\mathbf{P}$. An infinite sequence $p_1, p_2, \ldots$ of elements in P is called *good* if there exist positive integers i, j such that $i < j$ and $p_i \leq p_j$. Similarly, we say that a sequence is *bad* if $i < j$ implies $p_i \not\leq p_j$ for every i, j. It is easy to verify that $(A, \leq)$ is a wpo (or wqo) iff every infinite sequence of elements of A is good iff there are no bad sequences in A. The following result is an easy consequence of this characterization.

Lemma 4.1. *If* $\mathbf{P}$ *is a wpo, then* $(\mathscr{P}(P), \subseteq)$ *does not contain any infinite descending chain of downsets. Dually, if* $\mathbf{P}$ *is a dwpo, then* $(\mathscr{P}(P), \subseteq)$ *does not contain any infinite descending chain of upsets.*

Proof. Suppose that $\mathbf{P}$ is a wpo and there exists an infinite descending chain of downsets $D_1 \supset D_2 \supset \cdots$ in $\mathscr{P}(P)$. For each $i \in \mathbb{Z}^+$, we choose $d_i \in D_i \setminus D_{i+1}$ and show that $(d_i)_{i \in \mathbb{N}}$ is a bad sequence in P, contradicting the fact that $\mathbf{P}$ is a wpo. Indeed, by definition, $d_i \notin D_{i+1}$ for all i, so if $i < j$, then $d_i \notin D_j$. This implies that $d_i \not\leq^{\mathbf{P}} d_j$ because otherwise $d_i \leq^{\mathbf{P}} d_j \in D_j$ and the fact that D_j is a downset would imply that $d_i \in D_j$. The second part of the lemma follows by duality. □

Given a poset $\mathbf{P}$, we define an order $\preceq_{\exists}^{\forall}$ on $\mathscr{P}(P)$ by

$$A \preceq_{\exists}^{\forall} B \Leftrightarrow (\forall y \in B)(\exists x \in A)[x \leq y] \Leftrightarrow {\uparrow} A \supseteq B.$$

To characterize the conditions under which $(\mathscr{P}(P), \preceq_{\exists}^{\forall})$ is a wqo, we will utilize the concept of a *better quasi-ordered set* (bqo). The definition of bqo is presented below and the next lemma provides the majority of the results that we need to know about it.

By $[\omega]^{<\omega}$, we denote the set of finite, strictly increasing sequences with terms in ω, and we define a relation $\lhd$ on $[\omega]^{<\omega}$ as follows: $s \lhd t$ iff there is u such that s is a strict initial segment of u and $t = {}_*u$; here ${}_*u$ is obtained from the sequence u by omitting the first term.

A *block* is a subset B of $[\omega]^{<\omega}$ that contains an initial segment of every infinite subset of $\bigcup B$. Given a quasi-order Q, a *Q-pattern* is a function from a block B into Q. A Q-pattern $f\colon B \to Q$ is said to be *bad* if $f(s) \not\leq_Q f(t)$ for every pair $s, t \in B$ such that $s \lhd t$. Otherwise, f is *good.* A quasi-order Q is a bqo if there is no bad Q-pattern.

Better quasi-ordered sets were introduced as an strengthening of wqo's by Nash-Williams [15]. The following properties are known about them.

Lemma 4.2 ([15], [13], [12])**.**

(1) *Every bqo is a wqo.*
(2) *Finite partial orders and well ordered chains are bqo.*
(3) *If* $\mathbf{P}$ *and* $\mathbf{Q}$ *are bqo's, then their direct product* $\mathbf{P} \times \mathbf{Q}$ *and disjoint union* $\mathbf{P} \dot{\cup} \mathbf{Q}$ *are also bqo's.*
(4) $(Q, \leq)$ *is a bqo iff* $(\mathscr{P}(Q), \preceq_{\exists}^{\forall})$ *is a bqo.*
(5) *Every subposet of a bqo is a bqo.*

The following lemmas draw a connection between the construction $\mathscr{M}$, wpo's, dwpo's and bqo's.

Lemma 4.3. *If a pomonoid* $\mathbf{P}$ *is dually well partially ordered, then so is* $\mathscr{M}(\mathbf{P})$.

Proof. The order in $\mathscr{M}(\mathbf{P})$ is given by $A \leq_{\mathscr{M}} B$ iff $A = A \wedge B = A \cup B$, for $A, B \in \mathscr{M}(P)$, so the elements of $\mathscr{M}(P)$ are ordered under reverse inclusion. Since $\mathbf{P}$ is dually well partially ordered, all finitely generated upsets are actually finite, so there are no infinite ascending chains in $\mathscr{M}(\mathbf{P})$.

To show that there are no infinite antichains, we will prove that every antichain in $\mathscr{M}(\mathbf{P})$ would produce an antichain in a well known dwqo. We will define the order $\leq_{\mathscr{P}}$ on the set $\mathscr{P}_{\text{fin}}(P)$ of finite subsets of P. For $A, B \in \mathscr{P}_{\text{fin}}(P)$, we write $A \leq_{\mathscr{P}} B$ iff there exists an injective map $f\colon A \to B$ such that $a \geq^{\mathbf{P}} f(a)$ for all $a \in A$. It follows from (the dual version of) a result of Higman [9] (proved in the context of wqo's) that $(\mathscr{P}_{\text{fin}}(P), \leq_{\mathscr{P}})$ is also a dwpo.

We first show that for $A, B \in \mathscr{M}(P)$, $A \leq_{\mathscr{P}} B$ implies $A \subseteq B$. (Note that the converse is trivially true.) Indeed, $A \leq_{\mathscr{P}} B$ implies that there exists an injective function $f\colon A \to B$ such that $f(a) \leq^{\mathbf{P}} a$ for all $a \in mA$ and, since B is an upset, we obtain

$$A = \bigcup_{a \in mA} \uparrow\{a\} \subseteq \bigcup_{a \in mA} \uparrow\{f(a)\} \subseteq \bigcup_{b \in B} \uparrow\{b\} = B.$$

The contrapositive of $A \leq_{\mathscr{P}} B \Rightarrow A \subseteq B$ implies that if $A \not\subseteq B$ and $B \not\subseteq A$, then A and B are incomparable in $(\mathscr{P}_{\text{fin}}(P), \leq_{\mathscr{P}})$, so every antichain in $(\mathscr{M}(P), \leq_{\mathscr{M}})$ is an antichain in $(\mathscr{P}_{\text{fin}}(P), \leq_{\mathscr{P}})$. This implies $(\mathscr{M}(P), \leq_{\mathscr{M}})$ has no infinite antichains and is dually well partially ordered. □

While the previous result holds for arbitrary dwpo's, it is not true for arbitrary wpo's. The poset $\mathbf{R} = (\{(i, j) \in \mathbb{N}^2 : i < j\}, \leq_r)$, known as the *Rado* *structure*, where

$$(i_1, j_1) \leq_r (i_2, j_2) \Leftrightarrow (i_1 = i_2 \text{ and } j_1 < j_2) \text{ or } (j_1 < i_2),$$

is a wpo for which its finite subsets ordered by $\preceq^{\forall}_{\exists}$ have an infinite antichain (see [16] and [11]), hence $(\mathscr{M}(R), \leq_{\mathscr{M}})$ is not a wpo. However, a similar result holds under a stronger assumption.

Lemma 4.4. *If $(Q, \leq^{\mathbf{Q}})$ is a bqo, then $(\mathscr{M}(Q), \leq_{\mathscr{M}})$ is a bqo.*

Proof. Recall that the construction $\mathscr{M}$ orders upsets by reverse inclusion. For $A, B \in \mathscr{M}(Q)$, $A \supseteq B$ is equivalent to the condition that for all $y \in B$, there exists $x \in A$ such that $x \leq y$. Hence, the order $\preceq^{\forall}_{\exists}$ coincides with the one in $\mathscr{M}(Q)$ when we restrict our attention to finitely generated upsets. Therefore, $(\mathscr{M}(Q), \leq_{\mathscr{M}})$ is a substructure of $(\mathscr{P}(Q), \preceq^{\forall}_{\exists})$. Since the latter is a bqo by Lemma 4.2(4), it follows that so is the former. □

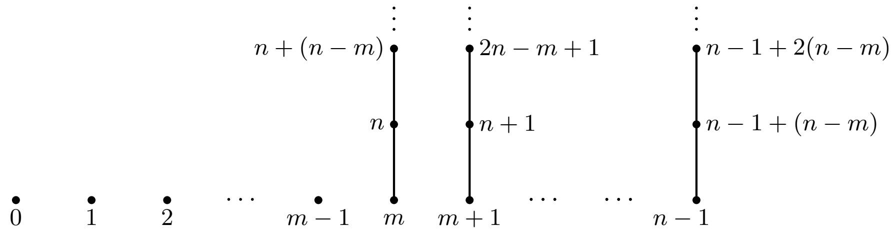
5. The FEP for $\mathcal{D}^n_m(a)$.

In [4], it is shown that given a knotted inequality $x^m \leq x^n$, an equation of the form (a) and a positive integer k, we can construct a k-generated pomonoid $\mathbf{H} = \mathbf{H}(a, m, n)$ that is free for the class of pomonoids that satisfy the knotted inequality and (a). It is further shown in [4] that $\mathbf{H}$ is dually well partially ordered when $m > n$, and a well partially ordered set when $m < n$. Let $X_k = \{x_1, \ldots, x_k\}$, where $k = |B|$; in this section, $\mathbf{A}$ and $\mathbf{B}$ are as in the definition of the FEP.

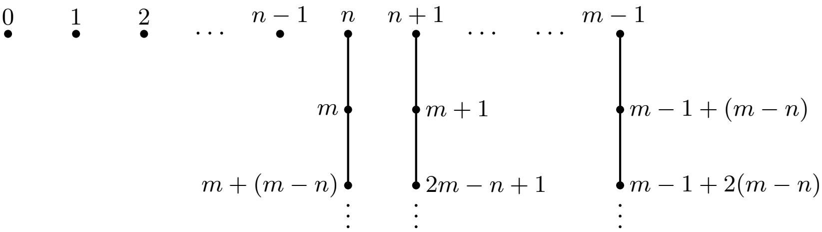
Here we summarize the construction of $\mathbf{H}$ given in [4], and we also mention some of its properties. For each $\ell = (\ell_0, \ell_1, \ell_2)$, with $\ell_1 > 0$, $\ell_1 + \ell_2 < \ell_0$, we set $d_\ell = \ell_0 - \ell_1 - \ell_2$. Given $s \in X_k^*$, we denote by $|s|_{x_i}$ the number of occurrences of x_i in s. We define $\alpha_N(s)$ to be the element of X_k^* obtained from s by moving next to the ℓ_1th occurrence of x_i the $(\ell_1 + 1)$th, the $(\ell_1 + 2)$th, and up to the $(|s|_{x_i} - \ell_2)$th occurrence of x_i, simultaneously for each x_i with at least ℓ_0-many occurrences in s. Thus, by collecting all these consecutive occurrences next to the ℓ_1th occurrence of x_i, we obtain a power of x_i. If we further truncate the exponent of this power to be at most d_ℓ, for each x_i, then we obtain the element $\alpha_D(s)$. In [4], it is shown that $\alpha_N(s)$ can be calculated iteratively by moving every x_i one at a time, while still being well defined.

We define $H = \alpha_N[X_k^*]$ with multiplication given by $\alpha_N(xy)$, for $x, y \in H$. It turns out that H is bijective with a subset of $\mathbb{N}^k \times \alpha_D[X_k^*]$, under the map $\varphi(s) = (|s|_{x_1}, \ldots, |s|_{x_k}, \alpha_D(s))$, where $|s|_x$ denotes the number of occurrences of x in s. Notice that $\alpha_D[X_k^*]$ is finite because for every $s \in X_k^*$, the word $\alpha_D(s)$ has length at most $k \cdot \ell_0$ (every $x_i \in X_k$ appears at most ℓ_0 times).

Given a knotted inequality $x^m \leq x^n$ and the above bijection, we can endow H with an order under which it becomes a pomonoid. In particular, the order on the component $\alpha_D[X_k^*]$ is discrete while the order $\leq^m_n$ (where $m < n$) on each component $\mathbb{N}$ of $\mathbb{N}^k$ depends on the knotted rule and is given as follows: $u \leq_n^m v$ if and only if $u = v$, or $m \leq u < v$ and $u \equiv v \pmod{n-m}$.



Notice that this order is the finite disjoint union of well ordered chains. Therefore, it is a bqo and a wpo by Lemma 4.2. In the case that $m > n$, we obtain the following order which is the dual of $\leq_m^n$; note that this gives a dwpo.



It is shown in [4] that for each decomposition (a) of the number $r+1$, there is an $\ell_{(a)}$ such that the equation $\alpha_N(s) = s$, for all $s \in X_k^*$, is a consequence of (a) in the theory of monoids. (Recall that the definition of α_N depends on $\ell_{(a)}$.) This is the key to the proof of freeness of $\mathbf{H}$, and even though $\mathbf{H}$ does not satisfy the equation (a), it does fulfill the above consequences.

Lemma 5.1 ([4])**.** *Given a pomonoid* $\mathbf{M} = (M, \leq^{\mathbf{M}}, \cdot^{\mathbf{M}}, 1)$ *that satisfies* (a) *and* $x^m \leq x^n$*, every map* $g_1 \colon X_k \to M$ *extends to an order-preserving monoid homomorphism* $g \colon \mathbf{H} \to \mathbf{M}$ *(where* $\mathbf{H} = \mathbf{H}(a, m, n)$*).*

Lemma 5.2. $\mathbf{H}$ *is a bqo for* $m < n$ *and a dwpo for* $m > n$*.*

Proof. For $m < n$, $(H, \leq^{\mathbf{H}})$ is a subposet of the disjoint union of $|\alpha_D[X_k^*]|$ copies of $(\mathbb{N}, \leq_n^m)^k$. Since $(\mathbb{N}, \leq_n^m)$ is both a wpo and a bqo, by Lemma 4.2 we obtain that $(H, \leq^{\mathbf{H}})$ is a bqo.

For $m > n$, $(H, \leq^{\mathbf{H}})$ is a subposet of product of $(\alpha_D[X_k^*], =) \times (\mathbb{N}, \geq_m^n)^k$. Since $(\mathbb{N}, \geq_m^n)$ is a dwpo, $(H, \leq^{\mathbf{H}})$ is also a dwpo, as a subposet of the finite product of dwpo's. □

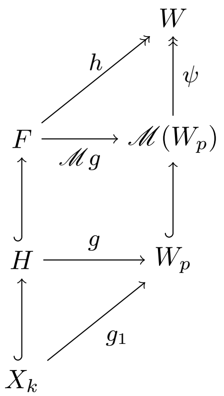
As a consequence of Lemmas 4.3, 4.4, and 5.2, we obtain the following corollary for $\mathbf{F} = \mathscr{M}(\mathbf{H})$.

Corollary 5.3. $\mathbf{F} = \mathscr{M}(\mathbf{H})$ *is a wpo for* $m < n$ *and a dwpo for* $m > n$*.*

By the k-freeness of $\mathbf{H}$, we obtain the following.

Lemma 5.4. *There is a surjective semilattice monoid homomorphism from* $\mathbf{F}$ *to* $\mathbf{W}$*. Hence,* $\mathbf{W}$ *is a wpo for* $m < n$ *and a dwpo for* $m > n$*.*

Proof. Let $g_1(x_i) = b_i$ for $i \in \{1, \ldots, k\}$, where $B = \{b_1, \ldots, b_k\}$. The following diagram is obtained by combining the results from Lemmas 3.3, 3.4, and 5.1.



By distributivity of multiplication over meet, we have that $W \subseteq \bigwedge B^*$. By the fact that elements of W are generated as meets of elements in B^* and by Lemma 3.5, we obtain that h is surjective. Given that h is order-preserving, we obtain that $\mathbf{W}$ is either a wpo or a dwpo. $\square$

Recall that to prove FEP, we only need to show $C_b = \{\{(u, b)\}^{\lhd} : u \in S_W\}$ is finite for every $b \in B$.

Lemma 5.5. *For each* $b \in B$*,* $(C_b, \subseteq)$ *is a dwpo if* $m < n$ *and a wpo if* $m > n$*.*

Proof. Recall that every $u \in S_W$ is of the form $u = (y \circ \_ \circ z) \curlywedge w$ for $y, z \in W$ and $w \in W \cup \{\top\}$, where $(y \circ \_ \circ z) \curlywedge \top$ stands for $y \circ \_ \circ z$. Based on this, we define $W_\top = W \cup \{\top\}$ and extend the order in $\leq^{\mathbf{W}}$ to include $\top$ by $x \leq^{\mathbf{W}} \top$ for all $x \in W_\top$.

It is easy to see that the map $\varphi \colon W^2 \times W_\top \to C_b$, where $\varphi(y, w, z) = \{(y \circ \_ \circ w \curlywedge z, b)\}^{\lhd}$, is surjective by considering the cases of z being $\top$ or not.

To show φ is order-reversing, let $(y_1, w_1, z_1), (y_2, w_2, z_2) \in W^2 \times W_\top$ with $(y_1, w_1, z_1) \leq^{\mathbf{W}} (y_2, w_2, z_2)$, and note that then, for all $x \in W$,

$$y_1 \circ x \circ w_1 \curlywedge z_1 \leq^{\mathbf{A}} y_2 \circ x \circ w_2 \curlywedge z_2.$$

So, if $x \in \varphi(y_2, w_2, z_2)$, that is, $y_2 \circ x \circ w_2 \curlywedge z_2 \leq^{\mathbf{A}} b$, then $y_1 \circ x \circ w_1 \curlywedge z_1 \leq^{\mathbf{A}} b$, which means that $x \in \varphi(y_1, w_1, z_1)$. So $\varphi(y_1, w_1, z_1) \supseteq \varphi(y_2, w_2, z_2)$.

By Lemma 5.4, $(W^2 \times W_\top, \leq^{\mathbf{W}})$ is a wpo if $m < n$ and a dwpo if $m > n$, therefore $(C_b, \subseteq)$ is a dwpo if $m < n$ and a wpo if $m > n$, as a surjective image of $(W^2 \times W_\top, \leq^{\mathbf{W}})$. $\square$

Lemma 5.6. $(C_b, \subseteq)$ *has no infinite descending chains for* $m < n$ *and no infinite ascending chains for* $m > n$*.*

Proof. For $m < n$, $\mathbf{W}$ is a wpo. The elements of C_b are downsets in $\mathscr{P}(W)$, so by Lemma 4.1, $(C_b, \subseteq)$ has no infinite descending chains.

For $m > n$, $\mathbf{W}$ is a dwpo. If there existed an infinite ascending chain of downsets in $(C_b, \subseteq)$, the set complements of these elements would form a descending chain of upsets, contradicting Lemma 4.1. Thus, $(C_b, \subseteq)$ has no infinite ascending chains. □

Theorem 5.7. $\mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ *is finite.*

Proof. It follows from Lemma 5.5 and Lemma 5.6 that for all $b \in B$, $(C_b, \subseteq)$ has no infinite ascending chains, descending chains, or antichains, so it is finite. □

By Lemma 2.1 and Theorem 5.7, we obtain the following main result of the paper. Note that, for example, the commutative case is covered by our result.

Theorem 5.8. *For any knotted inequality $x^m \leq x^n$ and any equation of the form (a), all subvarieties of $\mathcal{D}^n_m(a)$ axiomatized by equations over $\{\wedge, \vee, \cdot, 1\}$ have the FEP and their universal theories are decidable.*

We conclude by extending the above result to varieties of FL-algebras axiomatized by some further equations. An *FL-algebra* is an expansion of a residuated lattice with an extra constant 0, which is used to define negation operations ${\sim}x = x\backslash 0$ and $-x = 0/x$. An FL-algebra is called *cyclic* if it satisfies ${\sim}x = -x$; it will be called *pseudo-complemented* if it satisfies $x \wedge {\sim}x \leq 0$ and $x \wedge -x \leq 0$.

Theorem 5.9. *Let $\mathcal{V}$ be a subvariety of fully distributive FL-algebras axiomatized by a knotted inequality $x^m \leq x^n$, some equation of the form (a), and any combination of the following identities:*

(1) *cyclicity,*
(2) *pseudo complementation,*
(3) $0 = 1$,
(4) $0 \leq x$,
(5) *any identity over the language of* $\{\wedge, \vee, \cdot, 1\}$.

Then $\mathcal{V}$ has the FEP and its universal theory is decidable.

Proof. It follows from [7] that $\mathbf{D} = \mathbf{W}^+_{\mathbf{A},\mathbf{B}}$ can be expanded to an FL-algebra by interpreting 0 as $0_D = \{(\mathrm{id}, 0)\}^{\triangleleft}$, and if $\mathbf{A}$ is already an FL-algebra, then the embedding $b \mapsto \{(\mathrm{id}, b)\}^{\triangleleft}$ is also an FL-algebra embedding, where 0_A is mapped to 0_D, in case $0_A \in B$, which we can actually also assume without loss of generality since it preserves the finiteness of B.

We first prove that the construction preserves cyclicity. Let $X \in D$ and $z \in {\sim}X = X\backslash 0_D = \{y : X \circ y \subseteq 0_D\}$. We have that $X \circ z \subseteq 0_D$ iff for all $x \in X$, $x \circ z \leq^{\mathbf{A}} 0$. Cyclicity implies that for all $y \in A$, $y\backslash 0 = 0/y$, while residuation implies $xy \leq^{\mathbf{A}} 0 \Leftrightarrow x \leq^{\mathbf{A}} 0/y \Leftrightarrow x \leq^{\mathbf{A}} y\backslash 0 \Leftrightarrow yx \leq^{\mathbf{A}} 0$. Hence,

$$X \circ z \subseteq 0_D \;\Leftrightarrow\; \forall x \in X, x \circ z \leq^{\mathbf{A}} 0 \;\Leftrightarrow\; \forall x \in X, z \circ x \leq^{\mathbf{A}} 0 \;\Leftrightarrow\; z \circ X \subseteq 0_D.$$

The conclusion, $z \circ X \subseteq 0_D$, is equivalent to $z \in -X = 0_D/X$. Therefore, for all $X \in D$, $\sim X = -X$, which means that $\mathbf{D}$ is cyclic.

Now we show that pseudo-complementation is preserved. For $z \in X \cap \sim X$, we have $z \in X$ and $X \circ \{z\} \subseteq 0_D$. In particular, $z^2 \leq^{\mathbf{A}} 0$, which implies that $z \leq^{\mathbf{A}} \sim z$. We have that

$$z = z \wedge z \leq^{\mathbf{A}} z \wedge \sim z \leq^{\mathbf{A}} 0.$$

Thus, $X \cap \sim X \subseteq 0_D$ as desired. The other inequality is proven similarly.

The property of being zero-bounded ($0 \leq x$) is preserved as well. The verification is straightforward. The proofs of the previous lemmas rely on the same construction of $\mathbf{D}$, therefore they can be combined freely. □

As a final remark, we mention that our results also hold for the corresponding varieties of generalized bunched implication algebras, namely (conservative) extensions of distributive residuated lattices with the residual of meet and also with a top element; see [7] for more properties. This is because if $\mathbf{A}$ is a generalized bunched implication algebra, then the algebra $\mathbf{D}$ constructed here, based on the distributive residuated lattice reduct of $\mathbf{A}$, is finite and can be uniquely extended to a generalized bunched implication algebra; also, the results in [7] ensure that the resulting map is an embedding also with respect to the new operations.

References

[1] Blok, W.J., van Alten, C.J.: On the finite embeddability property for residuated lattices, pocrims and BCK-algebras. Rep. Math. Logic **34**, 159–165 (2000)

[2] Blok, W.J., van Alten, C.J.: The finite embeddability property for residuated lattices, pocrims and BCK-algebras. Algebra Universalis **48**, 253–271 (2002)

[3] Blok, W.J., van Alten, C.J.: On the finite embeddability property for residuated ordered groupoids. Trans. Amer. Math. Soc. **357**, 4141–4157 (2005)

[4] Cardona, R., Galatos, N.: The finite embeddability property for noncommutative knotted extensions of RL. Internat. J. Algebra Comput. **25**, 349–379 (2015)

[5] Galatos, N., Horčík, R.: Cayley's and Holland's theorems for idempotent semirings and their applications to residuated lattices. Semigroup Forum **87**, 569–589 (2013)

[6] Galatos, N., Jipsen, P.: Residuated frames with applications to decidability. Trans. Amer. Math. Soc. **365**, 1219–1249 (2013)

[7] Galatos, N., Jipsen, P.: Distributive residuated frames and generalized bunched implication algebras. Algebra Universalis (in press)

[8] Galatos, N., Jipsen, P., Kowalski, T., Ono, H.: Residuated Lattices: An Algebraic Glimpse at Substructural Logics. Studies in Logic and the Foundations of Mathematics, vol 151. Elsevier, Amsterdam (2007)

[9] Higman, G.: Ordering by divisibility in abstract algebras. Proc. London Math. Soc. **3**, 326–336 (1952)

[10] Horčík, R.: Word problem for knotted residuated lattices. J. Pure Appl. Algebra **219**, 1548–1563 (2015)

[11] Jančar, P.: A note on well quasi-orderings for powersets. Inform. Process. Lett. **72**, 155–160 (1999)

[12] Marcone, A.: Fine analysis of the quasi-orderings on the power set. Order **18**, 339–347 (2002)

[13] Milner, E.C.: Basic wqo- and bqo-theory. In: Graphs and order (Banff, Alta., 1984). NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 147, pp. 487–502. Reidel, Dordrecht (1985)
[14] Nash-Williams, C.St.J.A.: On well-quasi-ordering finite trees. Proc. Cambridge Philos. Soc. **59**, 833–835 (1963)
[15] Nash-Williams, C.St.J.A.: On well-quasi-ordering infinite trees. Proc. Cambridge Philos. Soc. **61**, 697–720 (1965)
[16] Rado, R.: Partial well-ordering of sets of vectors. Mathematika **1**, 89–95 (1954)
[17] van Alten, C.J.: The finite model property for knotted extensions of propositional linear logic. J. Symbolic Logic **70**, 84–98 (2005)

Riquelmi Cardona

Escuela de Matemática, Universidad de El Salvador. Ciudad Universitaria, Final de Av. Mártires y Héroes del 30 julio, San Salvador, El Salvador.
e-mail: `riquelmi.cardona@ues.edu.sv`

Nikolaos Galatos

Department of Mathematics, University of Denver, C.M. Knudson Hall, 2390 S. York St., Denver, CO 80208
e-mail: `ngalatos@du.edu`
URL: `http://web.cs.du.edu/~ngalatos/`