Weakening Relation Algebras and FL^2-algebras

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Abstract. FL^2-algebras are lattice-ordered algebras with two sets of residuated operators. The classes RA of relation algebras and GBI of generalized bunched implication algebras are subvarieties of FL^2-algebras. We prove that the congruences of FL^2-algebras are determined by the congruence class of the respective identity elements, and we characterize the subsets that correspond to this congruence class. For involutive GBI-algebras the characterization simplifies to a form similar to relation algebras.

For a positive idempotent element p in a relation algebra $\mathbf{A}$, the double division conucleus image $p\backslash\mathbf{A}/p$ is an (abstract) weakening relation algebra, and all representable weakening relation algebras (RWkRAs) are obtained in this way from representable relation algebras (RRAs). The class $S(\mathsf{dRA})$ of subalgebras of $\{p\backslash\mathbf{A}/p : A \in \mathsf{RA}, 1 \le p^2 = p \in A\}$ is a discriminator variety of cyclic involutive GBI-algebras that includes RA. We investigate $S(\mathsf{dRA})$ to find additional identities that are valid in all RWkRAs. A representable weakening relation algebra is determined by a chain if and only if it satisfies $0 \le 1$, and we prove that the identity $1 \le 0$ holds only in trivial members of $S(\mathsf{dRA})$.

Keywords: Relation algebras · Residuated lattices · Bunched implication algebras

1 Introduction

Tarski defined a relation algebra $(A, \wedge, \vee, \neg, \top, \bot, ;, \breve{}, 1, 0)$ to be an algebra that satisfies a short list of identities that hold in all algebras of binary relations on a set: $(A, \wedge, \vee, \neg, \top, \bot)$ is a Boolean algebra, $(A, ;, 1)$ is a monoid, ; and $\breve{}$ distribute over $\vee$, $x;\bot = \bot = \bot;x$, $0 = \neg 1$, $x^{\smile\smile} = x$, $(xy)^{\smile} = y^{\smile}x^{\smile}$ and $x^{\smile} \cdot \neg(xy) \le \neg y$.

An interesting generalization is to consider algebras of weakening closed binary relations on partially ordered sets $\mathbf{P}=(P, \le)$. A relation $R \subseteq P^2$ is *weakening closed* or a *weakening relation* if $x' \le x\ R\ y \le y'$ implies $x'\ R\ y'$, or equivalently, $\le;R;\le\ \subseteq R$. The collection of all weakening relations on $\mathbf{P}$ is denoted $\mathrm{Wk}(\mathbf{P})$. If R is weakening closed, so is its *complement-converse* $R^{c\smile} = \{(y, x) \mid (x, y) \notin R\}$. This unary operation is denoted by ${\sim}R$.

U. Fahrenberg et al. (Eds.): RAMiCS 2020, LNCS 12062, pp. 117–133, 2020.
https://doi.org/10.1007/978-3-030-43520-2_8

Weakening relations are also closed under union, intersection, Heyting implication $\to$ (= residual of intersection), relation composition ; and residuals $\backslash, /$ of composition. The partial order relation $\leq$ is a weakening relation and, since it is the identity of composition, it is denoted by 1. The complement-converse of 1 is denoted by 0. The *full weakening relation algebra* on a poset $\mathbf{P}$ is

$$\mathbf{Wk}(\mathbf{P}) = (\mathrm{Wk}(\mathbf{P}), \cap, \cup, \to, P^2, \varnothing, ;, \sim, 1, 0).$$

The residuals $\backslash, /$ are omitted since they are definable via $x \backslash y = \sim(\sim y; x)$ and $x / y = \sim(y; \sim x)$. The variety RWkRA of representable weakening relation algebras is generated by the class $\{\mathbf{Wk}(\mathbf{P}) \mid \mathbf{P}$ is a poset$\}$. When the poset is an antichain, or equivalently, when $\leq$ is the identity relation then $\mathbf{Wk}(\mathbf{P})$ is the usual *full relation algebra* $\mathbf{Rel}(P)$ since in this case $R \to \varnothing = R^c$ is the complement of R, and $\sim(R^c) = R^\smile$ is the converse of R. Hence RWkRA contains the variety RRA of all representable relation algebras (which is generated by all full relation algebras).

Some applications of weakening relation algebras were given by Stell [22,23] in the area of image processing and hypergraphs. Since the lattice reducts of weakening relation algebras are Heyting algebras rather than Boolean algebras, weakening relations can be thought of as intuitionistic relations.

The variety RWkRA retains many of the algebraic properties of RRA, as shown in [8] and reviewed in Sect. 3. The aim of this paper is to investigate the identities that hold in RWkRA. We do this in the more general context of generalized bunched implication algebras, residuated lattices, and FL^2-algebras (defined below) in order to point out some of the syntactic symmetries of weakening relation algebras and to relate this variety to some other well-studied classes of algebras.

A *residuated lattice* is of the form $(A, \wedge, \vee, \cdot, \backslash, /, 1)$ such that $(A, \wedge, \vee)$ is a lattice, $(A, \cdot, 1)$ is a monoid, and for all $x, y, z \in A$ the *residuation property* holds:

$$xy \leq z \iff x \leq z/y \iff y \leq x \backslash z.$$

A *full Lambek algebra* or *FL-algebra* $(A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$ is a residuated lattice with an additional constant 0, hence FL-algebras are also called *pointed residuated lattices*. The residuation property implies that $x(y \vee z) = xy \vee xz$ and $(x \vee y)z = xz \vee yz$ hence FL-algebras include idempotent semirings as reducts. In fact any finite idempotent semiring expands uniquely to an FL-algebra in which 0 is the bottom element. Hence FL-algebras are closely related to many computational algebraic theories, such as Kleene algebras, Kleene lattices and Pratt's action algebras.

FL-algebras and their reducts cover the algebraic semantics of a large number of logics, including classical propositional logic, intuitionistic logic, relevance logic, multi-valued logic, Hajek's basic logic, abelian logic, BCK-logic and many others. However they do not capture bunched implication logic or the logic of relation algebras (also known as arrow logic). Bunched implication logic is an integral part of separation logic, a Hoare logic for reasoning about pointer structures and concurrent programs [19–21]. Generalized bunched implication algebras were defined in [7] to provide a common algebraic version of bunched implication algebras and relation algebras.

In this paper we introduce FL^2-algebras in order to give a new definition of relation algebras and bunched implication algebras that exposes interesting symmetries of both algebraic theories. A *FL^2-algebra* is of the form $\mathbf{A} = (A, \wedge, \vee, \diamond, \to, \leftarrow, t, f, \cdot, \backslash, /, 1, 0)$ such that

$$\mathbf{A}_t = (A, \wedge, \vee, \diamond, \to, \leftarrow, t, f) \quad \text{and} \quad \mathbf{A}_1 = (A, \wedge, \vee, \cdot, \backslash, /, 1, 0)$$

are both FL-algebras. We call $\mathbf{A}_t$ the *logical reduct* and $\mathbf{A}_1$ the *dynamic reduct* of $\mathbf{A}$. The class of all FL-algebras can be defined by identities, hence it and the class FL^2-algebras are varieties, denoted by FL and FL^2 respectively. To reduce the number of parentheses, we adopt the convention that $\cdot$ binds stronger than $\backslash, /$ followed by $\diamond$, $\wedge, \vee$ and $\to, \leftarrow$.

Define $\neg x = x \to f$, $\ulcorner x = f \leftarrow x$, ${\sim}x = x\backslash 0$ and $-x = 0/x$. An FL^2-algebra is *involutive* if ${\sim}{-x} = x = -{\sim}x$, *f-involutive* if $\neg\ulcorner x = x = \ulcorner\neg x$, and *doubly involutive* if all four identities hold. An FL^2-algebra is *cyclic* if ${\sim}x = -x$, *f-cyclic* if $\neg x = \ulcorner x$, and *doubly cyclic* if both hold.

Relation algebras are well known examples of doubly cyclic FL^2-algebras. In fact they are term-equivalent to the subvariety defined by the identities $x \wedge y = x \diamond y$ (hence $y \leftarrow x = x \to y$ and A_t is a Boolean algebra) and $\neg{\sim}(xy) = (\neg{\sim}y)(\neg{\sim}x)$. The operation $\neg{\sim}x$ is the *converse* of relation algebras, usually written $x^{\smile}$.

A *generalized bunched implication* (GBI-)*algebra* $(A, \wedge, \vee, \to, \top, \cdot, \backslash, /, 1)$ is defined as a Brouwerian algebra $(A, \wedge, \vee, \to, \top)$ such that $(A, \wedge, \vee, \cdot, \backslash, /, 1)$ is a residuated lattice. Equivalently a GBI-algebra is an FL^2-algebra that satisfies $x \wedge y = x \diamond y$, $t = f$ and $0 = 1$. A *bunched implication algebra*, or *BI-algebra*, is a commutative GBI-algebra (i.e., $xy = yx$) that has been expanded by a constant $\bot$ denoting the least element of the lattice. Alternatively, it is an FL^2-algebra that satisfies $x \wedge y = x \diamond y$, $f \leq x$, $0 = 1$ and $xy = yx$. Since the logical constants t, f are the top and bottom elements in this algebra they are usually denoted by $\top$, $\bot$.

An interesting subclass of cyclic GBI-algebras is the variety of *symmetric Heyting relation algebras* or SHRAs [23], defined by adding the identity ${\sim}\neg(xy) \leq ({\sim}\neg y)({\sim}\neg x)$. This identity holds in all representable weakening relation algebras, hence RWkRA is a subvariety of SHRA.

Another subvariety of FL^2-algebras is the variety of *skew relation algebras* [6], defined in this setting as Boolean involutive FL^2-algebras. As mentioned before, the variety of relation algebras is obtained by adding the identity $(xy)^{\smile} = y^{\smile}x^{\smile}$ where $x^{\smile} = \neg{\sim}x$ ([6], Cor. 29).

We provide a simpler characterization of the congruences of GBI-algebras in Sect. 3 (using congruence terms that have only *one* parameter), which also reveals hidden symmetries in the description given in [8]. Towards that goal, we first provide this description in the more natural and symmetric setting of FL^2 in Sect. 2, and this is our main reason for introducing FL^2. Equivalent characterizations are provided in Lemma 3 (in a fully symmetric setting), which then specialize to two distinct characterizations in Corollaries 9 and 10 (one for the congruence filters of 1 and one for the congruence filters of $\top$).

All FL^2-algebras can be constructed by selecting two pointed residuated lattices that have a common underlying lattice. If the lattice is non-distributive, then the resulting FL^2-algebra is outside the variety of GBI-algebra. One of the appeals of GBI-algebras in computer science is the fact that they provide the means to study both the logical and the dynamic aspect of situations. Of course in GBI the logical part is restricted to intuitionistic logic, but FL^2 allows for considering cases where the logical part is any substructural logic, such as linear logic, relevance logic or a particular fuzzy logic. Methods for combining logics have been studied extensively, and FL^2 is an example of fusion of logics as described by Gabbay in [5]. The results in the first half of this paper provide some insight into the algebraic structure of the fusion of two substructural logics.

The models of relevance logic RW, namely distributive cyclic involutive residuated lattices, are exactly the implication subreducts of de Morgan BI-algebras, namely the extension of BI where the dynamic part is involutive. In [3] it is shown that the addition of a Boolean negation to de Morgan BI-algebras results in a non-conservative extension called *classical* BI. A display calculus for this logic shows remarkable symmetry between the classical logic part and the involutive dynamic part of this logic. The setting of FL^2 is well suited to studying weaker versions of this logic that omit some rules like contraction and/or weakening. It is also worth noting that classical BI-algebras coincide with commutative skew relation algebras (defined in [6]).

In Sect. 4 we recall the definition of discriminator variety and some results about weakening relation algebras from [8]. Finally, Sect. 5 defines the double-division conucleus construction and shows that the image of the variety of relation algebras under this construction produces a class of GBI-algebras that is a non-Boolean analogue to Tarski's variety of abstract relation algebras.

Throughout the paper we make use of elementary properties of the residuals, such as $x(x\backslash y) \le y$, $x \le xy / y$, $x(y \vee z)w = xyw \vee xzw$ and that residuals are order-reversing in the "denominator" or antecedent and order-preserving in the "numerator" or consequent.

2 Congruences of FL^2-algebras

An algebraic theory determines a category in which all models of the theory are objects and the morphisms are homomorphisms between the algebraic models. The kernel $\{(x, y) : h(x) = h(y)\}$ of a homomorphism $h : \mathbf{A} \to \mathbf{B}$ is a congruence relation (i.e., an equivalence relation that is preserved by all algebraic operations) on $\mathbf{A}$, and an important step in understanding the structure of the category is to be able to describe the lattice of congruences $\mathrm{Con}(\mathbf{A})$ on each object $\mathbf{A}$.

An FL^2-algebra $\mathbf{A}$ has two residuated lattices as reducts, hence any congruence on $\mathbf{A}$ is a residuated lattice congruence. The description of congruences in residuated lattices is due to Blount and Tsinakis [2]. Here we use a version of this result that appears in [9].

An algebra is said to be *c-regular* if c is a constant in the algebra and each congruence of the algebra is determined by its c-congruence class. Residuated lattices are 1-regular since for a congruence θ on a residuated lattice $\mathbf{L}$

$$x\,\theta\,y \iff x/y \wedge y/x \wedge 1 \in [1]_\theta$$

where $[1]_\theta = \{z \in L : z\theta 1\}$ is the 1-congruence class of θ. If we define $x \leq_\theta y$ by $x \leq z$ and $z\,\theta\,y$ for some $z \in L$, or equivalently by $x\,\theta\,w$ and $w \leq y$ for some $w \in L$, then $x\,\theta\,y \iff x \leq_\theta y$ and $y \leq_\theta x$, hence the above equivalence follows from the observation that

$$x \leq_\theta y \iff 1 \leq_\theta y/x \iff y/x \wedge 1\,\theta\,1.$$

Instead of the right residual / one could also use the left residual \ for this equivalence. Rather than working with 1-congruence classes, it is convenient to use certain filters.

Recall that a filter of a lattice L is a subset F such that $x \wedge y, a \vee x \in F$ for all $x, y \in F$ and $a \in L$. For $x \in X \subseteq L$ let $\uparrow x = \{y \in L : x \leq y\}$ be the principal (lattice) filter generated by x and $\uparrow X = \bigcup_{x \in X} \uparrow x$.

A *congruence filter* of a residuated lattice or FL-algebra is a subset of the form $F = \uparrow([1]_\theta)$. This is a lattice filter since the congruence class of 1 is closed under meet. It is also a union of θ-classes since $\leq_\theta$ is transitive. The class $[1]_\theta$ can be recovered from F since $[1]_\theta = \{x : x, 1/x \in F\}$.

It is easy to check that

$$1,\ xy,\ \lambda_a(x) := a\backslash xa,\ \rho_a(x) := ax/a \in F \quad \text{for all } x, y \in F \text{ and } a \in L.$$

Note that the closure of F under the *conjugation terms* $ax/a, a\backslash xa$ is equivalent to the following *normality conditions* (where quantifiers range over F):

$$(\lambda_a)\ \forall x \exists x',\ ax' \leq xa \qquad \text{and} \qquad (\rho_a)\ \forall x \exists x',\ x'a \leq ax.$$

A filter F is said to satisfy (λ) if (λ_a) holds for all $a \in L$ and likewise for (ρ). The set of congruence-filters of $\mathbf{L}$ is denoted by $\mathrm{CF}(\mathbf{L})$.

Theorem 1 ([9]). *For a residuated lattice or FL-algebra* $\mathbf{A}$*, a subset* F *is a congruence-filter if and only if* F *is a lattice filter and a submonoid of* $\mathbf{A}$ *that satisfies* (λ) *and* (ρ)*.*

Moreover, $\mathrm{Con}(\mathbf{A})$ *is isomorphic to the lattice* $\mathrm{CF}(\mathbf{A})$ *of congruence-filters via the bijection* $\theta \mapsto \uparrow([1]_\theta)$ *and* $F \mapsto \{(x, y) : x/y, y/x \in F\}$*.*

Since there are two signatures for FL-algebras, there are two ways to characterize the congruences of an FL^2-algebra, either by congruence 1*-filters* $\uparrow([1]_\theta)$ or by congruence t*-filters* $\uparrow([t]_\theta)$. We usually drop the prefix "congruence", and mostly work with 1-filters. However all results can be translated to t-filters by interchanging the operation symbols of the two signatures.

For FL^2 we need the following stronger t*-normality* conditions to determine the 1-filters (the quantifiers range over the filters). For any $a \in A$,

$$(U_a)\ \forall x \exists x_1,\ x_1 a \leq xt \diamond a, \qquad (U'_a)\ \forall x \exists x_2,\ ax_2 \leq a \diamond xt,$$
$$(V_a)\ \forall x \exists x_3,\ x_3 t \diamond a \leq ax, \qquad (V'_a)\ \forall x \exists x_4,\ a \diamond x_4 t \leq xa.$$

A filter satisfies (U) if (U_a) holds for all $a \in A$, and the same for (U'), (V) and (V'). The conjunction of these four conditions is referred to as (UV_a) or, if they hold for all a, as (UV).

Lemma 3 below shows that (UV_a) is indeed stronger than $(\lambda_a), (\rho_a)$. With the help of normality we can derive several other variants of the inequations in (UV_a) such as $\forall x \exists x', x'a \leq a \diamond tx$. We will use these variations occasionally in the following lemma about some useful two-parameter conditions. For $a, b \in A$ define

$$\forall x \exists x_1, x_2, \ \ x_1(a \diamond b) \leq xa \diamond b \quad \text{and} \quad x_2(a \diamond b) \leq a \diamond xb, \tag{$Q_{a,b}$}$$

$$\forall x \exists x_1, x_2, \ \ a \diamond x_1 b \leq xa \diamond b \quad \text{and} \quad x_2 a \diamond b \leq a \diamond xb, \tag{$R_{a,b}$}$$

$$\forall x \exists x_1, x_2, \ x_1(a \to b) \leq a \to xb \quad \text{and} \quad x_2(a \leftarrow b) \leq xa \leftarrow b. \tag{$S_{a,b}$}$$

Lemma 2. *The condition (UV) implies (Q), (R) and (S).*

Proof. We first derive $(R_{a,b})$ from (V_a') and (U_b). Given $x \in F$, there exist $x_1, x_4 \in F$ such that (reading from right to left)

$$a \diamond x_1 b \leq a \diamond (x_4 t \diamond b) = (a \diamond x_4 t) \diamond b \leq xa \diamond b.$$

By a similar calculation using (V_b') and (U_a'), there exist $x_2, x_4 \in F$ such that $x_2 a \diamond b \leq a \diamond x_4 t \diamond b \leq a \diamond xb$.

Next we derive $(Q_{a,b})$ from $(R_{t,a})$ and $(R_{t,a\diamond b})$. Given $x \in F$, there exist $x_1, x_2 \in F$ with

$$x_1(a \diamond b) = t \diamond x_1(a \diamond b) \leq x_2 t \diamond (a \diamond b) = (x_2 t \diamond a) \diamond b \leq (t \diamond xa) \diamond b = xa \diamond b.$$

For $(S_{a,b})$ the relevant calculation shows there exist $x_1, x_2, x_3, x_4 \in F$ such that

$$a \diamond x_4(a \to b) \leq a \diamond (a \to b)x_3 \diamond t \leq a \diamond (a \to b) \diamond x_1 t \leq b \diamond x_1 t \leq bx_2 \diamond t \leq xb,$$

hence for all $x \in F$ there exists $x_1 \in F$ such that $x_1(a \to b) \leq a \to xb$. The remaining inequalities are derived in a similar way. □

We also consider the conditions

$$(\lambda_a') \ \forall x \exists x', \ x't \diamond a \leq a \diamond xt \qquad \text{and} \qquad (\rho_a') \ \forall x \exists x', \ a \diamond x't \leq xt \diamond a.$$

As before, (λ') means that (λ_a') holds for all a, and likewise for (ρ').

Lemma 3. *We have the following implications between the above conditions.*

1. (U) *and* $(V) \Rightarrow (\rho)$
2. (U') *and* $(V') \Rightarrow (\lambda)$
3. (U') *and* $(V) \Rightarrow (\rho')$
4. (U) *and* $(V') \Rightarrow (\lambda')$.

Moreover, the following sets of conditions are equivalent:

5. $(U), (U'), (V), (V')$, *that is* (UV)
6. $(U), (V), (\lambda), (\lambda'), (\rho')$
7. $(U'), (V'), (\rho), (\lambda'), (\rho')$
8. $(U'), (V), (\lambda), (\rho), (\rho')$
9. $(U), (V'), (\lambda), (\rho), (\lambda')$.

Proof. For (1) we have for all $x \in F$ there exist $x_1, x_3 \in F$ such that $x_1 a \le x_3 t \diamond a \le ax$. For (2) there exist $x_2, x_4 \in F$ such that $ax_2 \le a \diamond x_4 t \le xa$. For (3), we have $x_3 t \diamond a \le ax_2 \le a \diamond xt$, while for (4) $a \diamond x_4 t \le x_1 a \le xt \diamond a$.

That (5) implies (6) follows from (1–4). For the converse, $(\lambda'), (U), (\lambda)$ imply $ax'' \le x_1 a \le x't \diamond a \le a \diamond xt$, giving (U'), and $(\lambda), (V), (\rho')$ imply $a \diamond x't \le x_3 t \diamond a \le ax \le xa$, yielding (V'). The equivalence of (5) and (7) is analogous.

That (5) implies (8) follows from (1–4). For the converse, $(\rho'), (U'), (\rho)$ show $x''a \le ax_2 \le a \diamond x't \le xt \diamond a$, yielding (U), and $(\lambda), (V), (\rho')$ show $a \diamond x't \le x_3 t \diamond a \le ax \le xa$ giving (V'). Likewise (5) and (9) are equivalent. □

Theorem 4. *For an FL^2-algebra* $\mathbf{A}$*, a subset* F *is the* 1*-filter of some congruence* θ *of* $\mathbf{A}$ *if and only if* F *is a lattice filter and a* $\cdot, 1$*-submonoid of* $\mathbf{A}$ *that satisfies* (UV)*, or any of the equivalent conditions 6.–9. of Lemma 3.*

Proof. Assume $F = \uparrow([1]_\theta)$ is the 1-filter of some FL^2-congruence θ. As observed earlier, F is a lattice filter that contains 1, so if $x, y \in F$ then there exist $u, v \in [1]_\theta$ with $u \le x, v \le y$ and $1 \cdot 1 \,\theta\, uv \le xy$, hence $xy \in F$ showing that F is a submonoid. Next we prove (U_a) $x_1 a \le xt \diamond a$. For $a \in A$ and $x \in F$ there exists $y \in F$ such that $y \le x$ and $y \in [1]_\theta$. Now

$$1 \,\theta\, y \;\Rightarrow\; t \,\theta\, ty \;\Rightarrow\; a = t \diamond a \,\theta\, yt \diamond a \;\Rightarrow\; 1 \le a/a \,\theta\, (yt \diamond a)/a \le (xt \diamond a)/a.$$

Hence $(xt \diamond a)/a \in F$ using the observation that if $1 \le u\theta v \le w$ then $w \in \uparrow([1]_\theta)$. Letting $x_1 = (xt \diamond a)/a$, we obtain $x_1 a \le xt \diamond a$.

For (V'_a) $a \diamond x_4 t \le xa$ we use the following calculation.

$$1 \,\theta\, y \Rightarrow a \,\theta\, ya \Rightarrow t \le a \to a \,\theta\, a \to ya \Rightarrow 1 \le t/t \,\theta\, (a \to ya)/t \le (a \to xa)/t,$$

hence $(a \to xa)/t \in F$, and choosing $x_4 = (a \to xa)/t$ implies $a \diamond x_4 t \le xa$. The conditions (U'_a) and (V) are proved in a similar way.

Conversely, assume F is a lattice filter with $1, xy \in F$ for all $x, y \in F$ and (UV) holds. Define $\theta = \{(a, b) : a/b, b/a \in F\}$. This relation is reflexive since $1 \in F$, transitive since $(x/y)(y/z) \le x/z$, and obviously symmetric. Assuming $a \,\theta\, b$, it suffices to show

$$(a \wedge c)/(b \wedge c), (a \vee c)/(b \vee c), (a \diamond c)/(b \diamond c), (c \diamond a)/(c \diamond b) \in F,$$
$$ac/bc, ca/cb, (a/c)/(b/c), (c/a)/(c/b), (a\backslash c)/(b\backslash c), (c\backslash a)/(c\backslash b) \in F \text{ and}$$
$$(a \to c)/(b \to c), (c \to a)/(c \to b), (a \leftarrow c)/(b \leftarrow c), (c \leftarrow a)/(c \leftarrow b) \in F$$

since interchanging a, b the same statements follow from $b \,\theta\, a$, hence θ is compatible with all FL^2 operations.

From $a \,\theta\, b$ we obtain $a/b, b/a \in F$ and since F is a filter $a/b \wedge 1 \in F$. The calculation for compatibility of meet is as follows:

$$(a/b \wedge 1)(b \wedge c) \le (a/b)b \wedge 1c \le a \wedge c$$

hence $a/b \wedge 1 \le (a \wedge c)/(b \wedge c) \in F$. The calculation for join is the same, using the distribution of $\cdot$ over $\vee$.

It is remarkable that all the remaining statements can be deduced from (UV). Lemma 3 shows that $(\lambda), (\rho)$ follow and, by Lemma 2 conditions (Q), (R), (S) also hold. The implication $a/b \in F \Rightarrow ac/bc \in F$ is easy since $(a/b)bc \le ac$ follows from $(a/b)b \le a$. However $a/b \in F \Rightarrow ca/cb \in F$ uses (ρ_c) with $x = a/b$, so there exists $x' \in F$ such that

$$x'cb = (x'c)b \le (c(a/b))b = c((a/b)b) \le ca$$

and therefore $x' \le ca/cb$ implies $ca/cb \in F$. Similarly the implication $a/b \in F \Rightarrow (a/c)/(b/c) \in F$ is easy since $a/b \le (a/c)/(b/c)$ holds, while for $b/a \in F \Rightarrow (c/a)/(c/b) \in F$ we use $(\rho_{c/b})$ with $x = b/a$ to get $x' \in F$ such that

$$x'(c/b)a \le (c/b)(b/a)a \le (c/b)b \le c$$

and then $x' \le (c/a)/(c/b)$ implies $(c/a)/(c/b) \in F$.

From $a/b \in F$ and $(Q_{b,c})$ we obtain $x_1 \in F$ such that $x_1(b \diamond c) \le (a\,/\,b)b \diamond c \le a \diamond c$ hence $(a \diamond c)/(b \diamond c) \in F$. Similarly $(S_{c,b})$ is used to find $x_1 \in F$ such that $x_1(c \to b) \le c \to (a/b)b \le c \to a$, which shows that $(c \to a)/(c \to b) \in F$. For $(a \to c)/(b \to c) \in F$ we use $(R_{a,b\to c})$ and $x = b/a \in F$ to get $x_1 \in F$ with

$$a \diamond x_1(b \to c) \le (b/a)a \diamond (b \to c) \le b \diamond (b \to c) \le c$$

hence $x_1 \le (a \to c)/(b \to c) \in F$. The remaining terms are shown to be in F by mirror-image arguments, so θ is a congruence for the FL^2-algebra $\mathbf{A}$.

It remains to show that $F = \uparrow([1]_\theta)$. If $a\,\theta\,1$ then by definition of θ, $a = a/1 \in F$ hence $\uparrow([1]_\theta) \subseteq F$. Conversely, given $a \in F$ we need to find $c \in F$ such that $1\,\theta\,c \le a$. By assumption $1 \in F$ so we can take $c = a \wedge 1 \in F$, in which case $1 \le 1/c$. It follows that $1/c$ and $c/1$ are in F, hence $1\,\theta\,c$. □

Note that join is only used to prove compatibility of join, hence the result generalizes to a meet-semilattice version of FL^2. The theorem also applies to the FL-algebra subvariety defined by $xy = x \diamond y$ (thus $1 = t$, $/ = \leftarrow$, $\backslash = \to$), hence the result implies Theorem 1. It is also possible to prove a congruence characterization for nonassociative FL^2-algebras using the techniques of [10].

Since relation algebras and bunched implication algebras are subvarieties of FL^2-algebras, the description of the congruence filters also applies to them. While the congruences of relation algebras have been well understood since the 1950s [16], for bunched implication algebras a description first appeared in [8]. However, the description and the proof given here are both simpler and more general. Because of the symmetry in the signature of FL^2-algebras, we immediately get the following result. Consider the conditions

$$(\bar{U}_a)\ \forall x \exists x_1,\ x_1{\diamond}a \le (x{\diamond}1)a, \qquad (\bar{U}'_a)\ \forall x \exists x_2,\ a{\diamond}x_2 \le a(x{\diamond}1),$$
$$(\bar{V}_a)\ \forall x \exists x_3,\ (x_3{\diamond}1)a \le a{\diamond}x, \qquad (\bar{V}'_a)\ \forall x \exists x_4,\ a(x_4{\diamond}1) \le x{\diamond}a.$$

Collectively we refer to them as $(\overline{UV_a})$ or, if they hold for all a, as $(\overline{UV})$. Similarly we have conditions $(\bar{\lambda}), (\bar{\rho}), (\bar{\lambda}'), (\bar{\rho}')$.

Corollary 5. *For an FL^2-algebra $\mathbf{A}$, a subset G is the t-filter of some congruence θ of $\mathbf{A}$ if and only if G is a lattice filter and a $\diamond, t$-submonoid of $\mathbf{A}$ that satisfies $(\overline{UV})$.*

Solving (U_a) for x_1 yields $x_1 \leq u_a(x) := (xt \diamond a)/a$. Given that F is assumed to be upward closed, demanding the existence of an element $x_1 \in F$ is equivalent to asking that $u_a(x)$ is in F. Translating the remaining three conditions, we obtain that they are equivalent to closure under the terms

$$u_a(x) = (xt \diamond a)/a, \qquad u'_a(x) = a\backslash(a \diamond xt),$$
$$v_a(x) = (ax \leftarrow a)/t, \qquad v'_a(x) = (a \rightarrow xa)/t.$$

As noted above, condition (UV_a) for a filter is equivalent to the filter being closed under the unary terms u_a, u'_a, v_a and v'_a. It is an interesting problem to determine if these terms can be applied in a specific order, and how they interact with submonoid generation and closure under meets. We leave this for future research.

The condition (λ_a) can be expressed in a concise way by noting that

$$\forall x \in F\ \ \exists x' \in F,\ ax' \leq xa \iff \forall x \in F,\ xa \in {\uparrow}(aF) \iff Fa \subseteq {\uparrow}(aF).$$

Hence $(\lambda_a), (\rho_a)$ are equivalent to ${\uparrow}(aF) = {\uparrow}(Fa)$. The same argument proves the following result.

Corollary 6. *A lattice filter F in an FL^2-algebra satisfies (UV_a) if and only if ${\uparrow}(a \diamond Ft) = {\uparrow}(Fa) = {\uparrow}(aF) = {\uparrow}(Ft \diamond a)$.*

Likewise, ${\uparrow}(a(F \diamond 1)) = {\uparrow}(F \diamond a) = {\uparrow}(a \diamond F) = {\uparrow}((F \diamond 1)a)$ is equivalent to the condition $(\overline{UV_a})$ holding for F.

The characterization of congruences by 1-filters simplifies a bit when applied to algebras where $[1]_\theta$ has a least element for all congruences, as is the case for finite algebras. An element $c \in A$ is *central* if $ca = ac$ for all $a \in A$, *negative* if $c \leq 1$ and *idempotent* if $cc = c$. A *congruence element* is a central negative idempotent element. The join and the product of two congruence elements is again a congruence element. It is a well known corollary of Theorem 1 that for a finite residuated lattice or FL-algebra, the congruence lattice is dually isomorphic to the lattice $(\mathrm{CE}(\mathbf{A}), \cdot, \vee)$ of congruence elements [9]. The dual isomorphism between $\mathrm{CE}(\mathbf{A})$ and the filter lattice $\mathrm{CF}(\mathbf{A})$ is given by $a \mapsto {\uparrow}a$ and $F \mapsto \bigwedge F$.

In an FL^2-algebra an element c is *t-central* if $a \diamond ct = ac = ca = ct \diamond a$ for all $a \in A$ and *1-central* if $a(c \diamond 1) = a \diamond c = c \diamond a = (c \diamond 1)a$. A 1-*congruence element* is a t-central negative idempotent element and a *t-congruence element* is a 1-central negative idempotent element.

Corollary 7. *For an FL^2-algebra $\mathbf{A}$ in which all 1-congruence classes have a least element, the lattice $CE_1(\mathbf{A})$ of 1-congruence elements is dually isomorphic to the lattice $CF_1(\mathbf{A})$ of 1-filter elements.*

Let $\mathbf{A}$ be an FL^2-algebra and $\mathbf{A}_1$ its FL-algebra reduct with $\wedge, \vee, \cdot, \backslash, /, 1, 0$. The isomorphism between the lattice $\mathrm{CF}(\mathbf{A}_1)$ of congruence filters and the lattice $\mathrm{Con}(\mathbf{A}_1)$ of congruences restricts to an isomorphism between lattice $\mathrm{CF}_1(\mathbf{A})$ of 1-filters of $\mathbf{A}$ and its congruence lattice $\mathrm{Con}(\mathbf{A})$. The above characterization also applies to the t-filters of $\mathbf{A}$, hence the lattice $\mathrm{CF}_t(\mathbf{A})$ of t-filters is isomorphic to $\mathrm{Con}(\mathbf{A})$ as well. The next result shows how to map between corresponding 1-filters and t-filters without having to construct the congruence relation. As in Lemma 2, the condition $(\overline{UV})$ has the following consequence:

$$\forall x \exists x_1, x_2,\ x_1 \diamond ab \le (x \diamond a)b \ \text{ and } \ x_2 \diamond ab \le a(x \diamond b). \qquad (Q'_{a,b})$$

Theorem 8. *For FL^2-algebras there is a one-one correspondence between 1-filters and t-filters via the mutually inverse lattice isomorphisms $F \mapsto {\uparrow}(Ft)$ and $G \mapsto {\uparrow}(G \diamond 1)$.*

Proof. Let G be a t-filter of an FL^2-algebra, and define $F = {\uparrow}(G \diamond 1)$. Then $1 \in F$ since $t \in G$, and for $u, v \in F$ there exist $x, y \in G$ such that $x \diamond 1 \le u$ and $y \diamond 1 \le v$. Using $(Q'_{1,y\diamond 1})$ there exists $x' \in G$ such that

$$x' \diamond y \diamond 1 = x' \diamond 1(y \diamond 1) \le (x \diamond 1)(y \diamond 1) \le uv,$$

and since G is closed under $\diamond$, $x' \diamond y \in G$ implies $uv \in F$. Next we show (U_a) holds for F. Since G is a t-filter, $(\overline{UV})$ holds for G. From $u \in F$ we obtain $x \in G$ such that $x \diamond 1 \le u$. By $(\bar{U}_t), (\bar{\lambda})$ and $(\bar{V}_a)$ there exist $x_1, x_3, x' \in G$ with

$$(x_3 \diamond 1)a \le a \diamond x' \le x_1 \diamond a = (x_1 \diamond t) \diamond a \le (x \diamond 1)t \diamond a \le ut \diamond a,$$

hence choosing $u_1 = x_3 \diamond 1$ we have found $u_1 \in F$ such that $u_1 a \le ut \diamond a$. The conditions $(U'), (V), (V')$ can be derived in a similar way. The proof that ${\uparrow}(Ft)$ is a t-filter for any 1-filter F follows by symmetry.

It remains to check that $F = {\uparrow}(G \diamond 1) \iff {\uparrow}(Ft) = G$. Assume $F = {\uparrow}(G \diamond 1)$ and let $x \in {\uparrow}(Ft)$. Then there exists $u \in F$ such that $ut \le x$. Since $u \in F$ we have $y \diamond 1 \le u$ for some $y \in G$. By $(\bar{U}_t)$ there exists $y_1 \in G$ such that $y_1 = y_1 \diamond t \le (y \diamond 1)t$. It follows that $(y \diamond 1)t \in G$, and since $(y \diamond 1)t \le ut \le x$ we have $x \in G$. This show ${\uparrow}(Ft) \subseteq G$. Now let $x \in G$, and note that by $(\bar{V}_t)$ there exists $x_3 \in G$ such that $(x_3 \diamond 1)t \le t \diamond x = x$. Taking $u = x_3 \diamond 1$ we have $u \in F$ and $ut \le x$, hence $x \in {\uparrow}(Ft)$. We conclude that ${\uparrow}(Ft) = G$. The reverse implication follows by symmetry of the signature. □

This correspondence restricts to a bijection between 1-congruence elements c and t-congruence elements d: $c \mapsto ct$ and $d \mapsto d \diamond 1$.

3 Congruences in GBI-algebras

The results in this section can be specialized to various subvarieties of FL^2. For example, for GBI-algebras, we can characterize the 1-filters by taking multiplication to be $\cdot$ and meet to be $\diamond$. Note that the constant t is denoted by $\top$ for

GBI-algebras because it is always the top element of the algebra. Since $\wedge$ is commutative, conditions $(\lambda'), (\rho')$ are automatically satisfied and (6) and (7) of Lemma 3 apply. We state the characterization explicitly.

Corollary 9. *The* 1*-filters of a GBI-algebra* $\mathbf{A}$ *are the filter submonoids that are closed under the terms*

$$u_a(x) = (x\top \wedge a)/a, \quad v_a(x) = (a \to ax)/\top \quad \textit{and} \quad \lambda_a(x) = a\backslash xa,$$

or equivalently by the terms

$$u'_a(x) = a\backslash(a \wedge x\top), \quad v'_a(x) = (a \to xa)/\top \quad \textit{and} \quad \rho_a(x) = ax/a.$$

Equivalently, they are the filter submonoids that satisfy, for all $a \in A$,

$$(U_a)\ \forall x \exists x_1,\ x_1 a \le x\top \wedge a, \quad (V_a)\ \forall x \exists x_3,\ x_3\top \wedge a \le ax, \quad (\lambda_a)\ \forall x \exists x',\ ax' \le xa$$

or equivalently the conditions

$$(U'_a)\ \forall x \exists x_2,\ ax_2 \le a \wedge x\top, \quad (V'_a)\ \forall x \exists x_4\ a \wedge x_4\top \le xa, \quad (\rho_a)\ \forall x \exists x',\ x'a \le ax.$$

Likewise, we can characterize the $\top$-filters by taking multiplication to be $\diamond$ and meet to be $\cdot$, in which case $(\lambda), (\rho)$ are automatically satisfied, the condition of F being a submonoids with respect to $\wedge$ holds, and (8) and (9) of Lemma 3 give short descriptions. To clarify that we are using a different interpretation of the operations $\cdot$ and $\diamond$, we place a bar over the terms and conditions. Conditions $(\bar{\lambda}), (\bar{\rho})$ are satisfied by the commutativity of meet.

Note that translating the FL^2 condition $(\bar{V}_a)$ to a term produces $\bar{v}_a(x) = 1 \to (a \wedge x)/a$ (in the GBI language). This simplifies to $\bar{v}_a(x) = 1 \to (x/a)$ since $1 \to (a \wedge x)/a = 1 \to (a/a \wedge x/a) = (1 \to a/a) \wedge (1 \to x/a)$ and $\top \le 1 \to (a/a)$.

Corollary 10. *The* $\top$*-filters of a GBI-algebra* $\mathbf{A}$ *are the filters that are closed under the terms*

$$\bar{u}'_a(x) = a \to a(x \wedge 1), \quad \bar{v}_a(x) = 1 \to (x/a) \quad \textit{and} \quad \bar{\lambda}'_a(x) = 1 \to a\backslash(x \wedge 1)a,$$

or equivalently by the terms

$$\bar{u}_a(x) = a \to (x \wedge 1)a, \quad \bar{v}'_a(x) = 1 \to (a\backslash x) \quad \textit{and} \quad \bar{\rho}'_a(x) = 1 \to a(x \wedge 1)/a.$$

Equivalently, they are the filter submonoids that satisfy, for all $a \in A$,

$$(\bar{U}_a)\ \forall x \exists x_1,\ x_1 \wedge a \le (x \wedge 1)a, \qquad (\bar{V}'_a)\ \forall x \exists x_4,\ a(x_4 \wedge 1) \le x \quad \textit{and}$$

$$(\bar{\rho}'_a)\ \forall x \exists x',\ (x' \wedge 1)a \le a(x \wedge 1),$$

or equivalently the conditions

$$(\bar{U}'_a)\ \forall x \exists x_2,\ a \wedge x_2 \le a(x \wedge 1), \qquad (\bar{V}_a)\ \forall x \exists x_3,\ (x_3 \wedge 1)a \le x \quad \textit{and}$$

$$(\bar{\lambda}'_a)\ \forall x \exists x',\ a(x' \wedge 1) \le (x \wedge 1)a.$$

In a GBI-algebra, by Theorem 4 and Corollary 6 the 1-filters are the submonoid filters F satisfying $\uparrow(Fa) = \uparrow(aF) = \uparrow(a \wedge F\top)$, for all $a \in A$; an element c is a 1-congruence element iff it is negative, idempotent and $\top$-central: $ca = ac = a \wedge c\top$, for all $a \in A$. Likewise, $\top$-filters are the filters G satisfying $\uparrow(a(G \wedge 1)) = \uparrow((G \wedge 1)a) = \uparrow(G \wedge a)$, for all $a \in A$; an element d is a $\top$-congruence element iff it is 1-central: $a(c \wedge 1) = (c \wedge 1)a = c \wedge a$, for all $a \in A$.

For involutive GBI-algebras the characterization simplifies even further. The following result from [8] shows that the characterization of t-filters does not require any parameters in this case.

Theorem 11. *For an involutive GBI-algebra, a lattice filter F is a $\top$-filter if and only if for all $x \in F$ it follows that $\neg{\sim}x, \neg{-}x, {\sim}(\top(-x)\top) \in F$.*

Involutive GBI-algebras include all relation algebras and all representable weakening relation algebras. Several results from relation algebras generalize to the setting of involutive GBI-algebras and other varieties of bunched implication algebras. For example, the term ${\sim}(\top(-x)\top)$ in the previous result is the dual of Tarski's term $\top x\top$ that is used to characterize congruence ideals in relation algebras.

4 Discriminator Varieties of GBI-algebras

Recall that an algebra is *subdirectly irreducible* if it has a smallest nontrivial congruence. For FL^2-algebras Theorem 4 and Corollary 5 imply that this property is the same as having a smallest nontrivial 1-filter or, equivalently, a smallest nontrivial t-filter. For example, this makes it easy to compute all finite subdirectly irreducible bunched implication algebras. Since they have lattice reducts, Jónsson's Lemma [14] implies that two nonisomorphic finite subdirectly irreducible BI-algebras generate distinct subvarieties, i.e., there exists an identity that holds in one of them and fails in the other. The same observations apply to finite FL^2-algebras.

Relation algebras form a discriminator variety, which means that the variety is generated by a class of algebras which have a *ternary discriminator term* $t(x, y, z)$ such that for all algebras in this generating class

$$t(x,y,z) = \begin{cases} x & \text{if } x \neq y \\ z & \text{otherwise.} \end{cases}$$

For relation algebras such a term is given by

$$t(x,y,z) = ((\top; (x \oplus y); \top) \wedge x) \vee (\neg(\top; (x \oplus y); \top) \wedge z)$$

where $x \oplus y = (x \vee y) \wedge \neg(x \wedge y)$ is the symmetric difference operation.

Discriminator varieties are well behaved in the sense that all subvarieties are also discriminator varieties (with the same term t) and all their subdirectly irreducible members are *simple*, i.e., the congruence lattice has only two elements, namely the identity congruence and the top congruence that relates all pairs. In addition, every subalgebra of a simple member is simple, and every finite member is a direct product of simple members. For relation algebras, simplicity is characterized by the Tarski rule $x \neq \bot \;\Rightarrow\; \top x \top = \top$.

An interesting question is whether there are other prominent subvarieties of FL^2-algebras that are discriminator varieties. This is not the case for the variety of BI-algebras since it contains the subvariety of Heyting algebras, defined relative to FL^2 by $x \wedge y = xy = x \diamond y$. Heyting algebras are not a discriminator variety because, e.g., the 3-element Heyting algebra is not simple.

The full weakening relation algebras $\mathbf{Wk}(\mathbf{P})$ for any poset $\mathbf{P}$ satisfy the Tarski rule (since composition and $\top$ are the same as for relation algebras), but the term $t(x, y, z)$ has to be constructed differently since negation is not classical. The following dual form has the required property:

$$t'(x, y, z) = (c(x \leftrightarrow y) \wedge z) \vee (\neg c(x \leftrightarrow y) \wedge x)$$

where $x \leftrightarrow y = (x \to y) \wedge (y \to x)$ and $c(x) = \top\backslash x/\top$. The term c is known as a (dual) unary discriminator [12] since it satisfies $c(\top) = \top$ and for $x \neq \top$, $c(x) = \bot$, i.e., it behaves like a dual Tarski rule, also known as a Baaz Delta [1] in fuzzy logic. Some concepts from relation algebra need to be dualized since in the theory of relation algebras ideals and atoms are more suitable concepts, but in the weaker (noninvolutive) theories of BI-algebras and FL^2-algebras, filters are needed to characterize the congruences. It is easy to check that t' is a discriminator in all full weakening relation algebras, hence the variety RWkRA generated by them is a discriminator variety.

In [8] it is shown that $\mathsf{RWkRA} = SP(\{\mathbf{Wk}(\mathbf{P}) : \mathbf{P} \text{ is a poset}\})$, hence every member of RWkRA is embedded in an algebra of relations and deserves to be called representable. In other words, RWkRA is analogous to the variety RRA of representable relation algebras. Since RRA is not finitely axiomatizable and can be defined from RWkRA by adding a single equation, it follows that RWkRA is also not finitely axiomatizable. A natural problem is to define a finitely based variety WkRA analogous to Tarski's variety RA of relation algebras. The variety SHRA defined in [23] is too large since it fails some short identities that hold in all full weakening relation algebras. It is also not known whether SHRA is a discriminator variety.

In the next section we recall a construction from [8] that generalizes the double coset construction of relation algebras. Applying this construction to RA leads to a variety $S(\mathsf{dRA})$ of cyclic involutive GBI-algebras that contains $\mathsf{RA} \cup \mathsf{RWkRA}$ and is properly contained in SHRA. Currently no (finite) axiomatization is known for $S(\mathsf{dRA})$ but we obtain several identities that hold in all its members.

5 The Double Division Conuclei Construction

The process of factoring a set by an equivalence relation is captured at the level of relation algebras by a construction described in [15]. In a relation algebra $\mathbf{A}$, let e be an idempotent ($ee = e$) symmetric ($e = e^{\smile}$) element and $eAe = \{exe : x \in A\}$.

Then $e\mathbf{A}e = (eAe, \wedge, \vee, \neg_e, e\top e, \bot, \cdot, \breve{}, e, \neg_e e)$ is a relation algebra, where $\neg_e x = \neg x \wedge e\top e$. For group relation algebras this construction is known as a double coset relation algebra, and in this case $e \geq 1$. In [8] this construction is generalized to residuated lattices and GBI-algebras for arbitrary positive idempotents $p{=}p^2 \geq 1$. Given such an element p, let $\delta_p(x) = p\backslash x/p$ and note that this *double division* operation is a *conucleus*, i.e., an interior operator that satisfies $\delta_p(x)\delta_p(y) \leq \delta_p(xy)$. This holds because $\delta_p(x) = \delta'(\delta''(x))$ where $\delta'(x) = p\backslash x$ and $\delta''(x) = x/p$, both of which are conuclei, and this property is preserved under composition. By a version of [9, Prop. 3.41] without the identity, the conucleus image $\delta(\mathbf{A})$ of a residuated lattice is a residuated lattice $(\delta(A), \wedge_\delta, \vee, \cdot, \backslash_\delta, /_\delta)$ possibly without an identity, where $x *_\delta y = \delta(x * y)$ for $* \in \{\wedge, \backslash, /\}$. For the conucleus image $\delta_p(\mathbf{A})$, the element p is the identity element: $p\backslash x/p \leq (p\backslash x/p)p$ since p is positive, and $(p\backslash x/p)pp = (p\backslash x/p)p \leq p\backslash x$ hence $(p\backslash x/p)p \leq p\backslash x/p$. An even easier way to show this is to make use of the result from [8] that $\delta_p(A) = \{pxp : x \in A\}$.

The double division conucleus δ_p is of special interest for relation algebras since a positive idempotent p in a full relation algebra $\mathrm{Rel}(P)$ on a set P is a *preorder* $\mathbf{P} = (P, \sqsubseteq)$ (i.e., $p = \sqsubseteq$ is reflexive and transitive). If we assume $p \wedge p\breve{} = 1$, then $\mathbf{P}$ is a poset and it follows that the full weakening relation algebra $\mathbf{Wk}(\mathbf{P})$ is equal to $\delta_p(\mathrm{Rel}(P))$. This shows that the variety RWkRA contains all double division conucleus images of members of RRA. For any class $\mathcal{K}$ of GBI-algebras we define $\mathsf{d}\mathcal{K} = \{\delta_p(\mathbf{A}) : \mathbf{A} \in \mathcal{K}, 1 \leq p^2 = p \in A\}$. In [8] it is proved that if $\mathcal{V}$ is a variety of bounded GBI-algebras with $\top\backslash x/\top$ as unary discriminator on the subdirectly irreducible members then $S(\mathsf{d}\mathcal{V})$ is a discriminator variety with the same unary discriminator term. Applying this result to the variety RA results in the discriminator variety $S(\mathsf{dRA})$ that contains both RA and RWkRA.

For an element x in a GBI-algebra, define the domain $d(x) = x\top \wedge 1$ and the range $r(x) = \top x \wedge 1$. In [13] it was shown that RWkRA satisfies the standard domain and range identities $d(x)x = x$ and $xr(x) = x$, as well as the identity $\top x \top x \top = \top x \top$. Stell's results in [23] about SHRA, together with the fact that the latter contains RWkRA, imply that RWkRA satisfies the inequality ${\sim}\neg(xy) \leq ({\sim}\neg y)({\sim}\neg x)$. Mace4 [18] shows that these identities do not hold in all cyclic involutive GBI-algebras. The 3-element Łukasiewicz algebra $L = \{0 < a < 1\}$ with $aa{=}0$ is a commutative (hence cyclic) involutive BI-algebra and taking $x{=}a$ gives counterexamples for the first three identities below. The last identity fails with $x{=}1$ in the 4-element Boolean commutative involutive BI-algebra $B{=}\{0{<}1, \neg 1{<}\top\}$ where $(\neg 1)(\neg 1) = 1$ and ${\sim}1 = 1$, ${\sim}\neg 1 = \neg 1$.

Theorem 12. *The identities*

$$d(x)x = x, \quad xr(x) = x, \quad \top x \top x \top = \top x \top \quad \textit{and} \quad {\sim}\neg(xy) \leq ({\sim}\neg y)({\sim}\neg x)$$

hold in $S(\mathsf{dRA})$*.*

Proof. Let x be an element in $\delta_p(A)$ for some relation algebra $\mathbf{A}$ and positive idempotent $p \in A$. The identity element of $\delta_p(\mathbf{A})$ is p, hence $d(x) = x\top \wedge p$. Since $p \geq 1$, $d(x)x \geq (x\top \wedge 1)x = x$ where the last equality holds because it already holds in RA. The opposite inequality holds since $d(x) \le p$, and p is the identity element. The proof for $r(x)$ is similar.

Although a conucleus can in principle map the top element of $\mathbf{A}$ to a smaller element, this is not the case for δ_p since $p \top p = \top$ is in the image of δ_p. Hence the third identity is true since it evaluates the same way in $\mathbf{A}$ as in $\delta_p(\mathbf{A})$.

For the fourth identity, let $x, y \in \delta_p(A)$. Applying ~ on both sides and reversing the inequality $\sim\neg(xy) \le (\sim\neg y)(\sim\neg x)$ we get the equivalent version $\neg x + \neg y \le \neg(xy)$, where $x+y = \sim((\sim y)(\sim x))$ is the *dual product*. Since $x \le \neg y \Leftrightarrow y \le \neg x$ holds in Heyting algebras, the inequality becomes $xy \le \neg(\neg x + \neg y)$. The definition of $\neg x$ in $\delta_p(\mathbf{A})$ is $\delta_p(x^c)$ where x^c is the complement in $\mathbf{A}$. Hence we get the equivalent version $xy \le \delta_p((\neg x + \neg y)^c) = p\backslash(\neg x + \neg y)^c/p$. Using residuation this is equivalent to $pxyp \le (\neg x + \neg y)^c$ and to $xy \le (\delta_p(x^c) + \delta_p(y^c))^c$ since $xy \in \delta_p(A)$, hence $pxyp = xy$. Using de Morgan's law $xy = (x^c + y^c)^c$ in RA and applying complements on both sides the equation is equivalent to $\delta_p(x^c) + \delta_p(y^c) \le x^c + y^c$, where the last inequality holds because δ_p is decreasing. □

These identities are easily derived from the equational basis of RA, but some of these derivations make use of identities that do not hold in all algebras of weakening relations. It would be interesting to find an equational basis for $S(\mathsf{dRA})$. The inequality in Theorem 12 might be part of such a basis, while the other three identities are perhaps derivable from other identities that still need to be discovered.

An example of an identity that holds in all relation algebras but is not preserved by double division conuclei is $(x \wedge 1)(y \wedge 1) = x \wedge y \wedge 1$. Some new identities have nontrivial models RWkRA. For example it is proved in [8] that $0 \le 1$ holds in $\mathbf{Wk}(\mathbf{P})$ if and only if $\mathbf{P}$ is a chain. Here we note that the opposite inequality cannot hold in $S(\mathsf{dRA})$.

Lemma 13. *If* $\mathbf{A} \in S(\mathsf{dRA})$ *satisfies* $1 \le 0$ *then* $\mathbf{A}$ *is trivial.*

Proof. Suppose $1 \le 0$ holds in $\delta_p(\mathbf{A})$ for some relation algebra $\mathbf{A}$ and positive idempotent $p \in A$. Then $p \le \sim p$ in $\mathbf{A}$. Applying complementation on both sides we get $p^{\smile} \le p^c$, or equivalently $p^{\smile} \wedge p = \bot$. Since p is positive, $1 \le p^{\smile}$ hence it follows that $1 = \bot$, forcing $\mathbf{A}$ to be trivial. □

This shows that the 3-element Sugihara chain [9] is not in $S(\mathsf{dRA})$ since it satisfies $0 = 1$. However the 4-element Sugihara chain is representable by the following 4 relations on the rationals $\mathbf{Q}$: $\{\varnothing, <, \le, \mathbf{Q}^2\}$.

Another problem is to find small algebras that are in $S(\mathsf{dRA})$ but not in RWkRA. Of course many small nonrepresentable relation algebras are known, but they must have at least 16 elements. It is currently not known if there are smaller examples in $S(\mathsf{dRA})$.

6 Conclusion

We have shown that several concepts from relation algebras can be lifted to more general settings where they apply to other classes of algebras that occur in logic and computer science. While the variety of FL^2-algebras is somewhat general, it is a convenient setting for results about congruences since the symmetry of the two sets of connectives allows for shorter proofs. Adapting the characterization of FL^2 congruences to GBI-algebras produces a description that is significantly simpler than the previous results in [8]. The variety RWkRA of representable weakening relation algebras is a subvariety of cyclic involutive FL^2 and generalizes RRA from relations over sets to weakening relations over posets. We defined a discriminator variety $S(\mathsf{dRA})$ of cyclic involutive GBI-algebras that contains $\mathsf{RA} \cup \mathsf{RWkRA}$ and showed that it satisfies some identities that hold in both relation algebras and weakening relation algebras.

We thank the referees for the interesting comments and corrections that have substantially improved this paper.

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