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From distributive ℓ-monoids to ℓ-groups, and back again

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Abstract

We prove that an inverse-free equation is valid in the variety LG of lattice-ordered groups (ℓ-groups) if and only if it is valid in the variety DLM of distributive lattice-ordered monoids (distributive ℓ-monoids). This contrasts with the fact that, as proved by Repnitskiĭ, there exist inverse-free equations that are valid in all Abelian ℓ-groups but not in all commutative distributive ℓ-monoids, and, as we prove here, there exist inverse-free equations that are valid in all totally ordered groups but not in all totally ordered monoids. We also prove that DLM has the finite model property and a decidable equational theory, establish a correspondence between the validity of equations in DLM and the existence of certain right orders on free monoids, and provide an effective method for reducing the validity of equations in LG to the validity of equations in DLM.

Keywords: Lattice-ordered groups, distributive lattice-ordered monoids, free

groups, free monoids.

1. Introduction

A *lattice-ordered group* (ℓ-*group*) is an algebraic structure $\langle L, \wedge, \vee, \cdot, {}^{-1}, e \rangle$ such that $\langle L, \cdot, {}^{-1}, e \rangle$ is a group, $\langle L, \wedge, \vee \rangle$ is a lattice, and the group multiplication preserves the lattice order, i.e., $a \leq b$ implies $cad \leq cbd$ for all $a, b, c, d \in L$, where $a \leq b \; :\Longleftrightarrow \; a \wedge b = a$. The class of ℓ-groups forms a variety (equational class) LG and admits the following Cayley-style representation theorem:

Theorem 1.1 (Holland [6]). *Every ℓ-group embeds into an ℓ-group* $\mathbf{Aut}(\langle \Omega, \leq \rangle)$ *consisting of the group of order-automorphisms of a totally ordered set (chain)* $\langle \Omega, \leq \rangle$ *equipped with the pointwise lattice order.*

Holland's theorem has provided the foundations for the development of a rich and extensive theory of ℓ-groups (see [2, 11] for details and references). In particular, it was proved by Holland [7] that an equation is valid in LG if and only if it is valid in $\mathbf{Aut}(\langle \mathbb{Q}, \leq \rangle)$, and by Holland and McCleary [8] that the equational theory of LG is decidable.

The inverse-free reduct of any ℓ-group is a *distributive lattice-ordered monoid* (*distributive ℓ-monoid*): an algebraic structure $\langle M, \wedge, \vee, \cdot, e \rangle$ such that $\langle M, \cdot, e \rangle$ is a monoid, $\langle M, \wedge, \vee \rangle$ is a distributive lattice, and the lattice operations distribute over the monoid multiplication, i.e., for all $a, b, c, d \in M$,

$$a(b \vee c)d = abd \vee acd \quad \text{and} \quad a(b \wedge c)d = abd \wedge acd.$$

The class of distributive ℓ-monoids also forms a variety DLM and admits a Cayley-style (or Holland-style) representation theorem:

Theorem 1.2 (Anderson and Edwards [1]). *Every distributive ℓ-monoid embeds into a distributive ℓ-monoid* $\mathbf{End}(\langle \Omega, \leq \rangle)$ *consisting of the monoid of order-endomorphisms of a chain* $\langle \Omega, \leq \rangle$ *equipped with the pointwise lattice order.*

Despite the obvious similarity of Theorem 1.2 to Theorem 1.1, the precise nature of the relationship between the varieties of distributive ℓ-monoids and ℓ-groups has remained unclear. It was proved by Repnitskiĭ in [13] that the variety of commutative distributive ℓ-monoids does not have the same equational theory as the class of inverse-free reducts of Abelian ℓ-groups, but the decidability of its equational theory remains an open problem. In this paper, we prove the following results for the general (noncommutative) case:

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Theorem 2.3. The variety of distributive ℓ-monoids has the finite model property.[3] More precisely, an equation is valid in all distributive ℓ-monoids if and only if it is valid in all distributive ℓ-monoids of order-endomorphisms of a finite chain.

Corollary 2.4. The equational theory of distributive ℓ-monoids is decidable.

Theorem 2.9. An inverse-free equation is valid in the variety of ℓ-groups if and only if it is valid in the variety of distributive ℓ-monoids.

Theorem 2.9 shows, by way of Birkhoff's variety theorem [3], that distributive ℓ-monoids are precisely the homomorphic images of the inverse-free subreducts of ℓ-groups. It also allows us, using a characterization of valid ℓ-group equations given in [4], to relate the validity of equations in distributive ℓ-monoids to the existence of certain right orders on free monoids. As a notable consequence of this correspondence, we obtain:

Corollary 3.4. Every right order on the free monoid over a set X extends to a right order on the free group over X.

To check whether an equation is valid in all distributive ℓ-monoids, it suffices, by Theorem 2.9, to check the validity of this same equation in all ℓ-groups. We prove here that a certain converse also holds, namely:

Theorem 4.2. Let ε be any ℓ-group equation with variables in a set X. A finite set of inverse-free equations Σ with variables in $X \cup Y$ for some finite set Y can be effectively constructed such that ε is valid in all ℓ-groups if and only if the equations in Σ are valid in all distributive ℓ-monoids.

Finally, we turn our attention to totally ordered groups and totally ordered monoids, that is, ℓ-groups and distributive ℓ-monoids with a total lattice order. We show that the variety generated by the class of totally ordered monoids can be axiomatized relative to DLM by a single equation (Proposition 5.4). However, analogously to the case of commutative distributive ℓ-monoids and unlike the case of DLM, we prove:

Theorem 5.7. There is an inverse-free equation that is valid in all totally ordered groups, but not in all totally ordered monoids.

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[3]Recall that a variety V has the *(strong) finite model property* if an equation (respectively, quasiequation) is valid in V if and only if it is valid in the finite members of V.

We also exhibit an inverse-free equation that is valid in all finite totally ordered monoids, but not in the ordered group of the integers (Proposition 5.8), witnessing the failure of the finite model property for the variety of commutative distributive ℓ-monoids and the varieties generated by totally ordered monoids and inverse-free reducts of totally ordered groups (Corollary 5.9).

2. From distributive ℓ-monoids to ℓ-groups

In this section, we establish the finite model property for the variety DLM of distributive ℓ-monoids (Theorem 2.3) and the decidability of its equational theory (Corollary 2.4). We then prove that an inverse-free equation is valid in DLM if and only if it is valid in the variety LG of ℓ-groups (Theorem 2.9). The key tool for obtaining these results is the notion of a total preorder on a set of monoid terms that is preserved under right multiplication, which bears some similarity to the notion of a diagram employed in [8]. In particular, the existence of such a preorder satisfying a given finite set of inequalities is related to the validity of a corresponding inverse-free equation in DLM or LG.

Let X be any set. We denote by $\mathbf{T}_m(X)$, $\mathbf{T}_g(X)$, $\mathbf{T}_d(X)$, and $\mathbf{T}_\ell(X)$ the term algebras over X for monoids, groups, distributive ℓ-monoids, and ℓ-groups, respectively, and by $\mathbf{F}_m(X)$, $\mathbf{F}_g(X)$, $\mathbf{F}_d(X)$, and $\mathbf{F}_\ell(X)$, the corresponding free algebras, assuming for convenience that $F_m(X) \subseteq T_m(X)$, $F_g(X) \subseteq T_g(X)$, $F_d(X) \subseteq T_d(X)$, and $F_\ell(X) \subseteq T_\ell(X)$. Given a set of ordered pairs of monoid terms $S \subseteq F_m(X)^2$, we define the set of *initial subterms* of S:

$$\mathrm{is}(S) := \{u \in F_m(X) \mid \exists s, t \in F_m(X) \colon \langle us, t\rangle \in S \text{ or } \langle s, ut\rangle \in S\}.$$

Note in particular that $s, t \in \mathrm{is}(S)$ for each $\langle s, t\rangle \in S$.

Recall now that a *preorder* $\preceq$ on a set P is a binary relation on P that is reflexive and transitive. We write $a \prec b$ to denote that $a \preceq b$ and $b \not\preceq a$, and call $\preceq$ *total* if $a \preceq b$ or $b \preceq a$ for all $a, b \in P$. Let $\preceq$ be a preorder on a set of monoid terms $P \subseteq F_m(X)$. We say that $\preceq$ is *right-X-invariant* if for all $x \in X$, whenever $u \preceq v$ and $ux, vx \in P$, also $ux \preceq vx$, and *strictly right-X-invariant* if it is right X-invariant and for all $x \in X$, whenever $u \prec v$ and $ux, vx \in P$, also $ux \prec vx$.

Following standard practice for ℓ-groups, we write $(p)f$ for the value of a (partial) map $f \colon \Omega \to \Omega$ defined at $p \in \Omega$. As a notational aid, we also often write φ_r to denote the value of a (partial) map φ defined for some element r.

Lemma 2.1. *Let $S \subseteq F_m(X)^2$ be a finite set of ordered pairs of monoid terms and let $\preceq$ be a total right-X-invariant preorder on $\mathrm{is}(S)$ satisfying $s \prec t$ for each $\langle s, t\rangle \in S$.*

(a) *There exists a chain $\langle \Omega, \leq \rangle$ satisfying $|\Omega| \leq |\mathrm{is}(S)|$, a homomorphism $\varphi \colon \mathbf{T}_d(X) \to \mathbf{End}(\langle \Omega, \leq \rangle)$, and some $p \in \Omega$ such that $(p)\varphi_s < (p)\varphi_t$ for each $\langle s, t \rangle \in S$.*

(b) *If $\preceq$ is also strictly right-X-invariant, then there exists a homomorphism $\psi \colon \mathbf{T}_\ell(X) \to \mathbf{Aut}(\langle \mathbb{Q}, \leq \rangle)$ and some $q \in \mathbb{Q}$ such that $(q)\psi_s < (q)\psi_t$ for each $\langle s, t \rangle \in S$.*

Proof. For (a), we let $[u] := \{v \in \mathrm{is}(S) \mid u \preceq v \text{ and } v \preceq u\}$ for each $u \in \mathrm{is}(S)$ and define $\Omega := \{[u] \mid u \in \mathrm{is}(S)\}$, noting that $|\Omega| \leq |\mathrm{is}(S)|$. If $[u] = [u']$, $[v] = [v']$, and $u \preceq v$, then $u' \preceq v'$, so we can define for $[u], [v] \in \Omega$,

$$[u] \leq [v] \;:\Longleftrightarrow\; u \preceq v.$$

Clearly, $\leq$ is a total order on Ω and $[s] < [t]$ for each $\langle s, t \rangle \in S$. Moreover, if $[u], [v] \in \Omega$, $x \in X$, and $ux, vx \in \mathrm{is}(S)$, then, using the right-$X$-invariance of $\preceq$,

$$[u] \leq [v] \implies [ux] \leq [vx].$$

In particular, if $[u] = [v] \in \Omega$, $x \in X$, and $ux, vx \in \mathrm{is}(S)$, then $[ux] = [vx]$. Hence for each $x \in X$, we obtain a partial order-endomorphism $\tilde{\varphi}_x \colon \Omega \to \Omega$ of $\langle \Omega, \leq \rangle$ by defining $([u])\tilde{\varphi}_x := [ux]$ whenever $[u] \in \Omega$ and $ux \in \mathrm{is}(S)$. Moreover, each of these partial maps $\tilde{\varphi}_x$ extends to an order-endomorphism $\varphi_x \colon \Omega \to \Omega$ of $\langle \Omega, \leq \rangle$. Now let $\varphi \colon \mathbf{T}_d(X) \to \mathbf{End}(\langle \Omega, \leq \rangle)$ be the homomorphism extending the assignment $x \mapsto \varphi_x$. Then $([\mathrm{e}])\varphi_u = [u]$ for every $u \in \mathrm{is}(S)$ and hence $([\mathrm{e}])\varphi_s < ([\mathrm{e}])\varphi_t$ for each $\langle s, t \rangle \in S$.

For (b), note that the set Ω defined in (a) is finite and, assuming that $\preceq$ is strictly right-X-invariant, the partial order-endomorphisms $\tilde{\varphi}_x \colon \Omega \to \Omega$ of $\langle \Omega, \leq \rangle$ for $x \in X$ are injective. Hence $\langle \Omega, \leq \rangle$ can be identified with a subchain of $\langle \mathbb{Q}, \leq \rangle$ and each $\tilde{\varphi}_x$ can be extended to an order-automorphism $\psi_x \colon \mathbb{Q} \to \mathbb{Q}$ of $\langle \mathbb{Q}, \leq \rangle$. As in (a), we obtain a homomorphism $\psi \colon \mathbf{T}_\ell(X) \to \mathbf{Aut}(\langle \mathbb{Q}, \leq \rangle)$ extending the assignment $x \mapsto \psi_x$ such that $([\mathrm{e}])\varphi_s < ([\mathrm{e}])\varphi_t$ for each $\langle s, t \rangle \in S$. $\qquad\square$

For $s, t \in T_\ell(X)$, we write $s \leq t$ as an abbreviation for the equation $s \wedge t \approx s$, noting that $s \approx t$ is valid in an ℓ-monoid or ℓ-group $\mathbf{L}$ if and only if $s \leq t$ and $t \leq s$ are valid in $\mathbf{L}$. It is easily seen that every ℓ-group (or ℓ-monoid) term is equivalent in LG (or DLM) to both a join of meets of group (monoid) terms and a meet of joins of group (monoid) terms. It follows that to check the validity of an (inverse-free) equation in LG (or DLM), it suffices to consider equations of the form $\bigwedge_{i=1}^{n} t_i \leq \bigvee_{j=1}^{m} s_j$ where $s_j, t_i \in F_g(X)$ (or $s_j, t_i \in F_m(X)$) for $1 \leq i \leq n$,

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$1 \leq j \leq m$. The next lemma relates the validity of an inverse-free equation of this form in LG or DLM to the existence of a total (strictly) right-X-invariant preorder on a corresponding set of initial subterms.

Lemma 2.2. *Let $\varepsilon = (\bigwedge_{i=1}^{n} t_i \leq \bigvee_{j=1}^{m} s_j)$ where $s_j, t_i \in F_m(X)$ for $1 \leq i \leq n$, $1 \leq j \leq m$, and let $S := \{\langle s_j, t_i \rangle \in F_m(X)^2 \mid 1 \leq i \leq n,\ 1 \leq j \leq m\}$.*

- (a) DLM $\models \varepsilon$ *if and only if there is no total right-X-invariant preorder $\preceq$ on* is(S) *satisfying $s \prec t$ for each $\langle s, t \rangle \in S$.*
- (b) LG $\models \varepsilon$ *if and only if there is no total strictly right-X-invariant preorder $\preceq$ on* is(S) *satisfying $s \prec t$ for each $\langle s, t \rangle \in S$.*

Proof. For the left-to-right direction of (a), suppose contrapositively that there exists a total right-X-invariant preorder $\preceq$ on is(S) satisfying $s \prec t$ for each $\langle s, t \rangle \in S$. By Lemma 2.1(a), there exist a chain $\langle \Omega, \leq \rangle$, a homomorphism $\varphi\colon \mathbf{T}_d(X) \to \mathbf{End}(\langle \Omega, \leq \rangle)$, and some $p \in \Omega$ such that $(p)\varphi_s < (p)\varphi_t$ for each $\langle s, t \rangle \in S$. So $(p)\varphi_{\bigwedge_{i=1}^{n} t_i} > (p)\varphi_{\bigvee_{j=1}^{m} s_j}$, and hence DLM $\not\models \varepsilon$. Similarly, for the left-to-right direction of (b), there exist, by Lemma 2.1(b), a homomorphism $\psi\colon \mathbf{T}_\ell(X) \to \mathbf{Aut}(\langle \mathbb{Q}, \leq \rangle)$ and some $q \in \mathbb{Q}$ such that $(q)\psi_{\bigwedge_{i=1}^{n} t_i} > (q)\psi_{\bigvee_{j=1}^{m} s_j}$ and hence LG $\not\models \varepsilon$.

For the right-to-left direction of (a), suppose contrapositively that DLM $\not\models \varepsilon$. By Theorem 1.2, there exist a chain $\langle \Omega, \leq \rangle$, a homomorphism $\varphi\colon \mathbf{T}_d(X) \to \mathbf{End}(\langle \Omega, \leq \rangle)$, and some $p \in \Omega$ such that $\bigwedge_{i=1}^{n}(p)\varphi_{t_i} > \bigvee_{j=1}^{m}(p)\varphi_{s_j}$. Then $(p)\varphi_t > (p)\varphi_s$ for each $\langle s, t \rangle \in S$ and we define for $u, v \in$ is(S),

$$u \preceq v :\Longleftrightarrow (p)\varphi_u \leq (p)\varphi_v.$$

Clearly $\preceq$ is a total preorder satisfying $s \prec t$ for each $\langle s, t \rangle \in S$. Moreover, since φ is a homomorphism, $\preceq$ is right-X-invariant on is(S).

For the right-to-left direction of (b), suppose that LG $\not\models \varepsilon$. By Theorem 1.1, there exist a chain $\langle \Omega, \leq \rangle$, a homomorphism $\psi\colon \mathbf{T}_\ell(X) \to \mathbf{Aut}(\langle \Omega, \leq \rangle)$, and $q \in \Omega$ such that $\bigwedge_{i=1}^{n}(q)\psi_{t_i} > \bigvee_{j=1}^{m}(q)\psi_{s_j}$. The proof then proceeds exactly as in the case of (a), except that we may observe finally that $\preceq$ is strictly right-X-invariant on is(S), using the fact that ψ_u is bijective for each $u \in$ is(S). $\square$

We now combine the first parts of the preceding lemmas to obtain:

Theorem 2.3. *The variety of distributive ℓ-monoids has the finite model property. More precisely, an equation is valid in all distributive ℓ-monoids if and only if it is valid in all distributive ℓ-monoids of order-endomorphisms of a finite chain.*

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Proof. It suffices to establish the result for an equation $\varepsilon = (\bigwedge_{i=1}^{n} t_i \leq \bigvee_{j=1}^{m} s_j)$, where $s_1, \ldots, s_m, t_1, \ldots, t_n \in F_m(X)$. Suppose that $\mathsf{DLM} \not\models \varepsilon$ and let $S := \{\langle s_j, t_i \rangle \mid 1 \leq i \leq n, \ 1 \leq j \leq m\}$. Combining Lemmas 2.2(a) and 2.1(a), there exist a finite chain $\langle \Omega, \leq \rangle$, a homomorphism $\varphi \colon \mathbf{T}_d(X) \to \mathbf{End}(\langle \Omega, \leq \rangle)$, and some $p \in \Omega$ such that $(p)\varphi_s < (p)\varphi_t$ for each $\langle s, t \rangle \in S$. But then $(p)\varphi_{\bigwedge_{i=1}^{n} t_i} > (p)\varphi_{\bigvee_{j=1}^{m} s_j}$, so $\mathbf{End}(\langle \Omega, \leq \rangle) \not\models \varepsilon$. $\qquad\square$

Since DLM is a finitely axiomatized variety, we also obtain:

Corollary 2.4. *The equational theory of distributive ℓ-monoids is decidable.*

Similarly, the second parts of Lemmas 2.1 and 2.2 can be used to show that an inverse-free equation is valid in all ℓ-groups if and only if it is valid in $\mathbf{Aut}(\langle \mathbb{Q}, \leq \rangle)$. Indeed, this correspondence is known to hold for all equations.

Theorem 2.5 ([7])**.** *An equation is valid in all ℓ-groups if and only if it is valid in $\mathbf{Aut}(\langle \mathbb{Q}, \leq \rangle)$.*

Lemma 2.7 below provides the key ingredient for showing that an inverse-free equation is valid in LG if and only if it is valid in DLM. First, we illustrate the rather involved construction in the proof of this lemma with a simple example.

Example 2.6. Let $\mathbf{End}(\mathbf{2})$ be the distributive ℓ-monoid of order-endomorphisms of the two-element chain $\mathbf{2} = \langle \{0, 1\}, \leq \rangle$, and let $\langle k_0, k_1 \rangle$ denote the member of $\mathbf{End}(\mathbf{2})$ with $0 \mapsto k_0$ and $1 \mapsto k_1$. The equation $yxy \leq xyx$ fails in $\mathbf{End}(\mathbf{2})$, since for the homomorphism $\varphi \colon \mathbf{T}_d(\{x, y\}) \to \mathbf{End}(\mathbf{2})$ extending the assignment $x \mapsto \varphi_x = \langle 0, 0 \rangle$ and $y \mapsto \varphi_y = \langle 1, 1 \rangle$, we obtain

$$(1)\varphi_{yxy} = (((1)\varphi_y)\varphi_x)\varphi_y = 1 > 0 = (((1)\varphi_x)\varphi_y)\varphi_x = (1)\varphi_{xyx}.$$

Let $S := \{\langle xyx, yxy \rangle\}$. Then φ yields a total right-$\{x, y\}$-invariant preorder $\preceq$ on $\mathrm{is}(S) = \{\mathrm{e}, x, y, xy, yx, xyx, yxy\}$ given by $x \sim yx \sim xyx \prec \mathrm{e} \sim y \sim xy \sim yxy$, since $(1)\varphi_x = (1)\varphi_{yx} = (1)\varphi_{xyx} = 0 < 1 = (1)\varphi_\mathrm{e} = (1)\varphi_y = (1)\varphi_{xy} = (1)\varphi_{yxy}$. Note that $\preceq$ is not strictly right-$\{x, y\}$-invariant, since $x \prec \mathrm{e}$, but $xy \sim y$; this corresponds to the fact that φ_y is not a partial bijective map on $\{0, 1\}$, as $0 < 1$ and $(0)\varphi_y = (1)\varphi_y$.

We describe a total strictly right-$\{x, y\}$-invariant preorder $\unlhd$ on $\mathrm{is}(S)$ such that $\prec \subseteq \lhd$. This corresponds to constructing partial bijections $\widehat{\varphi}_x$ and $\widehat{\varphi}_y$ on $\mathrm{is}(S)$ that extend φ_x and φ_y, respectively. The relation $\lhd$ can be computed directly using the definition given in Lemma 2.7, but to provide both a simpler description and

intuition for the construction, we identify each element $x_k \cdots x_1$ of $\mathrm{is}(S)$ with the sequence $((1)\varphi_e, (1)\varphi_{x_k}, \ldots, (1)\varphi_{x_k \cdots x_1})$, so $e = (1)$, $x = (1,0)$, $y = (1,1)$, $xy = (1,0,1)$, $yx = (1,1,0)$, $xyx = (1,0,1,0)$, and $yxy = (1,1,0,1)$. Note that these are the paths of elements of $\{0,1\}$ involved in the successive computation steps for each term at the point $p = 1$ and can be visualized as indicated in Figure 1.
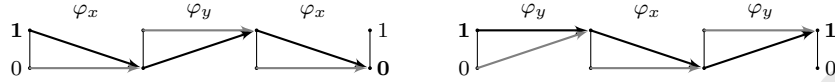

Figure 1: The paths for $xyx = (1,0,1,0)$ and $yxy = (1,1,0,1)$.

The relation $\lhd$ on these paths is simply the reverse lexicographic order:

$$(1,0) \lhd (1,0,1,0) \lhd (1,1,0) \lhd (1) \lhd (1,0,1) \lhd (1,1,0,1) \lhd (1,1),$$

where the first three elements serve as copies of 0 and the last four as copies of 1, so via the above identification we obtain

$$x \lhd xyx \lhd yx \lhd e \lhd xy \lhd yxy \lhd y.$$

It can be verified that this is a total strictly right-$\{x,y\}$-invariant (pre)order, or, more easily, that the corresponding partial order-endomorphisms $\widehat{\varphi}_x$ and $\widehat{\varphi}_y$ are partial bijections as shown in Figure 2.
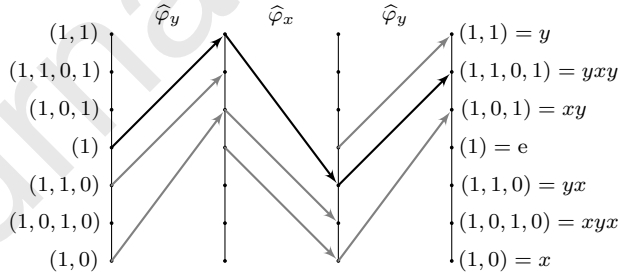

Figure 2: The partial bijections $\widehat{\varphi}_x$ and $\widehat{\varphi}_y$ and the evaluation of $\widehat{\varphi}_{xyx}$ at $(1) = e$.

Lemma 2.7. *For any $S \subseteq F_m(X)^2$ and total right-X-invariant preorder $\preceq$ on* $\mathrm{is}(S)$*, there exists a total strictly right-X-invariant preorder $\lhd$ on* $\mathrm{is}(S)$ *such that* $\prec \, \subseteq \, \lhd$*.*

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Proof. We define the following relations on $\mathrm{is}(S)$:

$$u \sim v \; :\Longleftrightarrow \; u \preceq v \text{ and } v \preceq u;$$

$$x_k \cdots x_1 \lhd y_l \cdots y_1 \; :\Longleftrightarrow \; \exists j \leq l+1 \colon x_k \cdots x_i \sim y_l \cdots y_i \text{ for all } i < j \text{ and}$$
$$(x_k \cdots x_j \prec y_l \cdots y_j \text{ or } j = k+2);$$

$$x_k \cdots x_1 \equiv y_l \cdots y_1 \; :\Longleftrightarrow \; k = l \text{ and } x_k \cdots x_i \sim y_l \cdots y_i \text{ for each } i \leq k;$$

$$u \trianglelefteq v \; :\Longleftrightarrow \; u \lhd v \text{ or } u \equiv v,$$

assuming that $x_k \cdots x_i$ is the empty product e for $i > k$.

Observe that setting $j = 1$ in the definition of $\lhd$ yields $\prec \subseteq \lhd$. Also $u \lhd v$ implies $u \not\equiv v$. The irreflexivity of $\lhd$ follows directly from the fact that $\prec$ is irreflexive. For the transitivity of $\lhd$, we consider $u, v, w \in \mathrm{is}(S)$ satisfying $u = x_k \cdots x_1$, $v = y_l \cdots y_1$, $w = z_m \cdots z_1$, $u \lhd v$, and $v \lhd w$. By definition, there exists a $j_1 \leq l+1$ such that $x_k \cdots x_i \sim y_l \cdots y_i$ for all $i < j_1$, and either $x_k \cdots x_{j_1} \prec y_l \cdots y_{j_1}$ or $j_1 = k+2$, and there exists a $j_2 \leq m+1$ such that $y_l \cdots y_i \sim z_m \cdots z_i$ for all $i < j_2$, and either $y_l \cdots y_{j_2} \prec z_m \cdots z_{j_2}$ or $j_2 = l+2$. There are four cases to check:

1. $x_k \cdots x_{j_1} \prec y_l \cdots y_{j_1}$ and $y_l \cdots y_{j_2} \prec z_m \cdots z_{j_2}$. If $j_2 \leq j_1$, then $x_k \cdots x_i \sim y_l \cdots y_i \sim z_m \cdots z_i$ for all $i < j_2$ and $x_k \cdots x_{j_2} \sim y_l \cdots y_{j_2} \prec z_m \cdots z_{j_2}$, so (since $\sim$ and $\preceq$ are transitive), $u \lhd w$. If $j_1 < j_2$, then $j_1 \leq m+1$ and $x_k \cdots x_i \sim y_l \cdots y_i \sim z_m \cdots z_i$ for all $i < j_1$ and $x_k \cdots x_{j_1} \prec y_l \cdots y_{j_1} \sim z_m \cdots z_{j_1}$, so $u \lhd w$.

2. $x_k \cdots x_{j_1} \prec y_l \cdots y_{j_1}$ and $j_2 = l+2$. Then $j_1 \leq l+1 < j_2 \leq m+1$, so $x_k \cdots x_i \sim y_l \cdots y_i \sim z_m \cdots z_i$ for all $i < j_1$ and $x_k \cdots x_{j_1} \prec y_l \cdots y_{j_1} \sim z_m \cdots z_{j_1}$. Hence $u \lhd w$.

3. $j_1 = k+2$ and $y_l \cdots y_{j_2} \prec z_m \cdots z_{j_2}$. If $j_2 < j_1$, then $x_k \cdots x_i \sim y_l \cdots y_i \sim z_m \cdots z_i$ for all $i < j_2$ and $x_k \cdots x_{j_2} \sim y_l \cdots y_{j_2} \prec z_m \cdots z_{j_2}$, so $u \lhd w$. If $j_1 \leq j_2$, then $j_1 \leq m+1$, $x_k \cdots x_i \sim y_l \cdots y_i \sim z_m \cdots z_i$ for all $i < j_1$, and $j_1 = k+2$, so $u \lhd w$.

4. $j_1 = k+2$ and $j_2 = l+2$. Then $j_1 \leq m+1$ and $x_k \cdots x_i \sim y_l \cdots y_i \sim z_m \cdots z_i$ for all $i < j_1$. Hence $u \lhd w$.

For the transitivity of $\trianglelefteq$, there are also several cases to check. Clearly, if $u \lhd v$ and $v \lhd w$, then $u \lhd w$, by the transitivity of $\lhd$. If $u \lhd v$ and $v \equiv w$, then $u \lhd w$, using the definition of $\lhd$ and $\equiv$ and the transitivity of $\sim$ and $\prec$. Similarly, if $u \equiv v$ and $v \lhd w$, then $u \lhd w$. Finally, if $u \equiv v$ and $v \equiv w$, then $u \equiv w$, by the

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transitivity of $\sim$. Moreover, $\unlhd$ is reflexive, since $u \equiv u$ for any $u \in \mathrm{is}(S)$, so $\unlhd$ is a preorder. Since $\preceq$ is total, $u \not\preceq v$ and $v \not\preceq u$ implies $u \equiv v$; so $\unlhd$ is total. Note also that $u \lhd v$ if and only if $u \unlhd v$ and $v \not\unlhd u$ as suggested by the notation.

To prove that $\unlhd$ is strictly right-X-invariant on $\mathrm{is}(S)$, consider $x \in X$ and $u, v \in \mathrm{is}(S)$ such that $u \unlhd v$ and $ux, vx \in \mathrm{is}(S)$. Suppose first that $u \equiv v$, so u and v have the same length and $u \sim v$. Then ux and vx have the same length and, since $\preceq$ is right-X-invariant, $ux \sim vx$. So $ux \equiv vx$ and hence $ux \unlhd vx$. Now suppose that $u \lhd v$. If $ux \prec vx$, then $ux \lhd vx$. Also, if $ux \sim vx$, then, since $u \lhd v$, the definition of $\lhd$ gives $ux \lhd vx$. Finally, suppose towards a contradiction that $ux \not\preceq vx$. Since $\preceq$ is right-X-invariant, $u \not\preceq v$. But then, since $\preceq$ is total, $v \prec u$ and so $v \lhd u$, contradicting $u \lhd v$. $\qquad \square$

Proposition 2.8. *An inverse-free equation is valid in all distributive ℓ-monoids if and only if it is valid in* $\mathbf{Aut}(\langle \mathbb{Q}, \leq \rangle)$.

Proof. The left-to-right direction follows directly from the fact that the inverse-free reduct of $\mathbf{Aut}(\langle \mathbb{Q}, \leq \rangle)$ is a distributive ℓ-monoid. For the converse, suppose without loss of generality that DLM $\not\models \bigwedge_{i=1}^{n} t_i \leq \bigvee_{j=1}^{m} s_j$, where $s_j, t_i \in F_m(X)$ for $1 \leq i \leq n$, $1 \leq j \leq m$, and let $S := \{\langle s_j, t_i \rangle \mid 1 \leq i \leq n, \ 1 \leq j \leq m\}$. By Lemma 2.2(a), there exists a total right-X-invariant preorder $\preceq$ on $\mathrm{is}(S)$ satisfying $s \prec t$ for each $\langle s, t \rangle \in S$. By Lemma 2.7, there exists a total strictly right-X-invariant preorder $\unlhd$ on $\mathrm{is}(S)$ such that $\prec \ \subseteq \ \lhd$. In particular, $s \lhd t$ for each $\langle s, t \rangle \in S$. Hence, by Lemma 2.1(b), there exist a homomorphism $\psi \colon \mathbf{T}_\ell(X) \to \mathbf{Aut}(\langle \mathbb{Q}, \leq \rangle)$ and $q \in \mathbb{Q}$ such that $(q)\psi_{s_j} < (q)\psi_{t_i}$ for $1 \leq i \leq n$, $1 \leq j \leq m$. So $\mathbf{Aut}(\langle \mathbb{Q}, \leq \rangle) \not\models \bigwedge_{i=1}^{n} t_i \leq \bigvee_{j=1}^{m} s_j$. $\qquad \square$

The main result of this section now follows directly from Proposition 2.8 and the fact that the inverse-free reduct of any ℓ-group is a distributive ℓ-monoid.

Theorem 2.9. *An inverse-free equation is valid in the variety of ℓ-groups if and only if it is valid in the variety of distributive ℓ-monoids.*

It follows by Birkhoff's variety theorem [3] that DLM is generated as a variety by the class of inverse-free reducts of ℓ-groups and hence that distributive ℓ-monoids are precisely the homomorphic images of the inverse-free subreducts of ℓ-groups.

Since the equational theories of the varieties of distributive lattices [10] and ℓ-groups [5] are co-NP-complete, we also obtain the following complexity result:

Corollary 2.10. *The equational theory of distributive ℓ-monoids is co-NP-complete.*

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The correspondence between ℓ-groups and distributive ℓ-monoids established in Theorem 2.9 does not extend to inverse-free quasiequations. In particular, the quasiequation $xz \approx yz \implies x \approx y$, describing right cancellativity, is valid in all ℓ-groups, but not in the distributive ℓ-monoid $\mathbf{End}(\mathbf{2})$. A further example is the quasiequation $xy \approx \mathrm{e} \implies yx \approx \mathrm{e}$, which is clearly valid in all ℓ-groups, but not in the distributive ℓ-monoid $\mathbf{End}(\langle \mathbb{N}, \leq \rangle)$. To see this, define $f, g \in \mathrm{End}(\langle \mathbb{N}, \leq \rangle)$ by $(n)f := n + 1$ and $(n)g := \max(n - 1, 0)$; then $(n)fg = n$ for all $n \in \mathbb{N}$, but $(0)gf = 1$. Let us also remark, however, that this quasiequation is valid in any finite distributive ℓ-monoid $\mathbf{L}$. If $ab = \mathrm{e}$ for some $a, b \in L$, then, by finiteness, $a^n = a^{n+k}$ for some $n, k \in \mathbb{N}^{>0}$, so $\mathrm{e} = a^n b^n = a^{n+k} b^n = a^k$ and $ba = a^k ba = a^{k-1}aba = a^k = \mathrm{e}$. Hence the variety of distributive ℓ-monoids does not have the strong finite model property.

3. Right orders on free groups and free monoids

In this section, we use Theorem 2.9 and a characterization of valid ℓ-group equations in LG given in [4] to relate the existence of a right order on a free monoid satisfying some finite set of inequalities to the validity of an equation in DLM (Theorem 3.3). In particular, it follows that any right order on the free monoid over a set X extends to a right order on the free group over X (Corollary 3.4).

Recall first that a *right order* on a monoid (or group) $\mathbf{M}$ is a total order $\leq$ on M such that $a \leq b$ implies $ac \leq bc$ for any $a, b, c \in M$; in this case, $\mathbf{M}$ is said to be *right-orderable*. Left orders and left-orderability are defined symmetrically.

The following result of [4] establishes a correspondence between the validity of an equation in LG and the existence of a right order on a free group with a negative cone (or, by duality, a positive cone) containing certain elements.

Theorem 3.1 ([4, Theorem 2]). *Let $s_1, \ldots, s_m \in F_g(X)$. Then $\mathsf{LG} \models \mathrm{e} \leq \bigvee_{j=1}^{m} s_j$ if and only if there is no right order $\leq$ on $\mathbf{F}_g(X)$ satisfying $s_j < \mathrm{e}$ for $1 \leq j \leq m$.*

Combining this result with Theorem 2.9, we obtain a correspondence between the validity of an equation in DLM and the existence of a right order on a free monoid satisfying certain corresponding inequalities.

Proposition 3.2. *Let $\varepsilon = (\bigwedge_{i=1}^{n} t_i \leq \bigvee_{j=1}^{m} s_j)$ where $s_j, t_i \in F_m(X)$ for $1 \leq i \leq n$, $1 \leq j \leq m$. Then $\mathsf{DLM} \models \varepsilon$ if and only if there is no right order $\leq$ on $\mathbf{F}_m(X)$ satisfying $s_j < t_i$ for $1 \leq i \leq n$, $1 \leq j \leq m$.*

Proof. For the left-to-right direction, suppose contrapositively that there exists a right order $\leq$ on $\mathbf{F}_m(X)$ satisfying $s_j < t_i$ for $1 \leq i \leq n$, $1 \leq j \leq m$. Then

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DLM $\not\models \varepsilon$ by Lemma 2.2(a). For the converse, suppose contrapositively that DLM $\not\models \varepsilon$. By Theorem 2.9, also LG $\not\models \varepsilon$ and, rewriting the equation,

$$\mathsf{LG} \not\models \mathrm{e} \leq \bigvee \{s_j t_i^{-1} \mid 1 \leq i \leq n,\, 1 \leq j \leq m\}.$$

By Theorem 3.1, there exists a right order $\leq$ on $\mathbf{F}_g(X)$ such that $s_j t_i^{-1} < \mathrm{e}$, or equivalently $s_j < t_i$, for $1 \leq i \leq n$, $1 \leq j \leq m$. The restriction of $\leq$ to $F_m(X)$ therefore provides the required right order on $\mathbf{F}_m(X)$. □

Proposition 3.2 relates the validity of an equation in DLM to the existence of a right order extending an associated set of inequalities on a free monoid. However, it does not relate the existence of a right order on a free monoid extending a given set of inequalities to the validity of some equation in DLM. The next result establishes such a relationship via the introduction of finitely many new variables.

Theorem 3.3. *Let $s_1, t_1 \ldots, s_n, t_n \in F_m(X)$. The following are equivalent:*

(1) *There exists a right order $\leq$ on $\mathbf{F}_g(X)$ satisfying $s_i < t_i$ for $1 \leq i \leq n$.*
(2) *There exists a right order $\leq$ on $\mathbf{F}_m(X)$ satisfying $s_i < t_i$ for $1 \leq i \leq n$.*
(3) DLM $\not\models \bigwedge_{i=1}^n t_i y_i \leq \bigvee_{i=1}^n s_i y_i$ *for any distinct $y_1, \ldots, y_n \notin X$.*

Proof. $(1) \Rightarrow (2)$. This follows directly from the fact that if $\leq$ is a right order on $\mathbf{F}_g(X)$, then the restriction of $\leq$ to $F_m(X)$ is a right order on $\mathbf{F}_m(X)$.

$(2) \Rightarrow (3)$. Let $\leq$ be a right order on $\mathbf{F}_m(X)$ satisfying $s_i < t_i$ for $1 \leq i \leq n$, assuming without loss of generality that X is finite. By Lemma 2.1(b), there exists a homomorphism $\psi \colon \mathbf{T}_\ell(X) \to \mathbf{Aut}(\langle \mathbb{Q}, \leq \rangle)$ and $q \in \mathbb{Q}$ such that $(q)\psi_{s_i} < (q)\psi_{t_i}$ for $1 \leq i \leq n$. So $\mathbf{Aut}(\langle \mathbb{Q}, \leq \rangle) \not\models \mathrm{e} \leq \bigvee_{i=1}^n s_i t_i^{-1}$ and clearly LG $\not\models \bigwedge_{i=1}^n t_i t_i^{-1} \leq \bigvee_{i=1}^n s_i t_i^{-1}$. But then for any distinct $y_1, \ldots, y_n \notin X$, we have LG $\not\models \bigwedge_{i=1}^n t_i y_i \leq \bigvee_{i=1}^n s_i y_i$ and therefore also DLM $\not\models \bigwedge_{i=1}^n t_i y_i \leq \bigvee_{i=1}^n s_i y_i$.

$(3) \Rightarrow (1)$. Suppose that DLM $\not\models \bigwedge_{i=1}^n t_i y_i \leq \bigvee_{i=1}^n s_i y_i$ for some distinct $y_1, \ldots, y_n \notin X$. By Theorem 2.9, also LG $\not\models \bigwedge_{i=1}^n t_i y_i \leq \bigvee_{i=1}^n s_i y_i$ and, by multiplying by the inverse of the left side, LG $\not\models \mathrm{e} \leq (\bigvee_{i=1}^n s_i y_i)(\bigvee_{i=1}^n y_i^{-1} t_i^{-1})$. But then, since LG $\models \bigvee_{i=1}^n s_i t_i^{-1} \leq (\bigvee_{i=1}^n s_i y_i)(\bigvee_{i=1}^n y_i^{-1} t_i^{-1})$, it follows that LG $\not\models \mathrm{e} \leq \bigvee_{i=1}^n s_i t_i^{-1}$. Hence, by Theorem 3.1, there exists a right order $\leq$ on $\mathbf{F}_g(X)$ satisfying $s_i t_i^{-1} < \mathrm{e}$, or equivalently $s_i < t_i$, for $1 \leq i \leq n$. □

For any group $\mathbf{G}$ and $N \subseteq G$, there exists a right order $\leq$ on $\mathbf{G}$ satisfying $a < \mathrm{e}$ for all $a \in N$ if and only if for every finite subset $N' \subseteq N$, there exists a right order $\leq'$ on $\mathbf{G}$ satisfying $a < \mathrm{e}$ for all $a \in N'$ (see, e.g., [11, Chapter 5, Lemma 1]). Theorem 3.3 therefore yields the following corollary:

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Corollary 3.4. *Every right order on the free monoid over a set X extends to a right order on the free group over X.*

Note also that by left-right duality, every left order on the free monoid over a set X extends to a left order on the free group over X.

We conclude this section with a brief discussion of the relationship between distributive ℓ-monoids and right-orderable monoids. It was proved in [9] that a group is right-orderable if and only if it is a subgroup of the group reduct of an ℓ-group, and claimed in [1] that an analogous theorem holds in the setting of distributive ℓ-monoids. Indeed, any monoid $\mathbf{M}$ that admits a right order $\leq$ embeds into the monoid reduct of the distributive ℓ-monoid $\mathbf{End}(\langle M, \leq \rangle)$ by mapping each $a \in M$ to the order-endomorphism $x \mapsto xa$. However, contrary to the claim made in [1], it is not the case that every submonoid of the monoid reduct of a distributive ℓ-monoid is right-orderable.

Proposition 3.5. *The monoid reduct of $\mathbf{End}(\langle \Omega, \leq \rangle)$ is not right-orderable for any chain $\langle \Omega, \leq \rangle$ with $|\Omega| \geq 3$.*

Proof. We first prove the claim for the distributive ℓ-monoid $\mathbf{End}(\mathbf{3})$ of order-endomorphisms of the three-element chain $\mathbf{3} = \langle \{0, 1, 2\}, \leq \rangle$, using the same notation for endomorphisms as in Example 2.6. Assume towards a contradiction that $\mathbf{End}(\mathbf{3})$ admits a right order $\leq$. Note that for any $a, b, c \in End(\mathbf{3})$, if $ba < ca$, then $b < c$, since otherwise $c \leq b$ would yield $ca \leq ba$. Suppose first that $\langle 0, 0, 2 \rangle < \langle 0, 1, 1 \rangle$. Then

$$\langle 0, 0, 1 \rangle = \langle 0, 0, 2 \rangle \circ \langle 0, 1, 1 \rangle \leq \langle 0, 1, 1 \rangle \circ \langle 0, 1, 1 \rangle = \langle 0, 1, 1 \rangle$$

and $\langle 0, 0, 1 \rangle \circ \langle 0, 1, 1 \rangle = \langle 0, 0, 1 \rangle < \langle 0, 1, 1 \rangle = \langle 0, 1, 2 \rangle \circ \langle 0, 1, 1 \rangle$. So $\langle 0, 0, 1 \rangle < \langle 0, 1, 2 \rangle$, yielding $\langle 0, 0, 0 \rangle = \langle 0, 0, 1 \rangle \circ \langle 0, 0, 1 \rangle \leq \langle 0, 1, 2 \rangle \circ \langle 0, 0, 1 \rangle = \langle 0, 0, 1 \rangle$. But $\langle 0, 0, 2 \rangle < \langle 0, 1, 1 \rangle$ also implies $\langle 0, 0, 1 \rangle = \langle 0, 0, 2 \rangle \circ \langle 0, 0, 1 \rangle \leq \langle 0, 1, 1 \rangle \circ \langle 0, 0, 1 \rangle = \langle 0, 0, 0 \rangle$. Hence $\langle 0, 0, 1 \rangle = \langle 0, 0, 0 \rangle$, a contradiction. By replacing $<$ with $>$ in the above argument, $\langle 0, 0, 2 \rangle > \langle 0, 1, 1 \rangle$ implies $\langle 0, 0, 1 \rangle = \langle 0, 0, 0 \rangle$, also a contradiction. So the monoid reduct of $\mathbf{End}(\mathbf{3})$ is not right-orderable.

Now let $\langle \Omega, \leq \rangle$ be any chain with $|\Omega| \geq 3$. Without loss of generality we can assume that $\mathbf{3}$ is a subchain of Ω. We define a map $\varphi \colon End(\mathbf{3}) \to End(\langle \Omega, \leq \rangle)$ by fixing for each $q \in \Omega$,

$$(q)\varphi_f := \begin{cases} (\lfloor q \rfloor)f & \text{if } 0 \leq q \\ q & \text{if } q < 0, \end{cases}$$

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where $\lfloor q \rfloor := \max\{k \in \{0,1,2\} \mid k \le q\}$. Observe that $\lfloor \cdot \rfloor$ is order-preserving, so $\varphi_f \in End(\langle \Omega, \le \rangle)$ for every $f \in End(\mathbf{3})$. Also φ is injective, since φ_f restricted to $\mathbf{3}$ is f for each $f \in End(\mathbf{3})$. Let $f, g \in End(3)$ and $q \in \Omega$. If $q < 0$, then $(q)\varphi_{f \circ g} = q = (q)(\varphi_f \circ \varphi_g)$. Otherwise $0 \le q$, so $(q)\varphi_{f \circ g} = ((\lfloor q \rfloor)f)g = (\lfloor (\lfloor q \rfloor)f \rfloor)g = (q)(\varphi_f \circ \varphi_g)$. Hence φ is a semigroup embedding. Finally, since the monoid reduct of $\mathbf{End}(\mathbf{3})$ is not right-orderable, it follows that the monoid reduct of $\mathbf{End}(\langle \Omega, \le \rangle)$ is not right-orderable. $\qquad\square$

Note that, although a group is left-orderable if and only if it is right-orderable, this is not the case in general for monoids, even when they are submonoids of groups [15]. Nevertheless, a very similar argument to the one given in the proof of Proposition 3.5 shows that also the monoid of endomorphisms of any chain with at least three elements cannot be left-orderable.

4. From ℓ-groups to distributive ℓ-monoids

The validity of an equation in the variety of Abelian ℓ-groups is equivalent to the validity of the inverse-free equation obtained by multiplying on both sides to remove inverses. Although this method fails for LG, we show here that inverses can still be effectively eliminated from equations, while preserving validity, via the introduction of new variables. Hence, by Theorem 2.9, the validity of an equation in LG is equivalent to the validity of finitely many effectively constructed inverse-free equations in DLM (Theorem 4.2).

The following lemma shows how to remove one occurrence of an inverse from an equation while preserving validity in LG.

Lemma 4.1. *Let* $r, s, t, u, v \in T_\ell(X)$ *and* $y \notin X$.

(a) $\mathsf{LG} \models \mathrm{e} \le v \vee st \iff \mathsf{LG} \models \mathrm{e} \le v \vee sy \vee y^{-1}t$.
(b) $\mathsf{LG} \models u \le v \vee sr^{-1}t \iff \mathsf{LG} \models ryu \le ryv \vee rysyu \vee t$.

Proof. The left-to-right direction of (a) follows from the validity in LG of the quasiequation $\mathrm{e} \le xy \vee z \implies \mathrm{e} \le x \vee y \vee z$ (cf. [5, Lemma 3.3]). For the converse, suppose that $\mathsf{LG} \not\models \mathrm{e} \le v \vee st$. Then $\mathbf{Aut}(\langle \mathbb{Q}, \le \rangle) \not\models \mathrm{e} \le v \vee st$, by Theorem 2.5. Hence there exist a homomorphism $\varphi \colon \mathbf{T}_\ell(X) \to \mathbf{Aut}(\langle \mathbb{Q}, \le \rangle)$ and $q \in \mathbb{Q}$ such that $(q)\varphi_v < q$ and $(q)\varphi_{st} < q$. Consider $p_1, p_2 \in \mathbb{Q}$ with $p_1 < q < p_2$. Since $(q)\varphi_s < (q)\varphi_{t^{-1}}$ and $p_1 < p_2$, there exists a partial order-embedding on $\mathbb{Q}$ mapping $(q)\varphi_s$ to p_1 and $(q)\varphi_{t^{-1}}$ to p_2 that extends to an order-preserving bijection $\widehat{\varphi}_y \in \mathrm{Aut}(\langle \mathbb{Q}, \le \rangle)$. Now let also $\widehat{\varphi}_x := \varphi_x$ for each $x \in X$ to

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obtain a homomorphism $\widehat{\varphi}\colon \mathbf{T}_\ell(X \cup \{y\}) \to \mathbf{Aut}(\langle \mathbb{Q}, \leq \rangle)$ satisfying $q > (q)\widehat{\varphi}_v$, $q > (q)\widehat{\varphi}_{sy}$, and $q > (q)\widehat{\varphi}_{y^{-1}t}$. Hence $\mathsf{LG} \not\models \mathrm{e} \leq v \vee sy \vee y^{-1}t$ as required.

For (b), we apply (a) to obtain

$$
\begin{aligned}
\mathsf{LG} \models u \leq v \vee sr^{-1}t \;&\Longleftrightarrow\; \mathsf{LG} \models \mathrm{e} \leq vu^{-1} \vee sr^{-1}tu^{-1} \\
&\Longleftrightarrow\; \mathsf{LG} \models \mathrm{e} \leq vu^{-1} \vee sy \vee y^{-1}r^{-1}tu^{-1} \\
&\Longleftrightarrow\; \mathsf{LG} \models ryu \leq ryv \vee rysyu \vee t. \qquad \square
\end{aligned}
$$

Eliminating variables as described in the proof of Lemma 4.1 yields an inverse-free equation that is valid in LG if and only if it is valid in DLM.

Theorem 4.2. *Let ε be any ℓ-group equation with variables in a set X. A finite set of inverse-free equations Σ with variables in $X \cup Y$ for some finite set Y can be effectively constructed such that ε is valid in all ℓ-groups if and only if the equations in Σ are valid in all distributive ℓ-monoids.*

Proof. Let ε be any equation with variables in a set X. Since $\mathsf{LG} \models s \approx t$ if and only if $\mathsf{LG} \models \mathrm{e} \leq s^{-1}t \wedge st^{-1}$ and every ℓ-group term is equivalent in LG to a meet of joins of group terms, we may assume that ε has the form $\mathrm{e} \leq u_1 \wedge \cdots \wedge u_k$ for some joins of group terms $u_1, \ldots, u_k$. Suppose now that for each $i \in \{1, \ldots, k\}$, a finite set of inverse-free equations Σ_i with variables in $X \cup Y_i$ for some finite set Y_i can be effectively constructed such that $\mathrm{e} \leq u_i$ is valid in all ℓ-groups if and only if the equations in Σ_i are valid in all distributive ℓ-monoids. Then $\Sigma := \Sigma_1 \cup \cdots \cup \Sigma_k$ with variables in $X \cup Y$, where $Y := Y_1 \cup \cdots \cup Y_k$ is the finite set of inverse-free equations required by the theorem.

Generalizing slightly for the sake of the proof, it therefore suffices to define an algorithm that given as input any $t_0 \in T_m(X)$ and $t_1, \ldots, t_n \in T_g(X)$ constructs $s_0, s_1, \ldots, s_m \in T_m(X \cup Y)$ for some finite set Y such that

$$
\mathsf{LG} \models t_0 \leq t_1 \vee \cdots \vee t_n \;\Longleftrightarrow\; \mathsf{DLM} \models s_0 \leq s_1 \vee \cdots \vee s_m.
$$

If $t_0 \leq t_1 \vee \cdots \vee t_n$ is an inverse-free equation, then the algorithm outputs the same equation, which satisfies the equivalence by Theorem 2.9. Otherwise, suppose without loss of generality that $t_1 = ux^{-1}v$. By Lemma 4.1, for any $y \notin X$,

$$
\mathsf{LG} \models t_0 \leq t_1 \vee \cdots \vee t_n \;\Longleftrightarrow\; \mathsf{LG} \models xyt_0 \leq xyuyt_0 \vee v \vee xyt_2 \vee \cdots \vee xyt_n.
$$

The equation $xyt_0 \leq xyuxt_0 \vee v \vee xyt_2 \vee \cdots \vee xyt_n$ contains fewer inverses than $t_0 \leq t_1 \vee \cdots \vee t_n$, so iterating this procedure produces an inverse-free equation after finitely many steps. $\square$

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Since the variety DLM has the finite model property (Theorem 2.3), the algorithm given in the proof of Theorem 4.2 provides an alternative proof of the decidability of the equational theory of ℓ-groups, first established in [8].

5. Totally ordered monoids

In this section, we turn our attention to totally ordered monoids and groups, that is, distributive ℓ-monoids and ℓ-groups where the lattice order is total. We show that the variety generated by the class OM of totally ordered monoids can be axiomatized relative to DLM by a single equation (Proposition 5.4), and that there exist inverse-free equations that are valid in the class OG of totally ordered groups but not in OM (Theorem 5.7). We also prove that there is an inverse-free equation that is valid in all finite totally ordered monoids, but not in the ordered group of the integers (Proposition 5.8), showing that the variety of commutative distributive ℓ-monoids and the varieties generated by totally ordered monoids and inverse-free reducts of totally ordered groups do not have the finite model property (Corollary 5.9). The proofs of these results build on earlier work on distributive ℓ-monoids by Merlier [12] and Repnitskiĭ [13, 14].

We begin by establishing a subdirect representation theorem for distributive ℓ-monoids. Note first that since every distributive ℓ-monoid $\mathbf{M}$ has a distributive lattice reduct, prime ideals of its lattice reduct exist. For a prime (lattice) ideal I of a distributive ℓ-monoid $\mathbf{M}$ and $a, b \in M$, define

$$\frac{I}{a} := \{\langle c, d \rangle \in M \times M \mid cad \in I\} \quad \text{and} \quad a \sim_I b \; :\Longleftrightarrow \; \frac{I}{a} = \frac{I}{b}.$$

Proposition 5.1 ([12])**.** *Let $\mathbf{M}$ be a distributive ℓ-monoid and let I be a prime lattice ideal of $\mathbf{M}$. Then $\sim_I$ is an ℓ-monoid congruence and the quotient $\mathbf{M}/I :=$ $\mathbf{M}/\sim_I$ is a distributive ℓ-monoid. Moreover, for any $a, b \in M$,*

$$[a]_{\sim_I} \leq [b]_{\sim_I} \iff \frac{I}{b} \subseteq \frac{I}{a}, \quad \frac{I}{a \vee b} = \frac{I}{a} \cap \frac{I}{b}, \quad \text{and} \quad \frac{I}{a \wedge b} = \frac{I}{a} \cup \frac{I}{b}.$$

In particular, $\mathbf{M}/I$ is totally ordered if and only if $\langle \{\frac{I}{a} \mid a \in M\}, \subseteq \rangle$ is a chain.

Proposition 5.2. *Every distributive ℓ-monoid $\mathbf{M}$ is a subdirect product of all the distributive ℓ-monoids of the form $\mathbf{M}/I$, where I is a prime ideal of $\mathbf{M}$.*

Proof. Let $\mathcal{I}$ be the set of all prime lattice ideals of $\mathbf{M}$. By Proposition 5.1, there exists a natural surjective homomorphism $\nu_I \colon \mathbf{M} \to \mathbf{M}/I; \; a \mapsto [a]_{\sim_I}$ for each $I \in \mathcal{I}$. Combining these maps, we obtain a homomorphism

$$\nu \colon \mathbf{M} \to \prod_{I \in \mathcal{I}} \mathbf{M}/I; \quad a \mapsto (\nu_I(a))_{I \in \mathcal{I}}.$$

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It remains to show that ν is injective. Let $a, b \in M$ with $a \neq b$. By the prime ideal separation theorem for distributive lattices, there exists an $I \in \mathcal{I}$ such that, without loss of generality, $a \in I$ and $b \notin I$, yielding $\langle e, e \rangle \in \frac{I}{a}$ and $\langle e, e \rangle \notin \frac{I}{b}$. But then $\nu_I(a) \neq \nu_I(b)$ and $\nu(a) \neq \nu(b)$. So ν is a subdirect embedding. $\qquad \square$

The following lemma provides a description of the prime lattice ideals I of a distributive ℓ-monoid M such that M/I is a totally ordered monoid.

Lemma 5.3. *Let* M *be a distributive ℓ-monoid and let I be a prime lattice ideal of* M. *Then* M/I *is totally ordered if and only if for all* $b_1, b_2, c_1, c_2, d_1, d_2 \in M$,

$$c_1 b_1 c_2 \in I \ \text{ and } \ d_1 b_2 d_2 \in I \ \implies \ c_1 b_2 c_2 \in I \ \text{ or } \ d_1 b_1 d_2 \in I.$$

Proof. Suppose first that M/I is totally ordered and hence, by Proposition 5.1, that $\frac{I}{b_1} \subseteq \frac{I}{b_2}$ or $\frac{I}{b_2} \subseteq \frac{I}{b_1}$ for all $b_1, b_2 \in M$. Then $c_1 b_1 c_2 \in I$ (i.e., $\langle c_1, c_2 \rangle \in \frac{I}{b_1}$) and $d_1 b_2 d_2 \in I$ (i.e., $\langle d_1, d_2 \rangle \in \frac{I}{b_2}$) must entail $c_1 b_2 c_2 \in I$ (i.e., $\langle c_1, c_2 \rangle \in \frac{I}{b_2}$) or $d_1 b_1 d_2 \in I$ (i.e., $\langle d_1, d_2 \rangle \in \frac{I}{b_1}$) as required. For the converse, suppose that M/I is not totally ordered. By Proposition 5.1, there exist $b_1, b_2 \in M$ such that $\frac{I}{b_1} \not\subseteq \frac{I}{b_2}$ and $\frac{I}{b_2} \not\subseteq \frac{I}{b_1}$. That is, there exist $c_1, c_2, d_1, d_2 \in M$ such that $c_1 b_1 c_2 \in I$ and $d_1 b_2 d_2 \in I$, but $c_1 b_2 c_2 \notin I$ and $d_1 b_1 d_2 \notin I$, as required. $\qquad \square$

An ℓ-group or a distributive ℓ-monoid is called *representable* if it is isomorphic to a subdirect product of members of OG or OM, respectively. The following result provides a characterization of representable distributive ℓ-monoids in terms of their prime lattice ideals, and an equation axiomatizing the variety of these algebras relative to DLM.

Proposition 5.4. *The following are equivalent for any distributive ℓ-monoid* M:

(1) M *is representable.*
(2) $M \models (x_1 \leq x_2 \vee z_1 y_1 z_2) \,\&\, (x_1 \leq x_2 \vee w_1 y_2 w_2) \implies x_1 \leq x_2 \vee z_1 y_2 z_2 \vee w_1 y_1 w_2.$
(3) $M \models z_1 y_1 z_2 \wedge w_1 y_2 w_2 \leq z_1 y_2 z_2 \vee w_1 y_1 w_2.$
(4) *For any prime lattice ideal I of* M, *the quotient* M/I *is totally ordered.*

Proof. $(1) \Rightarrow (2)$. Since quasiequations are preserved by taking direct products and subalgebras, it suffices to prove that (2) holds for the case where M is a totally ordered monoid. Let $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2 \in M$ satisfy $a_1 \leq a_2 \vee c_1 b_1 c_2$ and $a_1 \leq a_2 \vee d_1 b_2 d_2$. Since M is totally ordered, we can assume without loss

of generality that $b_1 \leq b_2$. It follows that $c_1 b_1 c_2 \leq c_1 b_2 c_2$ and therefore $a_1 \leq a_2 \vee c_1 b_1 c_2 \leq a_2 \vee c_1 b_2 c_2 \leq a_2 \vee c_1 b_2 c_2 \vee d_1 b_1 d_2$ as required.

$(2) \Rightarrow (3)$. Let $s_1 := z_1 y_1 z_2$, $s_2 := w_1 y_2 w_2$, $t_1 := z_1 y_2 z_2$, and $t_2 := w_1 y_1 w_2$, and suppose that $\mathbf{M} \models (x_1 \leq x_2 \vee s_1) \,\&\, (x_1 \leq x_2 \vee s_2) \implies x_1 \leq x_2 \vee t_1 \vee t_2$. Since $\mathbf{M} \models s_1 \wedge s_2 \leq t_1 \vee t_2 \vee s_1$ and $\mathbf{M} \models s_1 \wedge s_2 \leq t_1 \vee t_2 \vee s_2$, it follows that $\mathbf{M} \models s_1 \wedge s_2 \leq t_1 \vee t_2$ as required.

$(3) \Rightarrow (4)$. Assume (3) and suppose that $c_1 b_1 c_2 \in I$ and $d_1 b_2 d_2 \in I$ for some $b_1, b_2, c_1, c_2, d_1, d_1 \in M$. Since I is a lattice ideal, $c_1 b_1 c_2 \vee d_1 b_2 d_2 \in I$. By (3) and the downwards closure of I, also $c_1 b_2 c_2 \wedge d_1 b_1 d_2 \in I$. But then, since I is prime, it must be the case that either $c_1 b_2 c_2 \in I$ or $d_1 b_1 d_2 \in I$. Hence, by Lemma 5.3, the quotient $\mathbf{M}/I$ is totally ordered.

$(4) \Rightarrow (1)$. By (4), $\mathbf{M}/I$ is totally-ordered when I is a prime ideal of $\mathbf{M}$, so representability follows by Proposition 5.2. $\qquad\square$

It follows directly from Propositions 5.2 and 5.4 that the class of representable distributive ℓ-monoids is the variety generated by the class OM of totally ordered monoids. Similarly, it follows from these results that the class of representable ℓ-groups is the variety generated by the class OG of totally ordered groups and is axiomatized relative to LG by $z_1 y_1 z_2 \wedge w_1 y_2 w_2 \leq z_1 y_2 z_2 \vee w_1 y_1 w_2$. (Just observe that if the inverse-free reduct of an ℓ-group $\mathbf{L}$ is a subdirect product of totally ordered monoids, then each component is a homomorphic image of $\mathbf{L}$ and hence a totally ordered group.) Hence, an equation is valid in these varieties if and only if it is valid in their totally ordered members.

We also obtain the following known fact:

Corollary 5.5 ([12, Corollary 2]). *Commutative distributive ℓ-monoids are representable.*

Proof. By Proposition 5.4, it suffices to note that for any commutative distributive ℓ-monoid $\mathbf{M}$ and $b_1, b_2, c_1, c_2, d_1, d_2 \in M$,

$$
\begin{aligned}
c_1 b_1 c_2 \wedge d_1 b_2 d_2 &= c_1 c_2 b_1 \wedge d_1 d_2 b_2 \\
&\leq (c_1 c_2 \vee d_1 d_2) b_1 \wedge (c_1 c_2 \vee d_1 d_2) b_2 \\
&= (c_1 c_2 \vee d_1 d_2)(b_1 \wedge b_2) \\
&= c_1 c_2 (b_1 \wedge b_2) \vee d_1 d_2 (b_1 \wedge b_2) \\
&\leq c_1 c_2 b_2 \vee d_1 d_2 b_1 \\
&= c_1 b_2 c_2 \vee d_1 b_1 d_2. \qquad\qquad\square
\end{aligned}
$$

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It is shown in [13] that there are inverse-free equations that are valid in all totally ordered Abelian groups, but not in all totally ordered commutative monoids. We make use here of just one of these equations.

Lemma 5.6 ([13, Lemma 7]). *The following equation is valid in all totally ordered Abelian groups, but not in all totally ordered commutative monoids:*

$$x_1 x_2 x_3 \wedge x_4 x_5 x_6 \wedge x_7 x_8 x_9 \leq x_1 x_4 x_7 \vee x_2 x_5 x_8 \vee x_3 x_6 x_9.$$

We use this result to show that the same discrepancy holds when comparing the equational theories of OM and OG.

Theorem 5.7. *There is an inverse-free equation that is valid in all totally ordered groups, but not in all totally ordered monoids.*

Proof. Consider the inverse-free equation $t_1 \wedge t_2 \leq s_1 \vee s_2$, where

$$t_1 := x_1 x_2 x_3 \wedge x_5 x_4 x_6 \wedge x_9 x_7 x_8; \quad s_1 := x_1 x_4 x_7 \vee x_5 x_2 x_8 \vee x_9 x_6 x_3;$$
$$t_2 := x_1 x_3 x_2 \wedge x_5 x_6 x_4 \wedge x_9 x_8 x_7; \quad s_2 := x_1 x_7 x_4 \vee x_5 x_8 x_2 \vee x_9 x_3 x_6.$$

Clearly $t_1 \approx t_2$ and $s_1 \approx s_2$ are valid in all totally ordered commutative monoids, so $t_1 \wedge t_2 \leq s_1 \vee s_2$ fails in some totally ordered monoid by Lemma 5.6. It remains to show that this equation, or equivalently $\mathrm{e} \leq (t_1^{-1} \vee t_2^{-1})(s_1 \vee s_2)$, is valid in every totally ordered group. Recall first that (cf. [5, Lemma 3.3])

$$\mathsf{LG} \models \mathrm{e} \leq xy \vee z \implies \mathrm{e} \leq x \vee y \vee z. \tag{1}$$

Since $\mathsf{LG} \models \mathrm{e} \leq \mathrm{e} \vee x_8 x_3^{-1} x_8^{-1} x_3$, it follows using (1) that

$$\mathsf{LG} \models \mathrm{e} \leq x_3^{-1} x_8 x_3 x_8^{-1} \vee x_8 x_3^{-1} x_8^{-1} x_3. \tag{2}$$

An application of (1) with (2) as premise yields

$$\mathsf{LG} \models \mathrm{e} \leq x_3^{-1} x_8 x_6^{-1} x_7 \vee x_7^{-1} x_6 x_3 x_8^{-1} \vee x_8 x_3^{-1} x_8^{-1} x_3, \tag{3}$$

and then another application of (1) with (3) as premise yields

$$\mathsf{LG} \models \mathrm{e} \leq x_3^{-1} x_8 x_6^{-1} x_7 \vee x_7^{-1} x_6 x_3 x_8^{-1} \vee x_8 x_3^{-1} x_7 x_6^{-1} \vee x_6 x_7^{-1} x_8^{-1} x_3. \tag{4}$$

For any ordered group **L** and $a, b, c \in L$, if $\mathrm{e} \leq ab \vee c$, then either $\mathrm{e} \leq c$, or $a^{-1} \leq b$ and hence $\mathrm{e} \leq ba$, so $\mathrm{e} \leq ba \vee c$. Hence

$$\mathsf{OG} \models \mathrm{e} \leq xy \vee z \implies \mathrm{e} \leq yx \vee z. \tag{5}$$

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We apply (5) four times with (4) as the first premise to obtain

$$\mathsf{OG} \models \mathrm{e} \leq x_7 x_3^{-1} x_8 x_6^{-1} \vee x_8^{-1} x_7^{-1} x_6 x_3 \vee x_3^{-1} x_7 x_6^{-1} x_8 \vee x_7^{-1} x_8^{-1} x_3 x_6. \quad (6)$$

For convenience, let

$$u_1 := x_3^{-1} x_2^{-1} x_4 x_7; \qquad u_2 := x_6^{-1} x_4^{-1} x_2 x_8; \qquad u_3 := x_8^{-1} x_7^{-1} x_6 x_3;$$
$$u_4 := x_2^{-1} x_3^{-1} x_7 x_4; \qquad u_5 := x_4^{-1} x_6^{-1} x_8 x_2; \qquad u_6 := x_7^{-1} x_8^{-1} x_3 x_6.$$

An application of (1) with (6) as premise yields

$$\mathsf{OG} \models \mathrm{e} \leq x_7 x_3^{-1} x_2^{-1} x_4 \vee x_4^{-1} x_2 x_8 x_6^{-1} \vee u_3 \vee x_3^{-1} x_7 x_6^{-1} x_8 \vee u_6. \quad (7)$$

Applying (5) twice with (7) as the first premise, we obtain

$$\mathsf{OG} \models \mathrm{e} \leq u_1 \vee u_2 \vee u_3 \vee x_3^{-1} x_7 x_6^{-1} x_8 \vee u_6. \quad (8)$$

Another application of (1) with (8) as premise yields

$$\mathsf{OG} \models \mathrm{e} \leq u_1 \vee u_2 \vee u_3 \vee x_3^{-1} x_7 x_4 x_2^{-1} \vee x_2 x_4^{-1} x_6^{-1} x_8 \vee u_6. \quad (9)$$

Applying (5) twice with (9) as the first premise, we obtain

$$\mathsf{OG} \models \mathrm{e} \leq u_1 \vee u_2 \vee u_3 \vee u_4 \vee u_5 \vee u_6. \quad (10)$$

Observe now that for some joins of group terms u', u'',

$$\mathsf{OG} \models t_1^{-1} s_1 \approx u_1 \vee u_2 \vee u_3 \vee u' \quad \text{and} \quad \mathsf{OG} \models t_2^{-1} s_2 \approx u_4 \vee u_5 \vee u_6 \vee u''.$$

Hence, since $\mathsf{OG} \models (t_1^{-1} \vee t_2^{-1})(s_1 \vee s_2) \approx t_1^{-1} s_1 \vee t_1^{-1} s_2 \vee t_2^{-1} s_1 \vee t_2^{-1} s_2$, by (10),

$$\mathsf{OG} \models \mathrm{e} \leq (t_1^{-1} \vee t_2^{-1})(s_1 \vee s_2). \qquad \square$$

In [13], it is proved that the variety generated by the class of inverse-free reducts of Abelian ℓ-groups is not finitely based and can be axiomatized relative to DLM by the set of inverse-free equations $s_1 \wedge \cdots \wedge s_n \leq t_1 \vee \cdots \vee t_n$ such that $s_1, \ldots, s_n, t_1, \ldots, t_n \in T_m(X)$ and $s_1 \cdots s_n \approx t_1 \cdots t_n$ is valid in all commutative monoids. It is not known, however, if the variety generated by the class of inverse-free reducts of totally ordered groups is finitely based. Decidability in each case of the equational theories of commutative distributive ℓ-monoids, totally ordered monoids, and inverse-free reducts of totally ordered groups is also open. The following result shows, at least, that unlike DLM, the varieties generated by these classes do not have the finite model property.

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Proposition 5.8. *There is an equation that is valid in every finite totally ordered monoid, but not in* $\mathbf{Z} = \langle \mathbb{Z}, \min, \max, +, 0 \rangle$.

Proof. Consider the equation $xy^2 \leq e \vee x^2y^3$. Note that $\mathbf{Z} \not\models xy^2 \leq e \vee x^2y^3$, since $(-3) + 2 + 2 = 1 > 0 = 0 \vee ((-3) + (-3) + 2 + 2 + 2)$. We show that this equation holds in every finite totally ordered monoid M. Suppose towards a contradiction that $ab^2 > e \vee a^2b^3$ for some $a, b \in M$, i.e., $ab^2 > e$ and $ab^2 > a^2b^3$.

Observe first that, inductively, $ab^2 > a^{2+n}b^{3+n}$ for each $n \in \mathbb{N}$. The base case $n = 0$ holds by assumption, and for $n > 0$, assuming $ab^2 > a^{2+n-1}b^{3+n-1}$ yields $ab^2 > a^2b^3 = a(ab^2)b \geq a(a^{2+n-1}b^{3+n-1})b = a^{2+n}b^{3+n}$. Also, inductively, $a^nb^{2n} \geq ab^2$ for each $n \in \mathbb{N}^{>0}$. The base case $n = 1$ is clear, and for $n > 1$, assuming $a^{n-1}b^{2n-2} \geq ab^2$ yields (recalling that $ab^2 > e$),

$$a^nb^{2n} = a^{n-1}(ab^2)b^{2n-2} \geq a^{n-1}eb^{2n-2} = a^{n-1}b^{2n-2} \geq ab^2.$$

Finally, since M is finite and totally ordered, $a^{n+1} = a^n$ and $b^{n+1} = b^n$ for some $n \in \mathbb{N}$. But then $ab^2 > a^{2+n}b^{3+n} = a^nb^n = a^nb^{2n} \geq ab^2$, a contradiction. $\square$

Corollary 5.9. *The variety of commutative distributive ℓ-monoids and varieties generated by the classes of totally ordered monoids and inverse-free reducts of totally ordered groups do not have the finite model property.*

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