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Complemented MacNeille completions and algebras of fractions

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ABSTRACT

We introduce (ℓ-)bimonoids as ordered algebras consisting of two compatible monoidal structures on a partially ordered (lattice-ordered) set. Bimonoids form an appropriate framework for the study of a general notion of complementation, which subsumes both Boolean complements in bounded distributive lattices and multiplicative inverses in monoids. The central question of the paper is whether and how bimonoids can be embedded into complemented bimonoids, generalizing the embedding of cancellative commutative monoids into their groups of fractions and of bounded distributive lattices into their free Boolean extensions. We prove that each commutative (ℓ-)bimonoid embeds into a complete complemented commutative ℓ-bimonoid in a doubly dense way reminiscent of the Dedekind–MacNeille completion. Moreover, this complemented completion, which is term equivalent to a commutative involutive residuated lattice, sometimes contains a tighter complemented envelope analogous to the group of fractions. In the case of cancellative commutative monoids this algebra of fractions is precisely the familiar group of fractions, while in the case of Brouwerian (Heyting) algebras it is a (bounded) idempotent involutive commutative residuated lattice. This construction of the algebra of fractions in fact yields a categorical equivalence between varieties of integral and of involutive residuated structures which subsumes as special cases the known equivalences between Abelian ℓ-groups

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and their negative cones, and between Sugihara monoids and their negative cones.

1. Introduction

In various areas of mathematics, ordered algebraic structures naturally arise in which a partially ordered monoid (or *pomonoid*) $\langle A, \leq, \cdot, 1 \rangle$ is equipped with an order-inverting involution $x \mapsto \overline{x}$. This gives rise to another partially ordered monoid $\langle A, \geq, +, 0 \rangle$ over the same set, where $0 = \overline{1}$ and the two monoidal operations are related by De Morgan duality:

$$\overline{x \cdot y} = \overline{y} + \overline{x} \qquad \text{and} \qquad \overline{x + y} = \overline{y} \cdot \overline{x}.$$

Moreover, the order-inverting involution often relates the two monoids in a way that is reminiscent of Boolean algebras. Namely, in Boolean algebras, where we take the two monoids to be the meet and join semilattices, we have

$$x \leq z \vee \neg y \iff x \wedge y \leq z \iff y \leq \neg x \vee z.$$

Similarly, in partially ordered groups, where the two monoids coincide, we have

$$x \leq z \cdot y^{-1} \iff x \cdot y \leq z \iff y \leq x^{-1} \cdot z.$$

Finally, in MV-algebras we have

$$x \leq z \oplus \neg y \iff x \odot y \leq z \iff y \leq \neg x \oplus z.$$

Groups can and will be identified with partially ordered groups whose partial order is the equality relation.

The notion of an *involutive residuated pomonoid* was devised to subsume the above examples and more. Involutive residuated pomonoids can thus be thought of as ordered algebras of the form $\langle A, \leq, \cdot, 1, +, 0, ^{-} \rangle$ where each element has a Boolean-like or group-like complement and the two monoids are De Morgan duals of each other. *Involutive residuated lattices* moreover require that the partial order be a lattice.

This paper is devoted to the study of the *positive subreducts* of commutative involutive residuated pomonoids. These are defined as the ordered algebras of the form $\langle A, \leq, \cdot, 1, +, 0 \rangle$ which can be embedded into some commutative involutive residuated pomonoid. More generally, we consider the following problem:

Given a class K of involutive residuated pomonoids,
describe the positive subreducts of the structures in K.

We also consider the positive subreducts of commutative involutive residuated lattices, i.e. algebras of the form $\langle A, \vee, \wedge, \cdot, 1, +, 0\rangle$ which can be embedded into some commutative involutive residuated lattice. Although some of our definitions and constructions allow for non-commutative bimonoids, throughout the paper we focus on the commutative case.[1]

There are in fact at least two classical results of this form: each cancellative commutative monoid can be embedded into an Abelian group (the smallest such group is the *group of fractions* of the monoid) and each bounded distributive lattice can be embedded into a Boolean algebra (the smallest such algebra is the *free Boolean extension* of the bounded distributive lattice). Conversely, Abelian groups are precisely those cancellative commutative monoids where each element x has a multiplicative inverse x^{-1} and Boolean algebras are precisely those bounded distributive lattices where each element x has a Boolean complement $\neg x$. These results can be expressed more compactly as follows: bounded distributive lattices are precisely the positive subreducts of Boolean algebras and cancellative commutative monoids are precisely the positive subreducts of Abelian groups.

One of our goals is to unify and extend these two constructions, bringing out both their common features and their essential differences. Our first main result describes the positive subreducts of commutative involutive residuated pomonoids. We identify these subreducts as *commutative bimonoids*. These are pairs of commutative pomonoids over the same partially ordered set satisfying the compatibility condition

$$a \cdot (b + c) \leq (a \cdot b) + c,$$

which we call *hemidistributivity*.[2] Two elements a, b of a commutative bimonoid are called *complements* if

$$a \cdot b \leq 0 \qquad \text{and} \qquad 1 \leq a + b.$$

This notion subsumes both Boolean complements in bounded distributive lattices and multiplicative inverses in monoids. Complemented commutative bimonoids are in fact term equivalent to commutative involutive residuated pomonoids. The problem of describing the positive subreducts of involutive residuated structures therefore reduces to the following problem:

Given a bimonoid, embed it into a complemented bimonoid.

[1] Beyond the commutative case this problem is highly non-trivial already for groups. Although monoids which embed into a group of (right) fractions are precisely the so-called left-reversible cancellative monoids, the description of monoids which embed into *some* group, not necessarily a group of fractions, is substantially more complicated (see [9, Ch. 12]).

[2] The term "hemidistributivity" was first introduced for this condition by Dunn & Hardegree [11]. The term "bimonoid" is also used to denote a compatible pair of a monoid and a comonoid over a symmetric monoidal category (see [29]). Our usage of the term is essentially unrelated.

We show that this can always be done in the commutative case.

Specifically, we prove in Section 3 that each commutative bimonoid $\mathbf{A}$ embeds into what we call its *complemented Dedekind–MacNeille completion* $\mathbf{A}^{\Delta}$. This is a complete commutative involutive residuated lattice (unique up to a unique isomorphism which fixes $\mathbf{A}$) containing $\mathbf{A}$ as a sub-bimonoid where each element is a join of the form $\bigvee_{i \in I}(a_i \cdot \overline{b}_i)$, or equivalently a meet of the form $\bigwedge_{i \in I}(a_i + \overline{b}_i)$, for $a_i, b_i \in \mathbf{A}$. The construction of the complemented Dedekind–MacNeille completion $\mathbf{A}^{\Delta}$ relies heavily on the machinery of involutive residuated frames developed by Galatos & Jipsen [14].

If $\mathbf{A}$ is moreover a commutative ℓ-bimonoid, i.e. a commutative lattice-ordered bimonoid which satisfies the equations $x \cdot (y \vee z) \approx (x \cdot y) \vee (x \cdot z)$ and $x + (y \wedge z) \approx (x + y) \wedge (x + z)$, then the embedding of $\mathbf{A}$ into $\mathbf{A}^{\Delta}$ preserves finite meets and joins. Commutative ℓ-bimonoids are thus the positive subreducts of commutative involutive residuated lattices. Moreover, in Section 5 we show how to axiomatize the positive subreducts of any variety of commutative involutive residuated lattices defined by equations in the signature $\{\vee, \cdot, 1\}$.

The above construction provides a commutative complemented envelope for each commutative bimonoid. This is a satisfactory result, but observe that not all complemented extensions are created equal: a generic element of a free Boolean extension has the form $\bigvee_{i \in I}(a_i \wedge \neg b_i)$, while a generic element of a group of fractions has the simpler form $a \cdot b^{-1}$. Let us introduce a name for complemented extensions of this second, simpler kind: we shall say that a commutative involutive residuated pomonoid $\mathbf{B}$ which contains a commutative bimonoid $\mathbf{A}$ as a sub-bimonoid is a *commutative complemented bimonoid of fractions* of $\mathbf{A}$ if each element x of $\mathbf{B}$ has the form $x = a \cdot \overline{b}$, or equivalently $x = a + \overline{b}$, for some $a, b \in \mathbf{A}$. If a commutative complemented bimonoid of fractions exists, it is unique up to isomorphism and it embeds into $\mathbf{A}^{\Delta}$. We denote it $\mathbf{A}^{\div}$.

Although in the present paper we restrict our attention to complemented Dedekind–MacNeille completions and their subalgebras, this is not the only possible direction in which one can look for generalizations of free Boolean extensions. In particular, the problem of axiomatizing the ℓ-bimonoidal subreducts of MV-algebras was recently solved by Abbadini et al. [1] using a different kind of complemented envelope, where each element is a finite sum rather than a possibly infinite join of elements of the form $a \cdot \overline{b}$.

The second problem that we consider is then the question of which bimonoids admit such well-behaved complemented extensions. That is:

Which bimonoids have a complemented bimonoid of fractions?

We only consider this problem for commutative complemented bimonoids of fractions of commutative bimonoids. As we already observed above, cancellative commutative monoids admit a bimonoid of fractions, while distributive lattices typically do not. We show in Section 4 that commutative bimonoids which admit a commutative complemented bimonoid of fractions can be described by a certain first-order sentence. Moreover, for residuated bimonoids (bimonoids whose multiplicative pomonoid is residuated) the existence of a complemented bimonoid of fractions may be witnessed by a pair of terms satisfying certain inequalities. With the help of these terms we can construct the complemented bimonoid of fractions explicitly.

The construction of the complemented bimonoid of fractions covers some known cases as well as some new ones. In particular, we prove that all Brouwerian algebras, hence also all Heyting algebras, admit a complemented bimonoid of fractions, which is an idempotent involutive residuated lattice. This extends to the non-semilinear case of a result of Galatos & Raftery [17] for so-called relative Stone algebras, whose algebras of fractions are known as Sugihara monoids. More generally, complemented bimonoids of fractions can be constructed for Brouwerian semilattices and for Boolean-pointed Brouwerian algebras. The latter construction extends a result of Fussner & Galatos [13] for Boolean-pointed relative Stone algebras. The two reflection constructions of Galatos & Raftery [16] are also covered (if we allow for bimonoids of fractions of bisemigroups), as is of course the classical construction of the Abelian group of fractions of a cancellative commutative monoid.[3]

The importance of considering bimonoidal structure, instead of merely residuated lattice structure, when constructing complemented extensions can be illustrated on Heyting algebras. These can be seen as bimonoids in two different ways: if we take $0 = \bot$ and $x+y = x \vee y$, the smallest complemented extension is the free Boolean extension, whereas if we take $0 = \top$ and $x + y = x \wedge y$, we obtain a (non-integral) idempotent involutive residuated lattice. This latter algebra is what we referred to above as the bimonoid of fractions of the Heyting algebra.

Borrowing an idea used by Montagna & Tsinakis [26] in the context of groups of fractions, the bimonoid $\mathbf{A}$ may be identified inside of its complemented bimonoid of fractions $\mathbf{A}^{\div}$ as the image of a certain interior operator $\sigma^{\div}$, provided that multiplication in $\mathbf{A}$ is residuated. This sometimes allows us to extend the construction of the complemented bimonoid of fractions to a categorical equivalence. In one direction, we take a suitable residuated bimonoid $\mathbf{A}$ to a complemented bimonoid $\mathbf{A}^{\div}$ equipped with an interior operator $\sigma^{\div}$, while in the other direction we take a suitable complemented bimonoid $\mathbf{B}$ equipped with an interior operator σ to the image of this operator $\mathbf{B}_{\sigma}$, which is a sub-bimonoid of $\mathbf{B}$. This yields a new categorical equivalence subsuming several known equivalences: the restriction to the commutative case of the equivalence between lattice-ordered groups (ℓ-groups) and certain integral residuated lattices due to Bahls et al. [2], the restriction to the commutative case of the equivalence between certain ℓ-groups with a conucleus and certain cancellative residuated lattices due to Montagna & Tsinakis [26], the equivalence between odd Sugihara monoids and relative

[3] Let us recall the definitions of these algebras. Brouwerian algebras differ from Heyting algebras by removing the assumption that a bottom element exists. Relative Stone algebras are Brouwerian algebras which satisfy the equation $(x \to y) \vee (y \to x) \approx 1$; equivalently, they are subdirect products of Brouwerian chains. Relative Stone algebras with a bottom element are better known as Gödel algebras. Boolean-pointed Brouwerian algebras are Brouwerian algebras equipped with a constant 0 such that the interval $[0, 1]$ is a Boolean lattice. Brouwerian semilattices are unital meet semilattices equipped with relative pseudocomplementation $(x \to y)$. Finally, Sugihara monoids are distributive idempotent commutative involutive residuated lattices.

Stone algebras due to Galatos & Raftery [17], and its extension to Sugihara monoids and Boolean-pointed relative Stone algebras due to Fussner & Galatos [13].

Let us now outline the structure of the paper. In Section 2 we review some basic terminology concerning involutive residuated structures and introduce bimonoids, lattice-ordered bimonoids (ℓ-bimonoids), and related structures as an attempt to describe the positive subreducts of involutive residuated structures. We define complements in bimonoids and show that involutive residuated pomonoids (involutive residuated lattices) are term equivalent to bimonoids (ℓ-bimonoids) with complementation. We then provide some examples of bimonoids. In Section 3 we prove the first main result of the paper, namely the existence of commutative complemented Dedekind–MacNeille completions of commutative bimonoids. The existence of commutative complemented bimonoids of fractions is then studied in Section 4. This in particular yields new categorical equivalences between varieties of residuated structures, as well as uniform proofs of some known equivalences. In Section 5 we study the preservation of equations by the two constructions (complemented Dedekind–MacNeille completions and complemented bimonoids of fractions) and axiomatize the ℓ-bimonoidal subreducts of each variety of commutative involutive residuated lattices axiomatized by equations in the signature $\{\vee, \cdot, 1\}$. Finally, in Section 6 we list some problems which the paper leaves open.

2. Bimonoids

The main algebraic structures studied in the present paper, namely bimonoids and their lattice-ordered and residuated variants, are introduced in this section. We define complements in bimonoids and show that complemented bimonoids are term equivalent to involutive residuated pomonoids. We then introduce some basic terminology and provide examples of how bimonoids can be constructed from partially ordered monoids. Familiarity with the basic notions of universal algebra is assumed (see e.g. [8,24]).

2.1. Involutive residuated structures

The structures that we study in this paper all have at least a partially ordered semigroup reduct.

Definition 2.1 *(Partially ordered semigroups and sℓ-semigroups).* A *partially ordered semigroup* or *posemigroup* $\langle A, \leq, \cdot \rangle$ is a semigroup $\langle A, \cdot \rangle$ equipped with a partial order $\langle A, \leq \rangle$ which satisfies the implications

$$x \leq y \implies x \cdot z \leq y \cdot z \qquad \text{and} \qquad x \leq y \implies z \cdot x \leq z \cdot y.$$

A *semilattice-ordered semigroup* or *sℓ-semigroup* $\langle A, \vee, \cdot \rangle$ is a semigroup $\langle A, \cdot \rangle$ equipped with a (join) semilattice structure $\langle A, \vee \rangle$ which satisfies the equations

$$x \cdot (y \vee z) \approx (x \cdot y) \vee (x \cdot z) \qquad \text{and} \qquad (x \vee y) \cdot z \approx (x \cdot z) \vee (y \cdot z).$$

A *partially ordered monoid* or *pomonoid* $\langle A, \leq, \cdot, 1\rangle$ is a posemigroup $\langle A, \leq, \cdot\rangle$ with a multiplicative unit 1. A *semilattice-ordered monoid* or *sℓ-monoid* $\langle A, \vee, \cdot, 1\rangle$, also known as an *idempotent semiring*, is an sℓ-semigroup $\langle A, \vee, \cdot\rangle$ with a multiplicative unit 1.

The classes of sℓ-semigroups and sℓ-monoids are varieties of algebras, while the classes of posemigroups and pomonoids are varieties of ordered algebras in the sense of Pigozzi [28]. Each sℓ-semigroup can of course be seen as a posemigroup. Several important classes of posemigroups can be defined by inequalities.

Definition 2.2 *(Commutative, integral, and idempotent posemigroups).* A posemigroup is *commutative* if it satisfies $x \cdot y \approx y \cdot x$, it is *integral* if it satisfies the inequalities $x \cdot y \leq x$ and $x \cdot y \leq y$, and it is idempotent if it satisfies $x \cdot x \approx x$.

A pomonoid is integral if and only if $x \leq 1$ for each x. In addition to a partial order and a semigroup structure, the algebras that we study in the present paper are often also equipped with division-like operations called the left and right residual.

Definition 2.3 *(Residuated posemigroups and sℓ-semigroups).* A binary operation $x\backslash y$ on a posemigroup is called *left division* (or the *right residual* of multiplication) if it satisfies

$$x \cdot y \leq z \iff y \leq x\backslash z.$$

A binary operation x/y on a posemigroup is called *right division* (or the *left residual* of multiplication) if it satisfies

$$x \cdot y \leq z \iff x \leq z/y.$$

A *residuated posemigroup* $\langle A, \leq, \cdot, \backslash, /\rangle$ is a posemigroup $\langle A, \leq, \cdot\rangle$ equipped with the two binary operations $x\backslash y$ and x/y of left and right division. A *residuated sℓ-semigroup* $\langle A, \vee, \cdot, \backslash, /\rangle$ is an sℓ-semigroup $\langle A, \vee, \cdot\rangle$ which is also a residuated posemigroup $\langle A, \leq, \cdot, \backslash, /\rangle$ with respect to the semilattice order. A *residuated pomonoid (sℓ-monoid)* is a residuated posemigroup (sℓ-semigroup) equipped with a multiplicative unit 1.

Residuated posemigroups (pomonoids) form a variety of ordered algebras, while residuated sℓ-semigroups (sℓ-monoids) form an ordinary variety of algebras. In commutative residuated posemigroups $x\backslash y = y/x$. In that case we simplify the algebraic signature and use the notation $x \to y$ for the residual $x\backslash y = y/x$.

Definition 2.4 *(Admissible joins).* Let $\mathbf{A}$ be a posemigroup and $X \subseteq \mathbf{A}$. The join $\bigvee X$, if it exists, is called *admissible* if for each $y \in \mathbf{A}$

$$(\bigvee X) \cdot y = \bigvee\{x \cdot y \mid x \in X\},$$
$$y \cdot (\bigvee X) = \bigvee\{y \cdot x \mid x \in X\}.$$

Fact 2.5. *All existing joins are admissible in a residuated posemigroup.*

All finite non-empty joins are admissible in an sℓ-semigroup. The empty join, i.e. the bottom element $\bot$, is admissible if and only if $\bot \cdot x = \bot = x \cdot \bot$ for each x.

We shall mainly be interested in so-called involutive (residuated) posemigroups, where the binary division operations can be decomposed into a binary addition ($x + y$) and a unary complementation (x^ℓ or x^r). Examples of such algebras include ℓ-groups, Boolean algebras, MV-algebras, Sugihara monoids, and relation algebras (see e.g. [15, Section 2.3] for the definitions of these structures). Although these algebras only have one unary complementation operation (x^{-1} in for ℓ-groups, $\neg x$ for Boolean algebras and MV-algebras), in general one has to consider two distinct unary operations on a pointed residuated pomonoid, which we denote x^ℓ and x^r. This notation was chosen to be consistent with the existing notation for pregroups [22].

Definition 2.6 *(Involutive residuated posemigroups).* An *involutive residuated posemigroup* $\langle A, \leq, \cdot, ^\ell, ^r \rangle$ is a posemigroup $\langle A, \leq, \cdot \rangle$ with two antitone unary operations x^ℓ and x^r satisfying $x^{\ell r} \approx x \approx x^{r \ell}$ such that $\langle A, \leq, \cdot, \backslash, / \rangle$ is a residuated posemigroup, where

$$x \backslash y := (y^\ell \cdot x)^r, \qquad y/x := (x \cdot y^r)^\ell.$$

An *involutive residuated pomonoid* $\langle A, \leq, \cdot, 1, ^\ell, ^r \rangle$ is an involutive residuated posemigroup $\langle A, \leq, \cdot, ^\ell, ^r \rangle$ which is also a pomonoid $\langle A, \leq, \cdot, 1 \rangle$.

For the basic arithmetic of involutive residuated posemigroups see [15, Section 3.3]. For pomonoids the antitonicity of the two operations x^ℓ and x^r follows from the other conditions. In involutive residuated posemigroups we may introduce another semigroup operation as the De Morgan dual of multiplication:

$$x + y := (y^\ell \cdot x^\ell)^r = (y^r \cdot x^r)^\ell.$$

Using this operation we may express the two residuals as

$$x \backslash y := x^r + y, \qquad y/x := y + x^\ell.$$

Up to term equivalence we can therefore view involutive residuated posemigroups as ordered algebras of the form $\langle A, \leq, \cdot, +, ^\ell, ^r \rangle$. This is indeed how we shall treat them in the rest of this paper. We generally prefer to use the notation $x^r + y$ and $y + x^\ell$ for $x \backslash y$ and y/x in involutive residuated posemigroups.

In an involutive residuated pomonoid we may also introduce the constant

$$0 := 1^r = 1^\ell.$$

We have $x + 0 = x = 0 + x$. Using this constant we may express the two antitone operations x^ℓ and x^r as

$$x^\ell := 0/x \qquad \text{and} \qquad x^r := x\backslash 0.$$

The equations $x^{\ell r} \approx x \approx x^{r\ell}$ then transform into

$$(0/x)\backslash 0 \approx x \approx 0/(x\backslash 0).$$

Up to term equivalence we can therefore treat involutive residuated pomonoids as ordered algebras of the form $\langle A, \leq, \cdot, 1, +, 0, ^\ell, ^r \rangle$. Equivalently, involutive residuated pomonoids can be defined as pointed residuated pomonoids $\langle A, \leq, \cdot, 1, \backslash, /, 0\rangle$ which satisfy the equations $(0/x)\backslash 0 \approx x \approx 0/(x\backslash 0)$.

Definition 2.7 *(Involutive residuated lattices).* An *involutive residuated lattice* $\langle A, \vee, \wedge, \cdot, 1, +, 0, ^\ell, ^r \rangle$ is a lattice $\langle A, \vee, \wedge\rangle$ which is also an involutive residuated pomonoid $\langle A, \leq, \cdot, 1, +, 0, ^\ell, ^r \rangle$ with respect to the lattice order.

2.2. Bimonoids and ℓ-bimonoids: basic definitions

A natural problem is now to describe the *positive subreducts* of involutive residuated structures, i.e. subreducts without the antitone operations x^ℓ and x^r. For example, what are the subreducts of the form $\langle A, \leq, \cdot, +\rangle$ or $\langle A, \leq, \cdot, 1, +, 0\rangle$ of involutive residuated pomonoids or the subreducts of the form $\langle A, \vee, \wedge, \cdot, +\rangle$ or $\langle A, \vee, \wedge, \cdot, 1, +, 0\rangle$ of involutive residuated lattices? To describe them, we introduce bisemigroups and ℓ-bisemigroups and their unital counterparts, bimonoids and ℓ-bimonoids.

Definition 2.8 *(Bisemigroups and bimonoids).* A *bisemigroup* $\mathbf{A} = \langle A, \leq, \cdot, +\rangle$ is a pair of posemigroups

$$\mathbf{A}_\circ = \langle A, \leq, \cdot\rangle \qquad \text{and} \qquad \mathbf{A}_+ = \langle A, \geq, +\rangle$$

over dual orders, called the *multiplicative* and the *additive posemigroup* of $\mathbf{A}$, such that

$$x \cdot (y + z) \leq (x \cdot y) + z \qquad \text{and} \qquad (z + y) \cdot x \leq z + (y \cdot x).$$

A *bimonoid* $\mathbf{A} = \langle A, \leq, \cdot, 1, +, 0\rangle$ is a bisemigroup equipped with a multiplicative unit 1 and an additive unit 0. The *multiplicative* and *additive pomonoids* of $\mathbf{A}$ are then

$$\mathbf{A}_\circ = \langle A, \leq, \cdot, 1\rangle \qquad \text{and} \qquad \mathbf{A}_+ = \langle A, \geq, +, 0\rangle.$$

Although the notation is similar, the reader should not think of multiplication and addition in bisemigroups as analogues of the homonymous operations in rings or even semirings. Rather, one should think of splitting the single multiplication operation of partially ordered groups into a mirror pair of two operations.

Definition 2.9 *(Admissible joins and meets).* Let $\mathbf{A}$ be a posemigroup and $X \subseteq \mathbf{A}$. The join $\bigvee X$, if it exists, is called *admissible* if it is admissible in the multiplicative posemigroup $\mathbf{A}_\circ$, i.e. if for each $y \in \mathbf{A}$

$$(\bigvee X) \cdot y = \bigvee \{x \cdot y \mid x \in X\} \qquad \text{and} \qquad y \cdot (\bigvee X) = \bigvee \{y \cdot x \mid x \in X\}.$$

The meet $\bigwedge X$, if it exists, is called *admissible* if it is admissible in the additive posemigroup $\mathbf{A}_+$, i.e. if for each $y \in \mathbf{A}$

$$(\bigwedge X) + y = \bigwedge \{x + y \mid x \in X\} \qquad \text{and} \qquad y \cdot (\bigwedge X) = \bigwedge \{y + x \mid x \in X\}.$$

A *homomorphism* of bisemigroups (bimonoids) is an order-preserving homomorphism of multiplicative and additive semigroups (monoids). An *embedding* of bisemigroups (bimonoids) is a homomorphism which is an order embedding. A *complete embedding* moreover preserves all existing joins and meets.

Definition 2.10 *(Lattice-ordered bisemigroups and bimonoids).* A *lattice-ordered bisemigroup* or *ℓ-bisemigroup* $\mathbf{A} = \langle A, \vee, \wedge, \cdot, + \rangle$ is a pair of sℓ-semigroups $\mathbf{A}_\circ = \langle A, \vee, \cdot \rangle$ and $\mathbf{A}_+ = \langle A, \wedge, + \rangle$, called respectively the *multiplicative* and the *additive sℓ-semigroup* of $\mathbf{A}$, such that $\langle A, \vee, \wedge \rangle$ is a lattice with the lattice order $\leq$ and $\langle A, \leq, \cdot, + \rangle$ is a bisemigroup. A *lattice-ordered bimonoid* or *ℓ-bimonoid* $\mathbf{A} = \langle A, \vee, \wedge, \cdot, 1, +, 0 \rangle$ is a pair of sℓ-monoids $\mathbf{A}_\circ = \langle A, \vee, \cdot, 1 \rangle$ and $\mathbf{A}_+ = \langle A, \wedge, +, 0 \rangle$, called respectively the *multiplicative* and the *additive sℓ-monoid* of $\mathbf{A}$, such that $\langle A, \vee, \wedge \rangle$ is a lattice with the lattice order $\leq$ and $\langle A, \leq, \cdot, 1, +, 0 \rangle$ is a bimonoid.

In other words, ℓ-bisemigroups (ℓ-bimonoids) form a variety of algebras axiomatized by the lattice axioms, two sets of semigroup (monoid) axioms, the hemidistributivity axioms

$$x \cdot (y + z) \leq (x \cdot y) + z, \qquad (x + y) \cdot z \leq x + (y \cdot z),$$

and the axioms stating that finite non-empty joins and meets are admissible:

$$x \cdot (y \vee z) \approx (x \cdot y) \vee (x \cdot z), \qquad x + (y \wedge z) \approx (x + y) \wedge (x + z),$$
$$(x \vee y) \cdot z \approx (x \cdot z) \vee (y \cdot z), \qquad (x \wedge y) + z \approx (x + z) \wedge (y + z).$$

Bisemigroups and bimonoids exhibit an order duality which extends the order duality on posets. Namely, the following two bimonoids are said to be *order dual*:

$$\langle A, \leq, \cdot, 1, +, 0 \rangle \qquad \text{and} \qquad \langle A, \geq, +, 0, \cdot, 1 \rangle.$$

Similarly, the following two ℓ-bimonoids are said to be order dual:

$$\langle A, \vee, \wedge, \cdot, 1, +, 0\rangle \qquad \text{and} \qquad \langle A, \wedge, \vee, +, 0, \cdot, 1\rangle.$$

This duality involves exchanging the roles of addition and multiplication as well as inverting the partial order. The order dual of an (ℓ-)bimonoid $\mathbf{A}$ is an (ℓ-)bimonoid $\mathbf{A}^\partial$, thus an inequality holds in all (ℓ-)bimonoids if and only if its naturally defined order dual does. Like ordinary monoids, bimonoids and ℓ-bimonoids also exhibit a *left–right symmetry*, which consists in changing the operations $x \cdot y$ and $x + y$ to $y \cdot x$ and $y + x$. The above applies *mutatis mutandis* to bisemigroups and ℓ-bisemigroups too.

We use both of the notations $x \cdot y$ and xy for multiplication. The notation $x \cdot y$ will be preferred when we wish to emphasize that multiplication and addition are on an equal footing, while the tighter notation xy will be preferred when we wish to avoid writing too many parentheses. In particular, multiplication in such contexts is assumed to bind more tightly than other operations, e.g. we write $xy + z$ for $(x \cdot y) + z$. We also use the notation

$$x^n = \overbrace{x \cdot \ldots \cdot x}^{n \text{ times}} \qquad \text{and} \qquad nx = \overbrace{x + \ldots + x}^{n \text{ times}}.$$

The compatibility condition between addition and multiplication, which in the case of bimonoids can also be written as

$$x \cdot (y + z) \cdot w \leq (x \cdot y) + (z \cdot w),$$

will be called *hemidistributivity*. This term is due to Dunn & Hardegree [11], who studied this compatibility condition as an algebraic formulation of the multiple-conclusion cut rule. The condition itself, however, is older. To the best of our knowledge, the hemidistributive law was first explicitly written down in a paper of Meyer & Routley [25, p. 236].[4] It was also independently considered by Grishin [21] in his study of symmetric variants of the Lambek calculus. In this context the condition is called *mixed associativity*. A categorical version of hemidistributivity was studied by Cockett & Seely [10], who called it *weak distributivity*. Their weakly distributive categories in fact form the categorical counterpart of bimonoids, or conversely bimonoids are obtained from weakly distributive categories by restricting to partial orders.

The hemidistributive laws can also be thought of as an ordered version of the *interassociative laws* between two semigroup operations

$$x \cdot (y + z) \approx (x \cdot y) + z \qquad \text{and} \qquad (y + z) \cdot x \approx y + (z \cdot x),$$

introduced under this name for arbitrary binary operations by Zupnik [30] and later studied in the context of semigroups [7,20]. Note, however, that interassociativity binds

[4] We thank James Raftery for bringing this to our attention.

the two operations together very closely. If a multiplicative unit exists, addition can be defined in terms of multiplication as $x +_a y := x \cdot a \cdot y$ for some suitable a. This is because by interassociativity

$$x + y = (x \cdot 1) + (1 \cdot y) = x \cdot (1 + 1) \cdot y = x +_{(1+1)} y.$$

Conversely, the operation $x +_a y$ is interassociative with multiplication for each a. As we shall see, hemidistributivity allows for a much looser relationship between two monoidal operations than interassociativity.

It is in particular important to note that the two monoidal structures in an ℓ-bimonoid do not determine each other, in contrast to the case of lattices, where the join semilattice is uniquely determined by the meet semilattice and vice versa. For example, if $\langle L, \vee, \wedge, 0, 1\rangle$ is a bounded distributive lattice, then taking $x \cdot y = x \wedge y$ and $x + y = x \vee y$ yields an ℓ-bimonoid, but so does taking $x \cdot y = x \wedge y = x + y$ (with the appropriate multiplicative and additive units).

The bisemigroups that we shall consider in this paper will often be residuated, meaning that the multiplication has a right and a left residual. Note that addition will not be required to have a residual (or a dual residual) in such structures.

Definition 2.11 *(Residuated bisemigroups and ℓ-bisemigroups).* A *residuated bisemigroup* is an ordered algebra $\langle A, \leq, \cdot, \backslash, /, +\rangle$ such that $\langle A, \leq, \cdot, +\rangle$ is a bisemigroup and $\langle A, \leq, \cdot, \backslash, /\rangle$ is a residuated posemigroup. A *residuated ℓ-bisemigroup* is an algebra $\langle A, \vee, \wedge, \cdot, \backslash, /, +\rangle$ such that $\langle A, \vee, \wedge, \cdot, +\rangle$ is an ℓ-bisemigroup and $\langle A, \leq, \cdot, \backslash, /\rangle$ is a residuated posemigroup. A *residuated bimonoid* (*residuated ℓ-bimonoid*) is a bimonoid (ℓ-bimonoid) which is also a residuated bisemigroup (ℓ-bisemigroup).

2.3. Bimonoids and ℓ-bimonoids: complementation

Bimonoids form an appropriate setting for the study of a general notion of complementation, which in particular subsumes Boolean complements in distributive lattices and multiplicative inverses in pomonoids. We show that involutive residuated pomonoids (lattices) are up to term equivalence precisely bimonoids (ℓ-bimonoids) equipped with complementation.

Definition 2.12 *(Complements in bisemigroups).* Let x and y be elements of a bisemigroup $\mathbf{A}$. Then y is called a *left complement (ℓ-complement)* of x in $\mathbf{A}$ if the following inequalities hold for each $w \in \mathbf{A}$:

$$w + (y \cdot x) \leq w, \qquad w \leq w \cdot (x + y),$$
$$(y \cdot x) + w \leq w, \qquad w \leq (x + y) \cdot w.$$

It is called a *right complement (r-complement)* of x in $\mathbf{A}$ if the following inequalities hold for each $w \in \mathbf{A}$:

$$w + (x \cdot y) \leq w, \qquad w \leq w \cdot (y + x),$$
$$(x \cdot y) + w \leq w, \qquad w \leq (y + x) \cdot w.$$

A bisemigroup is *complemented* if each element has both a left complement and a right complement.

Surjective homomorphisms of bisemigroups preserve all existing left and right complements. In bimonoids, the definition of a complement can be simplified substantially.

Proposition 2.13 *(Complements in bimonoids). Let x and y be elements of a bimonoid $\mathbf{A}$. Then y is an ℓ-complement of x in $\mathbf{A}$ if and only if*

$$y \cdot x \leq 0 \qquad \text{and} \qquad 1 \leq x + y.$$

Likewise, y is an r-complement of x in $\mathbf{A}$ if and only if

$$x \cdot y \leq 0 \qquad \text{and} \qquad 1 \leq y + x.$$

In particular, 0 and 1 are both left and right complements of each other in a bimonoid. All homomorphisms of bimonoids preserve existing left and right complements. Crucially, left and right complements are unique whenever they exist. This holds even in a bisemigroup.

Proposition 2.14 *(Uniqueness of complements). Each element of a bisemigroup has at most one ℓ-complement (at most one r-complement).*

Proof. If y and z are ℓ-complements of x, then $y \leq y \cdot (x + z) \leq (y \cdot x) + z \leq z$. The other case is analogous. □

We use x^ℓ (x^r) to denote the unique left (right) complement of x whenever it exists. (We shall verify shortly that this is consistent with our previous usage of this notation.) In the commutative case clearly $x^\ell = x^r$ whenever these exist. We use the notation $\overline{x}$ in this case.

Proposition 2.15 *(Residuation laws for complements). The following residuation laws hold in each bisemigroup whenever the complements exist:*

$$x \cdot y \leq z \iff y \leq x^r + z, \qquad x \leq y + z \iff y^\ell \cdot x \leq z,$$
$$x \cdot y \leq z \iff x \leq z + y^\ell, \qquad x \leq y + z \iff x \cdot z^r \leq y.$$

Proof. If $x \cdot y \leq z$, then $y \leq (x^r + x) \cdot y \leq x^r + (x \cdot y) \leq x^r + z$. Conversely, if $y \leq x^r + z$, then $x \cdot y \leq x \cdot (x^r + z) \leq (x \cdot x^r) + z \leq z$. The other claims follow by left–right duality and order duality. □

Proposition 2.16 *(De Morgan laws for bisemigroups). Bisemigroups satisfy the De Morgan laws and double negation elimination whenever the complements on the right-hand side of the equation exist:*

$$(x \cdot y)^\ell = y^\ell + x^\ell, \qquad (x + y)^\ell = y^\ell \cdot x^\ell, \qquad x = x^{r\ell},$$
$$(x \cdot y)^r = y^r + x^r, \qquad (x + y)^r = y^r \cdot x^r, \qquad x = x^{\ell r}.$$

Complementation is antitone in bisemigroups whenever the complements exist:

$$x \leq y \implies y^\ell \leq x^\ell,$$
$$x \leq y \implies y^r \leq x^r.$$

Moreover, ℓ-bisemigroups satisfy the De Morgan laws for lattice connectives whenever the complements on the right-hand side of the equation exist:

$$(x \wedge y)^\ell = x^\ell \vee y^\ell, \qquad (x \vee y)^\ell = x^\ell \wedge y^\ell,$$
$$(x \wedge y)^r = x^r \vee y^r, \qquad (x \vee y)^r = x^r \wedge y^r.$$

Proof. The equalities $x = x^{\ell r}$ and $x = x^{r\ell}$ hold because by definition y is an ℓ-complement of x if and only if x is an r-complement of y. If $x \leq y$, then

$$y^\ell \leq y^\ell \cdot (x + x^\ell) \leq (y^\ell \cdot x) + x^\ell \leq (y^\ell \cdot y) + x^\ell \leq x^\ell,$$

and likewise $y^r \leq x^r$. The De Morgan laws for the lattice connectives follow from the antitonicity of x^ℓ and x^r and the equalities $x = x^{\ell r}$ and $x = x^{r\ell}$. The De Morgan laws for $(x \cdot y)^r$ and $(x + y)^r$ will follow from the De Morgan laws for $(x \cdot y)^\ell$ and $(x + y)^\ell$ by left–right symmetry. Likewise, the De Morgan law for $(x + y)^\ell$ follows from the De Morgan law for $(x \cdot y)^\ell$. It remains to prove that $(x \cdot y)^\ell$ exists and equals $y^\ell + x^\ell$. But

$$w + ((y^\ell + x^\ell) \cdot x \cdot y) \leq w + ((y^\ell + (x^\ell \cdot x)) \cdot y) \leq w + (y^\ell \cdot y) \leq w,$$
$$w \cdot ((x \cdot y) + y^\ell + x^\ell) \geq w \cdot ((x \cdot (y + y^\ell)) + x^\ell) \geq w \cdot (x + x^\ell) \geq w,$$

and likewise for the other two inequalities which define $(x \cdot y)^\ell$. □

The above propositions in fact show that complemented bisemigroups (bimonoids) are precisely involutive residuated posemigroups (pomonoids) presented in a slightly different way. For the purposes of the following proposition, the expansion of a complemented bisemigroup (bimonoid) by the two unary operations x^ℓ and x^r will be called a bisemigroup (bimonoid) *with complementation*. In other words, the class of all bisemigroups (bimonoids) with complementation forms an ordered variety which is axiomatized by adding the following inequalities to an axiomatization of bisemigroups (bimonoids):

$$w + (x^\ell \cdot x) \leq w, \qquad w + (x \cdot x^r) \leq w, \qquad w \leq w \cdot (x + x^\ell), \qquad w \leq w \cdot (x^r + x),$$
$$(x^\ell \cdot x) + w \leq w, \qquad (x \cdot x^r) + w \leq w, \qquad w \leq (x + x^\ell) \cdot w, \qquad w \leq (x^r + x) \cdot w.$$

Proposition 2.17 *(Involutive residuated posemigroups are complemented bisemigroups). Bimonoids with complementation are term equivalent to involutive residuated pomonoids. Bisemigroups with complementation are term equivalent to involutive residuated posemigroups which satisfy the inequalities*

$$x \leq x(y\backslash y), \qquad x \leq (y\backslash y)x,$$
$$x \leq x(y/y), \qquad x \leq (y/y)x.$$

Proof. The previous proposition shows that bisemigroups with complementation can be seen as involutive residuated posemigroups. Moreover, the above four inequalities (which are satisfied in each residuated pomonoid) are precisely four of the eight inequalities defining complements in bisemigroups. Conversely, given an involutive residuated posemigroup satisfying the above four inequalities, we need to show that x^ℓ and x^r are the left and right complements of x and that the hemidistributive law holds. Let us first prove hemidistributivity:

$$x \cdot (y + z) \leq (x \cdot y) + z \iff x \cdot (z^\ell \cdot y^\ell)^r \leq (z^\ell \cdot (x \cdot y)^\ell)^r \iff z^\ell \cdot (x \cdot y)^\ell \cdot x \leq z^\ell \cdot y^\ell.$$

The last inequality holds because $(x \cdot y)^\ell \cdot x \leq y^\ell$ by residuation. The four inequalities above now immediately yield four of the eight inequalities defining complements. The other inequalities follow by applying the operations $^\ell$ and ^{r}. For example, $w^\ell + (x^\ell \cdot x) \leq w^\ell$ implies that $w = w^{r\ell} \leq (w^\ell + (x^\ell \cdot x))^r = (x^r + x) \cdot w$. □

2.4. Examples of bisemigroups and bimonoids

We saw in the previous section that bisemigroups (bimonoids) occur as subreducts of involutive residuated posemigroups (pomonoids). Let us now consider several other ways of constructing bisemigroups and bimonoids in order to provide the reader with a stock of examples.

Definition 2.18 *(Commutative, integral, and idempotent bisemigroups).* A bisemigroup is *commutative* if its multiplicative and additive posemigroups are both commutative, it is *multiplicatively (additively) integral* if its multiplicative (additive) posemigroup is integral, it is *bi-integral* if it is both multiplicative and additively integral, and it is *idempotent* if its multiplicative and additive posemigroups are both idempotent.

In particular, a bimonoid is multiplicatively (additively) integral if it satisfies $x \leq 1$ ($0 \leq x$). Idempotent bi-integral bisemigroups are in fact very familiar objects.

Proposition 2.19 *(Distributive lattices as bisemigroups). Idempotent bi-integral bisemigroups (bimonoids) are precisely (bounded) distributive lattices equipped with the lattice order.*

Proof. Integral idempotent posemigroups (pomonoids) are known to be precisely (unital) meet semilattices, the partial order coinciding with the semilattice order. Integral idempotent bisemigroups (bimonoids) are thus (bounded) lattices, the partial order being the lattice order. Moreover, for lattices hemidistributivity is equivalent to distributivity. □

Secondly, each posemigroup (pomonoid) can be seen as a bisemigroup (bimonoid) if we take

$$x + y := x \cdot y \qquad \text{and} \qquad 0 := 1 \text{ (for pomonoids).}$$

In the same way, each lattice-ordered semigroup (monoid) where multiplication distributes over binary meets and joins can be seen as an ℓ-bisemigroup (ℓ-bimonoid). Although these examples may at first sight seem too trivial to be of any interest, we shall see that non-trivial bimonoids may be constructed from these as bimonoids of fractions. In particular, it will be useful to view ℓ-groups and Brouwerian algebras as residuated ℓ-bimonoids where the multiplicative and additive monoids coincide.

Thirdly, observe that in the presence of multiplicative integrality the inequalities

$$x \cdot (y + z) \leq (x \cdot y) + (x \cdot z) \qquad \text{and} \qquad (y + z) \cdot x \leq (y \cdot x) + (z \cdot x)$$

imply hemidistributivity. Examples of multiplicatively integral ℓ-bisemigroups (ℓ-bimonoids) thus include all integral sℓ-semigroups (with a bottom element $\bot$) if we take

$$x + y := x \vee y \qquad \text{and} \qquad 0 := \bot \text{ (if } \bot \text{ exists).}$$

A bisemigroup can also be obtained from any posemigroup with a top element $\top$ by taking the $\top$-drastic addition: $x + y := \top$ for all x and y. Similarly a bisemigroup can be obtained from a posemigroup with a bottom element $\bot$ such that $\bot \cdot x = x = x \cdot \bot$ for all x by taking the $\bot$-drastic addition: $x + y := \bot$ for all x and y. Only the first of these constructions extends to pomonoids in a reasonable way: if we start with an integral pomonoid with a bottom element $\bot$ such that

$$x \cdot y = \bot \iff x = \bot \text{ or } y = \bot,$$

then the following modified $\top$-drastic addition yields a bimonoid with $0 := \bot$:

$$x + y := \begin{cases} 1 \text{ if } x > \bot \text{ and } y > \bot, \\ x \text{ if } y = \bot, \\ y \text{ if } x = \bot. \end{cases}$$

Integral pomonoids which satisfy the required condition are in fact easy to come by: take an arbitrary integral pomonoid and append a new bottom element $\bot$ with

$$\bot \cdot x = \bot = x \cdot \bot.$$

Our next family of examples consists of bimonoids obtained from pointed Brouwerian algebras.

Definition 2.20 *(Brouwerian algebras and Heyting algebras).* A *Brouwerian algebra* $\langle A, \vee, \wedge, 1, \to \rangle$ is a distributive lattice $\langle A, \vee, \wedge \rangle$ with a top element 1 such that the binary operation $x \to y$ satisfies the equivalence

$$x \wedge y \leq z \iff y \leq x \to z.$$

A *pointed Brouwerian algebra* is one equipped with a constant 0. A *Heyting algebra* is a pointed Brouwerian algebra where 0 is the smallest element.

Equivalently, Brouwerian (Heyting) algebras are precisely (bounded) idempotent integral residuated lattices. Heyting algebras and (pointed) Brouwerian algebras are varieties, and the variety of Brouwerian algebras may be identified with the subvariety of pointed Brouwerian algebras which satisfy the equation $0 = 1$. Pointed Brouwerian algebras are also called Johansson algebras or j-algebras [27].

Proposition 2.21 *(Pointed Brouwerian algebras as bimonoids). The variety of pointed Brouwerian algebras and the variety of multiplicatively integral idempotent commutative residuated ℓ-bimonoids are term equivalent via the correspondence*

$$x \cdot y := x \wedge y \qquad \textit{and} \qquad x + y := (0 \to (x \wedge y)) \wedge (x \vee y).$$

Proof. Given such a commutative residuated ℓ-bimonoid $\langle A, \vee, \wedge, \cdot, 1, \to, +, 0 \rangle$, idempotence and multiplicative integrality imply that $x \cdot y = x \wedge y$, therefore $\langle A, \vee, \wedge, 1, \to, 0 \rangle$ is a pointed Brouwerian algebra. We now show that $x + y = (0 \to xy)(x \vee y)$. Clearly $x + y \leq (x \vee y) + (x \vee y) = x \vee y$, and $0(x + y) = 0^2(x + y) \leq 0x + 0y \leq 0xy$, since $0x + 0y \leq 0x + 0 = 0x$ and $0x + 0y \leq 0 + 0y \leq 0y$, therefore $x + y \leq 0 \to xy$. Conversely, to prove that $(0 \to xy)(x \vee y) \leq x + y$, it suffices to prove that $(0 \to xy)x \leq x + y$. But we have $(0 \to xy)x \leq x(0 \to y) = (x + 0)(0 \to y) \leq x + 0(0 \to y) \leq x + y$. Thus $x + y = (0 \to (x \wedge y))(x \vee y)$.

Conversely, given a pointed Brouwerian algebra, we have $x + y = y + x$ and $x + 0 = (0 \to x0)(x \vee 0) = (0 \to x)(x \vee 0) = (0 \to x)x \vee (0 \to x)0 = x \vee 0x = x$, since $x = x \cdot x \leq (0 \to x)x \leq 1 \cdot x = x$. Moreover,

$$x + (y + z) = (0 \to x(0 \to yz)(y \vee z))(x \vee (0 \to yz)(y \vee z))$$

$$= (0 \to xyz)(x \vee (0 \to yz)(y \vee z))$$
$$= (0 \to xyz)(x \vee y \vee z)),$$

so $x + (y + z) = z + (x + y) = (x + y) + z$. Finally, we need to verify hemidistributivity: $x(y + z) = x(0 \to yz)(y \vee z) \leq (0 \to xyz)(xy \vee z) = xy + z$. The addition operation therefore yields a bimonoid. □

In particular, a specific variety of pointed Brouwerian algebras will be important later.

Definition 2.22 *(Boolean-pointed Brouwerian algebras).* A pointed Brouwerian algebra is called *Boolean-pointed* if the interval $[0, 1]$ is a Boolean lattice, or equivalently if it satisfies the equation $x \vee (x \to 0) \approx 1$.

The following lemma will be used later to simplify the proof of Fact 4.31.

Lemma 2.23 *(Inequational validity in Boolean-pointed Brouwerian algebras). Consider terms $t(x_1, \dots, x_m, y_1, \dots, y_n)$ and $u(x_1, \dots, x_m, y_1, \dots, y_n)$ in the signature of Brouwerian algebras. Then the inequality $t(x_1, \dots, x_m, 0, \dots, 0) \leq u(x_1, \dots, x_m, 0, \dots, 0)$ holds in all Boolean-pointed Brouwerian algebras if and only if $t(x_1, \dots, x_m, 1, \dots, 1) \leq u(x_1, \dots, x_m, 1, \dots, 1)$ holds in all Brouwerian algebras and $t(x_1, \dots, x_m, 0, \dots, 0) \vee 0 \leq u(x_1, \dots, x_m, 0, \dots, 0) \vee 0$ holds in all Boolean-pointed Brouwerian algebras.*

Proof. The left-to-right implication is immediate. Conversely, suppose that $b = t(a_1, \dots, a_m, 0, \dots, 0) \nleq u(a_1, \dots, a_m, 0 \dots, 0) = c$ in some Boolean-pointed Brouwerian algebra $\mathbf{A}$. Then either $b \wedge 0 \nleq c \wedge 0$ or $b \vee 0 \nleq c \vee 0$. In the former case, let F be the filter generated by 0, and let $\pi\colon \mathbf{A} \to \mathbf{A}/F$ be the appropriate projection map, which is a homomorphism of Brouwerian algebras. Then $b \wedge 0 \nleq c \wedge 0$ implies that $\pi(b) \nleq \pi(c)$. It follows that the equality $t(x_1, \dots, x_m, 1, \dots, 1) \approx u(x_1, \dots, x_m, 1, \dots, 1)$ fails in the Brouwerian algebra $\mathbf{A}/F$ for $x_i = \pi(a_i)$, since $\pi(0) = \pi(1)$. In the latter case, $c \to 0 \nleq b \to 0$. □

Further examples of bi-integral bisemigroups can be obtained using an ordinal sum construction. Consider a family of bisemigroups $\mathbf{A}_i$ for $i \in I$, where I is a chain ordered by $\sqsubseteq$. The chain is called *non-trivial* if it has at least two elements. We define the *ordinal sum* of this family to be an ordered algebra $\mathbf{A} = \langle A, \leq, +, \cdot \rangle$ over the universe $A := \bigcup_{i \in I} A_i$. Let $a \in \mathbf{A}_i$ and $b \in \mathbf{A}_j$. The order on $\mathbf{A}$ is defined as follows:

$$a \leq b \iff \text{either } i \sqsubset j \text{ or } i = j \text{ and } a \leq b.$$

The operations of $\mathbf{A}$ are:

$$a \cdot b = \begin{cases} a & \text{if } i \sqsubseteq j, \\ a \cdot b & \text{if } i = j, \\ b & \text{if } i \sqsupset j, \end{cases} \qquad a + b = \begin{cases} b & \text{if } i \sqsubset j, \\ a + b & \text{if } i = j, \\ a & \text{if } i \sqsupset j. \end{cases}$$

Fact 2.24. *Let I be a non-trivial chain ordered by $\sqsubseteq$ and let $\mathbf{A}_i$ be a family of (commutative) bisemigroups. Then its ordinal sum $\mathbf{A}$ is a bisemigroup if and only if each bisemigroup $\mathbf{A}_i$ for $i \in I$ is bi-integral. In that case $\mathbf{A}$ is a (commutative) bi-integral bisemigroup.*

Proof. Given a family of bi-integral bisemigroups, the two operations are clearly associative. To show that they are isotone, consider $a \in \mathbf{A}_i$, $b \in \mathbf{A}_j$, $c \in \mathbf{A}_k$. If $i = j$ and $a \leq b$, then case analysis ($k \sqsubset i$, $k = i$, $k \sqsubset i$) shows that $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$. Suppose therefore that $i \sqsubset j$. If $k \sqsubset i$ or $j \sqsubset k$, the inequalities $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$ follow from $c \leq c$ and $i \sqsubset j$, respectively. If $k = i$, they reduce to the inequalities $a \cdot c \leq c$ and $c \cdot a \leq c$, which hold by the multiplicative integrality of $\mathbf{A}_i = \mathbf{A}_k$. If $k = j$, they follow from $i \sqsubset j$. The proof for addition is entirely analogous and involves additive integrality.

Let us now verify that $a \cdot (b + c) \leq (a \cdot b) + c$ for $a \in \mathbf{A}_i$, $b \in \mathbf{A}_j$, $c \in \mathbf{A}_k$. The proof of the other hemidistributive law is analogous. If i, j, k are distinct, then the inequality $a \cdot (b + c) \leq (a \cdot b) + c$ in $\mathbf{A}$ follows from the inequality $i \wedge (j \vee k) \sqsubseteq (i \wedge j) \vee k$ in I. If $i = j = k$, then the inequality follows from the same inequality in $\mathbf{A}_i = \mathbf{A}_j = \mathbf{A}_k$. If $i \sqsubset j = k$, then the inequality simplifies to $a \leq c$, which follows from $i \leq k$. If $j = k \sqsubset i$, then the inequality simplifies to $b + c \leq b + c$. The cases $i = j \sqsubset k$ and $k \sqsubset i = j$ are similar. Finally, if $i = k \sqsubset j$, then the inequality simplifies to $a \leq a + c$, while if $j \sqsubset i = k$, then the inequality simplifies to $a \cdot c \leq c$. These inequalities follow respectively from the additive integrality of $\mathbf{A}_i$ and the multiplicative integrality of $\mathbf{A}_k$.

The ordered algebra $\mathbf{A}$ is thus a bisemigroup, and it is easily seen to be bi-integral. Conversely, suppose that $\mathbf{A}_i$ is not multiplicatively integral for some $i \in I$, say $a \cdot b \nleq b$ for some $a, b \in \mathbf{A}_i$. (If $\mathbf{A}_i$ is not additively integral instead of multiplicatively integral, or if $a \cdot b \nleq a$ instead of $a \cdot b \nleq b$, the proof is entirely analogous.) Consider some $c \in \mathbf{A}_j$ for $j \neq i$. If $i \sqsubset j$, then $a \leq c$ but $a \cdot b \nleq b = c \cdot b$. If on the other hand $j \sqsubset i$, then $a \cdot (c + b) = a \cdot b \nleq b = (a \cdot c) + b$. □

If the chain I has an upper bound $\top$ and $\mathbf{A}_\top$ has a multiplicative unit 1, then clearly 1 is a multiplicative unit for the whole of $\mathbf{A}$. Likewise, if I has a lower bound $\bot$ and $\mathbf{A}_\bot$ has an additive unit 0, then 0 is an additive unit for the whole of $\mathbf{A}$.

One special case of this construction is worth mentioning: if $\mathbf{B}$ is a bi-integral bisemigroup, we may take $I = \{0, 1\}$ with $0 \sqsubset 1$, $\mathbf{A}_0 := \mathbf{B}$, and $\mathbf{A}_1 := \mathbf{B}^\partial$ (the order dual of $\mathbf{B}$). If $\mathbf{B}$ has an additive unit, then $\mathbf{A}$ has both a multiplicative and an additive unit, i.e. it is a bi-integral bimonoid. For example, consider the additive posemigroup $\mathbf{B} := \langle \mathbb{N}, \leq, + \rangle$ of non-negative integers with the usual ordering. Expanding it by the drastic multiplication $x \cdot y := 0$ yields a bi-integral bisemigroup with an additive unit. Applying the above construction yields a bi-integral bimonoid, which is order isomorphic to the Chang MV-algebra. However, the bimonoidal structure is quite different when multiplying elements from $\mathbf{A}_0$ and $\mathbf{A}_1$.

Finally, any bisemigroup may be extended to a bounded bisemigroup by adding new top and bottom elements $\top$ and $\bot$ such that

$$\bot \cdot x = \bot = x \cdot \bot, \qquad \top + x = \top = x + \top,$$

$$x + \bot = \bot + x = \begin{cases} \bot \text{ if } x < \top, \\ \top \text{ if } x = \top, \end{cases} \qquad x \cdot \top = \top \cdot x = \begin{cases} \top \text{ if } x > \bot, \\ \bot \text{ if } x = \bot. \end{cases}$$

On the other hand, the task of extending an arbitrary bisemigroup to a bimonoid is not quite so simple. The reader may for example consider the problem of extending the bisemigroup $\langle \mathbb{Z}_+, \geq, \cdot, + \rangle$ of positive integers with the usual multiplication and addition and the dual of the usual order to a bimonoid.

3. Complemented MacNeille completions of bimonoids and bisemigroups

We now establish the first main result of the present paper: each commutative (ℓ-)bimonoid $\mathbf{A}$ embeds into a complete complemented commutative ℓ-bimonoid $\mathbf{A}^{\triangle}$ in a certain doubly dense way akin to the Dedekind–MacNeille completion. The construction of this complemented completion is accomplished using so-called involutive residuated frames, introduced by Galatos & Jipsen [14].

3.1. Complemented MacNeille completions: definition

The Dedekind–MacNeille completion of a lattice $\mathbf{L}$ is defined as an embedding of $\mathbf{L}$ into a complete lattice where the image of $\mathbf{L}$ is both meet dense and join dense in a precise sense. This completion is unique up to a unique isomorphism which fixes (the image of) $\mathbf{L}$. We now define the *complemented* Dedekind–MacNeille (DM) completion of a bimonoid, and show that in the commutative case it is unique in the same sense.

Definition 3.1 *(Join density and meet density).* A subset X of a poset P is *meet dense (join dense)* in P if each element of P is a meet (join) of some subset of X. A subset X of a bimonoid $\mathbf{A}$ is *admissibly meet dense (admissibly join dense)* in $\mathbf{A}$ if each element of $\mathbf{A}$ is an admissible meet (admissible join) of some subset of X.

Equivalently, a set X is meet dense in P if and only if for all $a, b \in P$

$$a \leq b \iff (b \leq x \implies a \leq x \text{ for all } x \in X).$$

Likewise, X is join dense in P if and only if for all $a, b \in P$

$$a \leq b \iff (x \leq a \implies x \leq b \text{ for all } x \in X).$$

Recall that $\overline{a}$ denotes the complement of a in a commutative bimonoid (if it exists), i.e. $\overline{a} = a^\ell = a^r$.

Definition 3.2 *(Commutative Δ_1-extensions).* Let $\iota\colon \mathbf{A} \hookrightarrow \mathbf{B}$ be an embedding of a commutative bimonoids. We call the embedding ι, and by extension the ℓ-bimonoid $\mathbf{B}$ itself, a *commutative (admissible) Δ_1-extension* of $\mathbf{A}$ if

$$\text{elements of the form } \iota(a) \cdot \overline{\iota(b)} \text{ with } a, b \in \mathbf{A} \text{ are (admissibly) join dense in } \mathbf{B}$$

and

$$\text{elements of the form } \iota(a) + \overline{\iota(b)} \text{ with } a, b \in \mathbf{A} \text{ are (admissibly) meet dense in } \mathbf{B}.$$

The above definition does not assume that the complements $\overline{\iota(b)}$ exist for each $b \in \mathbf{A}$. In other words, a commutative Δ_1-extension need not be complemented. In particular, each commutative bimonoid is a commutative admissible Δ_1-extension of itself, since each of its elements a has the form $a \cdot \overline{0}$, or equivalently $a + \overline{1}$. In a complemented commutative Δ_1-extension each existing meet and join is admissible, therefore complemented commutative Δ_1-extensions are always admissible. Moreover, the meet density and join density conditions are equivalent in this case.

Definition 3.3 *(Complemented Dedekind–MacNeille completions).* A *commutative complemented Dedekind–MacNeille (DM) completion* of a commutative bimonoid is a complete commutative complemented Δ_1-extension.

A commutative complemented DM completion of a finite commutative bimonoid of cardinality n has cardinality at most $2^{(n^2)}$, since there are at most n^2 elements of the form $\iota(a) \cdot \overline{\iota(b)}$ and each element of the completion is a join of some set of these elements.

Fact 3.4. *Let $\iota\colon \mathbf{A} \hookrightarrow \overline{\mathbf{A}}$ be a commutative Δ_1-extension. Then ι preserves all admissible joins and meets which exist in $\mathbf{A}$. In particular, if $\mathbf{A}$ is an ℓ-bimonoid, then ι is an embedding of ℓ-bimonoids.*

Proof. We only prove the claim for joins. Suppose that $a = \bigvee_{i \in I} a_i$ is an admissible join in $\mathbf{A}$. Clearly $\iota(a_i) \leq \iota(a)$ for each $i \in I$. If $\iota(a_i) \leq \iota(c) + \overline{\iota(d)}$ for each $i \in I$, then $\iota(a_i \cdot d) = \iota(a_i) \cdot \iota(d) \leq \iota(c)$ for each $i \in I$, therefore $a_i \cdot d \leq c$ for each $i \in I$, and $a \cdot d \leq c$ by the admissibility of a. Thus $\iota(a) \cdot \iota(d) = \iota(a \cdot d) \leq \iota(c)$ and $\iota(a) \leq \iota(c) + \overline{\iota(d)}$. It now follows from the meet density of elements of the form $\iota(c) + \overline{\iota(d)}$ that $\iota(a) = \bigvee_{i \in I} \iota(a_i)$. □

We generally disregard the embedding $\iota\colon \mathbf{A} \hookrightarrow \overline{\mathbf{A}}$ and treat $\mathbf{A}$ as a sub-bimonoid of $\overline{\mathbf{A}}$. Each commutative Δ_1-extension $\overline{\mathbf{A}}$ of a commutative bimonoid $\mathbf{A}$ turns out to be an *essential extension* in the sense that each homomorphism of bimonoids $h\colon \overline{\mathbf{A}} \to \mathbf{B}$ is an embedding whenever its restriction to $\mathbf{A}$ is.

Proposition 3.5 *(Commutative Δ_1-extensions are essential). Each commutative Δ_1-extension of a bimonoid is essential.*

Proof. Let $\overline{\mathbf{A}}$ be commutative Δ_1-extension of $\mathbf{A}$ and $h\colon \overline{\mathbf{A}} \to \mathbf{B}$ be a homomorphism of bimonoids whose restriction to $\mathbf{A}$ is an embedding. Then h preserves all existing complements. Suppose that h is not an embedding. Then there are $a_i, b_i, c_j, d_j \in \mathbf{A}$ for $i \in I$ and $j \in J$ such that

$$h\Big(\bigvee_{i \in I} a_i \cdot \overline{b_i}\Big) \leq_{\mathbf{B}} h\Big(\bigwedge_{j \in J} c_j + \overline{d_j}\Big) \qquad \text{and} \qquad \bigvee_{i \in I} a_i \cdot \overline{b_i} \nleq_{\overline{\mathbf{A}}} \bigwedge_{j \in J} c_j + \overline{d_j}.$$

It follows that $h(a \cdot \overline{b}) \leq_{\mathbf{B}} h(c + \overline{d})$ and $a \cdot \overline{b} \nleq_{\overline{\mathbf{A}}} c + \overline{d}$ for some $a, b, c, d \in \mathbf{A}$. But then $h(a \cdot d) \leq_{\mathbf{B}} h(c + b)$ and $a \cdot d \nleq_{\overline{\mathbf{A}}} c + b$. Since $a \cdot d \in \mathbf{A}$ and $c + b \in \mathbf{A}$, the restriction of h to $\mathbf{A}$ is not an embedding. □

A given commutative bimonoid may have many distinct commutative Δ_1-extensions. However, there is up to isomorphism only one such extension which is both *complete* and *complemented*. Moreover, it is universal in the sense that any other commutative admissible Δ_1-extension embeds into it. This is a consequence of the following sequence of easy lemmas, where $\overline{\mathbf{A}}$ is a commutative Δ_1-extension of a commutative bimonoid $\mathbf{A}$ and $a_i, b_i, c_j, d_j \in \mathbf{A}$ for $i \in I$ and $j \in J$.

Lemma 3.6 *(Joins below meets). Whenever the join and meet exist in $\overline{\mathbf{A}}$,*

$$\bigvee_{i \in I} a_i \cdot \overline{b_i} \leq_{\overline{\mathbf{A}}} \bigwedge_{j \in J} \overline{c_j} + d_j \iff c_j \cdot a_i \leq_{\mathbf{A}} d_j + b_i \text{ for all } i \in I \text{ and } j \in J.$$

Proof. This follows from the residuation law for complements in bimonoids (Proposition 2.15). □

Lemma 3.7 *(Joins below joins). Whenever the two joins exist in $\overline{\mathbf{A}}$,*

$$\bigvee_{i \in I} a_i \cdot \overline{b_i} \leq_{\overline{\mathbf{A}}} \bigvee_{j \in J} c_j \cdot \overline{d_j}$$

if and only if for all $x, y \in \mathbf{A}$

$$x \cdot c_j \leq_{\mathbf{A}} y + d_j \text{ for all } j \in J \implies x \cdot a_i \leq_{\mathbf{A}} y + b_i \text{ for all } i \in I.$$

If $\mathbf{A}$ is residuated and the meet below exists in $\mathbf{A}$, then this is equivalent to

$$\bigwedge_{j \in J} (x + d_j)/c_j \leq_{\mathbf{A}} (x + b_i)/a_i \text{ for all } i \in I \text{ and } x \in \mathbf{A}.$$

In particular $a \cdot \overline{b} \leq_{\overline{\mathbf{A}}} c \cdot \overline{d}$ if and only if $(x + d)/c \leq_{\mathbf{A}} (x + b)/a$ for all $x \in \mathbf{A}$.

Proof. The inequality $\bigvee_{i\in I} a_i \cdot \overline{b_i} \leq_{\overline{\mathbf{A}}} \bigvee_{j\in J} c_j \cdot \overline{d_j}$ is equivalent to

$$\bigvee_{j\in J} c_j \cdot \overline{d_j} \leq_{\overline{\mathbf{A}}} \overline{x} + y \implies \bigvee_{i\in I} a_i \cdot \overline{b_i} \leq_{\overline{\mathbf{A}}} \overline{x} + y \text{ for all } x, y \in \mathbf{A}.$$

This amounts to

$$(\forall j \in J)\,(c_j \cdot \overline{d_j} \leq_{\overline{\mathbf{A}}} \overline{x} + y) \implies (\forall i \in I)\,(a_i \cdot \overline{b_i} \leq_{\overline{\mathbf{A}}} \overline{x} + y) \text{ for all } x, y \in \mathbf{A}.$$

But by the residuation law for complemented bimonoids (Proposition 2.15) and the fact that $\mathbf{A}$ is a sub-bimonoid of $\overline{\mathbf{A}}$ this is equivalent to

$$(\forall j \in J)\,(x \cdot c_j \leq_{\mathbf{A}} y + d_j) \implies (\forall i \in I)\,(x \cdot a_i \leq_{\mathbf{A}} y + b_i) \text{ for all } x, y \in \mathbf{A}.$$

Finally, if $\mathbf{A}$ is residuated, then the above condition is equivalent to

$$(\forall j \in J)\,(x \leq_{\mathbf{A}} (y + d_j)/c_j) \implies (\forall i \in I)\,(x \leq_{\mathbf{A}} (y + b_i)/a_i) \text{ for all } x, y \in \mathbf{A}.$$

If the meet $\bigwedge_{j\in J}(y + d_j)/c_j$ exists in $\mathbf{A}$, then this is equivalent to

$$\bigwedge_{j\in J} (y + d_j)/c_j \leq_{\mathbf{A}} (y + b_i)/a_i \text{ for each } i \in I. \quad \square$$

Lemma 3.8 *(Meets below joins). Whenever the join and meet exist in $\overline{\mathbf{A}}$, the inequality*

$$\bigwedge_{i\in I} \overline{a_i} + b_i \leq_{\overline{\mathbf{A}}} \bigvee_{j\in J} c_j \cdot \overline{d_j}$$

holds if and only if for all $u, v, x, y \in \mathbf{A}$

$$(\forall i \in I)\,(\forall j \in J)\,(u \cdot a_i \leq_{\mathbf{A}} v + b_i \;\&\; x \cdot c_j \leq y + d_j) \implies u \cdot x \leq v + y.$$

Proof. This equivalence is proved using the same method as the previous lemma, taking advantage of the observation that $x \leq y$ in a poset if and only if

$$a \leq x \;\&\; y \leq b \implies a \leq b$$

for each join generator a and each meet generator b. $\square$

Theorem 3.9 *(Universality of complemented DM completions). Let $\iota\colon \mathbf{A} \to \mathbf{B}$ be an isomorphism of commutative bimonoids and let $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ be commutative admissible Δ_1-extensions of $\mathbf{A}$ and $\mathbf{B}$. If $\overline{\mathbf{B}}$ is complete and complemented, then there is a unique complete embedding of bimonoids $\overline{\iota}\colon \overline{\mathbf{A}} \to \overline{\mathbf{B}}$ which extends the isomorphism ι.*

Proof. We define the map $\overline{\iota}_+ \colon \overline{\mathbf{A}} \to \overline{\mathbf{B}}$ as

$$\overline{\iota}_+ \colon \bigvee_{i \in I}^{\overline{\mathbf{A}}} a_i \overline{b_i} \mapsto \bigvee_{i \in I}^{\overline{\mathbf{B}}} \iota(a_i)\overline{\iota(b_i)}.$$

The element $\bigvee_{i \in I}^{\overline{\mathbf{B}}} \iota(a_i)\overline{\iota(b_i)}$ exists because $\overline{\mathbf{B}}$ is complete and complemented, and the map is a well-defined order-embedding by Lemma 3.7. It preserves arbitrary joins by definition, and moreover it preserves products (here we use the admissibility of the extension $\overline{\mathbf{A}}$):

$$\begin{aligned}
\overline{\iota}_+\Big(\bigvee_{i \in I}^{\overline{\mathbf{A}}} a_i\overline{b_i} \cdot \bigvee_{i \in I}^{\overline{\mathbf{A}}} c_i\overline{d_i}\Big) &= \overline{\iota}_+\Big(\bigvee_{i \in I}^{\overline{\mathbf{A}}} \bigvee_{j \in J}^{\overline{\mathbf{A}}} a_i\overline{b_i} \cdot c_j\overline{d_j}\Big) = \overline{\iota}_+\Big(\bigvee_{i \in I}^{\overline{\mathbf{A}}} \bigvee_{j \in J}^{\overline{\mathbf{A}}} a_i c_j \cdot \overline{d_j + b_i}\Big) = \\
&= \bigvee_{i \in I}^{\overline{\mathbf{B}}} \bigvee_{j \in J}^{\overline{\mathbf{B}}} \iota(a_i c_j) \cdot \overline{\iota(d_j + b_i)} = \\
&= \bigvee_{i \in I}^{\overline{\mathbf{B}}} \bigvee_{j \in J}^{\overline{\mathbf{B}}} \iota(a_i)\overline{\iota(b_i)} \cdot \iota(c_j)\overline{\iota(d_j)} = \\
&= \bigvee_{i \in I}^{\overline{\mathbf{B}}} \iota(a_i)\overline{\iota(b_i)} \cdot \bigvee_{j \in J}^{\overline{\mathbf{B}}} \iota(c_j)\overline{\iota(d_j)} = \\
&= \iota_+\Big(\bigvee_{i \in I}^{\overline{\mathbf{A}}} a_i\overline{b_i}\Big) \cdot \iota_+\Big(\bigvee_{i \in I}^{\overline{\mathbf{A}}} c_i\overline{d_i}\Big)
\end{aligned}$$

We can also define the map $\overline{\iota}_- \colon \overline{\mathbf{A}} \to \overline{\mathbf{B}}$ as

$$\overline{\iota}_- \colon \bigwedge_{i \in I}^{\overline{\mathbf{A}}} \overline{a_i} + b_i \mapsto \bigwedge_{i \in I}^{\overline{\mathbf{B}}} \overline{\iota(a_i)} + \iota(b_i).$$

This map is a well-defined order-embedding by the order dual of Lemma 3.7. It again preserves arbitrary meets by definition, and it preserves sums by the order dual of the above argument. Moreover, $\overline{\iota}_+(1) = \overline{\iota}_+(1\overline{0}) = \iota(1)\overline{\iota(0)} = \iota(1)\iota(1) = \iota(1)$. Similarly, $\overline{\iota}_-(0) = \iota(0)$.

We know that $\overline{\iota}_-(x) = \overline{\iota}_+(x)$ for each $x \in \overline{\mathbf{A}}$ by Lemmas 3.6 and 3.8. Let us denote this common value by $\overline{\iota}(x)$. Then $\overline{\iota}$ is an embedding of bimonoids and clearly $\overline{\iota}(a) = \iota(a)$ for $a \in \mathbf{A}$. Conversely, it is clear from its definition that $\overline{\iota}$ is the only complete embedding of $\overline{\mathbf{A}}$ into $\overline{\mathbf{B}}$ which extends ι. □

The proof of the universality of the commutative complemented DM completion relies substantially on commutativity. Without this assumption, one would have to impose a specific way of simplifying products of the form, say, $ab^r cd^r$ into xy^r, and simplifying sums of the form, say, $a + b^\ell + c + d^\ell$ into $x + y^\ell$.

Corollary 3.10 *(Uniqueness of complemented DM completions). Let* $\iota\colon \mathbf{A} \to \mathbf{B}$ *be an isomorphism of commutative bimonoids and* $\mathbf{A}^{\Delta}$ *and* $\mathbf{B}^{\Delta}$ *be commutative complemented DM completions of* $\mathbf{A}$ *and* $\mathbf{B}$*. Then there is a unique isomorphism* $\iota^{\Delta}\colon \mathbf{A}^{\Delta} \to \mathbf{B}^{\Delta}$ *which extends* ι*.*

Each complemented DM completion of $\mathbf{A}$ is in fact an ordinary DM completion of any complemented Δ_1-extension of $\mathbf{A}$. Here by an *(ordinary) DM completion* we mean an embedding of bimonoids $\iota\colon \mathbf{A} \hookrightarrow \mathbf{B}$ such that $\iota[\mathbf{A}]$ is both join and meet dense in $\mathbf{B}$. Note that (ordinary) DM completions of involutive residuated structures were already constructed in [14] using the machinery of involutive residuated frames.

Proposition 3.11 *(Ordinary DM completions of complemented* Δ_1*-extensions). Let* $\iota_1\colon \mathbf{A} \hookrightarrow \overline{\mathbf{A}}$ *be a commutative complemented* Δ_1*-extension of a commutative bimonoid* $\mathbf{A}$ *and* $\iota\colon \mathbf{A} \hookrightarrow \mathbf{A}^{\Delta}$ *be a commutative complemented DM completion of a* $\mathbf{A}$*. Then there is a unique ordinary DM completion* $\iota_2\colon \overline{\mathbf{A}} \hookrightarrow \mathbf{A}^{\Delta}$ *such that* $\iota = \iota_2 \circ \iota_1$*.*

Proof. By the universality of complemented DM completions, there is a unique complete embedding of bimonoids $\iota_2\colon \overline{\mathbf{A}} \hookrightarrow \mathbf{A}^{\Delta}$ such that $\iota = \iota_2 \circ \iota_1$. Each element $x \in \mathbf{A}^{\Delta}$ therefore has the form $x = \bigvee_{i\in I} \iota_2(\iota_1(a_i))\overline{\iota_2(\iota_1(b_i))} = \bigvee_{i\in I} \iota_2(\iota_1(a_i)\overline{\iota_1(b_i)})$ for some $a_i, b_i \in \mathbf{A}$, i.e. $\bigvee_{i\in I} \iota_2(y_i)$ for some $y_i \in \overline{\mathbf{A}}$. Similarly, each $x \in \mathbf{A}^{\Delta}$ has the form $\bigwedge_{i\in I} \iota_2(y_i)$ for some $y_i \in \overline{\mathbf{A}}$. Thus ι_2 is an ordinary DM completion. To prove uniqueness, observe that any ordinary DM completion $\iota_2\colon \overline{\mathbf{A}} \hookrightarrow \mathbf{A}^{\Delta}$ such that $\iota = \iota_2 \circ \iota_1$ agrees with this one on where it sends elements of the forms $\iota_1(a)\overline{\iota_1(b)}$ for $a, b \in \mathbf{A}$. Since each element of $\overline{\mathbf{A}}$ is a join of such elements, any two ordinary DM completions $\iota_2\colon \overline{\mathbf{A}} \hookrightarrow \mathbf{A}^{\Delta}$ such that $\iota = \iota_2 \circ \iota_1$ must coincide. □

Corollary 3.12 *(Complemented DM completions of complemented bimonoids). If* $\mathbf{A}$ *is a commutative complemented bimonoid, then the commutative complemented DM completion of* $\mathbf{A}$ *is the ordinary DM completion of* $\mathbf{A}$*.*

In particular, the commutative complemented DM completion of a bounded distributive lattice $\mathbf{A}$ is the DM completion of the free Boolean extension of $\mathbf{A}$ (the smallest Boolean algebra into which $\mathbf{A}$ embeds). Let us remark that this is known to be precisely the injective hull of $\mathbf{A}$ in the category of bounded distributive lattices (see [3]). In particular, it is a maximal essential extension (in the sense of Proposition 3.5).

The question now arises whether this categorical characterization of the complemented DM completion as the injective hull generalizes to some broader class of bimonoids. Without pursuing the topic further, let us merely observe that there are commutative ℓ-bimonoids $\mathbf{A}$ whose commutative complemented DM completion is not a maximal essential extension of $\mathbf{A}$ (and therefore it is not the injective hull of $\mathbf{A}$). Equivalently, there are complete commutative complemented ℓ-bimonoids with proper essential extensions.

For example, one may take the essential embedding of the additive ℓ-group of integers $\mathbb{Z}$ into the lexicographic product $\mathbb{Z} \overrightarrow{\times} \mathbb{Z}$: the group $\mathbb{Z} \times \mathbb{Z}$ equipped with the lexicographic order, where $\langle a, b\rangle \leq \langle c, d\rangle$ if and only if either $a \leq c$ or $a = c$ and $b \leq d$. The map $\iota\colon \mathbb{Z} \to \mathbb{Z} \overrightarrow{\times} \mathbb{Z}$ such that $\iota(n) := \langle n, 0\rangle$ is an essential embedding of $\mathbb{Z}$ into $\mathbb{Z} \overrightarrow{\times} \mathbb{Z}$. If we now extend $\mathbb{Z}$ and $\mathbb{Z} \overrightarrow{\times} \mathbb{Z}$ by a new top and bottom element $\top$ and $\bot$ and extend the embedding ι so that $\iota(\top) := \top$ and $\iota(\bot) := \bot$, we obtain a proper essential extension of a commutative complete complemented ℓ-bimonoid. The complemented DM completion of $\mathbb{Z}$, i.e. the extension of $\mathbb{Z}$ by the top and bottom elements, is therefore not an injective hull of $\mathbb{Z}$, because the composite embedding of $\mathbb{Z}$ into the extension of $\mathbb{Z} \overrightarrow{\times} \mathbb{Z}$ by bounds is also essential.

3.2. Involutive residuated frames

Having established the uniqueness of commutative complemented DM completions, the rest of this section is devoted to proving their existence using the *involutive (residuated) frames* of Galatos & Jipsen [14]. The definition given below in fact differs from the original definition of an involutive (Gentzen) frame in several respects. However, adapting the results of Galatos & Jipsen [14] to suit our current needs is a straightforward task, and the results of this subsection are not substantially novel.[5]

Before giving the formal definition of an involutive frame, let us first explain the purpose of introducing these structures. Our goal will be to construct a complete complemented ℓ-bimonoid given a monoid of join generators $\mathbf{L} = \langle L, \circ, 1_\circ\rangle$, a monoid of meet generators $\mathbf{R} = \langle R, \oplus, 0_\oplus\rangle$, and a link between these two monoids. This link consists of a binary relation $x \sqsubseteq y$, which tells us how to compare a join generator x and a meet generator y, and mutually inverse monoidal anti-isomorphisms, which tell us how complementation acts on these monoids. Some natural compatibility conditions are of course postulated. The above data constitute the frame from which we then construct a complete complemented bimonoid.

Moreover, we wish to embed a given bimonoid $\mathbf{A}$ into the complete complemented ℓ-bimonoid constructed from the frame. This embedding can be constructed from a pair of maps $\lambda\colon A \to L$ and $\rho\colon A \to R$ (not necessarily homomorphisms) embedding $\mathbf{A}$ into $\mathbf{L}$ and $\mathbf{R}$. This yields what we call an involutive $\mathbf{A}$-frame, shown in Fig. 1. If the embeddings λ and ρ reflect the order in a natural sense, we call the involutive $\mathbf{A}$-frame faithful. The key result is that an (ℓ-)bimonoid $\mathbf{A}$ embeds into the involutive residuated lattice obtained from every faithful involutive $\mathbf{A}$-frame, and moreover there is a natural way to build a faithful commutative involutive $\mathbf{A}$-frame given a commutative bimonoid $\mathbf{A}$.

[5] Let us briefly summarize the differences between the two frameworks. Firstly, we take $\mathbf{A}$ to be a bimonoid rather than a partial involutive residuated lattice (or some more general involutive residuated structure). Secondly, we do not assume that the positive or the negative sides of the frame are generated by $\mathbf{A}$ as a monoid. Thirdly, we allow the positive and the negative sides of the frame to be distinct sets. Finally, we assume that the monoidal operations on both sides of the frame are single-valued and associative, rather than satisfying the weaker conditions $((a \circ b) \circ c)^{\triangleright} = (a \circ (b \circ c))^{\triangleright}$ and $((a \oplus b) \oplus c)^{\triangleleft} = (a \oplus (b \oplus c))^{\triangleleft}$. Nothing in our proofs depends on this last assumption, it merely makes our definitions easier to state.

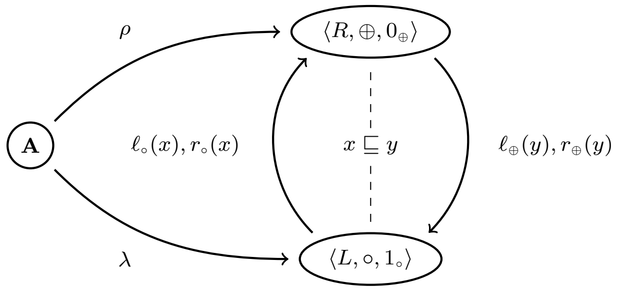

Fig. 1. An involutive **A**-frame.

All of the above builds on the Galois connection between a pair of sets L and R induced by a binary relation $\sqsubseteq\ \subseteq L \times R$. Such a triple $\langle L, R, \sqsubseteq\rangle$ is often called a polarity. For $X \subseteq L$, $x \in L$ and $Y \subseteq R$, $y \in R$ we use the notation

$$X \sqsubseteq y \iff x \sqsubseteq y \text{ for all } x \in X,$$
$$x \sqsubseteq Y \iff x \sqsubseteq y \text{ for all } y \in Y,$$
$$X^{\triangleright} := \{y \in R \mid X \sqsubseteq y\}, \qquad x^{\triangleright} := \{x\}^{\triangleright},$$
$$Y^{\triangleleft} := \{x \in L \mid x \sqsubseteq Y\}, \qquad y^{\triangleleft} := \{y\}^{\triangleleft}.$$

A set $X \subseteq L$ ($Y \subseteq R$) is called *Galois closed* if $X = X^{\triangleright\triangleleft}$ (if $Y = Y^{\triangleleft\triangleright}$). The Galois closed subsets of L and R ordered by inclusion form lattices which are anti-isomorphic via the maps $X \mapsto X^{\triangleright}$ and $Y \mapsto Y^{\triangleleft}$. Observe that $x \in X^{\triangleright\triangleleft}$ if and only if $X \sqsubseteq y$ implies $x \sqsubseteq y$ for all $y \in R$, and $y \in Y^{\triangleleft\triangleright}$ if and only if $x \sqsubseteq Y$ implies $x \sqsubseteq y$ for all $x \in L$. Consequently,

$$X^{\triangleright\triangleleft} \sqsubseteq y \iff X \sqsubseteq y \qquad \text{and} \qquad x \sqsubseteq Y^{\triangleleft\triangleright} \iff x \sqsubseteq Y.$$

This Galois connection is then expanded by a monoidal structure $\langle L, \circ, 1_\circ\rangle$ on L and a monoidal structure $\langle R, \oplus, 0_\oplus\rangle$ on R, as well as maps $\ell_\circ, r_\circ \colon L \to R$ and $\ell_\oplus, r_\oplus \colon R \to L$. We use the notation

$$X_1 \circ X_2 := \{x_1 \circ x_2 \mid x_1 \in X_1 \text{ and } x_2 \in X_2\},$$
$$Y_1 \oplus Y_2 := \{y_1 \oplus y_2 \mid y_1 \in Y_1 \text{ and } y_2 \in Y_2\},$$

with $x_1 \circ X_2 := \{x_1\} \circ X_2$ and likewise for $X_1 \circ x_2$, $x_1 \oplus X_2$, $X_1 \oplus x_2$.

Definition 3.13 *(Involutive frames).* A *(commutative) involutive frame* is a two-sorted structure consisting of two (commutative) monoids

$$\mathbf{L} = \langle L, \circ, 1_\circ\rangle,$$
$$\mathbf{R} = \langle R, \oplus, 0_\oplus\rangle,$$

two pairs of maps

$$\ell_\circ, r_\circ \colon L \to R,$$
$$\ell_\oplus, r_\oplus \colon R \to L,$$

and a relation $\sqsubseteq \subseteq L \times R$ satisfying the following form of residuation called *nuclearity*:

$$x \sqsubseteq z \oplus \ell_\circ(y) \iff x \circ y \sqsubseteq z \iff y \sqsubseteq r_\circ(x) \oplus z,$$
$$x \circ r_\oplus(z) \sqsubseteq y \iff x \sqsubseteq y \oplus z \iff \ell_\oplus(y) \circ x \sqsubseteq y.$$

Lemma 3.14. *In each involutive frame we have*

$$X_1^{\rhd\lhd} \circ X_2^{\rhd\lhd} \subseteq (X_1 \circ X_2)^{\rhd\lhd} \qquad \text{and} \qquad Y_1^{\lhd\rhd} \oplus Y_2^{\lhd\rhd} \subseteq (Y_1 \oplus Y_2)^{\lhd\rhd}$$

for all $X_1, X_2 \subseteq L$ and $Y_1, Y_2 \subseteq R$. Consequently,

$$(X_1^{\rhd\lhd} \circ X_2)^{\rhd} = (X_1 \circ X_2)^{\rhd} = (X_1 \circ X_2^{\rhd\lhd})^{\rhd} \qquad \text{and}$$
$$(Y_1^{\lhd\rhd} \oplus Y_2)^{\lhd} = (Y_1 \oplus Y_2)^{\lhd} = (Y_1 \oplus Y_2^{\lhd\rhd})^{\lhd}.$$

Proof. We only prove the claims for multiplication. The claims for addition then follow if we take the polarity $\langle R, L, \sqsupseteq\rangle$ instead of $\langle L, R\ \sqsubseteq\rangle$. The inclusion $X_1^{\rhd\lhd} \circ X_2^{\rhd\lhd} \subseteq (X_1 \circ X_2)^{\rhd\lhd}$ states that if $x_1 \in X_1^{\rhd\lhd}$, $x_2 \in X_2^{\rhd\lhd}$, and $X_1 \circ X_2 \sqsubseteq y$, then $x_1 \circ x_2 \sqsubseteq y$. But $X_1 \circ X_2 \sqsubseteq y$ implies $X_2 \sqsubseteq r_\circ[X_1] \oplus y$, hence $x_2 \in X_2^{\rhd\lhd} \sqsubseteq r_\circ[X_1] \oplus y$ and $X_1 \circ x_2 \sqsubseteq y$. Similarly, this implies $X_1 \sqsubseteq y \oplus \ell_\circ[X_2]$, hence $x_1 \in X_1^{\rhd\lhd} \sqsubseteq y \oplus \ell_\circ[X_2]$ and $x_1 \circ x_2 \sqsubseteq y$.

The other equalities now follow: $X_1 \subseteq X_1^{\rhd\lhd}$, hence $(X_1^{\rhd\lhd} \circ X_2)^{\rhd} \subseteq (X_1 \circ X_2)^{\rhd}$. Conversely, $X_1^{\rhd\lhd} \circ X_2 \subseteq X_1^{\rhd\lhd} \circ X_2^{\rhd\lhd} \subseteq (X_1 \circ X_2)^{\rhd\lhd}$, so $(X_1^{\rhd\lhd} \circ X_2)^{\rhd} \supseteq ((X_1 \circ X_2)^{\rhd\lhd})^{\rhd} = (X_1 \circ X_2)^{\rhd}$. □

Lemma 3.15 *(Hemidistributivity for involutive frames). In an involutive frame for each $X \subseteq L$ and $Y \subseteq R$ we have:*

$$u \circ X \sqsubseteq x \ \&\ v \sqsubseteq X^{\rhd} \oplus y \implies u \circ v \sqsubseteq x \oplus y,$$
$$u \circ Y^{\lhd} \sqsubseteq x \ \&\ v \sqsubseteq Y \oplus y \implies u \circ v \sqsubseteq x \oplus y,$$
$$u \sqsubseteq x \oplus X^{\rhd} \ \&\ X \circ v \sqsubseteq y \implies u \circ v \sqsubseteq x \oplus y,$$
$$u \sqsubseteq x \oplus Y \ \&\ Y^{\lhd} \circ v \sqsubseteq y \implies u \circ v \sqsubseteq x \oplus y.$$

Proof. If $u \circ X \sqsubseteq x$ and $v \sqsubseteq X^{\rhd} \oplus y$, then $X \sqsubseteq r_\circ(u) \oplus x$ and $v \circ r_\oplus(y) \sqsubseteq X^{\rhd}$, so $v \circ r_\oplus(y) \sqsubseteq r_\circ(u) \oplus x$ and $u \circ v \sqsubseteq x \oplus y$. The other implications are analogous. □

$$1_\circ \sqsubseteq \rho(1) \qquad \frac{1_\circ \sqsubseteq y}{\lambda(1) \sqsubseteq y} \qquad \frac{x \sqsubseteq 0_\oplus}{x \sqsubseteq \rho(0)} \qquad \lambda(0) \sqsubseteq 0_\oplus$$

$$\frac{\lambda(a) \sqsubseteq x \qquad \lambda(b) \sqsubseteq y}{\lambda(a+b) \sqsubseteq x \oplus y} \qquad \frac{x \sqsubseteq \rho(a) \qquad y \sqsubseteq \rho(b)}{x \circ y \sqsubseteq \rho(a \cdot b)}$$

$$\frac{\lambda(a) \circ \lambda(b) \sqsubseteq x}{\lambda(a \cdot b) \sqsubseteq x} \qquad \frac{x \sqsubseteq \rho(a) \oplus \rho(b)}{x \sqsubseteq \rho(a+b)}$$

$$\frac{\lambda(a) \sqsubseteq x \qquad \lambda(b) \sqsubseteq x}{\lambda(a \vee b) \sqsubseteq x} \qquad \frac{x \sqsubseteq \rho(a) \qquad x \sqsubseteq \rho(b)}{x \sqsubseteq \rho(a \wedge b)}$$

$$\frac{\lambda(a) \sqsubseteq x}{\lambda(a \wedge b) \sqsubseteq x} \qquad \frac{\lambda(b) \sqsubseteq x}{\lambda(a \wedge b) \sqsubseteq x} \qquad \frac{x \sqsubseteq \rho(a)}{x \sqsubseteq \rho(a \vee b)} \qquad \frac{x \sqsubseteq \rho(b)}{x \sqsubseteq \rho(a \vee b)}$$

Fig. 2. Gentzen conditions.

These implications can be seen as a form of the multiple-premise and multiple-conclusion Cut rule if we interpret the condition $u \circ v \sqsubseteq x \oplus y$ as expressing the provability of the sequent $u, v \vdash x, y$.

To embed an (ℓ-)bimonoid $\mathbf{A}$ into the algebra constructed from an involutive frame, we need to postulate some additional structure: a pair of maps $\lambda\colon A \to L$ and $\rho\colon A \to R$ which satisfy the conditions in Fig. 2. These will be called the *(ℓ-)bimonoidal Gentzen conditions*, depending on whether we include the conditions on meets and joins, due to their similarity to the logical rules in Gentzen calculi. They are to be interpreted as universally quantified implications: e.g. for all $a, b \in \mathbf{A}$ and $x, y \in R$ if $\lambda(a) \sqsubseteq x$ and $\lambda(b) \sqsubseteq y$, then $\lambda(a+b) \sqsubseteq x \oplus y$. The maps λ, ρ are not required to be homomorphisms and the maps $\ell_\circ, r_\circ, \ell_\oplus, r_\oplus$ are not required to be anti-homomorphisms, although in the frames constructed in the current paper they will be.

Definition 3.16 *(Involutive* $\mathbf{A}$*-frames).* Let $\mathbf{A}$ be an (ℓ-)bimonoid. An *involutive* $\mathbf{A}$*-frame* is then an involutive frame equipped with two maps $\lambda\colon A \to L$ and $\rho\colon A \to R$ which satisfy the (ℓ-)bimonoidal Gentzen conditions as well as Identity and Cut:

$$\lambda(a) \sqsubseteq \rho(a) \qquad\qquad \frac{x \sqsubseteq \rho(a) \qquad \lambda(a) \sqsubseteq y}{x \sqsubseteq y}$$

An (ℓ-)involutive $\mathbf{A}$-frame is called *faithful* if $\lambda(a) \sqsubseteq \rho(b)$ implies $a \leq b$ for all $a, b \in \mathbf{A}$.

Definition 3.17 *(The Galois algebra of an involutive frame).* The *Galois algebra* F^+ of an involutive frame F consists of the Galois closed subsets of L equipped with the following operations:

$$\begin{aligned} 1 &:= \{1_\circ\}^{\rhd\lhd}, & X_1 \cdot X_2 &:= (X_1 \circ X_2)^{\rhd\lhd}, \\ X_1 \wedge X_2 &:= X_1 \cap X_2, & X^\ell &:= (\ell_\circ[X])^{\lhd}, \\ 0 &:= \{0_\oplus\}^{\lhd}, & X_1 + X_2 &:= (X_1^{\rhd} \oplus X_2^{\rhd})^{\lhd}, \\ X_1 \vee X_2 &:= (X_1 \cup X_2)^{\rhd\lhd}, & X^r &:= (r_\circ[X])^{\lhd}. \end{aligned}$$

We now show that the Galois algebra of an involutive frame is a complete complemented ℓ-bimonoid, and moreover $\mathbf{A}$ embeds into the Galois algebra of a faithful involutive $\mathbf{A}$-frame. A map λ satisfying the conditions of the following lemma was called a quasi-homomorphism in [5].

Lemma 3.18 *(Quasi-homomorphism Lemma). In each involutive* $\mathbf{A}$*-frame*

$$\lambda(1) \in \{1_\circ\}^{\rhd\lhd} \sqsubseteq \rho(1),$$
$$\lambda(0) \in \{0_\oplus\}^{\lhd} \sqsubseteq \rho(0).$$

If X and Y are Galois closed subsets of L such that

$$\lambda(a) \in X \sqsubseteq \rho(a),$$
$$\lambda(b) \in Y \sqsubseteq \rho(b),$$

then we have

$$\begin{aligned} \lambda(a \cdot b) &\in (X \circ Y)^{\rhd\lhd} && \sqsubseteq \rho(a \cdot b), \\ \lambda(a + b) &\in (X^{\rhd} \oplus Y^{\rhd})^{\lhd} && \sqsubseteq \rho(a+b), \\ \lambda(a \vee b) &\in (X \cup Y)^{\rhd\lhd} && \sqsubseteq \rho(a \vee b), \\ \lambda(a \wedge b) &\in X \cap Y && \sqsubseteq \rho(a \wedge b). \end{aligned}$$

Proof. We divide the Gentzen conditions in Fig. 2 in left and right rules, depending on whether the conclusion contains the map λ or the map ρ. For example, by the left rule for addition we mean the Gentzen condition whose conclusion is $\lambda(a + b) \sqsubseteq x \oplus y$. Observe that in order to prove that $Z^{\rhd\lhd} \sqsubseteq y$ it suffices to prove that $Z \sqsubseteq y$: if $x \in Z^{\rhd\lhd}$ and $Z \sqsubseteq y$, then $x \sqsubseteq Z^{\rhd}$ and $y \in Z^{\rhd}$, hence $x \sqsubseteq y$.

The claim that $\lambda(1) \in \{1_\circ\}^{\rhd\lhd}$ is precisely the left rule for 1. The right rule states that $1_\circ \sqsubseteq \rho(1)$, therefore, as observed above, $\{1_\circ\}^{\rhd\lhd} \sqsubseteq \rho(1)$. The proof that $\lambda(0) \in \{0_\oplus\}^{\lhd} \sqsubseteq \rho(0)$ is analogous.

If $X \circ Y \sqsubseteq z$, then $\lambda(a) \circ \lambda(b) \sqsubseteq z$, hence $\lambda(a \cdot b) \sqsubseteq z$ by the left rule for multiplication. Thus $\lambda(a \cdot b) \in (X \cdot Y)^{\triangleright\triangleleft}$. For every $x \in X$ and $y \in Y$ we have $x \circ y \sqsubseteq \rho(a \cdot b)$ by the right rule for multiplication, thus $X \circ Y \sqsubseteq \rho(a \cdot b)$. The proof that $\lambda(a + b) \in (X^{\triangleright} \oplus Y^{\triangleright})^{\triangleleft} \sqsubseteq \rho(a + b)$ is analogous.

If $X \cup Y \sqsubseteq z$, then $\lambda(a) \sqsubseteq z$ and $\lambda(b) \sqsubseteq z$, hence by the left rule for joins $\lambda(a \vee b) \sqsubseteq z$. Thus $\lambda(a \vee b) \in (X \cup Y)^{\triangleright\triangleleft}$. If $X \sqsubseteq \rho(a)$ and $Y \sqsubseteq \rho(b)$, then by the right rule for joins $X \sqsubseteq \rho(a \vee b)$ and $Y \sqsubseteq \rho(a \vee b)$, hence $X \cup Y \sqsubseteq \rho(a \vee b)$ and $(X \cup Y)^{\triangleright\triangleleft} \sqsubseteq \rho(a \vee b)$. The proof of $\lambda(a \wedge b) \in X \cap Y \sqsubseteq \rho(a \wedge b)$ is analogous. □

Theorem 3.19 *(Galois algebras are involutive residuated lattices). Let F be a (commutative) involutive frame and $\mathbf{A}$ be an (ℓ-)bimonoid. Then the Galois algebra F^+ is a complete (commutative) involutive residuated lattice. If F is moreover an involutive $\mathbf{A}$-frame, then the map $a \mapsto \rho(a)^{\triangleleft} = \lambda(a)^{\triangleright\triangleleft}$ is a homomorphism from $\mathbf{A}$ to F^+. It is an embedding if F is faithful.*

Proof. In the following $X, Y, Z \subseteq L$ will be Galois closed sets. We first prove that multiplication in F^+ is associative. The proof that addition is associative is analogous. By Lemma 3.14 we have

$$\begin{aligned}(X \cdot Y) \cdot Z &= ((X \circ Y)^{\triangleright\triangleleft} \circ Z)^{\triangleright\triangleleft} \\ &= ((X \circ Y) \circ Z)^{\triangleright\triangleleft\triangleright\triangleleft} \\ &= (X \circ (Y \circ Z))^{\triangleright\triangleleft} \\ &= (X \circ (Y \circ Z)^{\triangleright\triangleleft})^{\triangleright\triangleleft} \\ &= X \cdot (Y \cdot Z).\end{aligned}$$

Next, we prove that 1 is a unit element with respect to multiplication. The proof that 0 is a unit element with respect to addition is analogous. Again by Lemma 3.14

$$\begin{aligned}X \cdot 1 &= (X \circ \{1_\circ\}^{\triangleright\triangleleft})^{\triangleright\triangleleft} = (X \circ 1_\circ)^{\triangleright\triangleleft} = X^{\triangleright\triangleleft} = X, \\ 1 \cdot X &= (\{1_\circ\}^{\triangleright\triangleleft} \circ X)^{\triangleright\triangleleft} = (1_\circ \circ X)^{\triangleright\triangleleft} = X^{\triangleright\triangleleft} = X.\end{aligned}$$

To prove the inclusion $X^{\ell} \cdot X \subseteq 0$, observe that

$$\begin{aligned}X^{\ell} \cdot X \subseteq 0 &\iff ((\ell_\circ[X])^{\triangleleft} \circ X)^{\triangleright\triangleleft} \subseteq \{0_\oplus\}^{\triangleleft} \\ &\iff (\ell_\circ[X])^{\triangleleft} \circ X \subseteq \{0_\oplus\}^{\triangleleft} \\ &\iff (\ell_\circ[X])^{\triangleleft} \circ X \sqsubseteq 0_\oplus \\ &\iff (x_1 \sqsubseteq \ell_\circ[X] \text{ and } x_2 \in X) \text{ implies } x_1 \circ x_2 \sqsubseteq 0_\oplus.\end{aligned}$$

But the last implication holds by nuclearity, since

$$x_1 \sqsubseteq \ell_\circ[X] \iff x_1 \sqsubseteq 0_\oplus \oplus \ell_\circ[X] \iff x_1 \circ X \sqsubseteq 0_\oplus.$$

To prove the inclusion $1 \subseteq X + X^\ell$, by Lemma 3.14 we have

$$\begin{aligned} 1 \subseteq X + X^\ell &\iff \{1_\circ\}^{\rhd\lhd} \subseteq (X^\rhd \oplus (\ell_\circ[X])^{\lhd\rhd})^\lhd \\ &\iff \{1_\circ\} \subseteq (X^\rhd \oplus (\ell_\circ[X])^{\lhd\rhd})^\lhd \\ &\iff \{1_\circ\} \subseteq (X^\rhd \oplus \ell_\circ[X])^\lhd \\ &\iff 1_\circ \sqsubseteq X^\rhd \oplus \ell_\circ[X] \\ &\iff 1_\circ \circ X \sqsubseteq X^\rhd \\ &\iff X \sqsubseteq X^\rhd. \end{aligned}$$

The proofs of the inclusions $X \cdot X^r \subseteq 0$ and $1 \subseteq X^r + X$ are entirely analogous.

It remains to prove hemidistributivity. We have

$$\begin{aligned} X \cdot (Y + Z) \subseteq (X \cdot Y) + Z &\iff X \circ (Y^\rhd \oplus Z^\rhd)^\lhd \subseteq ((X \circ Y)^\rhd \oplus Z^\rhd)^\lhd \\ &\iff X \circ (Y^\rhd \oplus Z^\rhd)^\lhd \sqsubseteq (X \circ Y)^\rhd \oplus Z^\rhd. \end{aligned}$$

Since $(X \circ Y)^\rhd = \bigcap_{x \in X} (x \circ Y)^\rhd$ and $(Y^\rhd \oplus Z^\rhd)^\lhd = \bigcap_{z \in Z^\rhd} (Y^\rhd \oplus z)^\lhd$, it suffices to prove that $x \circ (Y^\rhd \oplus z)^\lhd \sqsubseteq (x \circ Y)^\rhd \oplus z$ for all $x \in L$ and $z \in R$. But this follows if we take $X := Y$, $y := z$, and $u := x$ in the first condition of Lemma 3.15.

The above proves that the Galois algebra F^+ is a complemented ℓ-bimonoid. It is complete because the join of each family $X_i \in F^+$ for $i \in I$ is $\left(\bigcup_{i \in I} X_i\right)^{\rhd\lhd}$. Moreover, if the monoids **L** and **R** are commutative, then so is F^+. Now suppose that F is an **A**-frame. Then $\rho(a)^\lhd = \lambda(a)^{\rhd\lhd}$ by Identity and Cut. To prove that the map $a \mapsto \rho(a)^\lhd$ is a homomorphism from **A** to F^+, by Lemma 3.18 it suffices to show that in an involutive **A**-frame the only Galois closed set X such that $\lambda(a) \in X \sqsubseteq \rho(a)$ is $\rho(a)^\lhd$.

Suppose that $\lambda(a) \in X \sqsubseteq \rho(a)$ for $X \subseteq L$ Galois closed. Then $X \subseteq \rho(a)^\lhd$. Proving that $\rho(a)^\lhd \subseteq X$ amounts to proving that $x \sqsubseteq \rho(a)$ implies $x \in X = X^{\rhd\lhd}$, i.e. to proving the implication $X \sqsubseteq y \implies x \sqsubseteq y$. But $X \sqsubseteq y$ implies $\lambda(a) \sqsubseteq y$ and applying Cut to $x \sqsubseteq \rho(a)$ and $\lambda(a) \sqsubseteq y$ yields that $x \sqsubseteq y$.

Finally, let F be a faithful involutive **A**-frame. Then $\rho(a)^\lhd \subseteq \rho(b)^\lhd$ implies that $\lambda(a) \in \rho(b)^\lhd$, i.e. $\lambda(a) \sqsubseteq \rho(b)$, hence $a \leq b$ by faithfulness. □

Observe that in each involutive frame F we may define the following equivalence relations on **L** and **R**:

$$\langle x, x' \rangle \in \theta_L \text{ if and only if } (x \sqsubseteq y \iff x' \sqsubseteq y) \text{ for each } y \in R,$$
$$\langle y, y' \rangle \in \theta_R \text{ if and only if } (x \sqsubseteq y \iff x \sqsubseteq y') \text{ for each } x \in L.$$

These are in fact congruences on F. More precisely, if $\langle x_1, x_1' \rangle \in \theta_L$, $\langle x_2, x_2' \rangle \in \theta_L$, and $\langle y, y' \rangle \in \theta_R$, then $\langle x_1 \circ x_2, x_1' \circ x_2' \rangle \in \theta_L$ and $\langle \ell_\circ(x_1), \ell_\circ(x_1') \rangle, \langle r_\circ(x_1), r_\circ(x_1') \rangle \in \theta_R$ (and likewise with L and R exchanged). Moreover, $x \sqsubseteq y$ if and only if $x' \sqsubseteq y'$. It follows that the Galois algebra of the involutive frame F is isomorphic to the Galois algebra of the involutive frame F/θ consisting of $\mathbf{L}/\theta_L$ and $\mathbf{R}/\theta_R$ connected by $\sqsubseteq$ and $r_\circ, \ell_\circ, r_\oplus, \ell_\oplus$ via the map $X \mapsto \{[x]_{\theta_L} \mid x \in X\}$.

3.3. Complemented MacNeille completions: existence

We now use the tools introduced in the previous subsection to construct the commutative complemented DM completion of an arbitrary commutative bimonoid $\mathbf{A}$ as the Galois algebra of an involutive $\mathbf{A}$-frame.

We define the structure $F_{\mathbf{A}}$, which we claim to be a faithful commutative involutive $\mathbf{A}$-frame, as follows. The monoids $\mathbf{L}$ and $\mathbf{R}$ both have the same universe A^2, but elements of $\mathbf{L}$ are denoted $\langle a, b\rangle_\bullet$, while elements of $\mathbf{R}$ are denoted $\langle a, b\rangle_\oplus$. These are intended to correspond to $a \cdot \overline{b}$ and $a + \overline{b}$. For such pairs we define the monoidal operations

$$\langle a, b\rangle_\bullet \circ \langle c, d\rangle_\bullet := \langle a \cdot c, b + d\rangle_\bullet,$$
$$\langle a, b\rangle_\oplus \oplus \langle c, d\rangle_\oplus := \langle a + c, b \cdot d\rangle_\oplus,$$

with units

$$1_\circ := \langle 1, 0\rangle_\bullet,$$
$$0_\oplus := \langle 0, 1\rangle_\oplus,$$

and the maps

$$\ell_\circ(\langle a, b\rangle_\bullet) = \langle b, a\rangle_\oplus = r_\circ(\langle a, b\rangle_\bullet),$$
$$\ell_\oplus(\langle a, b\rangle_\oplus) = \langle b, a\rangle_\bullet = r_\oplus(\langle a, b\rangle_\oplus).$$

The relation connecting the two monoids is defined as

$$\langle a, b\rangle_\bullet \sqsubseteq \langle c, d\rangle_\oplus \iff a \cdot d \leq_{\mathbf{A}} b + c.$$

If we interpret $\langle a, b\rangle_\bullet$ and $\langle c, d\rangle_\oplus$ in the intended way, this equivalence is precisely what residuation yields. Finally, $\mathbf{A}$ embeds into $\mathbf{L}$ and $\mathbf{R}$ via the maps

$$\lambda(a) := \langle a, 0\rangle_\bullet,$$
$$\rho(a) := \langle a, 1\rangle_\oplus.$$

We now verify that this yields a faithful commutative involutive $\mathbf{A}$-frame.

Lemma 3.20 *(Involutive frames from bimonoids). $F_{\mathbf{A}}$ is a faithful commutative involutive* $\mathbf{A}$*-frame.*

Proof. Checking that $F_{\mathbf{A}}$ is a commutative monoidal pair is straightforward. Nuclearity states that $\langle a,b\rangle_\bullet \circ \langle c,d\rangle_\bullet = \langle a\cdot c, b+d\rangle_\bullet \sqsubseteq \langle e,f\rangle_\oplus$ is equivalent to $\langle a,b\rangle_\bullet \sqsubseteq \langle d+e, c\cdot f\rangle_\oplus = \langle d,c\rangle_\oplus \oplus \langle e,f\rangle_\oplus$. But this holds, as both of these conditions amount to $a\cdot c\cdot f \leq b+d+e$. Moreover, $\lambda(a) = \langle a,0\rangle_\bullet \sqsubseteq \langle b,1\rangle_\oplus = \rho(b)$ implies that $a = a\cdot 1 \leq b+0 = b$, therefore $F_{\mathbf{A}}$ is faithful. It remains to verify that the Gentzen conditions of Fig. 2 hold in $F_{\mathbf{A}}$. We only deal with the left rules, since the right rules follow the same pattern as the left rules for the dual connective.

The left rules for 0 and 1 clearly hold. To prove the left rule for multiplication, observe that $\lambda(a)\circ\lambda(b) = \langle a,0\rangle_\bullet \circ \langle b,0\rangle_\bullet = \langle a\cdot b,0\rangle_\bullet = \lambda(a\cdot b)$. To prove the left rule for join, observe that $\lambda(a) \sqsubseteq \langle c,d\rangle_\oplus$ and $\lambda(b) \sqsubseteq \langle c,d\rangle_\oplus$ imply that $a\cdot d \leq c$ and $b\cdot d \leq c$. By the distributivity of products over joins we have $(a\vee b)\cdot d \leq c$ and $\lambda(a\vee b) \sqsubseteq \langle c,d\rangle_\oplus$. The left rule for meet follows from the monotonicity of multiplication, i.e. $a\cdot d\leq c$ implies $(a\wedge b)\cdot d \leq c$, as does $b\cdot d\leq c$. Finally, we prove the left rule for addition. If $\lambda(a) \sqsubseteq \langle c,d\rangle_\oplus$, then $a\cdot d \leq c$. If $\lambda(b) \sqsubseteq \langle e,f\rangle_\oplus$, then $b\cdot f \leq e$. It follows by hemidistributivity that $(a+b)df \leq ad+bf \leq c+e$, i.e. $\lambda(a+b) = \langle a+b,0\rangle_\bullet \sqsubseteq \langle c+e, d\cdot f\rangle_\oplus = \langle c,d\rangle_\oplus \oplus \langle e,f\rangle_\oplus$. $\square$

Theorem 3.21 *(Complemented commutative DM completions exist). Each commutative (ℓ-)bimonoid* $\mathbf{A}$ *has a commutative complemented DM completion, namely the Galois algebra $F_{\mathbf{A}}^+$ of the involutive* $\mathbf{A}$*-frame $F_{\mathbf{A}}$.*

Proof. The Galois algebra of $F_{\mathbf{A}}$ is by Theorem 3.19 and Lemma 3.20 a complete involutive commutative residuated lattice into which $\mathbf{A}$ embeds via the map $a \mapsto \rho(a)^{\lhd} = \lambda(a)^{\rhd\lhd}$. It remains to prove that elements of the form

$$\lambda(a)^{\rhd\lhd}\cdot\overline{\lambda(b)^{\rhd\lhd}} = \lambda(a)^{\rhd\lhd}\cdot \ell_\circ[\lambda(b)^{\rhd\lhd}]^{\lhd}$$

are join dense in $F_{\mathbf{A}}$. Because the elements $\langle a,b\rangle_\bullet^{\rhd\lhd}$ are join dense in $F_{\mathbf{A}}$ and

$$\langle a,b\rangle_\bullet^{\rhd\lhd} = (\langle a,0\rangle_\bullet \circ \langle 1,b\rangle_\bullet)^{\rhd\lhd} = (\lambda(a)\circ \ell_\oplus(\rho(b)))^{\rhd\lhd} = \lambda(a)^{\rhd\lhd}\cdot \ell_\oplus(\rho(b))^{\rhd\lhd},$$

it suffices to prove that $\ell_\circ[\lambda(b)^{\rhd\lhd}] = \ell_\oplus(\rho(b))^{\rhd}$. But this holds because

$$\begin{aligned} y \in \ell_\oplus(\rho(b))^{\rhd} &\iff \ell_\oplus(\rho(b)) \sqsubseteq y \\ &\iff 1_\circ \sqsubseteq y \oplus \rho(b) \\ &\iff \ell_\oplus(y) \sqsubseteq \rho(b) \\ &\iff \ell_\oplus(y) \in \rho(b)^{\lhd} \\ &\iff \ell_\oplus(y) \in \lambda(b)^{\rhd\lhd} \\ &\iff y \in \ell_\circ[\lambda(b)^{\rhd\lhd}], \end{aligned}$$

using the fact that $\ell_\oplus(y) \in X \iff y \in \ell_\circ[X]$, by the definitions of $\ell_\oplus$ and $\ell_\circ$. □

The existence of the commutative complemented DM completion immediately yields a bimonoidal version of Funayama's theorem for distributive lattices [12]. This theorem, in its stronger form [6], states that a distributive lattice has an embedding into a complete Boolean algebra which preserves all existing joins if and only if the distributive lattice satisfies the join-infinite distributive law for all existing joins.[6]

Theorem 3.22 *(Funayama's theorem for bimonoids). Let $\iota\colon \mathbf{A} \hookrightarrow \mathbf{A}^\triangle$ be a commutative complemented DM completion of a commutative bimonoid $\mathbf{A}$. Then the following are equivalent:*

- *(i) each join in $\mathbf{A}$ is admissible,*
- *(ii) the embedding ι preserves all existing joins,*
- *(iii) some embedding of $\mathbf{A}$ into a commutative complete complemented bimonoid preserves all existing joins.*

Proof. If an embedding of $\mathbf{A}$ into a commutative complete complemented bimonoid $\mathbf{B}$ preserves the join $\bigvee X$, then $\bigvee X$ is admissible in $\mathbf{B}$ (since each join is admissible in an involutive residuated lattice), and therefore also in $\mathbf{A}$. Conversely, ι preserves each admissible join which exists in $\mathbf{A}$ by Fact 3.4, therefore if each join in $\mathbf{A}$ is admissible, then each join in $\mathbf{A}$ is preserved by ι. □

Let us now illustrate how one can find the complemented DM completion of a small bimonoid. Consider the multiplicative reduct of the three-element MV-chain $\mathbf{1} > \mathsf{a} > \mathsf{b}$ viewed as a bimonoid. That is, $\mathsf{a}\cdot\mathsf{a} = \mathsf{b}$, $x\cdot\mathsf{b} = \mathsf{b} = \mathsf{b}\cdot x$, $x\cdot\mathbf{1} = x = \mathbf{1}\cdot x$, and $x+y := x\cdot y$. Let us call this bimonoid $\mathbf{Ł_3}$. Although multiplication and addition coincide in $\mathbf{Ł_3}$, they come apart in $\mathbf{Ł_3^\triangle}$. In other words, even though the construction of expanding a pomonoid by $x + y := x \cdot y$ and $0 := 1$ is trivial on its own, it provides a way of constructing non-trivial bimonoids when combined with the complemented DM completion.

To describe the algebra $\mathbf{Ł_3^\triangle}$, we may first observe that $\mathsf{b} = \mathsf{b}\overline{\mathbf{1}} = \mathsf{b}\overline{\mathsf{a}} = \mathsf{b}\overline{\mathsf{b}}$. Of course, $\mathsf{a} = \mathsf{a}\overline{\mathbf{1}}$ and $\mathbf{1} = \mathbf{1}\overline{\mathbf{1}}$, since $\mathbf{1}$ (not b!) is the additive unit of $\mathbf{Ł_3}$. Each element of $\mathbf{Ł_3^\triangle}$ is now a join of a subset of $L := \{\mathsf{b}, \mathsf{a}, \mathbf{1}, \mathsf{a}\overline{\mathsf{a}}, \mathsf{a}\overline{\mathsf{b}}, \overline{\mathsf{a}}, \overline{\mathsf{b}}\}$ as well as a meet of a subset of $R := \{\mathsf{b}, \mathsf{a}, \mathbf{1}, \mathsf{b} + \overline{\mathsf{a}}, \mathsf{a} + \overline{\mathsf{a}}, \overline{\mathsf{a}}, \overline{\mathsf{b}}\}$. To find which joins of subsets of L are distinct, we list subsets of L of the form $\{x \in L \mid x \leq y\}$ for $y \in R$:

$$\{\mathsf{b}\}, \{\mathsf{b}, \mathsf{a}\}, \{\mathsf{b}, \mathsf{a}, \mathsf{a}\overline{\mathsf{a}}, \mathbf{1}\}, \{\mathsf{b}, \mathsf{a}, \mathsf{a}\overline{\mathsf{a}}, \mathsf{a}\overline{\mathsf{b}}\}, \{\mathsf{b}, \mathsf{a}, \mathsf{a}\overline{\mathsf{a}}, \mathbf{1}, \mathsf{a}\overline{\mathsf{b}}\}, \{\mathsf{b}, \mathsf{a}, \mathsf{a}\overline{\mathsf{a}}, \mathbf{1}, \mathsf{a}\overline{\mathsf{b}}, \overline{\mathsf{a}}\}, \{\mathsf{b}, \mathsf{a}, \mathsf{a}\overline{\mathsf{a}}, \mathbf{1}, \mathsf{a}\overline{\mathsf{b}}, \overline{\mathsf{a}}, \overline{\mathsf{b}}\}$$

Taking intersection of these yields one more set: $\{\mathsf{b}, \mathsf{a}, \mathsf{a}\overline{\mathsf{a}}\}$. These 8 sets correspond to the distinct joins which exist in $\mathbf{Ł_3^\triangle}$. It follows that the algebra $\mathbf{Ł_3^\triangle}$ has precisely the structure

[6] We thank Guram Bezhanishvili for bringing this theorem to our attention.

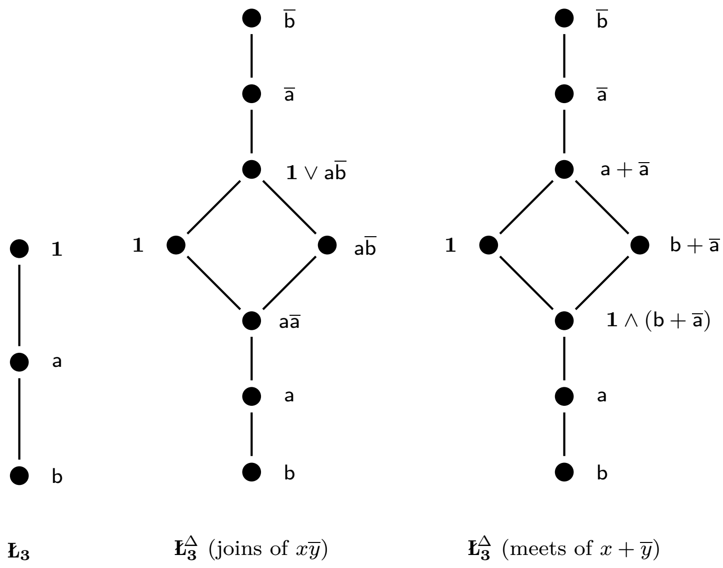

Fig. 3. The algebras $\mathbf{Ł_3}$ and $\mathbf{Ł_3^\triangle}$.

shown in the middle part of Fig. 3. In particular, the complemented DM completion of a linear bimonoid need not be linear. The calculation is also facilitated by the fact that the elements of L are ordered by $\mathsf{b} < \mathsf{a} < \mathsf{a}\overline{\mathsf{a}} < \mathbf{1}, \mathsf{a}\overline{\mathsf{b}} < \overline{\mathsf{a}} < \overline{\mathsf{b}}$ and that Galois closed sets have to be downsets of L.

This poset has two order-inverting involutions, depending on the behavior of $\mathbf{1}$ and $\mathsf{a}\overline{\mathsf{b}}$. However, we know that $\overline{\mathbf{1}} = \mathbf{1}$ (in each bimonoid $\overline{1} = 0$), hence the map $x \mapsto \overline{x}$ on $\mathbf{Ł_3^\triangle}$ is the unique involution with two fixpoints. This allows us to describe $\mathbf{Ł_3}$ in terms of its meet generators, as shown in the right part of Fig. 3. Multiplication and addition is now fully determined by the formulas $a\overline{b} \cdot c\overline{d} = (a \cdot c)\overline{(b + d)}$ and $a + \overline{b} + c + \overline{d} = (a + c) + \overline{b \cdot d}$ and the fact that multiplication distributes over joins and addition over meets. For example, $\mathsf{a}\overline{\mathsf{b}} \cdot \mathsf{a}\overline{\mathsf{b}} = \mathsf{a}\mathsf{a} \cdot \overline{\mathsf{b} + \mathsf{b}} = \mathsf{b}\overline{\mathsf{b}} = \mathsf{b}$ and $\overline{\mathsf{a}} \cdot (\mathbf{1} \vee \mathsf{a}\overline{\mathsf{b}}) = \mathbf{1}\overline{\mathsf{a}} \vee \mathsf{a}\overline{\mathsf{a} + \mathsf{b}} = \overline{\mathsf{a}} \vee \mathsf{a}\overline{\mathsf{b}} = \overline{\mathsf{a}}$. Note that there is no need to verify that this algebra is indeed a commutative complemented ℓ-bimonoid: we merely transformed the abstract definition of $\mathbf{Ł_3^\triangle}$ as the commutative complemented DM completion of $\mathbf{Ł_3}$ (we already know that $\mathbf{Ł_3^\triangle}$ exists) into a more tangible form.

3.4. Complemented MacNeille completions: bisemigroups

The construction of the commutative complemented DM completion of a commutative bimonoid may be extended to bisemigroups. This does not involve any substantial conceptual difficulty: we merely admit join generators of the forms 1, a, $\overline{b}$ in addition to $a\overline{b}$, as well as meet generators of the forms 0, c, $\overline{d}$ in addition to $c + \overline{d}$. Given data which specifies under what conditions $a \leq 0$, $1 \leq b$, and $1 \leq 0$, the appropriate analogue of the involutive frame $F_{\mathbf{A}}$ is defined in the "obvious" way. The proof that this construction works is a routine modification of the proof for bimonoids, however, it involves a lot of tedious case analysis. We therefore merely sketch some of its parts.

Let $\mathbf{A}$ be a commutative bisemigroup (an ℓ-bisemigroup) in this subsection. Let F be an upset of $\mathbf{A}$ (a lattice filter of $\mathbf{A}$), I be a downset of $\mathbf{A}$ (a lattice ideal of $\mathbf{A}$), and let $\alpha \in \{+, -\}$. (We admit the empty set as a lattice filter and a lattice ideal here.) We impose the following compatibility conditions on F, I, and α:

- if $f \in F$, then $a \leq a \cdot f$ for each $a \in \mathbf{A}$,
- if $i \in I$, then $a + i \leq a$ for each $a \in \mathbf{A}$,
- if $F \cap I$ is non-empty, then $\alpha = +$.

Such F, I, and α always exist: we may always take $F = I = \emptyset$. Relative to this data, the *unital* commutative complemented DM completion of $\mathbf{A}$ is the unique complete commutative complemented bimonoid $\mathbf{A}^{\Delta}$ where

- elements of the forms 1, a, $\overline{b}$, $a\overline{b}$ for $a, b \in \mathbf{A}$ are join dense,
- elements of the forms 0, a, $\overline{b}$, $a + \overline{b}$ for $a, b \in \mathbf{A}$ are meet dense,
- $1 \leq a$ for $a \in \mathbf{A}$ if and only if $a \in F$,
- $a \leq 0$ for $a \in \mathbf{A}$ if and only if $a \in I$, and
- $1 \leq 0$ if and only if $\alpha = +$.

The proof of uniqueness (indeed, universality) of $\mathbf{A}^{\Delta}$ carries over almost verbatim from the bimonoidal case, with some tedious case analysis thrown in. This completion again preserves all admissible meets and joins.

The complemented DM completion $\mathbf{A}^{\Delta}$ is again obtained as the Galois algebra of a certain involutive $\mathbf{A}$-frame $F_{\mathbf{A}}$. Of course, if $\mathbf{A}$ is a bisemigroup, we have to disregard the Gentzen conditions for 1 and 0 in the definition of an involutive $\mathbf{A}$-frame, since these are not part of the signature of $\mathbf{A}$.[7]

The definition of the frame $F_{\mathbf{A}}$ needs to be modified as follows. The set of join generators L will consist of elements of four types, representing respectively 1, a, $\overline{b}$, and $a\overline{b}$ for $a, b \in \mathbf{A}$. For the sake of simplicity, we shall simply write these as 1, a, $\overline{b}$, and $a\overline{b}$, with the understanding that $a\overline{b}$ is to be interpreted as a formal pair consisting of a and b, $\overline{b}$ as a formal pair consisting of b and the sign $-$, and a as a formal pair consisting of a and the sign $+$. The set of meet generators R will consist of elements of the form 0, a, $\overline{b}$, or $a + \overline{b}$.

We equip L and R with a monoidal structure in the obvious way: for example, $a \circ 1 = a$, $a \circ c = ac$, $a \circ \overline{d} = a\overline{d}$, and $a \circ c\overline{d} = ac\overline{d}$ etc. The maps $\ell_{\circ} = r_{\circ} \colon L \to R$ and $\ell_{\oplus} = r_{\oplus} \colon R \to L$ are defined as expected:

[7] The intermediate case where $\mathbf{A}$ has a multiplicative unit but not an additive unit or vice versa can be handled similarly.

$$\ell_\circ(1) = 0, \qquad \ell_\circ(a) = \overline{a}, \qquad \ell_\circ(\overline{b}) = b, \qquad \ell_\circ(a\overline{b}) = b + \overline{a},$$
$$\ell_\oplus(0) = 1, \qquad \ell_\oplus(c) = \overline{c}, \qquad \ell_\oplus(\overline{d}) = d, \qquad \ell_\oplus(c + \overline{d}) = d\overline{c}.$$

The relation $\sqsubseteq$ between L and R is then defined as follows:

$$\begin{aligned}
1 \sqsubseteq 0 &\iff \alpha = +, & a \sqsubseteq 0 &\iff a \in I,\\
\overline{b} \sqsubseteq 0 &\iff b \in F, & a\overline{b} \sqsubseteq 0 &\iff a \leq b,\\
1 \sqsubseteq c &\iff c \in F, & a \sqsubseteq c &\iff a \leq c,\\
\overline{b} \sqsubseteq c &\iff b + c \in F, & a\overline{b} \sqsubseteq c &\iff a \leq b + c,\\
1 \sqsubseteq \overline{d} &\iff d \in I, & a \sqsubseteq \overline{d} &\iff ad \in I,\\
\overline{b} \sqsubseteq \overline{d} &\iff d \leq b, & a\overline{b} \sqsubseteq \overline{d} &\iff ad \leq b,\\
1 \sqsubseteq c + \overline{d} &\iff d \leq c, & a \sqsubseteq c + \overline{d} &\iff ad \leq c,\\
\overline{b} \sqsubseteq c + \overline{d} &\iff d \leq b + c, & a\overline{b} \sqsubseteq c + \overline{d} &\iff ad \leq b + c.
\end{aligned}$$

The above definitions yield a commutative involutive frame $F_{\mathbf{A}}$. To turn it into an involutive $\mathbf{A}$-frame, we equip it with the maps $\lambda\colon A \to L$ and $\rho\colon A \to R$ such that $\lambda(a) = a$ and $\rho(a) = a$.

Lemma 3.23 *(Involutive frames from bisemigroups). $F_{\mathbf{A}}$ is a faithful commutative involutive $\mathbf{A}$-frame.*

Proof. The algebras $\langle L, \circ, 1\rangle$ and $\langle R, \oplus, 0\rangle$ are clearly commutative monoids, and $\lambda(a) \sqsubseteq \rho(b)$ if and only if $a \leq b$ by definition, in particular $\lambda(a) \sqsubseteq \rho(a)$. It remains to check nuclearity, Cut, and the Gentzen conditions (excluding the conditions for the two units). This is a tedious but completely routine case analysis. We only rehearse the proof that Cut holds in $F_{\mathbf{A}}$.

Suppose that $x \sqsubseteq g$ and $g \sqsubseteq y$ for some $g \in \mathbf{A}$. (i) Suppose that $x = 1$. If $y = 0$, then $g \in F \cap I$, so $\alpha = +$ and $x \sqsubseteq y$. If $y = c \in \mathbf{A}$, then $g \in F$ and $g \leq c$, so $c \in F$ and $x \sqsubseteq y$. If $y = \overline{d}$, then $a \in F$ and $ad \in I$, so $d \in I$ and $x \sqsubseteq y$. If $y = c + \overline{d}$, then $g \in F$ and $dg \leq c$, so $d \leq c$ and $1 \sqsubseteq x + y$.

(ii) Suppose that $x = a$ for some $a \in \mathbf{A}$. If $y = 0$, then $a \leq g \in I$, so $a \in I$ and $x \sqsubseteq y$. If $y = c$, then $a \leq g \leq c$, so $x \sqsubseteq y$. If $y = \overline{d}$, then $a \leq g$ and $gd \in I$, so $ad \in I$ and $x \sqsubseteq y$. If $y = c + \overline{d}$, then $a \leq g$ and $gd \leq c$, so $ad \leq c$ and $x \sqsubseteq y$.

(iii) Suppose that $x = \overline{b}$ for some $b \in \mathbf{A}$. If $y = 0$, then $b + g \in F$ and $g \in I$, so $b \in F$ and $x \sqsubseteq y$. If $y = c$, then $b + g \in F$ and $g \leq c$, so $b + c \in F$ and $x \sqsubseteq y$. If $y = \overline{d}$, then $g + b \in F$ and $d \cdot g \in I$, so $d \leq d \cdot (g + b) \leq (d \cdot g) + b \leq b$ and $x \sqsubseteq y$. If $y = c + \overline{d}$, then $g + b \in F$ and $dg \leq c$, so $d \leq d \cdot (g + b) \leq (d \cdot g) + b \leq c + b$ and $x \sqsubseteq y$.

(iv) Suppose that $x = a\overline{b}$ for some $a, b \in \mathbf{A}$. If $y = 0$, then $a \leq b + g$ and $g \in I$, so $a \leq b$ and $x \sqsubseteq y$. If $y = c$, then $a \leq b + g$ and $g \leq c$, so $a \leq b + c$ and $x \sqsubseteq y$. If $y = \overline{d}$, then $a \leq b + g$ and $d \cdot g \in I$, so $a \cdot d \leq d \cdot (g + b) \leq (d \cdot g) + b \leq b$ and $x \sqsubseteq y$. If $y = c + \overline{d}$, then $a \leq b + g$ and $d \cdot g \leq c$, so $a \cdot d \leq (b + g) \cdot d \leq b + (d \cdot g) \leq b + c$. □

Theorem 3.24 *(Unital complemented commutative DM completions exist). Each commutative (ℓ-)bisemigroup* $\mathbf{A}$ *has a unital complemented commutative DM completion, relative to a choice of* F, I, α, *namely the Galois algebra* $F^{+}_{\mathbf{A}}$ *of the involutive* $\mathbf{A}$*-frame* $F_{\mathbf{A}}$.

Corollary 3.25 *(Embedding commutative bisemigroups into bimonoids). Each commutative (ℓ-)bisemigroup embeds into a commutative (ℓ-)bimonoid.*

4. Bimonoids of fractions

We saw in the previous section that each commutative bimonoid $\mathbf{A}$ has a commutative complemented Dedekind–MacNeille (DM) completion $\mathbf{A}^{\Delta}$, where each element is a join of elements of the form $a\overline{b}$ for $a, b \in \mathbf{A}$. This completion is unique up to isomorphism, it contains each commutative admissible Δ_1-extension of $\mathbf{A}$, and it preserves all admissible meets and joins (Theorems 3.9 and 3.21 and Fact 3.4).

This answers the question of whether commutative bimonoids have complemented extensions. However, some bimonoids enjoy better-behaved complemented extensions than others: each element in the group of fractions of a cancellative commutative monoid has the form $a\overline{b}$, rather than merely being a join of such elements. We call such complemented extensions *complemented bimonoids of fractions*. In this section, we determine which commutative bimonoids have a commutative complemented bimonoid of fractions. We also show how to construct complemented bimonoids of fractions if they exist. Moreover, we prove that this construction sometimes yields a categorical equivalence between a class of residuated commutative bimonoids and a class of commutative complemented bimonoids with an interior operator.

4.1. Definition and existence of bimonoids of fractions

We define a bimonoid of fractions as a special kind of complemented admissible Δ_1-extension.

Definition 4.1 *(Bimonoids of fractions).* A commutative bimonoid $\mathbf{B}$ is called a *commutative bimonoid of fractions* of a commutative bimonoid $\mathbf{A}$ if there is an embedding of bimonoids $\iota\colon \mathbf{A} \hookrightarrow \mathbf{B}$ such that each element of $\mathbf{B}$ has the form $\iota(a) \cdot \overline{\iota(b)}$ as well as $\iota(c) + \overline{\iota(d)}$, for some $a, b, c, d \in \mathbf{A}$.

In the above definition, we do not assume that $\overline{\iota(b)}$ exists for each $b \in \mathbf{A}$. As with other Δ_1-extensions, we generally disregard the embedding ι and treat $\mathbf{A}$ as a sub-bimonoid of $\mathbf{B}$. The basic facts about commutative admissible Δ_1-extensions and commutative complemented DM completions proved in Subsection 3.1 also apply to bimonoids of fractions and complemented bimonoids of fractions, either as a direct corollary of results about Δ_1-extensions, or by an analogous proof.

In particular, the embedding ι preserves all admissible joins and meets. Consequently, if $\mathbf{A}$ and $\mathbf{B}$ are ℓ-bimonoids, then ι is an embedding of ℓ-bimonoids. Moreover, commutative complemented bimonoids of fractions are universal among all commutative bimonoids of fractions, and thus unique up to isomorphism.

Theorem 4.2 *(Universality of complemented bimonoids of fractions). Let $\iota\colon \mathbf{A} \to \mathbf{B}$ be an isomorphism of commutative bimonoids and let $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ be commutative bimonoids of fractions of $\mathbf{A}$ and $\mathbf{B}$. If $\overline{\mathbf{B}}$ is complemented, then there is a unique embedding of bimonoids $\overline{\iota}\colon \overline{\mathbf{A}} \to \overline{\mathbf{B}}$ which extends the isomorphism ι.*

Proof. The proof is entirely analogous to the proof of the universality of complemented DM completions (Theorem 3.9). We define the maps $\overline{\iota}_{\pm}\colon \overline{\mathbf{A}} \to \overline{\mathbf{B}}$ as

$$\overline{\iota}_{+}\colon a\overline{b} \mapsto \iota(a)\overline{\iota(b)}, \qquad \overline{\iota}_{-}\colon a + \overline{b} \mapsto \iota(a) + \overline{\iota(b)}.$$

These maps are well-defined order-embeddings by Lemma 3.7 (and its order dual). The map $\overline{\iota}_{+}$ preserves products (and the multiplicative identity) by definition and the map $\overline{\iota}_{-}$ preserves sums (and the additive identity) by definition. But $\overline{\iota}_{+}(x) = \overline{\iota}_{-}(x)$ for each $x \in \overline{\mathbf{A}}$ by Lemmas 3.6 and 3.8. Let us denote this common value by $\overline{\iota}(x)$. Then $\overline{\iota}$ is an embedding of bimonoids. Conversely, it is clear from its definition that $\overline{\iota}$ is the only embedding of $\overline{\mathbf{A}}$ into $\overline{\mathbf{B}}$ which extends ι. □

Corollary 4.3 *(Uniqueness of complemented bimonoids of fractions). Let $\iota\colon \mathbf{A} \to \mathbf{B}$ be an isomorphism of commutative bimonoids and $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ be commutative complemented bimonoids of fractions of $\mathbf{A}$ and $\mathbf{B}$. Then there is a unique isomorphism $\overline{\iota}\colon \overline{\mathbf{A}} \to \overline{\mathbf{B}}$ which extends ι.*

In particular, if $\mathbf{A}$ has a commutative complemented bimonoid of fractions, then it is unique up to isomorphism and we denote it $\mathbf{A}^{\div}$. Throughout the following, $\mathbf{A}$ denotes a commutative bimonoid, and $\mathbf{A}^{\Delta}$ denotes a commutative complemented DM completion of $\mathbf{A}$ (unique up to isomorphism).

Proposition 4.4 *(Existence of bimonoids of fractions). The following are equivalent for each commutative bimonoid $\mathbf{A}$:*

- *(i) $\mathbf{A}$ has a commutative complemented bimonoid of fractions,*
- *(ii) the elements of the form $a \cdot \overline{b}$ form a complemented sub-bimonoid of $\mathbf{A}^{\Delta}$,*
- *(iii) the elements of the form $a + \overline{b}$ form a complemented sub-bimonoid of $\mathbf{A}^{\Delta}$,*
- *(iv) for each $a, b \in \mathbf{A}$ there are $x, y \in \mathbf{A}$ such that $a \cdot \overline{b} = x + \overline{y}$ in $\mathbf{A}^{\Delta}$,*
- *(v) for each $a, b \in \mathbf{A}$ there are $x, y \in \mathbf{A}$ such that $a + \overline{b} = x \cdot \overline{y}$ in $\mathbf{A}^{\Delta}$.*

Proof. For the implication from (i) to (ii) see the previous proof. The converse implication holds by the definition of a bimonoid of fractions. The equivalence between (ii) and (iii), as well as between (iv) and (v), holds because $\overline{a \cdot \overline{b}} = b + \overline{a}$ and $\overline{a + \overline{b}} = b \cdot \overline{a}$. The last equivalence holds because elements of the form $a + \overline{b}$ are closed under addition and contain the additive unit, while elements of the form $a \cdot \overline{b}$ are closed under multiplication and contain the multiplicative unit. □

The canonical example of a complemented bimonoid of fractions is the group of fractions of a cancellative commutative monoid (the partial order being the equality relation). On the other hand, many bimonoids do not have complemented bimonoids of fractions. If $\mathbf{A}$ is a finite distributive lattice, then $\mathbf{A}^{\Delta}$ is the free Boolean extension of $\mathbf{A}$. In general the elements of the form $a \wedge \overline{b}$ do not form a subalgebra of $\mathbf{A}^{\Delta}$. For example, if $\mathbf{A}$ is the four-element chain $\mathsf{p} < \mathsf{q} < \mathsf{r} < \mathsf{s}$, then $\mathsf{q} \vee \overline{\mathsf{r}}$ does not have the form $a \wedge \overline{b}$ in $\mathbf{A}^{\Delta}$.

In order to use the previous proposition to establish the existence of complemented bimonoids of fractions for certain classes of bimonoids, we need to be able to describe the condition $a \cdot \overline{b} = x + \overline{y}$ directly in terms of $\mathbf{A}$ rather than $\mathbf{A}^{\Delta}$. The following fact shows that this condition can be expressed by a universal sentence in the language of bimonoids. The condition that a commutative bimonoid has a commutative complemented bimonoid of fractions can therefore be expressed by an elementary Π_3-sentence in the language of bimonoids. This may seem like a rather complex description, but note that Heyting algebras are described by a sentence of the same logical complexity in the language of distributive lattices:

$$(\forall ab \in \mathbf{A})\,(\exists x \in \mathbf{A})\,(\forall c \in \mathbf{A})\,(a \wedge c \leq b \implies c \leq x).$$

Fact 4.5. *Let $\mathbf{A}$ be a commutative bimonoid and $a, b, x, y \in \mathbf{A}$. Then $a \cdot \overline{b} = x + \overline{y}$ in $\mathbf{A}^{\Delta}$ if and only if $a \cdot y \leq b + x$ and*

$$(\forall pquv \in \mathbf{A})\,(u \cdot y \leq v + x \ \&\ a \cdot q \leq b + p \implies u \cdot q \leq v + p).$$

If $\mathbf{A}$ is residuated, this is equivalent to $y \leq a \to (b + x)$ and

$$(\forall pq \in \mathbf{A})\,(a \to (b + p)) \cdot (y \to (x + q)) \leq p + q.$$

If some y satisfies these conditions, then in particular $y := a \to (b + x)$ does.

Proof. The inequality $a\overline{b} \leq x + \overline{y}$ in $\mathbf{A}^{\Delta}$ is equivalent to $ay \leq b + x$ and

$$\begin{aligned} x + \overline{y} \leq a\overline{b} &\iff (\forall uv \in \mathbf{A})\,\big(u\overline{v} \leq x + \overline{y} \implies u\overline{v} \leq a\overline{b}\big) \\ &\iff (\forall uv \in \mathbf{A})\,(uy \leq v + x \implies (\forall pq \in \mathbf{A})\,(aq \leq b + p \implies uq \leq v + p)) \\ &\iff (\forall pquv \in \mathbf{A})\,(uy \leq v + x \ \&\ aq \leq b + p \implies uq \leq v + p) \\ &\iff (\forall pquv \in \mathbf{A})\,(uy \leq v + x \ \&\ q \leq a \to (b + p) \implies q \leq u \to (v + p)) \end{aligned}$$

$$\iff (\forall puv \in \mathbf{A})\,(uy \leq v + x \implies a \to (b+p) \leq u \to (v+p))$$
$$\iff (\forall puv \in \mathbf{A})\,(u \leq y \to (v+x) \implies u \leq (a \to (b+p)) \to (v+p))$$
$$\iff (\forall pv \in \mathbf{A})\,(y \to (v+x) \leq (a \to (b+p)) \to (v+p))$$
$$\iff (\forall pv \in \mathbf{A})\,(a \to (b+p)) \cdot (y \to (v+x)) \leq v + p$$
$$\iff (\forall pq \in \mathbf{A})\,(a \to (b+p)) \cdot (y \to (x+q)) \leq p + q,$$

provided that the appropriate residuals exist. □

Let us now use this criterion to prove the existence of commutative complemented bimonoids of fractions for various classes of commutative bimonoids. For unital meet semilattices, where $a + b = a \wedge b = a \cdot b$, the above condition for $a \cdot \overline{b}$ to be equal to $x + \overline{y}$ can be simplified substantially.

Fact 4.6. *Let* $\mathbf{A}$ *be a unital meet semilattice. Then the following are equivalent for* $a, b, y \in \mathbf{A}$*:*

(i) $a \cdot \overline{b} = x + \overline{y}$ *in* $\mathbf{A}^{\Delta}$ *for some* $x \in \mathbf{A}$*,*
(ii) $a \cdot \overline{b} = a + \overline{y}$ *in* $\mathbf{A}^{\Delta}$*,*
(iii) $a \cdot \overline{ab} = a + \overline{y}$ *in* $\mathbf{A}^{\Delta}$*,*
(iv) $ay = ab$ *and moreover*

$$(\forall pq \in \mathbf{A})\,(py \leq ab \;\&\; aq \leq ab \implies pq \leq ab)\,.$$

Proof. We know that $a \cdot \overline{b} = x + \overline{y}$ if and only if $ay \leq bx$ and

$$(\forall pquv \in \mathbf{A})\,(uy \leq vx \;\&\; aq \leq bp \implies uq \leq vp)\,.$$

But then $ay \leq x$, therefore the implication also holds for $x := ay$, as does the inequality $ay \leq bx$. Moreover, the implication holds for $x := ay$ if and only if it holds for $x := a$, and the inequality $ay \leq bx$ also holds for $x := a$. Therefore, if some x satisfies these conditions, then $x := a$ does too. For $x = a$, the two conditions become $ay \leq b$ and

$$(\forall pquv \in \mathbf{A})\,(uy \leq va \;\&\; aq \leq bp \implies uq \leq vp)\,.$$

These are satisfied by b if and only if they are satisfied by $b := ab$. Moreover, the implication is equivalent to the conjunction of

$$(\forall pquv \in \mathbf{A})\,(uy \leq v \;\&\; uy \leq a \;\&\; aq \leq bp \implies uq \leq v)\,,$$
$$(\forall pquv \in \mathbf{A})\,(uy \leq v \;\&\; uy \leq a \;\&\; aq \leq bp \implies uq \leq p)\,.$$

These are equivalent respectively to

$$(\forall quv \in \mathbf{A})\,(uy \leq v \;\&\; uy \leq a \;\&\; aq \leq b \implies uq \leq v)\,,$$
$$(\forall pqu \in \mathbf{A})\,(uy \leq a \;\&\; aq \leq b \;\&\; aq \leq p \implies uq \leq p)\,.$$

The implication $(\forall v \in \mathbf{A})\,(uy \leq v \implies uq \leq v)$ is equivalent to $uq \leq uy$, i.e. to $uq \leq y$, and the implication $(\forall p)\,(aq \leq p \implies uq \leq p)$ is equivalent to $uq \leq aq$, i.e. $uq \leq a$. The two implications are thus equivalent to

$$(\forall qu \in \mathbf{A})\,(uy \leq a \;\&\; aq \leq b \implies uq \leq y)\,,$$
$$(\forall qu \in \mathbf{A})\,(uy \leq a \;\&\; aq \leq b \implies uq \leq a)\,,$$

or in other words, changing the variable u to p,

$$(\forall pq \in \mathbf{A})\,(py \leq a \;\&\; aq \leq b \implies pq \leq ay)\,.$$

Taking $p := a$ and $q := b$ yields that $ab \leq ay$, and we already know that $ay \leq ab$. Moreover, $py \leq a$ is equivalent to $py \leq ab$, since $ay \leq b$. □

This immediately yields the existence of bimonoids of fractions of Brouwerian semilattices.

Definition 4.7 *(Brouwerian semilattices).* A *Brouwerian semilattice* is an integral idempotent residuated pomonoid, or equivalently a unital meet semilattice $\langle A, \wedge, 1\rangle$ equipped with a binary operation $x \to y$ such that

$$x \wedge y \leq z \iff y \leq x \to z.$$

Fact 4.8. *Each Brouwerian semilattice has a commutative complemented bimonoid of fractions where*

$$a \cdot \overline{b} = a + \overline{a \to b}.$$

Proof. Taking $y := a \to b$ in the previous fact yields that indeed $ay = ab$. Moreover, if $py \leq ab$ and $aq \leq ab$, then $p \leq y \to ab$ and $q \leq a \to ab \leq a \to b$. But then $pq \leq ((a \to b) \to ab)(a \to b)) \leq b$. □

By contrast, the semilattices $\mathbf{M_3}$ and $\mathbf{N_5}$ do not have commutative complemented bimonoids of fractions. We do not know of any non-distributive semilattice which has such a bimonoid of fractions.

Recall the discussion of Boolean-pointed Brouwerian algebras from Subsection 2.4: they are Brouwerian algebras equipped with a constant 0 such that the interval $[0, 1]$ is Boolean.

Fact 4.9. *Each Boolean-pointed Brouwerian algebra has a commutative complemented bimonoid of fractions where*

$$a \cdot \overline{b} = 0a + \overline{a \to b}.$$

Proof. By Fact 4.5 it suffices to show that $a(a \to b) \leq b + 0a$ and for all p, q

$$a(a \to (b + p))((a \to b) \to (0a + q)) \leq p + q.$$

Clearly $a(a \to b) = ab = (b+0)a \leq b+0a$. To prove the other inequality, by Lemma 2.23 it suffices to prove that $(a \to bp)((a \to b) \to aq) \leq pq$ holds in all Brouwerian lattices and that

$$(a \to (b + p))((a \to b) \to (0a + q)) \leq (p + q) \vee 0$$

holds in all Boolean-pointed Brouwerian lattices. The former inequality is routine to prove, and it implies that $(a \to bp)((a \to b) \to aq) \leq 0 \to pq$, therefore it only remains to prove that

$$(a \to (b + p))((a \to b) \to (0a + q)) \leq p \vee q \vee 0.$$

Since $p \vee q \vee 0 = ((p \vee q) \to 0) \to 0$, this is equivalent to

$$(a \to (b + p))((a \to b) \to (0a + q))(p \to 0)(q \to 0) \leq 0.$$

But $(a \to (b + p))(p \to 0) \leq a \to (b + p(p \to 0)) \leq a \to (b + 0) = a \to b$. Likewise, $(a \to b)((a \to b) \to (0a + q))(q \to 0) \leq (0a + q)(q \to 0) \leq 0a + 0 = 0a \leq 0$. □

Finally, let us directly verify that our criterion covers groups of fractions.

Definition 4.10 *(Order-cancellative monoids).* A pomonoid is called *order-cancellative* if it satisfies the implications

$$a \cdot x \leq b \cdot x \implies a \leq b,$$
$$x \cdot a \leq x \cdot b \implies a \leq b.$$

Fact 4.11. *A residuated pomonoid is order-cancellative if and only if it satisfies the equations $xy\backslash xz \approx y\backslash z$ and $xz/yz \approx x/y$. An $s\ell$-monoid is order-cancellative if and only if it is* cancellative, *i.e. if and only if $xy \approx xz$ implies $y \approx z$ and moreover $xz \approx yz$ implies $x \approx y$.*

Fact 4.12. *Each order-cancellative commutative monoid has a commutative complemented bimonoid of fractions where*

$$a \cdot \overline{b} = a + \overline{b}.$$

Proof. The condition $a\overline{b} = x + \overline{y}$ is equivalent to $ay \leq bx$ and

$$(\forall pquv \in \mathbf{A})\,(uy \leq vx \;\&\; aq \leq bp \implies uq \leq vp)\,.$$

Take $x := a$ and $y := b$. Then $ay \leq bx$, and if $ub \leq va$ and $aq \leq bp$, then $ubq \leq vaq \leq vbp$, hence by order-cancellativity $uq \leq vp$. □

As our final example of complemented bimonoids of fractions, let us consider the two constructions of Galatos & Raftery [16], which embed certain residuated pomonoids into cyclic involutive residuated pomonoids. Here *cyclicity* means that $x^\ell = x^r$ for each x. This common value will be denoted $\overline{x}$.

The construction comes in two flavors. If $\mathbf{A} = \langle A, \leq, \cdot, 1, \backslash, / \rangle$ is a residuated pomonoid with an upper bound $\top$, then we construct an involutive residuated pomonoid $\mathbf{A}^*$ consisting of two disjoint copies of A. The two copies will be denoted A and A', and the elements of A' will be denoted a' for $a \in \mathbf{A}$. The order of $\mathbf{A}^*$ extends the order of $\mathbf{A}$ as follows for $a, b \in A$:

$$a' \leq b, \qquad a \nleq b', \qquad a' \leq b' \iff b \leq a.$$

Multiplication in $\mathbf{A}^*$ extends multiplication in $\mathbf{A}$ by:

$$a' \cdot b := (a\backslash b)', \qquad a \cdot b' := (b/a)', \qquad a' \cdot b' := \top'.$$

On the other hand, if $\mathbf{A}$ is a residuated pomonoid with a lower bound $\bot$ (in which case it also has an upper bound $\top := \bot\backslash\bot$), then we construct the involutive residuated pomonoid $\mathbf{A}_*$ over $A \cup A'$ in a similar way. The order on $\mathbf{A}_*$ extends the order on $\mathbf{A}$:

$$a' \nleq b, \qquad a \leq b', \qquad a' \leq b' \iff b \leq a.$$

Multiplication in $\mathbf{A}_*$ extends multiplication in $\mathbf{A}$:

$$a' \cdot b := (a\backslash b)', \qquad a \cdot b' := (b/a)', \qquad a' \cdot b' := \bot'.$$

That is, $\mathbf{A}^*$ adds a mirror copy of $\mathbf{A}$ below $\mathbf{A}$, and $\mathbf{A}_*$ adds a mirror copy of $\mathbf{A}$ above $\mathbf{A}$. Both of these constructions yield a bounded cyclic involutive residuated pomonoid where $\overline{a} = a'$ and $\overline{a'} = a$. Also, if $\mathbf{A}$ is lattice-ordered, then so are $\mathbf{A}^*$ and $\mathbf{A}_*$.

We can now interpret these involutive residuated pomonoids as complemented bimonoids of fractions of $\mathbf{A}$, if we suitably expand $\mathbf{A}$ to a bisemigroup. In the first case, we expand $\mathbf{A}$ to a bisemigroup with a multiplicative unit by taking what we called the $\top$-drastic addition in Subsection 2.4: $x + y := \top$. In the second case, we expand $\mathbf{A}$ to a bisemigroup with a multiplicative unit by taking the $\bot$-drastic addition: $x+y := \bot$. (This yields a bisemigroup because $x \cdot \bot = \bot = \bot \cdot x$ in each bounded residuated pomonoid.) But observe that this is precisely the addition of $\mathbf{A}^*$ and $\mathbf{A}_*$ restricted to $\mathbf{A}$, thanks to the definitions $a' \cdot b' := \top'$ and $a' \cdot b' := \bot'$. The bisemigroup $\mathbf{A}$ is therefore a sub-bisemigroup of $\mathbf{A}^*$ and $\mathbf{A}_*$, and moreover this embedding preserves the multiplicative unit (in the $\top$-drastic case, it also preserves residuals).

If we extend our definition of a bimonoid of fractions to allow for cyclic bimonoids of fractions of bisemigroups, then the cyclic complemented bimonoids $\mathbf{A}^*$ and $\mathbf{A}_*$ are right (as well as left) cyclic bimonoids of fractions of $\mathbf{A}$: each element has one of the forms 1, a, $\overline{b}$, or $a\overline{b}$ ($\overline{b}a$), as well as one of the forms 0, a, $\overline{b}$, or $a + \overline{b}$ ($\overline{b} + a$) for some $a, b \in \mathbf{A}$. Indeed, in this case each element simply has the form a or $\overline{b}$.

4.2. Constructing complemented bimonoids of fractions

In the following, let $\mathbf{A}$ be a commutative bimonoid with a commutative complemented bimonoid of fractions. We show how to construct this bimonoid of fractions, first as a quotient of a preordered algebra on A^2, then (in some cases) directly as an ordered algebra on a subset of A^2. The input for this construction consists of a pair of functions α, β which specify how to solve the equation $a \cdot \overline{b} = x + \overline{y}$ for x and y in $\mathbf{A}^{\triangle}$. Recall that the condition that $a \cdot \overline{b} = x + \overline{y}$ can be stated directly in terms of $\mathbf{A}$ as a universal sentence in the language of commutative bimonoids (Fact 4.5). Moreover, if $\mathbf{A}$ is residuated, it can be stated in terms of $\mathbf{A}$ as a set of universally quantified inequalities in the language of commutative residuated bimonoids.

Definition 4.13 *(Transformation functions).* The functions $\alpha, \beta \colon A^2 \to A^2$ are called *transformation functions* for $\mathbf{A}$ if

$$a \cdot \overline{b} = \alpha(a, b) + \overline{\beta(a, b)} \text{ for all } a, b \in \mathbf{A}.$$

We assume below that α and β are transformation functions for $\mathbf{A}$. These are not uniquely determined by $\mathbf{A}$, therefore our constructions of complemented bimonoids of fractions will be relative to some choice of α and β. However, the resulting complemented bimonoids of fractions will of course be isomorphic.

Theorem 4.14 *(Constructing complemented bimonoids of fractions). Let $\mathbf{A}$ be a commutative bimonoid with transformation functions α, β. We define $\mathbf{A}^{\div}_{\mathrm{pre}}$ to be the algebra on A^2 with the operations*

$$\overline{\langle a, b\rangle_{\bullet}} := \langle \beta(a, b), \alpha(a, b)\rangle_{\bullet},$$
$$\langle a, b\rangle_{\bullet} \circ \langle c, d\rangle_{\bullet} := \langle a \cdot c, b + d\rangle_{\bullet},$$
$$\langle a, b\rangle_{\bullet} \oplus \langle c, d\rangle_{\bullet} := \langle \beta(\beta(a, b) \cdot \beta(c, d), \alpha(a, b) + \alpha(c, d)), \alpha(\beta(a, b) \cdot \beta(c, d), \alpha(a, b) + \alpha(c, d))\rangle_{\bullet},$$

and the constants $1^{\div} := \langle 1, 0\rangle_{\bullet}$ and $0^{\div} := \langle 0, 0\rangle_{\bullet}$. Let $\preccurlyeq$ be the preorder

$$\langle a, b\rangle_{\bullet} \preccurlyeq \langle c, d\rangle_{\bullet} \iff (\forall x, y \in \mathbf{A})(x \cdot c \leq y + d \implies x \cdot a \leq y + b).$$

The equivalence relation θ induced by this preorder is a congruence on $\mathbf{A}^{\div}_{\mathrm{pre}}$. Then the ordered algebra $\mathbf{A}^{\div} := \langle \mathbf{A}^{\div}_{\mathrm{pre}}/\theta, \preccurlyeq \rangle$ of equivalence classes of this congruence ordered by $\preccurlyeq$ is a commutative complemented bimonoid of fractions of $\mathbf{A}$ relative to the embedding $\iota^{\div}\colon a \mapsto [\langle a, 0\rangle_{\bullet}]_{\theta}$. Moreover, $\iota^{\div}(a) \circ \overline{\iota^{\div}(b)} = [\langle a, b\rangle_{\bullet}]_{\theta}$.

Proof. Because $\mathbf{A}$ has a pair of transformation functions, the sub-bimonoid $\mathbf{B}$ of $\mathbf{A}^{\Delta}$ consisting of elements of the form $a\overline{b}$ is a commutative complemented bimonoid of fractions of $\mathbf{A}$ (Proposition 4.4). Now consider the surjective map $\varepsilon\colon \langle a, b\rangle_{\bullet} \mapsto a\overline{b} \in \mathbf{B}$. By Lemma 3.7, $\langle a, b\rangle_{\bullet} \preccurlyeq \langle c, d\rangle_{\bullet}$ if and only if $\varepsilon\langle a, b\rangle_{\bullet} \leq \varepsilon\langle c, d\rangle_{\bullet}$. Moreover, ε is a homomorphism of bimonoids: $\varepsilon(1^{\div}) = 1\overline{0} = 1$ and $\varepsilon(0^{\div}) = 0\overline{0} = 0$ for the two units, $\varepsilon(\langle a, b\rangle_{\bullet} \circ \langle c, d\rangle_{\bullet}) = \varepsilon\langle a \cdot c, b + d\rangle_{\bullet} = ac \cdot \overline{b+d} = a\overline{b} \cdot c\overline{d} = \varepsilon\langle a, b\rangle_{\bullet} \cdot \varepsilon\langle c, d\rangle_{\bullet}$ for multiplication, and

$$
\begin{aligned}
\varepsilon\langle a, b\rangle_{\bullet} + \varepsilon\langle c, d\rangle_{\bullet} &= a\overline{b} + c\overline{d} \\
&= \alpha(a, b) + \overline{\beta(a, b)} + \alpha(c, d) + \overline{\beta(c, d)} \\
&= \alpha(a, b) + \alpha(c, d) + \overline{\beta(a, b) \cdot \beta(c, d)} \\
&= \beta(\beta(a, b) \cdot \beta(c, d), \alpha(a, b) + \alpha(c, d)) \\
&\quad \cdot \overline{\alpha(\beta(a, b) \cdot \beta(c, d), \alpha(a, b) + \alpha(c, d))} \\
&= \varepsilon(\langle a, b\rangle_{\bullet} \oplus \langle c, d\rangle_{\bullet}).
\end{aligned}
$$

The kernel of this map is precisely θ, therefore θ is a congruence and the map $[\langle a, b\rangle_{\bullet}]_{\theta} \mapsto a\overline{b}$ is a well-defined isomorphism of bimonoids between $\mathbf{A}^{\div}$ and $\mathbf{B}$. Since $\overline{a\overline{b}} = \overline{\alpha(a, b) + \overline{\beta(a, b)}} = \beta(a, b) \cdot \overline{\alpha(a, b)}$, it follows by this isomorphism that the complement of $[\langle a, b\rangle_{\bullet}]_{\theta}$ in $\mathbf{A}^{\div}$ is $[\langle \beta(a, b), \alpha(a, b)\rangle_{\bullet}]$. We also have $\varepsilon(\langle c, d\rangle_{\bullet}) = a$ if and only if $[\langle c, d\rangle_{\bullet}]_{\theta} = [\langle a, 0\rangle_{\bullet}]_{\theta}$. Since $\mathbf{B}$ is a commutative complemented bimonoid of fractions of $\mathbf{A}$ relative to the inclusion of $\mathbf{A}$ into $\mathbf{B}$ and ε is an isomorphism between $\mathbf{A}^{\div}$ and $\mathbf{B}$, this implies that $\mathbf{A}^{\div}$ is a commutative complemented bimonoid of fractions of $\mathbf{A}$ relative to the map $a \mapsto [\langle a, 0\rangle_{\bullet}]_{\theta}$. Finally, $\varepsilon(\iota^{\div}(a) \circ \overline{\iota^{\div}(b)}) = \varepsilon(\iota^{\div}(a)) \cdot \overline{\varepsilon(\iota^{\div}(b))} = a\overline{b} = \varepsilon(\langle a, b\rangle_{\bullet})$, so $\iota^{\div}(a) \circ \overline{\iota^{\div}(b)} = [\langle a, b\rangle_{\bullet}]_{\theta}$. □

For order-cancellative bimonoids, where we can take $\alpha(a, b) := a$ and $\beta(a, b) = b$, the above construction simplifies to the usual quotient construction of the group of fractions.

There is no reason to expect the above construction to be functorial. Suppose that $\mathbf{A}$ and $\mathbf{B}$ are commutative bimonoids such that $\mathbf{A}^{\div}$ and $\mathbf{B}^{\div}$ exist. Given a homomorphism of bimonoids $h\colon \mathbf{A} \to \mathbf{B}$, it is natural to try to define a map $h^{\div}\colon \mathbf{A}^{\div} \to \mathbf{B}^{\div}$ as $h^{\div}\colon \langle a, b\rangle_{\bullet} \mapsto \langle h(a), h(b)\rangle_{\bullet}$. However, we have no reason to expect this map to be well-defined in general, much less order preserving. Owing to the parameters x and y in the definition of the preorder on $\mathbf{A}^{\div}_{\mathrm{pre}}$, the pairs $\langle a, b\rangle_{\bullet}$ and $\langle c, d\rangle_{\bullet}$ might be equivalent in $\mathbf{A}^{\div}_{\mathrm{pre}}$ without $\langle h(a), h(b)\rangle_{\bullet}$ and $\langle h(c), h(d)\rangle_{\bullet}$ being equivalent in $\mathbf{B}^{\div}_{\mathrm{pre}}$.

The construction becomes functorial if we restrict to order-cancellative pomonoids, because the condition for $\langle a, b\rangle_{\bullet} \leq \langle c, d\rangle_{\bullet}$ in $\mathbf{A}^{\div}_{\mathrm{pre}}$ is then equivalent to $d \cdot a \leq c \cdot b$, i.e. the parameters x and y can be eliminated. However, even in this case the functor fails to be full: there may be homomorphisms $h\colon \mathbf{A}^{\div} \to \mathbf{B}^{\div}$ which do not restrict to maps from $\mathbf{A}$ to $\mathbf{B}$. For example, the non-trivial automorphism of the group of integers $\mathbb{Z} \cong \mathbb{N}^{\div}$ does not arise from any endomorphism of $\mathbb{N}$.

To resolve these issues, we adopt the solution that Montagna & Tsinakis [26] used in the context of groups of fractions of cancellative residuated pomonoids: we extend $\mathbf{A}^{\div}$ by an interior operator which allows us to recover $\mathbf{A}$ as its image. This requires us to move to the setting of *residuated* commutative bimonoids. Let us therefore assume from now on that the commutative bimonoid $\mathbf{A}$ is residuated.

Proposition 4.15 *(The interior operator $\sigma^{\div}$ on bimonoids of fractions). Let $\mathbf{B}$ be a commutative bimonoid of fractions of a residuated commutative bimonoid $\mathbf{A}$. Then the following map is a well-defined interior operator on $\mathbf{B}$ whose image is precisely $\mathbf{A}$:*

$$\sigma^{\div}(a + \overline{b}) := b \to a \text{ for } a, b \in \mathbf{A}.$$

Proof. If $a + \overline{b} \leq c + \overline{d}$, then $b \to (a + x) \leq d \to (c + x)$ for each x, hence $\sigma^{\div}(a + \overline{b}) = b \to a \leq d \to c = \sigma^{\div}(c + \overline{d})$. The map $\sigma^{\div}$ is thus well-defined ($a + \overline{b} = c + \overline{d}$ implies $b \to a = d \to c$) and isotone. It is also decreasing map because $\sigma^{\div}(a + \overline{b}) \leq a + \overline{b} \iff b \to a \leq a + \overline{b} \iff b \cdot (b \to a) \leq a$. It is an idempotent map because $\sigma^{\div}(\sigma^{\div}(a + \overline{b})) = \sigma^{\div}(b \to a) = \sigma^{\div}((b \to a) + \overline{1}) = 1 \to (b \to a) = b \to a = \sigma^{\div}(a + \overline{b})$. Finally, $\sigma^{\div}[B] = A$ since $\sigma^{\div}(a + \overline{b}) = b \to a \in \mathbf{A}$ and $\sigma^{\div}(a) = a$ for $a \in \mathbf{A}$. □

Proposition 4.16 *(The interior operator $\sigma^{\div}$ on Δ_1-extensions). Let $\mathbf{B}$ be a commutative Δ_1-extension of a complete commutative ℓ-bimonoid $\mathbf{A}$. Then the following map is a well-defined interior operator on $\mathbf{B}$ whose image is precisely $\mathbf{A}$:*

$$\sigma^{\div}\left(\bigwedge_{i \in I}(a_i + \overline{b}_i)\right) := \bigwedge_{i \in I}(b_i \to a_i) \text{ for } a, b \in \mathbf{A}.$$

Proof. Suppose that $\bigwedge_{i \in I}(a_i + \overline{b}_i) \leq \bigwedge_{j \in J}(c_j + \overline{d}_j)$ in $\mathbf{B}$ for $a_i, b_i, c_j, d_j \in \mathbf{A}$. To prove that the map $\sigma^{\div}$ is well-defined and isotone, we must show that $\bigwedge_{i \in I}(b_i \to a_i) \leq \bigwedge_{j \in J}(d_j \to c_j)$. But the former inequality is equivalent to the condition that for each $x, y \in \mathbf{A}$ we have: $xb_i \leq a_i + y$ for each $i \in I$ implies $xd_j \leq c_j + y$ for each $j \in J$. Taking $y := 0$ yields the latter inequality. The other conditions are proved as in the previous proposition. □

Of course, instead of requiring that $\mathbf{A}$ be complete, we may alternatively require that each element of $\mathbf{B}$ be a finite meet of elements of the form $a + \overline{b}$ for $a, b \in \mathbf{A}$.

Expanding bimonoids of fractions by the unary operation $\sigma^{\div}$ eliminates all homomorphisms $h\colon \mathbf{A}^{\div} \to \mathbf{B}^{\div}$ which do not restrict to homomorphisms from $\mathbf{A}$ to $\mathbf{B}$: such maps do not commute with $\sigma^{\div}$, since the images $\sigma^{\div}[\mathbf{A}^{\div}]$ and $\sigma^{\div}[\mathbf{B}^{\div}]$ are precisely $\mathbf{A}$ and $\mathbf{B}$.

To obtain a functorial construction, we must now restrict to complemented bimonoids of fractions where each element has a certain canonical or *normal* representation.

Definition 4.17 *(Normal representations).* Let $\mathbf{B}$ be a commutative complemented bimonoid of fractions of a commutative residuated bimonoid $\mathbf{A}$. Then an element $x \in \mathbf{B}$ is *normal* if

$$x = \sigma^{\div}(x) \cdot \overline{\sigma^{\div}(\overline{x})}.$$

A pair $\langle a, b\rangle_\bullet \in \mathbf{A}^2$ is *normal* if

$$\langle a, b\rangle_\bullet = \langle \sigma^{\div}(x), \sigma^{\div}(\overline{x})\rangle_\bullet \text{ for some } x \in \mathbf{B}.$$

Such a pair $\langle a, b\rangle_\bullet$ will be called the *normal representation* of x. If each $x \in \mathbf{B}$ is normal, we call $\mathbf{B}$ itself a *normal* commutative bimonoid of fractions. Transformation functions for $\mathbf{A}$ will be called *normal* transformation functions if $\mathbf{B}$ is normal.

Fact 4.18. *If $\langle a, b\rangle_\bullet$ is a normal pair, then so is $\langle b, a\rangle_\bullet$.*

Proof. If $\langle a, b\rangle_\bullet = \langle \sigma^{\div}(x), \sigma^{\div}(\overline{x})\rangle_\bullet$, then $\langle b, a\rangle_\bullet = \langle \sigma^{\div}(y), \sigma^{\div}(\overline{y})\rangle_\bullet$ for $y := \overline{x}$. □

Comparing normal representations is much easier than comparing general representations.

Lemma 4.19 *(Comparing normal representations). Let $\langle a, b\rangle_\bullet$ and $\langle c, d\rangle_\bullet$ be normal representations of x and y in a commutative complemented bimonoid of fractions $\mathbf{B}$ of $\mathbf{A}$. Then $x \leq y$ in $\mathbf{B}$ if and only if $a \leq c$ and $d \leq b$ in $\mathbf{A}$.*

Proof. If $a \leq c$ and $d \leq b$, then $x = a\overline{b} \leq c\overline{d} = y$ by monotonicity of multiplication. Conversely, if $x \leq y$, then $a = \sigma^{\div}(x) \leq \sigma^{\div}(y) = c$ and $d = \sigma^{\div}(\overline{y}) \leq \sigma^{\div}(\overline{x}) = b$ by the monotonicity of $\sigma^{\div}$. □

A crucial observation is that if $\mathbf{B}$ is normal, then

$$x = \sigma^{\div}(x) \cdot \overline{\sigma^{\div}(\overline{x})} = \sigma^{\div}(x) + \overline{\sigma^{\div}(\overline{x})},$$

since $\overline{x} = \sigma^{\div}(\overline{x}) \cdot \overline{\sigma^{\div}(x)}$ implies $x = \sigma^{\div}(x) + \overline{\sigma^{\div}(\overline{x})}$. That is, the pair $\langle \sigma^{\div}(x), \sigma^{\div}(\overline{x})\rangle$ works *both* as a multiplicative and an additive representation of x. Both multiplying and adding two normal representations can therefore be done naïvely: if $\langle a, b\rangle$ and $\langle c, d\rangle$ are normal representations of x and y, then

$$x \cdot y = (a \cdot c) \cdot \overline{(b + d)}, \qquad \text{and} \qquad x + y = (a + c) + \overline{(b \cdot d)}.$$

That is, $\langle a,b\rangle \cdot \langle c,d\rangle = \langle a\cdot c, b+d\rangle$ is a multiplicative representation of $x\cdot y$ and $\langle a,b\rangle + \langle c,d\rangle = \langle a+c, b\cdot d\rangle$ is an additive representation of $x+y$. The only catch here is that $\langle a\cdot c, b+d\rangle$ and $\langle a+c, b\cdot d\rangle$ need not be normal representations. To obtain normal representations of $x\cdot y$ and $x+y$, we need to further project them onto the set of normal representations.

Fact 4.20. *If $x = a\cdot \overline{b}$ in $\mathbf{A}^{\Delta}$, then $\langle \sigma^{\div}(x), \sigma^{\div}(\overline{x})\rangle = \langle \beta(a,b)\to\alpha(a,b), a\to b\rangle$.*

Proof. This follows immediately from the definitions of the maps α, β, and $\sigma^{\div}$, since $a\cdot\overline{b} = \alpha(a,b) + \overline{\beta(a,b)}$ and $\sigma^{\div}(a+\overline{b}) = b\to a$. □

Let us therefore define the map $\pi\colon \mathbf{A}^2\to\mathbf{A}^2$ as

$$\pi\langle a,b\rangle_{\bullet} := \langle \beta(a,b)\to\alpha(a,b), a\to b\rangle_{\bullet}.$$

In other words, $\pi\langle a,b\rangle_{\bullet}$ is a normal representation of $a\cdot\overline{b}$. Moreover, $\langle a,b\rangle_{\bullet}$ is normal if and only if $\langle a,b\rangle_{\bullet} = \pi\langle a,b\rangle_{\bullet}$. It follows that normal pairs are definable by equations.

Fact 4.21. *The transformation functions α, β for $\mathbf{A}$ are normal if and only if for all $a,b,x\in\mathbf{A}$*

$$a\to(b+x) = (\beta(a,b)\to\alpha(a,b))\to((a\to b)+x).$$

Proof. This is precisely what the condition that $\langle a,b\rangle_{\bullet}$ and $\pi\langle a,b\rangle_{\bullet}$ represent the same element of $\mathbf{A}^{\Delta}$, i.e. $a\cdot\overline{b} = (\beta(a,b)\to\alpha(a,b))\cdot\overline{a\to b}$, comes down to according to Lemma 3.7. □

Unlike general transformation functions, normal transformation functions are always unique if they exist. In a variety, (normal) transformation functions are always witnessed by certain terms.

Definition 4.22 *(Normal transformation terms).* Let K be a class of commutative residuated (ℓ-)bimonoids. A pair of *(normal) transformation terms* for K is a pair of terms $t(x,y)$ and $u(x,y)$ in the language of commutative residuated (ℓ-)bimonoids such that their interpretation on each $\mathbf{A}\in\mathsf{K}$ is a pair of (normal) transformation functions for $\mathbf{A}$.

Fact 4.23. *Let K be an ordered variety of commutative residuated bimonoids (a variety of commutative residuated ℓ-bimonoids). Every $\mathbf{A}\in\mathsf{K}$ has a pair of (normal) transformation functions if and only if K has a pair of (normal) transformation terms.*

Proof. The right-to-left implication is trivial. Conversely, let $\mathbf{F}$ be a K-free (ℓ-)bimonoid over 6 or more generators, among them x and y. Then $\mathbf{F}$ has certain (normal) transformation functions α, β. Applying them to x and y yields elements $\alpha(x,y), \beta(x,y)\in\mathbf{F}$.

These can be obtained by applying certain terms t and u to x and y: $\alpha(x, y) = t(x, y)$ and $\beta(x, y) = u(x, y)$. Because α and β are (normal) transformation functions, these satisfy the inequalities of Fact 4.5 (and Fact 4.21) if we take a, b, p, q to be some of the other generators of $\mathbf{F}$. Since these inequalities hold in the K-free algebra $\mathbf{F}$, they hold in every algebra of K, therefore t and u are (normal) transformation terms. □

We now show how to explicitly construct a commutative complemented bimonoid of fractions of $\mathbf{A}$ as a bimonoid on the set of all normal pairs in $\mathbf{A}^2$ using a pair of normal transformation functions α and β.

Theorem 4.24 *(Constructing normal complemented bimonoids of fractions). Let $\mathbf{A}$ be a commutative residuated bimonoid with normal transformation functions α and β. We define $\mathbf{A}^{\div}$ to be the ordered algebra over the set of all normal pairs*

$$\{\langle a, b\rangle_{\bullet} \in A^2 \mid \langle a, b\rangle_{\bullet} = \pi\langle a, b\rangle_{\bullet}\} = \{\langle \beta(a, b) \to \alpha(a, b), a \to b\rangle_{\bullet} \mid a, b \in \mathbf{A}\}$$

with the operations

$$\begin{aligned} \overline{\langle a, b\rangle_{\bullet}} &:= \langle b, a\rangle_{\bullet}, \\ \langle a, b\rangle_{\bullet} \circ \langle c, d\rangle_{\bullet} &:= \pi\langle a \cdot c, b + d\rangle_{\bullet}, \\ \langle a, b\rangle_{\bullet} \oplus \langle c, d\rangle_{\bullet} &:= \overline{\pi\langle b \cdot d, a + c\rangle_{\bullet}}, \end{aligned}$$

the constants $1^{\div} := \pi\langle 1, 0\rangle_{\bullet}$ and $0^{\div} := \pi\langle 0, 0\rangle_{\bullet}$, and the partial order

$$\langle a, b\rangle_{\bullet} \preccurlyeq \langle c, d\rangle_{\bullet} \iff a \leq c \text{ and } d \leq b.$$

If $\mathbf{A}$ is an ℓ-bimonoid, then we also equip $\mathbf{A}^{\div}$ with the operations

$$\begin{aligned} \langle a, b\rangle_{\bullet} \vee \langle c, d\rangle_{\bullet} &:= \pi\langle a \vee c, b \wedge d\rangle_{\bullet}, \\ \langle a, b\rangle_{\bullet} \wedge \langle c, d\rangle_{\bullet} &:= \overline{\pi\langle b \wedge d, a \vee c\rangle_{\bullet}}. \end{aligned}$$

Then $\mathbf{A}^{\div}$ is a normal commutative complemented (ℓ-)bimonoid of fractions of $\mathbf{A}$ relative to the embedding $\iota^{\div} : a \mapsto \pi\langle a, 0\rangle_{\bullet}$. Moreover, $\iota^{\div}(a) \circ \overline{\iota^{\div}(b)} = \pi\langle a, b\rangle_{\bullet}$.

Proof. We have already observed (Fact 4.18) that if $\langle a, b\rangle_{\bullet}$ is normal, then so is $\langle b, a\rangle_{\bullet}$, hence the operations of $\mathbf{A}^{\div}$ indeed yield normal pairs. Because $\mathbf{A}$ has a pair of normal transformation functions, the sub-bimonoid $\mathbf{B}$ of $\mathbf{A}^{\Delta}$ consisting of elements of the form $a\overline{b}$ is a normal commutative complemented bimonoid of fractions of $\mathbf{A}$ (Proposition 4.4). Now let us consider the map $\varepsilon \colon \mathbf{A}^{\div} \to \mathbf{B}$ such that $\varepsilon \colon \langle a, b\rangle_{\bullet} \mapsto a\overline{b} \in \mathbf{B}$.

This map is surjective, since each $x \in \mathbf{B}$ has a normal representation. By Lemma 4.19, $\langle a, b\rangle_{\bullet} \preccurlyeq \langle c, d\rangle_{\bullet}$ if and only if $\varepsilon\langle a, b\rangle_{\bullet} \leq \varepsilon\langle c, d\rangle_{\bullet}$. Observe that $\varepsilon(\pi\langle a, b\rangle_{\bullet}) = a\overline{b}$:

$\varepsilon(\pi\langle a,b\rangle_\bullet) = \varepsilon\langle\beta(a,b) \to \alpha(a,b), a \to b\rangle_\bullet = (\beta(a,b) \to \alpha(a,b)) \cdot \overline{a \to b} = a\overline{b}$ by Fact 4.20. The map ε is therefore a homomorphism: $\varepsilon(1^{\div}) = \varepsilon(\pi\langle 1,0\rangle_\bullet) = 1\overline{0} = 1$ and $\varepsilon(0^{\div}) = \varepsilon(\pi\langle 0,0\rangle_\bullet) = 0\overline{0} = 0$ for the two units, $\varepsilon(\langle a,b\rangle_\bullet \circ \langle c,d\rangle_\bullet) = \varepsilon(\pi\langle a\cdot c, b+d\rangle_\bullet) = ac \cdot \overline{b+d} = a\overline{b} \cdot c\overline{d} = \varepsilon\langle a,b\rangle_\bullet \cdot \varepsilon\langle c,d\rangle_\bullet$ for multiplication, and

$$\begin{aligned}
\varepsilon\langle a,b\rangle_\bullet + \varepsilon\langle c,d\rangle_\bullet &= a\overline{b} + c\overline{d} = a + \overline{b} + c + \overline{d} \\
&= (a+c) + \overline{(b\cdot d)} \\
&= ((b\cdot d) \to (a+c)) \cdot \overline{(\beta(b\cdot d, a+c) \to \alpha(b\cdot d, a+c))} \\
&= \varepsilon(\langle (b\cdot d) \to (a+c), \beta(b\cdot d, a+c) \to \alpha(b\cdot d, a+c)\rangle_\bullet) \\
&= \varepsilon(\langle a,b\rangle_\bullet \oplus \langle c,d\rangle_\bullet).
\end{aligned}$$

Here we used Fact 4.20 and the fact that for normal pairs $\langle a,b\rangle_\bullet$ we have $a\cdot\overline{b} = a+\overline{b}$ in $\mathbf{A}^{\Delta}$. It follows that the map $\varepsilon\colon \mathbf{A}^{\div} \to \mathbf{B}$ is an isomorphism of bimonoids. The complement of $\langle a,b\rangle_\bullet$ in $\mathbf{A}^{\div}$ is therefore $\varepsilon^{-1}(\overline{\varepsilon(\langle a,b\rangle_\bullet)}) = \varepsilon^{-1}(b+\overline{a}) = \varepsilon^{-1}(b\overline{a}) = \langle b,a\rangle_\bullet$.

Observe that $\varepsilon^{-1}(a)$ is by definition the unique normal representation of $a \in \mathbf{B}$, that is, $\pi\langle a,b\rangle_\bullet$. Since $\mathbf{B}$ is a commutative complemented bimonoid of fractions of $\mathbf{A}$ relative to the inclusion of $\mathbf{A}$ into $\mathbf{B}$ and ε is an isomorphism between $\mathbf{A}^{\div}$ and $\mathbf{B}$, this implies that $\mathbf{A}^{\div}$ is a commutative complemented bimonoid of fractions of $\mathbf{A}$ relative to the map $a \mapsto \pi\langle a,0\rangle_\bullet$. Finally, $\varepsilon(\iota^{\div}(a) \circ \overline{\iota^{\div}(b)}) = \varepsilon(\iota^{\div}(a)) \cdot \overline{\varepsilon(\iota^{\div}(b))} = a\overline{b} = \varepsilon(\pi\langle a,b\rangle_\bullet)$, so $\iota^{\div}(a) \circ \overline{\iota^{\div}(b)} = \pi\langle a,b\rangle_\bullet$. □

Corollary 4.25. *Let* $\mathbf{A}$ *be* ℓ*-bimonoid with a normal commutative complemented bimonoid of fractions. Then* $\mathbf{A}^{\div}$ *is lattice-ordered. In fact,* $\mathbf{A}^{\div}$ *is a sublattice of* $\mathbf{A}^{\Delta}$ *and* $\mathbf{A}^{\Delta}$ *is the DM-completion of* $\mathbf{A}^{\div}$*. In particular, if* $\mathbf{A}$ *is moreover finite, then* $\mathbf{A}^{\Delta} = \mathbf{A}^{\div}$*.*

If K is a class of commutative residuated (ℓ-)bimonoids with normal transformation terms, let $\mathsf{K}^{\div}$ be the class of all commutative complemented (ℓ-)bimonoids $\mathbf{A}^{\div}$ equipped with the unary operation $\sigma^{\div}$ for $\mathbf{A} \in \mathsf{K}$:

$$\mathsf{K}^{\div} := \{\langle \mathbf{A}^{\div}, \sigma^{\div}\rangle \mid \mathbf{A} \in \mathsf{K}\}.$$

The class of all structures isomorphic to those in $\mathsf{K}^{\div}$ will be denoted $\mathbb{I}(\mathsf{K}^{\div})$. This is exactly the class of all (ℓ-)bimonoids of fractions of (ℓ-)bimonoids in K, equipped with $\sigma^{\div}$.

Definition 4.26 *(Normal complemented bimonoids with an interior operator).* A commutative complemented (ℓ-)bimonoid equipped with an interior operator $\langle \mathbf{A}, \sigma\rangle$, and by extension the interior operator σ itself, is called *normal* if the image of σ is a sub-(ℓ-)bimonoid $\sigma[\mathbf{A}]$ of $\mathbf{A}$ and moreover $a = \sigma(a) \cdot \sigma(\overline{a})$ for each $a \in \mathbf{A}$.

Fact 4.27. *If* K *is an ordered variety (a variety) of commutative residuated (*ℓ*-)bimonoids with normal bimonoids of fractions, then* $\mathbb{I}(\mathsf{K}^{\div})$ *is an ordered subvariety (a subvariety)* *of the ordered variety (the variety) of commutative complemented (ℓ-)bimonoids with a normal interior operator.*

Proof. The condition of being a normal interior operator is definable by a set of inequalities. Now suppose that K is axiomatized relative to the class of commutative residuated (ℓ-)bimonoids with normal bimonoids of fractions by the bimonoidal inequalities $t_i(x_1, \dots, x_n) \leq u_i(x_1, \dots, x_n)$ for $i \in I$. Then $\mathbb{I}(\mathsf{K}^{\div})$ is the ordered subvariety of the ordered variety of normal commutative complemented bimonoids with an interior operator axiomatized by $t_i(\sigma(x_1), \dots, \sigma(x_n)) \leq u_i(\sigma(x_1), \dots, \sigma(x_n))$ for $i \in I$. The proof for ℓ-bimonoids is identical. □

We now define the functor $\div\colon \mathsf{K} \to \mathbb{I}(\mathsf{K}^{\div})$ as $\div(\mathbf{A}) := \langle \mathbf{A}^{\div}, \sigma^{\div} \rangle$ on objects and for each homomorphism $h\colon \mathbf{A} \to \mathbf{B}$ in K we take $\div(h)\colon \div(\mathbf{A}) \to \div(\mathbf{B})$ to be the map $h^{\div}\colon \langle a, b \rangle \mapsto \langle h(a), h(b) \rangle$. (This is map is well-defined because being a normal pair is defined by an equational condition and it is a homomorphism because the operations of $\mathbf{A}^{\div}$ and $\mathbf{B}^{\div}$ correspond to pairs of (ℓ-)bimonoidal terms.)

Conversely, there is a functor Σ which to each commutative residuated (ℓ-)bimonoid $\mathbf{A}$ with a normal interior operator σ assigns the (ℓ-)bimonoid $\sigma[\mathbf{A}]$ and to each homomorphism $h\colon \langle \mathbf{A}, \sigma \rangle \to \langle \mathbf{B}, \tau \rangle$ of such structures assigns its restriction to $\sigma[\mathbf{A}]$. Note that the functor Σ is defined on the class of all normal commutative complemented (ℓ-)bimonoids of fractions of commutative residuated (ℓ-)bimonoids, while the functor $\div$ is only defined on the class of all commutative residuated (ℓ-)bimonoids which have commutative complemented bimonoids of fractions witnessed by a specific pair of transformation terms.

Theorem 4.28 *(Categorical equivalences for bimonoids of fractions). Let K be an ordered variety of commutative residuated bimonoids with normal commutative complemented bimonoids of fractions. Then the functors $\div\colon \mathsf{K} \to \mathbb{I}(\mathsf{K}^{\div})$ and $\Sigma\colon \mathbb{I}(\mathsf{K}^{\div}) \to \mathsf{K}$ form a categorical equivalence between K and $\mathbb{I}(\mathsf{K}^{\div})$, the unit being the map $a \mapsto \pi\langle a, 0 \rangle_{\bullet}$ and the counit being the map $\langle a, b \rangle_{\bullet} \mapsto a\overline{b}$.*

Proof. Let K be axiomatized by the bimonoidal inequalities $t_i(x_1, \dots, x_n) \leq u_i(x_1, \dots, x_n)$ for $i \in I$. Then $\mathbb{I}(\mathsf{K})$ is the ordered subvariety of the ordered variety of normal commutative complemented bimonoids with an interior operator axiomatized by $t_i(\sigma(x_1), \dots, \sigma(x_n) \leq u_i(\sigma(x_1), \dots, \sigma(x_n))$ for $i \in I$. By Fact 4.23 there are normal transformation terms for K, therefore $\div\colon \mathsf{K} \to \mathbb{I}(\mathsf{K}^{\div})$ is indeed a functor. We already know that Σ is a functor. It now suffices to provide natural isomorphisms between $\mathbf{A}$ and $\Sigma(\div(\mathbf{A}))$ for $\mathbf{A} \in \mathsf{K}$ and between $\div(\Sigma(\langle \mathbf{B}, \sigma \rangle))$ and $\langle \mathbf{B}, \sigma \rangle$ for $\langle \mathbf{B}, \sigma \rangle \in \mathbb{I}(\mathsf{K}^{\div})$.

Let $\mathbf{A} \in \mathsf{K}$. The map $a \mapsto \pi\langle a, 0 \rangle_{\bullet}$ is a bimonoidal embedding of $\mathbf{A}$ into $\mathbf{A}^{\div}$ by Theorem 4.24. Its image coincides with the image of $\sigma^{\div}$, i.e. with $\mathbf{A}$ as a sub-bimonoid of $\mathbf{A}^{\div}$, therefore this map is a (natural) isomorphism between the bimonoids $\mathbf{A}$ and $\sigma^{\div}[\mathbf{A}^{\div}]$. On the other hand, each element of $\mathbb{I}(\mathsf{K}^{\div})$ is a normal commutative complemented bimonoid

$\mathbf{B}$ with an interior operator σ such that $\sigma[\mathbf{B}] \in \mathsf{K}$. Let $\mathbf{A} := \sigma[\mathbf{B}]$. By Theorem 4.24 the map $\langle a, b\rangle_{\bullet} \mapsto a\overline{b}$ is a (natural) isomorphism between $\sigma[\mathbf{B}]^{\div}$ and $\mathbf{B}$. □

The same proof yields an analogous theorem for varieties of commutative residuated ℓ-bimonoids.

Theorem 4.29 *(Categorical equivalences for ℓ-bimonoids of fractions). Let* K *be a variety of commutative residuated ℓ-bimonoids with normal commutative complemented ℓ-bimonoids of fractions. Then the functors* $\div\colon \mathsf{K} \to \mathbb{I}(\mathsf{K}^{\div})$ *and* $\Sigma\colon \mathbb{I}(\mathsf{K}^{\div}) \to \mathsf{K}$ *form a categorical equivalence between* K *and* $\mathbb{I}(\mathsf{K}^{\div})$*, the unit being the map* $a \mapsto \pi\langle a, 0\rangle_{\bullet}$ *and the counit being the map* $\langle a, b\rangle_{\bullet} \mapsto x$*.*

4.3. Applications

We now apply the theory developed in the previous section to obtain uniform proofs of some new categorical equivalences, as well as alternative proofs of some known ones. We first use Theorem 4.24 to show that certain varieties K of commutative residuated (ℓ-)bimonoids have normal complemented (ℓ-)bimonoids of fractions. Then in each case we find an intrinsic inequational description of $\mathbb{I}(\mathsf{K}^{\div})$. Finally, this will yield a categorical equivalence between certain ordered varieties (varieties) by Theorem 4.28 (Theorem 4.29).

We consider three cases: Brouwerian semilattices, Boolean-pointed Brouwerian algebras, and a certain ordered variety of cancellative residuated pomonoids. We already saw that all of these classes of commutative residuated bimonoids have transformation terms (Facts 4.8, 4.9, and 4.12). Moreover, in all three cases we can in fact use the *same* transformation terms:

$$a \cdot \overline{b} = 0a + \overline{a \to ab}.$$

This is because in the first two cases $a \to ab = a \to b$, in the first and last case $0a = a$, and in the last case $a \to ab = b$. We now show that these transformation terms are normal. This amounts to verifying the following equality in each of these classes (Fact 4.21):

$$a \to (b + x) \approx ((a \to ab) \to 0a) \to ((a \to b) + x).$$

Fact 4.30. *Each Brouwerian semilattice has a normal commutative complemented bimonoid of fractions with normal transformation functions* $\alpha(a, b) = a$ *and* $\beta(a, b) = a \to b$*.*

Proof. It suffices to verify that $a \to bx = ((a \to b) \to a) \to (a \to b)x$. This is routine. □

Fact 4.31. *Each Boolean-pointed Brouwerian algebra has a normal commutative complemented bimonoid of fractions with normal transformation functions* $\alpha(a, b) = 0a$ *and* $\beta(a, b) = a \to b$*.*

Proof. We need to show that $a \to (b+x) = ((a \to b) \to 0a) \to ((a \to b) + x)$. Let l and r be the left- and the right-hand side of this equation. By Lemma 2.23 it suffices to show that $a \to bx = ((a \to b) \to a) \to (a \to b)x$ holds in all Brouwerian algebras and $l \vee 0 = r \vee 0$ holds in all Boolean-pointed Brouwerian algebras. The former inequality is routine to prove. The inequality $l \leq r \vee 0$ is equivalent to the conjunction of the inequalities

$$(0a \to bx)(a \to (b \vee x))((a \to b) \to 0a) \leq (0 \to (a \to b)x) \vee 0,$$
$$(0a \to bx)(a \to (b \vee x))((a \to b) \to 0a) \leq (a \to b) \vee x \vee 0.$$

The first is routine and the second is equivalent to $(a \to (b+x))((a \to b) \to 0a)((a \to b) \to 0)(x \to 0) \leq 0$, since $y \vee 0 = (y \to 0) \to 0$ in Boolean-pointed Brouwerian algebras. We have $(a \to b) \to 0a \leq b \to 0$ and $(b \to 0)(a \to (b \vee x)) \leq a \to (0 \vee x)$ and $(x \to 0)(a \to (0 \vee x)) \leq a \to 0$. But $(a \to 0)(0a \to bx) \leq a \to b$ and $(a \to b)((a \to b) \to 0) \leq 0$.

It remains to prove that $r \leq l \vee 0$. This is equivalent to $(b \to 0) \to (b+x) \leq (b+x) \vee 0$, i.e. to the conjunction of $((b \to 0) \to (0 \to bx)(b \vee x)) \leq (0 \to bx) \vee 0$ and $((b \to 0) \to (0 \to bx)(b \vee x)) \leq b \vee x \vee 0$. The first inequality is routine, and the second is equivalent to $((b \to 0) \to (0 \to bx)(b \vee x))(b \to 0)(x \to 0) \leq 0$. But $((b \to 0) \to (0 \to bx)(b \vee x))(b \to 0)(x \to 0) \leq (b \vee x)(b \to 0)(x \to 0) \leq 0$. □

Finally, while every commutative order-cancellative pomonoid (viewed as a bimonoid with $x + y := x \cdot y$ and $0 := 1$) has a commutative complemented bimonoid of fractions, namely its partially ordered group of fractions, to obtain a normal bimonoid of fractions we need to restrict to a certain subclass of order-cancellative pomonoids. A residuated pomonoid is called *divisible* if it satisfies the implication

$$y \leq x \implies x \cdot (x \backslash y) \approx y \approx (y/x) \cdot x.$$

An integral residuated pomonoid is divisible if and only if $x \cdot (x \backslash y) = x \wedge y = (y/x) \cdot x$ for all x and y. This condition can be expressed by a set of inequalities, so divisible integral residuated pomonoids form an ordered subvariety of the ordered variety of integral residuated pomonoids. In this context, $x \wedge y$ may thus be treated as an abbreviation for $x \cdot (x \backslash y) = (y/x) \cdot x$. Each divisible integral residuated pomonoid satisfies the equations $x(y \wedge z) \approx xy \wedge xz$ and $(x \wedge y)z \approx xz \wedge yz$ (see [19]). It follows that a divisible integral residuated pomonoid is order-cancellative if and only if it is cancellative.

Fact 4.32. *Each cancellative divisible integral commutative residuated pomonoid has a normal commutative complemented bimonoid of fractions with normal transformation functions $\alpha(a,b) = a$ and $\beta(a,b) = b$.*

Proof. We need to verify that $a \to bx = (b \to a) \to ((a \to b)x)$:

$$a(b \to a)(a \to bx) = (b \to a)(a \wedge bx)$$

$$
\begin{aligned}
&= (b \to a)a \wedge (b \to a)bx \\
&= (b \to a)a \wedge (a \wedge b)x \\
&= (b \to a)a \wedge a(a \to b)x \\
&= a((b \to a) \wedge (a \to b)x) \\
&= a(b \to a)((b \to a) \to (a \to b)x).
\end{aligned}
$$

The required equation now follows by cancellativity. □

Up to term equivalence these pomonoids in fact form a variety of residuated lattices known as integral cancellative commutative GMV-algebras (see [19]).

Let us now provide an intrinsic inequational description of the complemented bimonoids of fractions obtained from the above three classes of (ℓ-)bimonoids.

Fact 4.33. *The commutative complemented bimonoids of fractions of Brouwerian semilattices are precisely the idempotent involutive commutative residuated pomonoids equipped with the map* $\sigma(x) := 1 \wedge x$ *which satisfy* $x \approx \sigma(x) \cdot \overline{\sigma(\overline{x})}$.

Proof. Let $\mathbf{A}$ be a Brouwerian semilattice. Then $\mathbf{A}^{\div}$ is idempotent: $a\overline{b}a\overline{b} = aa \cdot \overline{b + b} = a\overline{b}$. To show that $\sigma^{\div}(x) = 1 \wedge x$, it suffices to prove that the bimonoidal embedding $a \mapsto \pi\langle a, 0\rangle_{\bullet} = \langle a, 1\rangle_{\bullet}$ maps $\mathbf{A}$ onto the negative cone of $\mathbf{A}^{\div}$, which is the principal filter generated by $\langle 1, 1\rangle_{\bullet}$. But $\langle a, b\rangle_{\bullet} \leq \langle 1, 1\rangle_{\bullet}$ if and only if $a \leq 1$ and $1 \leq b$, or equivalently if and only if $b = 1$. Such pairs by definition form the image of $a \mapsto \langle a, 1\rangle_{\bullet}$.

Conversely, let $\mathbf{B}$ be an idempotent involutive commutative residuated pomonoid equipped with the map $\sigma(x) := 1 \wedge x$ which satisfies $x \approx \sigma(x) \cdot \overline{\sigma(\overline{x})}$. This equation states that $\mathbf{B}$ is a commutative bimonoid of fractions of its negative cone, which is an integral idempotent residuated pomonoid with respect to both multiplication and addition, since $0 = 1$. □

Fact 4.34. *The commutative complemented bimonoids of fractions of Boolean-pointed Brouwerian algebras are precisely the idempotent involutive commutative residuated lattices which satisfy* $x \approx (1 \wedge x)(0 \vee x)$, *expanded by* $\sigma(x) := 1 \wedge x$.

Proof. Let $\mathbf{A}$ be a Boolean-pointed Brouwerian algebra. Then $\mathbf{A}^{\div}$ is again idempotent. To show that $\sigma^{\div}(x) = 1 \wedge x$, it suffices to prove that the bimonoidal embedding $a \mapsto \pi\langle a, 0\rangle_{\bullet}$ maps $\mathbf{A}$ onto the negative cone of $\mathbf{A}^{\div}$, which is the principal filter generated by $\langle 1, 0\rangle_{\bullet}$. The into part of this claim follows from the integrality of $\mathbf{A}$. To prove the onto part, suppose that $\langle a, b\rangle_{\bullet} \leq \langle 1, 0\rangle_{\bullet}$, i.e. $a \leq 1$ and $0 \leq b$. Since the interval $[0, 1]$ is Boolean, $\overline{b} \in \mathbf{A}^{\Delta}$ is in fact an element of $\mathbf{A}$ (which is a sub-bimonoid of $\mathbf{A}^{\Delta}$), say $\overline{b} = c \in \mathbf{A}$. But then $a\overline{b} = ac \in \mathbf{A}$, so $\langle a, b\rangle_{\bullet}$ represents the element $ac \in \mathbf{A}$. That is, $\langle a, b\rangle_{\bullet} = \pi\langle ac, 0\rangle_{\bullet}$. The equation $x \approx (1 \wedge x)(0 \vee x)$ now follows from the fact that $x = \sigma^{\div}(x) \cdot \overline{\sigma^{\div}(\overline{x})}$.

Conversely, let $\mathbf{B}$ be an idempotent involutive commutative residuated lattice satisfying $x \approx (1 \wedge x)(0 \vee x)$. This equation states precisely that $\mathbf{B}$ is a normal bimonoid of fractions of its negative cone. Moreover, $1 = (1 \wedge 1)(1 \vee 0) = (1 \vee 0)$, so 0 lies in the negative cone. The negative cone of $\mathbf{B}$ is closed under multiplication because $1 + 1 = 1$, it is therefore an integral idempotent residuated ℓ-bimonoid, i.e. a pointed Brouwerian algebra. The interval $[0, 1]$ of $\mathbf{B}$ is then a complemented bi-integral idempotent bimonoid, i.e. a Boolean lattice. □

Our construction of complemented bimonoids of fractions of Boolean-pointed Brouwerian algebras extends a construction of Fussner & Galatos [13] for semilinear Boolean-pointed Brouwerian algebras. Recall that a (Boolean-pointed) Brouwerian algebra is *semilinear* if it is a subdirect product of (Boolean-pointed) Brouwerian chains, or equivalently if it satisfies the equation $(x \to y) \vee (y \to x) \approx 1$. Such algebras are called *relative Stone algebras* in [13]. In our terminology, Fussner & Galatos prove the following fact.

Fact 4.35. *The commutative complemented bimonoids of fractions of semilinear Boolean-pointed Brouwerian algebras are precisely Sugihara monoids (distributive commutative idempotent involutive residuated lattices).*

For cancellative commutative residuated pomonoids, we obtain precisely (the restriction to the commutative case of) the result of Montagna & Tsinakis [26]. Recall that a conucleus on a pomonoid $\mathbf{A}$ is an interior operator σ such that $\sigma(x) \cdot \sigma(y) \leq \sigma(x \cdot y)$ and $\sigma(1) = 1$. The image $\sigma[\mathbf{A}]$ of a conucleus is therefore a submonoid of $\mathbf{A}$. Moreover, if $\mathbf{A}$ is a residuated pomonoid, then so is $\sigma[\mathbf{A}]$, the residuals in $\sigma[\mathbf{A}]$ being the σ-images of residuals in $\mathbf{A}$: $a \backslash_{\sigma[\mathbf{A}]} b = \sigma(a \backslash_{\mathbf{A}} b)$ and $a /_{\sigma[\mathbf{A}]} b = \sigma(a /_{\mathbf{A}} b)$.

Fact 4.36. *The commutative complemented bimonoids of fractions of cancellative commutative residuated pomonoids (residuated lattices) are precisely Abelian pogroups (ℓ-groups) with a conucleus σ which are groups of fractions of the image of σ.*

Proof. We have already verified that the commutative complemented bimonoid of fractions is a partially ordered Abelian group in this case (Fact 4.12). We only need to verify that $\sigma^{\div}(x) \cdot \sigma^{\div}(y) \leq \sigma^{\div}(x \cdot y)$:

$$\sigma^{\div}(a\overline{b}) \cdot \sigma^{\div}(c\overline{d}) = (b \to a)(d \to c) \leq bd \to ac = \sigma^{\div}(ac \cdot \overline{bd}) = \sigma^{\div}(ac \cdot \overline{b + d})$$
$$= \sigma^{\div}(a\overline{b} \cdot c\overline{d}). \quad \square$$

The reader will immediately observe that this result has a less satisfactory form than the previous ones for Brouwerian semilattices and Boolean-pointed algebra. This is because complemented bimonoids of fractions of cancellative commutative residuated pomonoids need not be normal. One need not consider any exotic examples to see this: the complemented bimonoid of fractions of an Abelian ℓ-group $\mathbf{G}$ is of course $\mathbf{G}$ itself with the identity map as σ. But this is never a normal bimonoid of fractions, unless $\mathbf{G}$ is trivial: the normality equation $x \approx \sigma(x) \cdot \overline{\sigma(\overline{x})}$ becomes $x \approx x \cdot x$. To obtain normal groups of fractions, we must restrict to the following case.

Fact 4.37. *Let* $\mathbf{G}$ *be an Abelian* ℓ*-group equipped with a conucleus* σ*. Then* $\mathbf{G}$ *is a normal commutative complemented bimonoid of fractions of* $\sigma[\mathbf{G}]$ *with respect to the inclusion in* $\mathbf{G}$ *if and only if* $\sigma(x) = 1 \wedge x$*.*

Proof. Every ℓ-group satisfies the equation $x \approx (1 \wedge x)(1 \wedge x^{-1})^{-1}$. Conversely, the normality condition states that $ab^{-1} = (b \to a)(a \to b)^{-1}$, i.e. $a(a \to b) = b(b \to a)$, for $a, b \in \sigma[\mathbf{G}]$, where the residuals are taken in $\sigma[\mathbf{G}]$. But then $b(b \to a) \leq b$, so $b \to a \leq b \to b$ and taking $b := 1$ yields that $a \leq 1$ for each $a \in \sigma[\mathbf{G}]$. Conversely, consider some $x = ab^{-1} \leq 1$. Then $x = \sigma(x)\sigma(x^{-1})^{-1} = (b \to a)(a \to b)^{-1}$. But $ab^{-1} \leq 1$ implies $a \leq b$, so $a \to b = 1$ in $\sigma[\mathbf{G}]$. Thus $x = b \to a \in \sigma[\mathbf{G}]$. This proves that $\sigma[\mathbf{G}]$ is precisely the negative cone of $\mathbf{G}$. The claim that $\sigma(x) = 1 \wedge x$ now follows. □

The negative cones of ℓ-groups, images of ℓ-groups with respect to the conucleus $\sigma(x) = 1 \wedge x$, were already described by Bahls et al. [2] as integral cancellative divisible residuated lattices. Recall that an integral residuated lattice is *divisible* if it satisfies the equations $x \cdot (x \backslash y) = x \wedge y = (y/x) \cdot x$.

Fact 4.38. *The commutative complemented bimonoids of fractions of cancellative integral divisible commutative residuated lattices are precisely Abelian* ℓ*-groups equipped with the map* $\sigma(x) := 1 \wedge x$*.*

Proof. By Fact 4.32 such bimonoids of fractions are normal complemented bimonoids of fractions and moreover they are ℓ-groups. By Fact 4.37 normality implies that $\sigma(x) = 1 \wedge x$. Conversely, one merely verifies the negative cone of each Abelian ℓ-group is indeed a cancellative and divisible residuated lattice. □

Since in each case the interior operator $\sigma^{\div}$ was simply the projection onto the negative cone, we immediately obtain the following corollaries. The last one was already proved in [2].

Corollary 4.39 *(Brouwerian semilattices as negative cones). Brouwerian semilattices are precisely the negative cones of idempotent involutive commutative residuated pomonoids satisfying* $0 \approx 1$ *where* $1 \wedge x$ *exists for each* x*.*

Corollary 4.40 *(Boolean-pointed Brouwerian algebras as negative cones). Boolean-pointed Brouwerian algebras (Brouwerian algebras) are precisely the negative cones of idempotent involutive commutative residuated lattices (satisfying* $0 \approx 1$*).*

Proof. It only remains to observe that the negative cone of an idempotent involutive commutative residuated lattice (satisfying $0 \approx 1$), not necessarily satisfying $x \approx (1 \wedge x) \cdot (0 \vee x)$, is a Boolean-pointed Brouwerian algebra (a Brouwerian algebra). □

Corollary 4.41 *(Negative cones of ℓ-groups). The negative cones of ℓ-groups are precisely the cancellative divisible integral commutative residuated lattices.*

Finally, let us state the categorical equivalences that Theorems 4.28 and 4.29 now yield.

Theorem 4.42 *(Categorical equivalence: Brouwerian semilattices). The variety of Brouwerian semilattices is categorically equivalent to the variety of commutative idempotent involutive residuated pomonoids which satisfy* $0 \approx 1$ *equipped with the map* $\sigma(x) := 1 \wedge x$.

Theorem 4.43 *(Categorical equivalence: Boolean-pointed Brouwerian algebras). The variety of Boolean-pointed Brouwerian algebras (Brouwerian algebras) is categorically equivalent to the variety of commutative idempotent involutive residuated lattices which satisfy (*$0 \approx 1$ *and)* $x \approx (1 \wedge x) \cdot (0 \vee x)$.

Theorem 4.44 *(Categorical equivalence: Abelian ℓ-groups). The variety of Abelian ℓ-groups is categorically equivalent to the variety of cancellative divisible integral commutative residuated lattices.*

In the case of Brouwerian semilattices and Boolean-pointed Brouwerian algebras, one can easily formulate bounded variants of the above results, adding on each side the constant $\bot$ and $\top$ interpreted as the bottom and top element. It is entirely routine to verify that all of our proofs then still go through. Thus we obtain a similar equivalence between Boolean-pointed *Heyting* algebras and *bounded* idempotent involutive residuated lattices which satisfy $x \approx (1 \wedge x) \cdot (0 \vee x)$.

The equivalence for Abelian ℓ-groups is a restriction to the commutative case of the equivalence between cancellative divisible integral residuated lattices and arbitrary ℓ-groups due to Bahls et al. [2]. On the other hand, the equivalence for Boolean-pointed Brouwerian algebras is an extension of the equivalence between semilinear Boolean-pointed Brouwerian algebras and Sugihara monoids due to Fussner & Galatos [13], which in turn extends a previous equivalence between semilinear Brouwerian algebras and odd Sugihara monoids due to Galatos & Raftery [17]. (A Sugihara monoid is called *odd* if it satisfies $0 \approx 1$.)

The equivalences for Boolean-pointed Brouwerian algebras and Abelian ℓ-groups are in fact restrictions of a *single* equivalence between a variety of commutative residuated ℓ-bimonoids and a variety of commutative involutive residuated lattices. The variety of commutative residuated ℓ-bimonoids is axiomatized by the equations stating that $\alpha(a, b) := 0 \cdot a$ and $\beta(a, b) := a \to ab$ are normal transformation functions.

(These equations are described in Facts 4.21 and 4.5.) The variety of commutative involutive residuated lattices is axiomatized by the equation $x \approx (1 \wedge x) \cdot \overline{(1 \wedge \overline{x})}$, i.e. $x \approx (1 \wedge x) \cdot (0 \vee x)$. Similarly, the categorical equivalences for Brouwerian semilattices and Abelian ℓ-groups are restrictions of a single equivalence between an ordered variety of commutative residuated bimonoids and an ordered variety of commutative involutive residuated pomonoids equipped with a normal interior operator σ. The former ordered variety is again axiomatized by the inequalities expressing that $\alpha(a, b) := 0 \cdot a$ and $\beta(a, b) := a \rightarrow ab$ are normal transformation functions. The latter is axiomatized by $x \approx \sigma(x) \cdot \overline{\sigma(x)}$ and by inequalities stating that $\sigma(x)$ is the meet of 1 and x.

This is somewhat remarkable, since Brouwerian algebras and Abelian ℓ-groups in a sense represent opposite ends of the residuated lattice spectrum: ℓ-groups do not contain any non-trivial idempotents, while every element is idempotent in a Brouwerian algebra.

Note that Abelian ℓ-groups play a dual role here: we can either see them as involutive residuated lattices or as involutive residuated pomonoids equipped with a map σ which projects onto the negative cone. These are term equivalent ways of looking at Abelian ℓ-groups, since $x \wedge y = x \cdot \sigma(x^{-1}y)$.

The above categorical equivalences allow us to transfer categorical properties from the well-studied category of Brouwerian (Heyting) algebras to the category (bounded) commutative idempotent involutive residuated lattices which satisfy $0 \approx 1$ and $x \approx (1 \wedge x) \cdot (0 \vee x)$. For example, Maksimova [23] proved that there are exactly three (seven) non-trivial varieties of Brouwerian (Heyting) algebras with the amalgamation property. There are thus also exactly three (seven) non-trivial subvarieties of the above variety of (bounded) involutive residuated lattices which enjoy the amalgamation property, namely those whose negative cone lies in the varieties described by Maksimova.

4.4. Examples of complemented bimonoids of fractions

We now describe some examples of complemented bimonoids of fractions of Boolean-pointed Brouwerian algebras. Consider the smallest non-trivial Brouwerian algebra: the two-element chain $\bot < \mathbf{1}$. Taking $0 := \bot$ yields a bimonoid with $x \cdot y = x \wedge y$ and $x + y = x \vee y$, which is already complemented. Taking $0 := \mathbf{1}$ yields a bimonoid with operations $x \cdot y = x \wedge y = x + y$. This bimonoids has precisely three normal pairs, ordered as follows: $\langle \bot, \mathbf{1} \rangle_\bullet \preccurlyeq \langle \mathbf{1}, \mathbf{1} \rangle_\bullet \preccurlyeq \langle \mathbf{1}, \bot \rangle_\bullet$. The algebras of fractions is thus a linearly ordered commutative idempotent involutive residuated lattice, in other words a Sugihara chain. It is isomorphic to the three-element Sugihara chain with the universe $-1 < 0 < 1$ and the operations $\overline{x} = -x$ and

$$x \cdot y := \begin{cases} x \text{ if } |x| > |y| \text{ or } (|x| = |y| \text{ and } x \leq y), \\ y \text{ if } |x| < |y| \text{ or } (|x| = |y| \text{ and } x \geq y), \end{cases}$$

$$x + y := \begin{cases} x \text{ if } |x| > |y| \text{ or } (|x| = |y| \text{ and } x \geq y), \\ y \text{ if } |x| < |y| \text{ or } (|x| = |y| \text{ and } x \leq y). \end{cases}$$

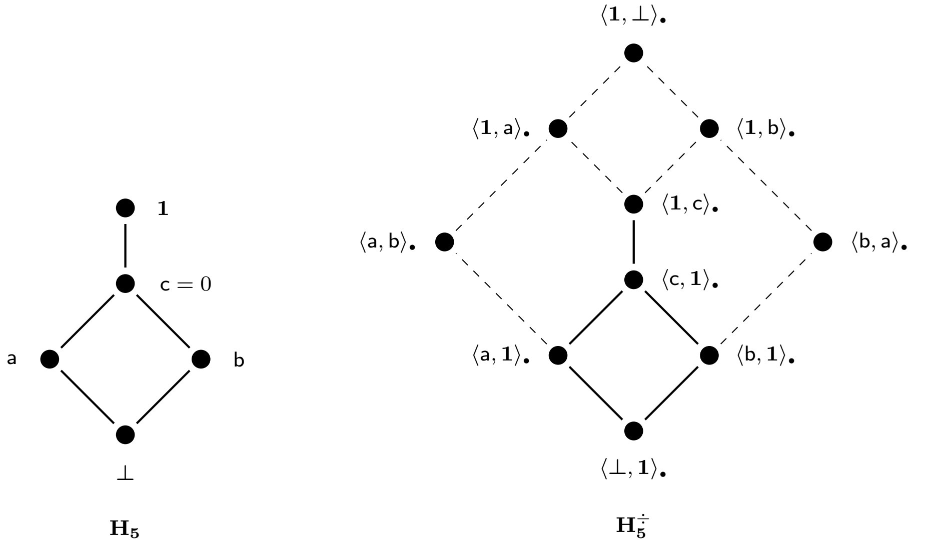

Fig. 4. The Boolean-pointed Brouwerian algebra $\mathbf{H_5}$ and its complemented bimonoid of fractions.

More generally, the algebra of fractions of the n-element Brouwerian chain where 0 is the top element is the Sugihara chain $-n < \cdots < -1 < 0 < 1 < \cdots < n$ with the above operations. The Sugihara chains $-n < \cdots < -1 < 1 < \cdots < n$ with the same operations are obtained as bimonoids of fractions of n-element Brouwerian chains where 0 is the coatom (corresponding to the element -1) rather than the top element. These examples, however, are already covered by the existing constructions of Galatos & Raftery [17,18] and Fussner & Galatos [13] for semilinear (Boolean-pointed) Brouwerian algebras.

The smallest Brouwerian lattice not covered by these existing constructions, i.e. the smallest non-semilinear Brouwerian lattice, is shown in Fig. 4. It can be expanded into a Boolean-pointed Brouwerian lattice in two different ways: either $0 := \mathsf{c}$ or $0 := \mathbf{1}$. Let us consider the first of these expansions. We shall call the resulting commutative residuated ℓ-bimonoid $\mathbf{H_5}$. Recall that addition in $\mathbf{H_5}$ is defined as $x + y := (0 \to xy)(x \vee y)$. In particular, $x + y = x \wedge y$ for $x, y \leq 0$ and $x + y = x \vee y$ for $x, y \geq 0$. The only values not covered by these two clauses are the sums $1 + x$ or $x + 1$ for $x < 0$: in that case $x + 1 = 1 + x = x$.

The elements of $\mathbf{H}_5^{\div}$ are precisely the normal pairs $\langle a, b\rangle_\bullet \in \mathbf{H}_5^2$, i.e. pairs such that $\langle a, b\rangle_\bullet = \pi\langle a, b\rangle_\bullet := \langle \beta(a, b) \to \alpha(a, b), a \to b\rangle_\bullet$, where $\alpha(a, b) = 0a$ and $\beta(a, b) = a \to b$. The first step in describing $\mathbf{H}_5^{\div}$ is therefore to find all pairs $\langle a, b\rangle_\bullet$ such that

$$a = (a \to b) \to 0a,$$
$$b = a \to b.$$

These are the pairs shown in Fig. 4, with $\langle a, b\rangle_\bullet \preccurlyeq \langle c, d\rangle_\bullet$ if and only if $a \leq c$ and $d \leq b$. The question is now how to succinctly describe the operations of $\mathbf{H}_5^{\div}$ on these pairs.

Table 1
Projection onto normal pairs in $\mathbf{H_5}$.

$\pi\langle x,y\rangle_{\bullet}$	$\bot$	a	b	c	$\mathbf{1}$
$\bot$	$\langle\bot,\mathbf{1}\rangle_{\bullet}$	$\langle\bot,\mathbf{1}\rangle_{\bullet}$	$\langle\bot,\mathbf{1}\rangle_{\bullet}$	$\langle\bot,\mathbf{1}\rangle_{\bullet}$	$\langle\bot,\mathbf{1}\rangle_{\bullet}$
a	$\langle\bot,\mathbf{1}\rangle_{\bullet}$	$\langle\mathsf{a},\mathbf{1}\rangle_{\bullet}$	$\langle\mathsf{a},\mathsf{b}\rangle_{\bullet}$	$\langle\mathsf{a},\mathbf{1}\rangle_{\bullet}$	$\langle\mathsf{a},\mathbf{1}\rangle_{\bullet}$
b	$\langle\mathsf{b},\mathsf{a}\rangle_{\bullet}$	$\langle\mathsf{b},\mathsf{a}\rangle_{\bullet}$	$\langle\mathsf{b},\mathbf{1}\rangle_{\bullet}$	$\langle\mathsf{b},\mathbf{1}\rangle_{\bullet}$	$\langle\mathsf{b},\mathbf{1}\rangle_{\bullet}$
c	$\langle\mathbf{1},\bot\rangle_{\bullet}$	$\langle\mathbf{1},\mathsf{a}\rangle_{\bullet}$	$\langle\mathbf{1},\mathsf{b}\rangle_{\bullet}$	$\langle\mathsf{c},\mathbf{1}\rangle_{\bullet}$	$\langle\mathsf{c},\mathbf{1}\rangle_{\bullet}$
$\mathbf{1}$	$\langle\mathbf{1},\bot\rangle_{\bullet}$	$\langle\mathbf{1},\mathsf{a}\rangle_{\bullet}$	$\langle\mathbf{1},\mathsf{b}\rangle_{\bullet}$	$\langle\mathbf{1},\mathsf{c}\rangle_{\bullet}$	$\langle\mathsf{c},\mathbf{1}\rangle_{\bullet}$

The lattice operations, of course, are determined by the order, and complementation is simply the map $\langle a,b\rangle_{\bullet} \mapsto \langle b,a\rangle_{\bullet}$. The monoidal operations are defined in terms of the monoidal operations of $\mathbf{H_5}$ and the projection map π. The information in Table 1 therefore suffices to compute any product or sum in $\mathbf{H}_{\mathbf{5}}^{\div}$. (The vertical axis represents x and the horizontal axis represents y.) For example, $\langle\mathsf{a},\mathsf{b}\rangle_{\bullet} \circ \langle\mathsf{b},\mathsf{a}\rangle_{\bullet} = \pi\langle\mathsf{a}\cdot\mathsf{b},\mathsf{b}+\mathsf{a}\rangle_{\bullet} = \pi\langle\mathsf{a}\wedge\mathsf{b},\mathsf{a}\wedge\mathsf{b}\rangle_{\bullet} = \pi\langle\bot,\bot\rangle_{\bullet} = \langle\bot,\mathbf{1}\rangle_{\bullet}$, while $\langle\mathsf{a},\mathsf{b}\rangle_{\bullet} \oplus \langle\mathbf{1},\mathsf{c}\rangle_{\bullet} = \overline{\pi\langle\mathsf{b}\cdot\mathsf{c},\mathsf{a}+\mathbf{1}\rangle_{\bullet}} = \overline{\pi\langle\mathsf{b},\mathsf{a}\rangle_{\bullet}} = \overline{\langle\mathsf{b},\mathsf{a}\rangle_{\bullet}} = \langle\mathsf{a},\mathsf{b}\rangle_{\bullet}$ and $\langle\mathsf{a},\mathsf{b}\rangle_{\bullet} \circ \langle\mathsf{a},\mathbf{1}\rangle_{\bullet} = \pi\langle\mathsf{a}\cdot\mathsf{c},\mathsf{b}+\mathbf{1}\rangle_{\bullet} = \pi\langle\mathsf{a},\mathsf{b}\rangle_{\bullet} = \langle\mathsf{a},\mathsf{b}\rangle_{\bullet}$.

Recall that the complemented bimonoid of fractions of a semilinear Boolean-pointed Brouwerian algebra is distributive, by the results of Fussner & Galatos [13]. On the other hand, even the simplest non-semilinear Boolean-pointed Brouwerian algebra $\mathbf{H_5}$ has a non-modular complemented bimonoid of fractions $\mathbf{H}_{\mathbf{5}}^{\div}$. It thus seems natural to ask whether this holds for all Boolean-pointed Brouwerian algebras. That is, is the complemented bimonoid of fractions of a Boolean-pointed Brouwerian algebras modular if and only if it is distributive, perhaps by virtue of complemented bimonoids of fractions of Boolean-pointed Brouwerian algebras being semidistributive? We do not know.

We can, however, use the algebra $\mathbf{H_5}$ for another purpose, namely to construct an idempotent involutive residuated lattice in which the equation $x \approx (1\wedge x)\cdot(0\vee x)$ fails. This equation is thus not valid in all commutative idempotent involutive residuated lattices, although it is true in all distributive ones.[8]

The universe of this idempotent involutive residuated lattice is $\{\bot,\mathsf{a},\mathsf{b},\mathsf{c},\mathbf{1}\}$. Multiplication coincides with meets in $\mathbf{H_5}$. Thus e.g. $\mathsf{a}\cdot\mathsf{b} = \bot$. In particular, $\mathbf{1}$ is indeed the multiplicative unit. The lattice structure, however, is the one shown in Fig. 5. Finally, taking $0 := \mathbf{1}$ yields the complementation $\overline{\mathsf{a}} = \mathsf{b}$, $\overline{\mathsf{b}} = \mathsf{a}$, $\overline{\mathbf{1}} = \mathbf{1}$, $\overline{\bot} = \mathsf{c}$, $\overline{\mathsf{c}} = \bot$. It is now routine to verify that this indeed yields a commutative idempotent involutive residuated lattice where $(\mathbf{1}\wedge\mathsf{a})\cdot(\mathbf{1}\vee\mathsf{a}) = \bot \neq \mathsf{a}$. Observe that the negative cone of this residuated lattice is precisely the two-element bimonoid $\bot < \mathbf{1}$ with $x\cdot y = x\wedge y = x+y$. We therefore have two non-isomorphic commutative idempotent involutive residuated lattices with the same negative cone.

Finally, let us show that while being Brouwerian is a sufficient condition for a unital meet semilattice to have a complemented bimonoid of fractions, it is not necessary. Consider the semilattices $\mathbf{2} := \{0,1\}$ and $\boldsymbol{\omega}+\mathbf{1} := \{0,1,2,\dots,\infty\}$ with the usual order. Let $\mathbf{T} := \mathbf{2}\times(\boldsymbol{\omega}+\mathbf{1})$ and let $\mathbf{S}$ be the subalgebra obtained by removing the element

[8] We are grateful to José Gil-Férez for pointing out this example to us.

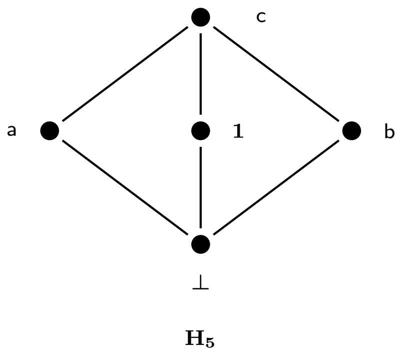

Fig. 5. A commutative idempotent involutive residuated lattice which fails $x \approx (1 \wedge x)(0 \vee x)$.

$\langle 0, \infty \rangle$. Then $\mathbf{T}$ is a Brouwerian semilattice but $\mathbf{S}$ is not: there is no largest $x \in \mathbf{S}$ such that $x \wedge \langle 1, 0 \rangle \leq \langle 0, 0 \rangle$. We show that $\mathbf{S}$ has a commutative complemented bimonoid of fractions. Note that while $\mathbf{S}$ is not a Brouwerian semilattice, it is still distributive.

By Proposition 4.4 it suffices to show that for each $a, b \in \mathbf{S}$ there are $x, y \in \mathbf{S}$ such that $a\overline{b} = x + \overline{y}$ in $\mathbf{S}^{\Delta}$. By Fact 4.6 there are $x, y \in \mathbf{S}$ such that $a \cdot \overline{b} = x + \overline{y}$ if and only if there is $y \in \mathbf{S}$ such that $ay = ab$ and

$$(\forall pq \in \mathbf{S})\, (py \leq ab \ \& \ aq \leq ab \implies pq \leq ab)\,.$$

Moreover, $a \cdot \overline{b} = a \cdot \overline{ab}$, therefore we may assume without loss of generality that $b \leq a$. By Fact 4.8 we know that in $\mathbf{T}$ we may take $y := a \to b$. It follows that for each $a, b \in \mathbf{S}$ such that $a \to b \in \mathbf{S}$ (the element $a \to b$ being computed in $\mathbf{T}$), there is $y \in \mathbf{S}$ such that $a \cdot \overline{b} = a + \overline{y}$. The only cases we have to consider are therefore $a, b \in \mathbf{S}$ with $b \leq a$ such that $a \to b = \langle 0, \infty \rangle$. The only such elements have the form $a = \langle 1, i \rangle$ and $b = \langle 0, i \rangle$ for some $i \in \omega$. But then $y := \langle 0, i+1 \rangle$ does the job: $ay = ab$, and if $p\langle 0, i+1 \rangle \leq \langle 0, i \rangle$ and $\langle 1, i \rangle q \leq \langle 0, i \rangle$, then $p \leq \langle 1, i \rangle$ and $q \leq \langle 0, \infty \rangle$, so $pq \leq \langle 0, i \rangle = ab$.

5. Bimonoidal subreducts of positive universal classes of involutive residuated lattices

In this section, we study the bimonoidal and ℓ-bimonoidal subreducts of positive universal classes of commutative residuated pomonoids and lattices. Throughout the section we use the abbreviations CRL and CRP for commutative residuated lattices and pomonoids. The existence of a commutative complemented DM completion (Theorem 3.21) settles the question of what the bimonoidal and ℓ-bimonoidal subreducts of involutive CRPs and CRLs are: they are precisely commutative bimonoids and ℓ-bimonoids.[9] The same problem, however, arises more generally for other classes of involutive CRLs and CRPs.

[9] Recall that an algebra $\mathbf{A}$ is a *reduct* of an algebra $\mathbf{B}$ if it is obtained from $\mathbf{B}$ by forgetting part of the signature of $\mathbf{B}$. It is a *subreduct* of $\mathbf{B}$ if it embeds into a reduct of $\mathbf{B}$.

In this section, we provide an algorithm for axiomatizing the ℓ-bimonoidal subreducts of each class of involutive CRLs axiomatized by sℓ-monoidal positive universal clauses, i.e. universally quantified finite disjunctions of equations in the signature $\{\vee, \cdot, 1\}$. The first step of this algorithm is to transform a given set of positive universal clauses into positive universal clauses of special form.

Definition 5.1 *(Linear inequalities).* An sℓ-monoidal inequality is *linear* if it has the form $t \leq u$ where t is a *linear term*, i.e. a product of distinct variables, and u is a join of products of variables. An sℓ-monoidal positive universal clause is *linear* if it is a universally quantified finite disjunction of linear sℓ-monoidal inequalities.

The following lemma was essentially proved in [14, p. 1232].

Lemma 5.2 *(Linearization). Each sℓ-monoidal equation (positive universal clause) is equivalent over sℓ-monoids to a set of linear inequalities (linear positive universal clauses).*

For example, the sℓ-monoidal inequality $x^2 \leq x$ is equivalent to the linear sℓ-monoidal inequality $x \cdot y \leq x \vee y$. Recall now that admissible joins are joins over which multiplication distributes (Definition 2.4).

Definition 5.3 *(Preservation of inequalities).* Let $\mathbf{A}$ be an sℓ-monoid. An sℓ-monoidal inequality *holds in* $X \subseteq \mathbf{A}$ if it holds in all valuations where variables take values in X. It is *preserved under products* in $\mathbf{A}$ if it holds in the monoid generated by X whenever it holds in X. It is *preserved under admissible joins* in $\mathbf{A}$ if it holds in the set of all admissible joins of elements of X whenever it holds in X.

Lemma 5.4 *(Preservation of linear inequalities). Linear sℓ-monoidal inequalities are preserved under admissible joins. The inequality $x \leq x^n$ is preserved under products in commutative pomonoids.*

Proof. The latter claim is immediate, since $xy \leq x^n y^n = (xy)^n$ if $x \leq x^n$ and $y \leq y^n$. As for the former, consider a linear sℓ-monoidal inequality $x_1 \cdot \ldots \cdot x_m \leq u(x_1, \ldots, x_m, y_1, \ldots, y_n)$. Suppose that $x_i := \bigvee_{k \in I_i} x_{i,k}$ for $1 \leq i \leq m$ and $x_{i,k} \in X$, and likewise $y_j := \bigvee_{l \in J_j} y_{j,l}$ for $1 \leq j \leq n$ and $y_{j,l} \in X$, where all of these joins are admissible. Then $x_1 \cdot \ldots \cdot x_m = \bigvee \{x_{1,k_1} \cdot \ldots \cdot x_{m,k_m} \mid k_1 \in I_1, \ldots, k_m \in I_m\}$. If the inequality in question holds in X, then $x_{1,k_1} \cdot \ldots \cdot x_{n,k_n} \leq u(x_{1,k_1}, \ldots, x_{m,k_m}, y_{1,l_1}, \ldots, y_{n,l_n})$ for all $k_i \in I_i$ and $l_j \in J_j$. But $x_{i,k_i} \leq x_i$ and $y_{j,l_j} \leq y_j$, therefore $x_{1,k_1} \cdot \ldots \cdot x_{m,k_m} \leq u(x_1, \ldots, x_m, y_1, \ldots, y_n)$ for all $k_i \in I_i$, and $x_1 \cdot \ldots x_m \leq u(x_1, \ldots, x_m, y_1, \ldots, y_n)$. $\square$

We now show how to obtain an axiomatization of the ℓ-bimonoidal subreducts of involutive CRLs which satisfy e.g. the inequality $x \cdot y \leq x \vee y$ obtained by linearizing $x^2 \leq x$. By the previous lemma, this inequality holds in $\mathbf{A}^{\Delta}$ whenever it holds in a join dense subset of $\mathbf{A}^{\Delta}$, in particular whenever it holds for all x and y of the form $a\overline{b}$ for $a, b \in \mathbf{A}$. But by the meet density of elements of the form $e+\overline{f}$ for $e, f \in \mathbf{A}$, the inequality $a\overline{b} \cdot c\overline{d} \leq a\overline{b} \vee c\overline{d}$ is equivalent to the implication

$$a\overline{b} \vee c\overline{d} \leq e + \overline{f} \implies a\overline{b} \cdot c\overline{d} \leq e + \overline{f},$$

or equivalently

$$a\overline{b} \leq e + \overline{f} \;\&\; c\overline{d} \leq e + \overline{f} \implies ac\overline{d + b} \leq e + \overline{f},$$

where a, b, c, d, e, f range over $\mathbf{A}$. Finally, this inequality may be expressed using the residuation laws as

$$a \cdot f \leq b + e \;\&\; c \cdot f \leq d + e \implies a \cdot c \cdot f \leq b + d + e.$$

Thus $\mathbf{A}^{\Delta}$ satisfies $x^2 \leq x$ if and only if $\mathbf{A}$ satisfies the quasiequation above. In particular, each commutative ℓ-bimonoid $\mathbf{A}$ which satisfies this quasiequation is a subreduct of an involutive CRL which satisfies $x^2 \leq x$.

Conversely, if an involutive CRL satisfies the inequality $x \cdot y \leq x \vee y$, then it satisfies the implication

$$x \leq z \;\&\; y \leq z \implies x \cdot y \leq z.$$

In particular, it satisfies

$$a\overline{b} \vee c\overline{d} \leq e + \overline{f} \implies a\overline{b} \cdot c\overline{d} \leq e + \overline{f},$$

and as before, this implication is equivalent to

$$a \cdot f \leq b + e \;\&\; c \cdot f \leq d + e \implies a \cdot c \cdot f \leq b + d + e.$$

This completes the proof of the following fact.

Fact 5.5. *The following are equivalent for each commutative ℓ-bimonoid $\mathbf{A}$:*

(i) $\mathbf{A}$ *is subreduct of an involutive CRL with $x^2 \leq x$,*
(ii) $\mathbf{A}^{\Delta}$ *satisfies $x^2 \leq x$,*
(iii) $\mathbf{A}$ *satisfies the bimonoidal quasiequation $a \cdot f \leq b + e \;\&\; c \cdot f \leq d + e \implies a \cdot c \cdot f \leq b + d + e$.*

The above reasoning works equally well for any set of sℓ-monoidal equations, or indeed any set of sℓ-monoidal positive universal clauses. However, although the procedure itself is straightforward, it would be somewhat cumbersome to try to describe the resulting ℓ-bimonoidal quasiequation or universal clause in full generality. Instead of an explicit proof, let us therefore consider one more example. We axiomatize the bimonoidal subreducts of linear (totally ordered) involutive CRPs.

Linearity is expressed by the positive universal clause

$$x \leq y \text{ or } y \leq x.$$

This positive clause is linear (in a different sense of the word), therefore it holds in $\mathbf{A}^{\Delta}$ whenever it holds in a join dense subset of $\mathbf{A}^{\Delta}$, in particular whenever it holds for all x and y of the form $a\overline{b}$. By the meet density of elements of the form $e + \overline{f}$, the linearity of $\mathbf{A}^{\Delta}$ is equivalent to the following disjunction of universally quantified implications in $\mathbf{A}$:

$$(\forall e, f)(c\overline{d} \leq e + \overline{f} \implies a\overline{b} \leq e + \overline{f}) \text{ or } (\forall e, f)(a\overline{b} \leq e + \overline{f} \implies c\overline{d} \leq e + \overline{f}).$$

Renaming the variables in order to transform the above condition into a universal clause yields the universally quantified sentence

$$(c\overline{d} \leq e + \overline{f} \implies a\overline{b} \leq e + \overline{f}) \text{ or } (a\overline{b} \leq g + \overline{h} \implies c\overline{d} \leq g + \overline{h}),$$

or equivalently the universal clause

$$a\overline{b} \leq g + \overline{h} \ \& \ c\overline{d} \leq e + \overline{f} \implies a\overline{b} \leq e + \overline{f} \text{ or } c\overline{d} \leq g + \overline{h}.$$

Applying the residuation laws now yields the bimonoidal universal clause

$$a \cdot h \leq b + g \ \& \ c \cdot f \leq d + e \implies a \cdot f \leq b + e \text{ or } c \cdot h \leq d + g.$$

The algebra $\mathbf{A}^{\Delta}$ is therefore linear if and only if $\mathbf{A}$ satisfies the universal sentence above. Conversely, if an involutive CRP is linear, then it satisfies the implication

$$(\forall z)(y \leq z \implies x \leq z) \text{ or } (\forall z)(x \leq z \implies y \leq z).$$

Taking $x = a\overline{b}$, $y = c\overline{d}$, $z = e + \overline{f}$, we may now repeat the above reasoning to show that our linear involutive CRP satisfies the desired bimonoidal universal clause. This completes the proof of the following fact.

Fact 5.6. *The following are equivalent for each commutative bimonoid* $\mathbf{A}$*:*

(i) $\mathbf{A}$ *is subreduct of a linear involutive CRP,*
(ii) $\mathbf{A}^{\Delta}$ *is linear,*
(iii) $\mathbf{A}$ *satisfies the universal clause*

$$a \cdot h \leq b + g \ \& \ c \cdot f \leq d + e \implies a \cdot f \leq b + e \textit{ or } c \cdot h \leq d + g.$$

It should be clear enough that the procedure outlined above applies in full generality to any set of pomonoidal or sℓ-monoidal *linear* positive universal clauses. We therefore obtain the following theorems.

Theorem 5.7 *(Subreducts of involutive CRLs). Let* $\mathbf{A}$ *be a commutative* ℓ*-bimonoid and* Π *be a set of positive universal clauses in the signature* $\{\vee, \cdot, 1\}$*. Then the following are equivalent:*

(i) $\mathbf{A}^{\triangle}$ *satisfies* Π*,*
(ii) $\mathbf{A}$ *is a subreduct of an involutive CRL satisfying* Π*,*
(iii) for each $\pi \in \Pi$*,* $\mathbf{A}$ *is a subreduct of an involutive CRL satisfying* π*.*

Theorem 5.8 *(Subreducts of involutive CRPs). Let* $\mathbf{A}$ *be a commutative bimonoid and* Π *be a set of linear positive universal clauses in the signature* $\{\cdot, 1\}$*. Then the following are equivalent:*

(i) $\mathbf{A}^{\triangle}$ *satisfies* Π*,*
(ii) $\mathbf{A}$ *is a subreduct of an involutive CRP satisfying* Π*,*
(iii) for each $\pi \in \Pi$*,* $\mathbf{A}$ *is a subreduct of an involutive CRP satisfying* π*.*

The only difference between the two cases is that for pomonoidal clauses we must assume linearity, while if joins are available each positive universal clause may be linearized by Lemma 5.2. For example, although $x^2 \leq x$ is a perfectly good pomonoidal inequality, the above algorithm does not tell us how to axiomatize the bimonoidal subreducts of involutive residuated pomonoids which satisfy this inequality.

The reader can verify that the universal clause axiomatizing bimonoidal subreducts of linear involutive CRPs fails in the three-element Łukasiewicz chain $\mathbf{1} > \mathsf{a} > \mathsf{b}$ considered in Subsection 3.3 (with $x + y = x \cdot y$). This is witnessed by the following valuation:

$$a := \mathbf{1} \qquad c := \mathsf{a} \qquad e := \mathsf{b} \qquad g := \mathbf{1}$$
$$b := \mathbf{1} \qquad d := \mathsf{b} \qquad f := \mathsf{a} \qquad h := \mathbf{1}$$

More generally, replacing a and b by elements x and y such that $x \nleq y$ but $x^2 \leq y^2$, the same valuation shows that no pomonoid (with $x + y := x \cdot y$ and $0 := 1$) with such elements x and y satisfies the above universal (inequational) clause. In particular, this holds for each pomonoid with a bottom element $\bot$ and a non-trivial nilpotent element, i.e. an a such that $a^n = \bot$ for some n.

In certain cases, the quasiequations obtained by the above procedure can be simplified to inequalities. This is the case with linear inequalities which are preserved under products, in particular with $x \leq x^n$.

Theorem 5.9 *(Subreducts of involutive CRLs with $x \leq x^n$). The ℓ-bimonoidal subreducts of involutive CRLs satisfying a set of inequalities of the form $x \leq x^n$ form a variety of ℓ-bimonoids axiomatized by the corresponding set of inequalities $x \leq x^n$ and $nx \leq x$.*

Proof. The inequality $nx \leq x$ holds in each involutive CRL where $x \leq x^n$ holds. Conversely, recall that the inequality $x \leq x^n$ is preserved under both joins and products. It therefore holds in $\mathbf{A}^{\Delta}$ if and only if it holds for each x of the form a or $\overline{a}$ for $a \in \mathbf{A}$. But $\overline{a} \leq (\overline{a})^n$ if and only if $na \leq a$. □

The same proof yields an analogous result for involutive CRPs.

Theorem 5.10 *(Subreducts of involutive CRPs with $x \leq x^n$). The bimonoidal subreducts of involutive CRPs satisfying a set of inequalities of the forms $x \leq x^n$ form a class of bimonoids axiomatized by the corresponding set of inequalities $x \leq x^n$ and $nx \leq x$.*

6. Open problems

Let us end the paper with a list of unresolved questions which arose in the course of the paper. The main task left open is to construct complemented DM completions for non-commutative bimonoids.

Problem 1. Can we embed an arbitrary (not necessarily commutative) bimonoid into a complemented one? In particular, can we find a non-commutative generalization of complemented DM completions?

We can also consider the same embedding problem for the categorical version of commutative bimonoids, so-called symmetric weakly distributive categories. The role of complemented commutative bimonoids is then played by so-called $*$-autonomous categories (see [4,10]).

Problem 2. Does each (small) symmetric weakly distributive category embed into a $*$-autonomous category?

There is also space for other kinds of complemented envelopes intermediate between bimonoids of fractions and complemented DM completions. Bounded distributive lattices are an example: their most natural complemented envelopes are their free Boolean extensions, where each element is a *finite* join of elements of the form $a \cdot \overline{b}$. One can also consider cases where each element has either the form $a \cdot \overline{b}$ or the form $a + \overline{b}$, as in the case of the algebra $\mathbf{Ł}_3^{\Delta}$ in Subsection 3.3.

Problem 3. Investigate other kinds of complemented Δ_1-extensions of commutative bimonoids which are more general than bimonoids of fractions but more restrictive than complemented DM completions.

Within the variety of bounded distributive lattices, complemented DM completions can be characterized in purely categorical terms: they are precisely the injective hulls in this category. We saw in Subsection 3.1 that this does not hold in the category of commutative ℓ-bimonoids. Nonetheless, it may be the case that this categorical characterization of complemented DM completions at least holds in some wider variety of commutative ℓ-bimonoids than the variety of bounded distributive lattices.

Problem 4. In the variety of bounded distributive lattices, complemented DM completions coincide with injective hulls. Does this extend to some larger variety of commutative ℓ-bimonoids?

We saw in Section 5 that the class of ℓ-bimonoidal subreducts of involutive commutative residuated lattices axiomatized by inequalities of the form $x \leq x^n$ or $1 \leq x^n$ is the variety of commutative ℓ-bimonoids axiomatized by the inequalities $x \leq x^n$ and $nx \leq x$ or $1 \leq x^n$ and $nx \leq 1$. Are there other knotted varieties of involutive commutative residuated lattices, i.e. classes axiomatized by inequalities of the form $x^m \leq x^n$ for $m, n \geq 1$, whose ℓ-bimonoidal subreducts form a variety? The same question can of course be asked for knotted partially ordered varieties of involutive commutative residuated pomonoids.

Problem 5. Is it the case that the only knotted varieties of involutive commutative residuated lattices whose ℓ-bimonoidal subreducts form a variety are axiomatized by inequalities of the form $x \leq x^n$ or $1 \leq x^n$?

Data availability

No data was used for the research described in the article.

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